The Boundary Limits of Certain Integrals in Scattering Theory*

by

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Abstract

Two integrals that arise in scattering theory are discussed. In contrast to recent statements in the literature it is shown that the correct interpretation is not as a Hadamard finite part, and this has implications conceptually as well as numerically.

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I. Introduction

In the application of integral equation methods in scattering theory it is necessary to determine the boundary limits of the integrals involved, and in some cases this requires special care. Factors that can influence the limits are the kernel itself, the smoothness of the boundary geometry, and the smoothness of the density function whose values can be obtained only by solution of the integral equation. For a density function in a particular space, the boundary limit may or may not exist depending on the kernel and the boundary, but since the kernel is given, it is the interplay of the smoothness of the boundary and the density function that we are concerned with. As a general rule, the smoother the boundary, the less stringent the conditions on the density function.

Two integrals of interest are

$$I_1(\bar{r}_0) = \frac{\partial}{\partial n_0} \iint_\Gamma \frac{\partial}{\partial n'} G(kR_0) f(\bar{r}') dS' \tag{1}$$

$$\bar{I}_2(\bar{r}_0) = \hat{n}_0 \times \iint_\Gamma \bar{J}(\bar{r}') \cdot \nabla' \nabla_0 G(kR_0) dS' \tag{2}$$

where the integration is over the closed boundary surface Γ of a finite body, $R_0 = |\bar{r}_0 - \bar{r}'|$ witn $\bar{r}_0, \bar{r}' \in \Gamma$, $\hat{n}_0 = \hat{n}(\bar{r}_0)$ and $\hat{n}' = \hat{n}(\bar{r}')$ are unit vector normals to Γ directed into the exterior region Ω, $\bar{J}(\bar{r}')$ is a tangential vector function, and $G(kR_0) = e^{ikR_0}/(4\pi R_0)$ is the free space Green's function. The first integral was used by Davis and Mittra [1] in the analysis of a thin scatterer problem, and the two-dimensional

analogue of (2) was considered by Bolomey and Tabbara [2] and Mautz and Harrington [3]. In all cases the integrals were interpreted in the sense of a Hadamard finite part, but this is incorrect practically as well as on strictly formal grounds. The Hadamard finite part integral is a singular integral of a specific type in which the integrand is known and has a fixed singular point. With (1) and (2), however, the singularities are not fixed, but spread over the entire domain of integration, and in addition the integrands are unknown, since they contain the density function whose determination is the objective of the integral equation technique.

The purpose of this note is to show the circumstances under which (1) and (2) exist and to give the meaning that the integrals then have. The main results are contained in the theorems of the next two sections, and we then examine the two-dimensional analogues of the integrals for which the 'self cell' contributions are trivially obtainable.

II. The Integral $I_1(\bar{r}_0)$

The integral is the normal derivative of a (generalized) double-layer potential in scattering theory and its properties can be deduced from those of the corresponding one in potential theory [4,5,6]. In particular, if the exterior limit of the integral exists, so does the interior one, and both limits are equal. In other words, if the integral exists, it is continuous across Γ. Schauder [4] has shown that for $\Gamma \in C_{1+\nu}(0 < \nu < 1)$ the integral operator in potential theory corresponding to that in (1) maps $C_{1+\mu}$ into C_{μ}, $0 < \mu < 1$.

Henceforth we assume that Γ is closed and in C_2. This assumption is more restrictive than Schauder's and implies that the curvature is defined everywhere on Γ. Let $w(Q)$, $Q \in \Gamma$, denote the direct value* of the double-layer potential

$$w(Q) = \int\int_{\Gamma} \frac{\partial}{\partial n_P} G(kr_{PQ}) f(P) dS_P \tag{3}$$

where the integral is not a principled-valued one as frequently supposed, but one defined in the Riemann sense.

Theorem 1: If $\Gamma \in C_2$, the integral operator $M(Q,P;k)$ such that

$$I_1(Q) = \frac{\partial}{\partial n_Q} w(Q) = \int\int_{\Gamma} \frac{\partial}{\partial n_Q} \frac{\partial}{\partial n_P} G(kr_{PQ}) f(P) dS_P \equiv M(Q,P;k) f(P) \tag{4}$$

maps C_1 into C_1 for all $Q \in \Gamma$.

