SCATTERING BY A FINITE RESISTIVE PLATE

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A widely used method for the solution of scattering problems is to formulate an integral equation for the current or some appropriate component of the field at the surface of the body and to solve the equation numerically. In some instances, however, even the formulation of a valid and effective integral equation is a non-trivial task, and this is so for a resistive plate of infinitesimal thickness and finite transverse dimensions illuminated by an electromagnetic wave.

A number of integral equation formulations are considered, based either on the standard boundary conditions

$$\hat{n} \wedge \underline{E}\Big|_-^+ = 0 \ , \qquad \underline{J} = \hat{n} \wedge \underline{H}\Big|_-^+ = -\frac{1}{R}\hat{n} \wedge (\hat{n} \wedge \underline{E})$$

where $\hat{n}$ is the unit outward normal to the upper face (+ sign), R is the resistivity of the plate in ohms per square, and $\underline{J}$ is the total induced electric current, or on the alternative conditions

$$\frac{\partial E_n}{\partial n}\Big|_-^+ = 0 \ , \qquad \frac{\partial E_n}{\partial n} + ik\frac{R}{Z}E_n\Big|_-^+ = 0$$

$$H_n\Big|_-^+ = 0 \ , \qquad \frac{\partial H_n}{\partial n}\Big|_-^+ + ik\frac{Z}{R}H_n = 0$$

valid for a planar sheet, where Z is the intrinsic impedance of free space. We remark that

$$\hat{n} \cdot \nabla \wedge \underline{J} = ik\frac{Z}{R}H_n \ , \qquad \nabla \cdot \underline{J} = -\frac{1}{R}\frac{\partial E_n}{\partial n}$$

The case R = 0 corresponds to perfect conductivity, but the simplified approach that Y. Rahmat-Samii and R. Mittra (IEEE Trans. AP-22, 608-610, 1974) have developed for a perfectly conducting plate has no analogue when $R \neq 0$. Various integral equations resulting from the application of the above conditions are presented, and their properties discussed.

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by

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At the last 3 U.S. URSI meetings I spoke about the scattering by imperfectly conducting strips, either resistive or ones having an impedance boundary condition imposed at the two faces, and derived asymptotic expressions for the bistatic scattered field which are uniform in angle and remarkably accurate even for strip widths down to a quarter of a wavelength.

The motivation for these studies was the need to develop cross section reduction techniques that are applicable to electrically thin and roughly planar structures like the tail fin of an aircraft. Such structures are, of course, finite in dimension, and in the case of a perfectly conducting plate it is well known that a segment of a strip provides at best a rather so-so means for estimating the scattering from the plate. We can see this in the case of backscattering from a rectangular plate in a plane perpendicular to two of the edges. The plate shown has a length 1.653 λ, a width 1.446 λ, and the field is polarized with electric vector parallel to the surface. The solid line is a measured cross section and the broken line is that obtained using a strip as a model. As the incidence approaches grazing, the discrepancies become more evident, and the fact is that we can no longer simply ignore the side edges and the currents they support. For

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incidence and/or observation out of the symmetry plane for a rectangular plate, and for a general polygonal plate under all conditions of incidence and observation, the strip approximation is largely irrelevant.

We would expect the same to be true for an imperfectly conducting plate, but there is a point to be noted here. Because imperfect conductivity reduces the edge currents and their scattering, there are instances where a strip provides a better approximation than it did for perfect conductivity, for example, the case of a rectangular plate with incidence and observation in a principal plane. The figure shows the backscattered field for a plate of resistivity 210 Ω/square under the same conditions as before. The agreement between the measured values and the strip approximation is better at near-grazing angles, but it is still true that for general angles of incidence and observation the strip solution is of no help.

We now turn to the actual problem of a finite plate which, for simplicity, I will assume to be resistive. Our objective is to find a formulation amenable to a numerical solution, and this naturally suggests an integral equation approach. It is realized that in practice we may be limited to plates no more than about a square wavelength in area, but even for plates of this size the solution could be helpful in the development of analytic approximations.

Consider first the boundary conditions for a plate of infinitesimal thickness having a resistivity R Ω/sq., possibly nonuniform over the plate. Since the plate supports only an electric current,

$$\hat{n} \wedge \underline{E}\Big|_{-}^{+} = 0$$

across the plate. It now follows trivially that

$$H_n\Big|_{-}^{+} = 0$$

and, with the possible exception of the edges of the plate,

$$\frac{\partial E_n}{\partial n}\Big|_{-}^{+} = 0$$

where E_n, H_n are the normal components. The only components whose discontinuities are not yet established are those of the tangential magnetic field and, of course,

$$\hat{n} \wedge \underline{H}\Big|_{-}^{+} = \underline{J}$$

where $\underline{J}$ is the total electric current.

