RL 650

# Diffraction by a Discontinuity in Curvature

by

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Abstract - Using the known asymptotic expansion of the surface field on a body consisting of two half-parabolic cylinders of different latera recta joined at the front when illuminated by a plane electromagnetic wave, uniform expressions for the far field amplitude are obtained for E and H polarizations. The results are accurate to order $k^{-1}$ where $k$ is the wave number, and incorporate the effects of specular reflection as well as diffraction by the discontinuity in curvature.

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In a previous paper [1] a model consisting of two half-parabolic cylinders of different latera recta joined at the front was used to derive an expression for the diffraction coefficient for a line discontinuity in curvature. The result was obtained by expanding the surface field in inverse powers of the wavenumber k and then evaluating the integral expression for the far field over a portion of the surface including the join. Since the objective was to determine only the join contribution, terms attributable to specular scattering off the parabolic cylinders were excluded, leading to a failure of the expression when the join coincides with the specular point. Although this is not a major drawback for most applications of the formula, the case has recently become of interest, e.g. in specifying construction tolerances for low radar cross section shapes, and the purpose of this note is to derive an expression for the diffraction coefficient which is uniform in angle.

We consider a two-dimensional perfectly conducting surface consisting of two half-parabolic cylinders whose equation in Cartesian coordinates (x,y,z) is

$$x = -\frac{1}{2}ay^2$$

where $a = a_2$ $(y > 0)$ and $a = a_1$ $(y < 0)$. The join coincides with the z axis and, if the frontal curvatures $a_2$ and $a_1$ of the respective cylinders are not the same, constitutes a line discontinuity in curvature. A plane electromagnetic wave is incident with its propagation vector lying in the xy plane and making an angle $\alpha$ with the y axis. In the case of H polarization (magnetic vector in the z direction), we choose

$$\underline{H}^i = \hat{z}\, e^{-ik(x\sin\alpha + y\cos\alpha)}$$

where a time factor $e^{-i\omega t}$ has been assumed and suppressed. For E polarization, $\underline{E}^i$ is given by the same expression, but in order to present the method it is sufficient to confine attention to H polarization.

Following the procedure described in [1], the far field amplitude is written as

$$P_H(\alpha,\theta) = \frac{1}{4}\int_{-\infty}^{\infty} (\sin\theta + a\,\frac{\zeta}{k}\cos\theta)u(\frac{\zeta}{k})\ \exp\{-i\zeta\cos\theta + ia\,\frac{\zeta^2}{2k}\sin\theta\}d\zeta \tag{3}$$

where $u(\zeta/k)$ is the total field $H_z$ on the surface of the cylinder. If $0 < \alpha < \pi$ so that the join is directly illuminated,

$$u(\zeta/k) = u^{p.o.}(\zeta/k) + u^{(1)}(\zeta/k) + u^{(2)}(\zeta/k) + O(k^{-2}) \tag{4}$$

where

$$u^{p.o.}(\zeta/k) = 2\ \exp\{-i\zeta\cos\alpha + ia\,\frac{\zeta^2}{2k}\sin\alpha\} \tag{5}$$

is the physical optics expression for the surface field and

$$u^{(1)}(\zeta/k) = -\ i\,\frac{a}{k}(\sin\alpha + a\,\frac{\zeta}{k}\cos\alpha)^{-3}\ \exp\{-i\zeta\cos\alpha + ia\,\frac{\zeta^2}{2k}\cos\alpha\} \tag{6}$$

is a correction obtained by application of the Luneberg-Kline method to a uniform parabolic surface. The sum of these is the field that would exist on each section of the surface were the whole a continuation of it, and by asymptotic solution of an integral equation, the perturbation field created by the join is

$$u^{(2)}(\zeta/k) = \pm\,\frac{i}{k}\,(a_2 - a_1)\ e^{-i\zeta\cos\alpha}\ L(\zeta) \tag{7}$$

with the upper and lower signs for the upper and lower cylinders respectively. $L(\zeta)$ is expressible as an infinite integral involving a Hankel function.