Omitting details which are a direct extension of those in [4], an outline of the proof is as follows. Divide Γ into Γ' and Γ_ε where Γ_ε is a sufficiently small open surface about a fixed point $Q_1 \in \Gamma_\varepsilon$. For $Q_2 \in \Gamma_\varepsilon$

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*A double-layer potential defined at $\bar{r} \in \Omega$ yields an exterior limit which is the sum of $1/2f(Q)$ and $w(Q)$. The integral (3) is called the direct value in potential theory.

$$\frac{\partial}{\partial n_{Q_1}} w(Q_1) - \frac{\partial}{\partial n_{Q_2}} w(Q_2) = \int\int_{\Gamma' \cup \Gamma_\varepsilon} \frac{\partial}{\partial n_{Q_1}} \frac{\partial}{\partial n_P} G(kr_{PQ_1}) \{f(P) - f(Q_1)\} dS_P$$

$$- \int\int_{\Gamma' \cup \Gamma_\varepsilon} \frac{\partial}{\partial n_{Q_2}} \frac{\partial}{\partial n_P} G(kr_{PQ_2}) \{f(P) - f(Q_2)\} dS_P$$

$$+ \left[f(Q_1) \int\int_\Gamma \frac{\partial}{\partial n_{Q_1}} \frac{\partial}{\partial n_P} G(kr_{PQ_1}) dS_P - f(Q_2) \int\int_\Gamma \frac{\partial}{\partial n_{Q_2}} \frac{\partial}{\partial n_P} g(kr_{PQ_2}) dS_P \right] . \tag{5}$$

Since f is in C_1, the terms in square brackets tend to zero as $Q_2 \to Q_1$. The integrals over Γ_ε are

$$I(Q_1,Q_2) = \int\int_{\Gamma_\varepsilon} \frac{\partial}{\partial n_{Q_1}} \frac{\partial}{\partial n_P} G(kr_{PQ_1}) \{f(P) - f(Q_1)\} dS_P$$

$$- \int\int_{\Gamma_\varepsilon} \frac{\partial}{\partial n_{Q_2}} \frac{\partial}{\partial n_P} G(kr_{PQ_2}) \{f(P) - f(Q_2)\} dS_P$$

where

$$\frac{\partial}{\partial n_Q} \frac{\partial}{\partial n_P} G(kr_{PQ}) = \frac{1}{4\pi r_{PQ}^3} \{\hat{n}_P \cdot \hat{n}_Q - 3(\hat{r}_{PQ} \cdot \hat{n}_P)(\hat{r}_{PQ} \cdot \hat{n}_Q)\} ,$$

and since $\Gamma \in C_2$,

$$|I(Q_1,Q_2)| \leq AB r_{Q_1Q_2} \tag{6}$$

where A and B are some constants independent of Q_1 and Q_2, but dependent on the smoothness of Γ and f. The integrals over the open surface Γ' are similarly bounded, and from these estimates it follows that $I_1(Q)$ is in C_1.

It is easily shown that if the density function is merely continuous on a boundary of class $C_{1+\nu}$, $0 < \nu \leq 1$, $I_1(Q)$ is unbounded and hence does not exist. In particular, the theorem fails if Γ has a surface singularity, but if Γ is in C_2, $I_1(Q)$ is in C_1, at variance with a Hadamard finite part interpretation.

III. The Integral $\bar{I}_2(\bar{r}_0)$

Theorem 2: If $\Gamma \in C_2$ the integral operator $\bar{N}(Q,P;k)$ defined by

$$\bar{I}_2(Q) = \hat{n}_Q \times \int_\Gamma\int \bar{J}(P) \cdot \nabla_P\nabla_Q G(kr_{PQ})dS_P = \hat{s}_Q\bar{N}(Q,P;k) \cdot \bar{J}(P) \qquad (7)$$

maps $C_{1+\mu}$ into C_μ, $0 < \mu < 1$, where $\hat{s}_Q$ is a unit tangent vector at $Q \in \Gamma$.