The tangential derivatives of $\underline{J}$ are related to the normal components of the field. From Maxwell's equations and the definition of $\underline{J}$

$$E_n\Big|_{-}^{+} = \frac{Z}{ik} \nabla_s \cdot \underline{J}$$

$$\frac{\partial H_n}{\partial n}\Big|_{-}^{+} = -\hat{n} \cdot \nabla \wedge \underline{J}$$

where Z is the free space impedance and $\nabla_s \cdot$ is the surface divergence.

So far we have used no information about the plate other than that it supports only a planar electric current. The above results are therefore satisfied automatically by any representation of the field in terms of an electric current alone. The additional boundary condition which defines a resistive plate is

$$\hat{n} \wedge (\underline{E} - R\underline{J}) = 0 \ .$$

If R = constant, it can be shown that the condition implies

$$\left.\frac{\partial H_n}{\partial n}\right|_-^+ + ik\frac{Z}{R}H_n = 0 \ , \quad \frac{\partial E_n}{\partial n} + ik\frac{R}{Z}\left.E_n\right|_-^+ = 0 \ .$$

These are <u>scalar</u> boundary conditions in the sense of involving one field component each, and are equivalent but alternative to the original ones

$$\hat{n} \wedge \left.\underline{E}\right|_-^+ = 0 \ , \quad \hat{n} \wedge \left.\underline{H}\right|_-^+ = -\frac{1}{R}\hat{n} \wedge (\hat{n} \wedge \underline{E}) \ .$$

In the special case of perfect conductivity (R = 0), the first pair give

$$H_n = 0 \ , \quad \frac{\partial E_n}{\partial n} = 0$$

in place of the more common ones

$$E_{tan} = 0 \ .$$

The boundary conditions involving the normal components of the field are attractive in holding out the possibility of producing uncoupled integral equations for a plate. Unfortunately, it does not work out this way. In addition, if $R \neq$ constant, the conditions become

$$\left.\frac{\partial H_n}{\partial n}\right|_-^+ + ik \frac{Z}{R} H_n = -ik \frac{Z}{R} \hat{n} \cdot (\underline{J} \wedge \nabla R)$$

$$\left.\frac{\partial E_n}{\partial n} + ik \frac{R}{Z} E_n\right|_-^+ = -\underline{J} \cdot \nabla R$$

and are no longer scalar. Because of our interest in nonuniform resistivities, the standard boundary conditions turn out to be the more convenient ones to use.

We now develop some integral equation formulations for the problem. Since the plate can support only an electric current, the Franz method leads immediately to the following expressions for the scattered field:

$$\underline{H}^s = - \iint \underline{J} \wedge \nabla\Phi \, dx'dy'$$

$$\underline{E}^s = \frac{iZ}{k} \iint \{k^2 \underline{J}\Phi + (\underline{J} \cdot \nabla)\nabla\Phi\} \, dx'dy'$$

$$\equiv \frac{iZ}{k} (k^2 + \nabla\nabla.) \iint \underline{J}\Phi \, dx'dy'$$

where

$$\Phi = \frac{e^{ik|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|}$$

is the free space Green's function. Of the six possible integral equations corresponding to the 6 field components, those resulting from E_n and H_{tan} are merely identities. The obvious and usable integral equations then follow from a consideration of the tangential electric field components and are

$$RJ_x = E_x^i + ikZ \iint J_x \Phi \, dx'dy' + i\frac{Z}{k} \iint \left(J_x \frac{\partial^2 \Phi}{\partial x^2} + J_y \frac{\partial^2 \Phi}{\partial x \partial y} \right) dx'dy'$$

$$RJ_y = E_y^i + ikZ \iint J_y \Phi \, dx'dy' + i\frac{Z}{k} \iint \left(J_x \frac{\partial^2 \Phi}{\partial x \partial y} + J_y \frac{\partial^2 \Phi}{\partial y^2} \right) dx'dy' \qquad \text{(I)}$$

where x,y,z are Cartesian coordinates with z normal to the plate. These are two coupled integral equations for the current components. The kernels are far from attractive, and because of the derivatives involved, the integrals require special consideration either in terms of distribution theory or functions of high order continuity, the numerical implications of which are not obvious. Moreover, as Rahmat-Samii and Mittra (1974) have noted, in at least the special case R = 0 the equations are numerically unstable.