The contribution of $u^{p.o.}(\zeta/k)$ to $P_H(\alpha,\theta)$ is

$$P_H^{p.o.}(\alpha,\theta) = \frac{1}{2}\int_{-\infty}^{\infty}(\sin\theta + a\,\frac{\zeta}{k}\cos\theta)\exp\{-ip\zeta + ia\,\frac{\zeta^2}{2k}(\sin\alpha + \sin\theta)\}d\zeta \tag{8}$$

where $p = \cos\alpha + \cos\theta$. If $\theta \neq \pi - \alpha$ so that $p \neq 0$, the specular point is distinct from the join and the exponential factor associated with the second term in the exponent can be expanded for small $\zeta$.

A straightforward analysis then shows [1]

$$P_H^{p.o.}(\alpha,\theta) = -\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha-\theta)}{p^3} + O(k^{-2}), \tag{9}$$

which is just the physical optics expression for the join contribution, and is infinite when $p = 0$. However, if $\theta = \pi - \alpha$ implying $p = 0$, the specular point coincides with the join. The second term in the exponent in (8) is now the dominant (and only) one, and the resulting expression for the far field coefficient is

$$P_H^{p.o.}(\alpha,\pi-\alpha) = \frac{1}{4}\sqrt{\pi k \sin\alpha}\left(\frac{1}{\sqrt{a_2}} + \frac{1}{\sqrt{a_1}}\right)e^{i\pi/4}. \tag{10}$$

It is therefore evident that an expression for $P_H^{p.o.}(\alpha,\theta)$ which is uniform in angle must include the specular return as well as the join contribution, and in view of the above analyses, all terms in the integrand of (8) must be retained to produce even the terms shown for $p = 0$ and $p \neq 0$. Fortunately, an exact evaluation of the integral is possible, and yields

$$P_H^{p.o.}(\alpha,\theta) = -\frac{1+\cos(\alpha-\theta)}{p(\sin\alpha+\sin\theta)}\{G(-\tau_2) - G(\tau_1)\} \tag{11}$$

where

$$\tau_i = p\sqrt{\frac{k}{2a_i(\sin\alpha+\sin\theta)}}, \qquad i = 1,2 \tag{12}$$

and $G(\tau)$ is the modified Fresnel integral

$$G(\tau) = \tau e^{-i\tau^2}\int_\tau^\infty e^{it^2}\,dt. \tag{13}$$

From the partial field $u^{(1)}(\zeta/k)$ we have

$$P_H^{(1)}(\alpha,\theta) = -\frac{i}{4k}\int_{-\infty}^{\infty} a\,\frac{\sin\theta + a\,\zeta/4\,\cos\theta}{(\sin\alpha + a\,\zeta/k\,\cos\alpha)^3}\exp\{-ip\zeta + ia\,\frac{\zeta^2}{2k}(\sin\alpha + \sin\theta)\}d\zeta \tag{14}$$

and an exact analytical evaluation of this integral is impossible. If $p \neq 0$, however, the join contribution is easily determined and is

$$P_H^{(1)}(\alpha,\theta) = -\frac{a_2-a_1}{4k}\,\frac{\sin\theta\operatorname{cosec}^3\alpha}{p} + O(k^{-2}), \tag{15}$$

whereas if $p = 0$

$$P_H^{(1)}(\alpha,\pi-\alpha) = \frac{1}{8}\sqrt{\frac{\pi\sin\alpha}{k}}\operatorname{cosec}^3\alpha\,(\sqrt{a_2}+\sqrt{a_1})e^{-i\pi/4} - \frac{a_2-a_1}{2k}\cos\alpha\operatorname{cosec}^4\alpha + O(k^{-3/2}), \tag{16}$$

and a uniform expression is

$$P_H^{(1)}(\alpha,\theta) = \frac{i}{2kp}\left\{\frac{\sin\alpha+\sin\theta}{1+\cos(\alpha-\theta)}\right\}^2\left\{a_2G(-\tau_2) - a_1G(\tau_1) - \frac{i}{2}(a_2-a_1)\right\} - \frac{a_2-a_1}{4k}\,\frac{\sin\theta\operatorname{cosec}^3\alpha}{p} + O(k^{-3/2}). \tag{17}$$