The proof is trivial. For $Q \in \Omega$ and $R = |QP|$

$$\int_\Gamma\int \bar{J}(P) \cdot \nabla_P\nabla_Q G(kR)dS_P = -\int_\Gamma\int \nabla_Q G(kR)\nabla_P \cdot \bar{J}(P)dS_P$$

provided, of course, $\bar{J} \in C_1(\Gamma)$. Hence, for $Q \in \Gamma$,

$$\bar{I}_2(Q) = -\hat{n}_Q \times \int_\Gamma\int \nabla_Q G(kr_{PQ})\nabla_P \cdot \bar{J}(P)dS_P \qquad (8)$$

which is simply the tangential derivative of a single layer potential with density function $\hat{s}_Q\nabla_P \cdot \bar{J}(P)$. From the fact well known in potential

theory that this is continuous across a boundary $\Gamma \in C_2$ and is in C_μ on the boundary if the density function is in C_μ, $0 < \mu < 1$, the theorem follows. Here again, therefore, the interpretation as a Hadamard finite part is incorrect.

In the numerical solution of an integral equation containing either of the integrals (1) or (2), the error affects the self cell contribution and, as a result, the diagonal terms in the matrix. The determination of the self cell contributions is most easily considered in two dimensions where (1) is directly related to an integral similar in form to (2).

IV. Two-Dimensional Case

The two-dimensional analogue of (8) and, hence, (2) is

$$-\frac{i}{4}\hat{n}_Q \times \oint_\gamma \nabla_Q H_0^{(1)}(kr_{PQ}) \frac{\partial}{\partial s_P} J(s_P) ds_P \tag{9}$$

where s is arclength along a smooth closed boundary γ, $H_0^{(1)}$ is the Hankel function of the first kind of zero order, and the density function $\bar{J}(s_P) = \hat{s}_P J(s_P)$. If $(\hat{n}_Q, \hat{s}_Q, \hat{z}_Q)$ is a right-handed rectangular coordinate system at $Q \in \gamma$, the scalar product of (9) with $\hat{z}_Q$ yields

$$-\frac{i}{4} \oint_\gamma \frac{\partial}{\partial s_P} J(s_P) \frac{\partial}{\partial s_Q} H_0^{(1)}(kr_{PQ}) ds_P \ . \tag{10}$$

By a tedious but straightforward analysis, it can also be shown that

$$\frac{i}{4}\oint_\gamma J(s_P)\,\frac{\partial}{\partial n_Q}\,\frac{\partial}{\partial n_P}\,H_o^{(1)}(kr_{PQ})ds_P$$

$$= \frac{i}{4}k^2\oint_\gamma J(s_P)\,(\hat{s}_P\cdot\hat{s}_Q)H_o^{(1)}(kr_{PQ})ds_P$$

$$+\frac{i}{4}\oint_\gamma \frac{\partial}{\partial s_P}J(s_P)\,\frac{\partial}{\partial s_Q}\,H_o^{(1)}(kr_{PQ})ds_P \qquad (11)$$

and this not only relates the two-dimensional versions of (1) and (2) explicitly, but can serve to give a proper interpretation of the integral on the left.

In particular, in any numerical evaluation of the integrals on the two sides of (11), the self cell contributions must be equal. The first integral on the right-hand side poses no problem and its contribution is $\alpha J(s_Q)$ where α tends to zero with the cell size Δ, assumed small compared with the wavelength. The second integral produces $(-\Delta/2\pi)(\partial^2/\partial s_Q^2)J(s_Q)$ which also tends to zero with Δ, but when, in the moment method, the second derivative is expressed in terms of J via finite differencing, we obtain a contribution proportional to $(1/\pi\Delta)J(s_Q)$. This is therefore the dominant term in the expressions for the self cell contributions of the integrals on both the left- and right-hand sides of (11). It is a term which would be omitted were the first and third integrals treated as Hadamard finite parts, but one whose retention is implied by our stricter interpretation.

It must be admitted that the retention of these terms does not seem to have a major impact on a numerical solution of an integral equation in which either of the integrals appears. Mathematically the self cell

contributions do tend to zero with decreasing cell size, and the solution does remain stable as Δ decreases. At most, therefore, the omission of the terms $O(1/\Delta)$ would produce some loss of efficiency in the program by forcing a smaller cell size than otherwise necessary to achieve the same accuracy of solution.

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