The remaining contribution to the far field amplitude is

$$P_H^{(2)'}(\alpha,\theta) = \frac{i}{8k}(a_2-a_1)\left(\int_0^\infty - \int_{-\infty}^0\right)(\sin\theta + a\zeta/k\cos\theta)\,L(\zeta)\exp\{-ip\zeta + ia\frac{\zeta^2}{2k}\sin\theta\}d\zeta \tag{18}$$

and as shown in [1]

$$P_H^{(2)}(\alpha,\theta) = -\frac{a_2-a_1}{2kp^3}\{1+\cos(\alpha+\theta) - \frac{1}{2}p^2\sin\theta\operatorname{cosec}^3\alpha\} + O(k^{-2}) \tag{19}$$

This is finite when $p = 0$ and, indeed,

$$P_H^{(2)}(\alpha,\pi-\alpha) = \frac{a_2-a_1}{2k}\cos\alpha\operatorname{cosec}^4\alpha + O(k^{-2}) \tag{20}$$

For the complete far field amplitude a uniform expression is obtained by adding (11), (17) and (19), and can be written as

$$P_H(\alpha,\theta) = U + V + O(k^{-3/2}) \tag{21}$$

where

$$U = -\frac{1 + \cos(\alpha - \theta)}{p(\sin\alpha + \sin\theta)}\left\{G(-\tau_2) - G(\tau_1)\right\} \tag{22}$$

is the physical optics expression, and

$$V = \frac{i}{2kp}\left\{\frac{\sin\alpha + \sin\theta}{1 + \cos(\alpha - \theta)}\right\}^2 \left\{a_2[G(-\tau_2) - {}^i/2] - a_1[G(\tau_1) - {}^i/2]\right\}$$
$$- \frac{a_2 - a_1}{2k}\ \frac{1 + \cos(\alpha + \theta)}{p^3}\ . \tag{23}$$

We observe that U and V are symmetrical in $\alpha$ and $\theta$ , and $P_H(\alpha,\theta)$ therefore satisfies the reciprocity theorem concerning the interchange of transmitter and receiver. For $p > 0$ and bounded away from zero, insertion of the asymptotic expansion of the modified Fresnel integral gives

$$P_H(\alpha,\theta) = \sqrt{\frac{\pi k}{2a_2}}\ \frac{1 + \cos(\alpha - \theta)}{(\sin\alpha + \sin\theta)^{3/2}}\ \left\{1 - \frac{i}{2k}\left[\frac{\sin\alpha + \sin\theta}{1 + \cos(\alpha - \theta)}\right]^3\right\}$$
$$- \frac{a_2 - a_1}{2k}\ \frac{1 + \cos(\alpha - \theta)}{p^3} - \frac{a_2 - a_1}{2k}\ \frac{1 + \cos(\alpha + \theta)}{p^3}$$
$$+ O(k^{-3/2}). \tag{24}$$

The first term is produced by reflection off the parabolic cylinder in $y > 0$ and the last two terms represent the join contribution. If $p < 0$ implying $\tau_1, \tau_2 < 0$, $a_2$ must be replaced by $a_1$ in the first term of (24), corresponding to reflection off the cylinder in $y < 0$, but if $p = 0$ so that the specular point and join coincide,

$$P_H(\alpha,\pi - \alpha) = \frac{1}{4}\sqrt{\pi k\sin\alpha}\ \left(\frac{1}{\sqrt{a_2}} + \frac{1}{\sqrt{a_1}}\right)\left\{1 - i\ \frac{\sqrt{a_2 a_1}}{2k}\ \mathrm{cosec}^3\alpha\right\}e^{i\pi/4} + O(k^{-3/2}) \tag{25}$$

and the terms $O(k^{-1})$ have cancelled one another.

The analysis for E polarization is similar, and the resulting uniform expression for the far field amplitude is

$$P_E(\alpha,\theta) = - U + V$$

where $- U$ is the physical optics expression and U, V are as defined in (22) and (23).

## Acknowledgement

The author is grateful to D. K. M. Mitzner for suggesting this problem, and to the Air Force Office of Scientific Research for support under Grant 77-3188.

## Reference

[1] T. B. A. Senior, "The Diffraction Matrix for a Discontinuity in Curvature," IEEE Trans. Ant. and Prop., Vol. 20, No. 3, pp. 326 - 333, May 1972. [The following typographical errors should be noted: in eq. (32) the last two factors $\cos\alpha$ should be replaced by $\sin\alpha$, and on the right hand sides of (42) and (43), $\theta$ should be replaced by $-\theta$].

Legend for Figure

Fig. 1: Model for discontinuity in curvature.

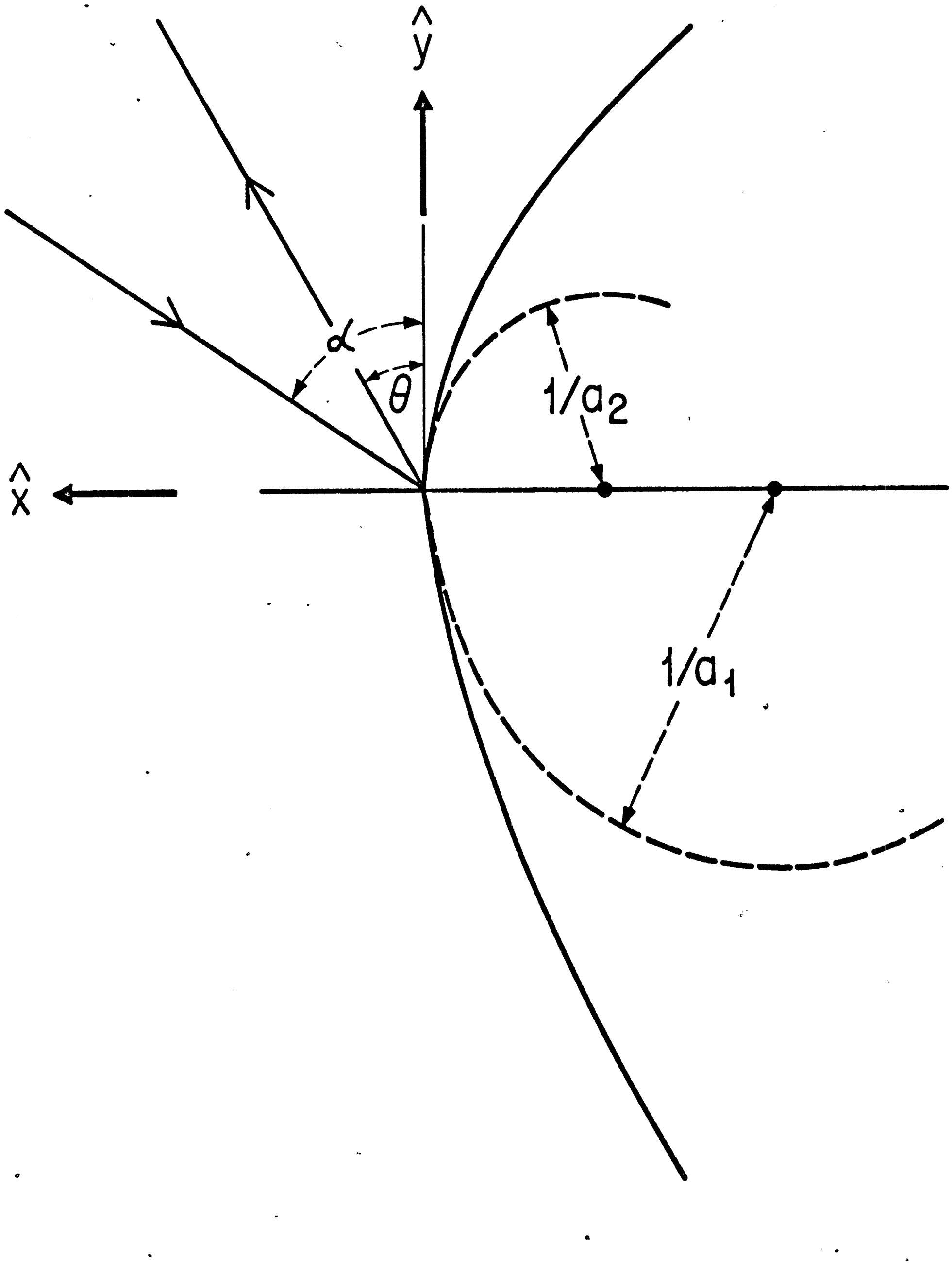


Fig. 1. The diffraction coefficient model for a discontinuity in curvature