We still have available the normal component of the magnetic field and by differentiating this tangentially we can combine it with the representations for the tangential electric field to obtain the following integral equations:

$$\left(k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(RJ_x - E_x^i - ikZ \iint J_x \Phi \, dx'dy'\right) = \frac{\partial}{\partial x}\left(R\nabla_s \cdot \underline{J} - \nabla_s \cdot \underline{E}^i\right)$$

$$\left(k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(RJ_y - E_y^i - ikZ \iint J_y \Phi \, dx'dy'\right) = \frac{\partial}{\partial y}\left(R\nabla_s \cdot \underline{J} - \nabla_s \cdot \underline{E}^i\right)$$

The derivation is directly analogous to that used by Rahmat-Samii and Mittra and, in the case of a perfectly conducting plate ($R = 0$), the results are very advantageous. The right-hand sides then involve only the incident field, and we can therefore integrate out the derivatives on the left at the expense of introducing a solution of the homogeneous differential equation. Unfortunately, this is not possible if $R \neq 0$, and since the equations in their present form are integro-differential, they offer no advantages over (I).

Nevertheless, the form of the differential operator on the left and the fact that it is just $-(\partial^2/\partial z^2)$ off the plate, suggests that it could be fruitful to follow the procedure that led to the above prior to going down to the plate. The task is rather straightforward and gives

$$\begin{aligned} \iint J_n \frac{\partial^2 \Phi}{\partial z^2} \, dx'dy' &= -\frac{\partial H_y^i}{\partial z} + ik\frac{R}{Z}\left\{J_x - \frac{1}{k^2}\frac{\partial}{\partial y}\left(\hat{z} \cdot \nabla \wedge \underline{J}\right)\right\} \\ \iint J_y \frac{\partial^2 \Phi}{\partial z^2} \, dx'dy' &= \frac{\partial H_x^i}{\partial z} + ik\frac{R}{Z}\left\{J_y + \frac{1}{k^2}\frac{\partial}{\partial x}\left(\hat{z} \cdot \nabla \wedge \underline{J}\right)\right\} \end{aligned} \qquad \text{(II)}$$

We have now reduced the singularity somewhat and made it the same for both equations at the expense of having integro-differential equations involving the second derivatives of the unknowns. These terms provide the coupling and do so explicitly rather than via the integrals.

To try reducing the singularities still further, one way is to seek integral equations for some components of the fields rather than the current. The only components for which this is feasible are the normal ones, and equations can be obtained for H_z and $\partial E_z/\partial z$ in which the coupling is provided by line integrals of the currents (or, alternatively, the field components) around the perimeter of the plate. However, these equations have some difficulties associated with them and I will not discuss them further.

Each of the above pairs I and II of equations has its own advantages and disadvantages. The first pair are pure integral equations and because the currents are finite everywhere if $R \neq 0$, it could be that the numerical instability previously found does not now occur. The second pair have somewhat more attractive kernels, but are integro-differential equations, and for this reason our initial tendency was to concentrate on the first. In both cases, however, the correct interpretation of the integrals is still a matter of debate, particularly as regards a numerical treatment.

To provide an interpretation, one method is to regularize the equations in a manner similar to what Kleinman and others have used over the last 10 years or so; in other words, to write

$$\iint J(\underline{r}')K(\underline{r},\underline{r}')\,dx'dy' = \iint \{J(\underline{r}') - J(\underline{r})\}\,K(\underline{r},\underline{r}')\,dx'dy'$$

$$+ J(\underline{r}) \iint \{K(\underline{r},\underline{r}') - K_0(\underline{r},\underline{r}')\}\,dx'dy'$$

$$+ J(\underline{r}) \iint K_0(\underline{r},\underline{r}')\,dx'dy'$$

where K_0 is the static limit of the kernel K. The problem of interpretation is now concentrated in the last integral, and has been reduced to its simplest form. This suggests that some insight could be gained by considering the low frequency (or Rayleigh) scattering problem for the plate.

In the limit when the wavenumber $k \to 0$ a resistive plate becomes perfectly conducting. The scattered field is attributable to induced electric and magnetic dipoles whose components (or, if you prefer, the elements of the polarizability tensors) are expressible as weighted integrals of certain potential functions. The only components of concern are the tangential components of the electric dipole moment and the normal component of the magnetic dipole. For the electric dipole the potential problems are entirely standard, and the resulting integral equations have a simple 1/R kernel. For the magnetic dipole, however, the straightforward approach requires the determination of a potential ψ satisfying

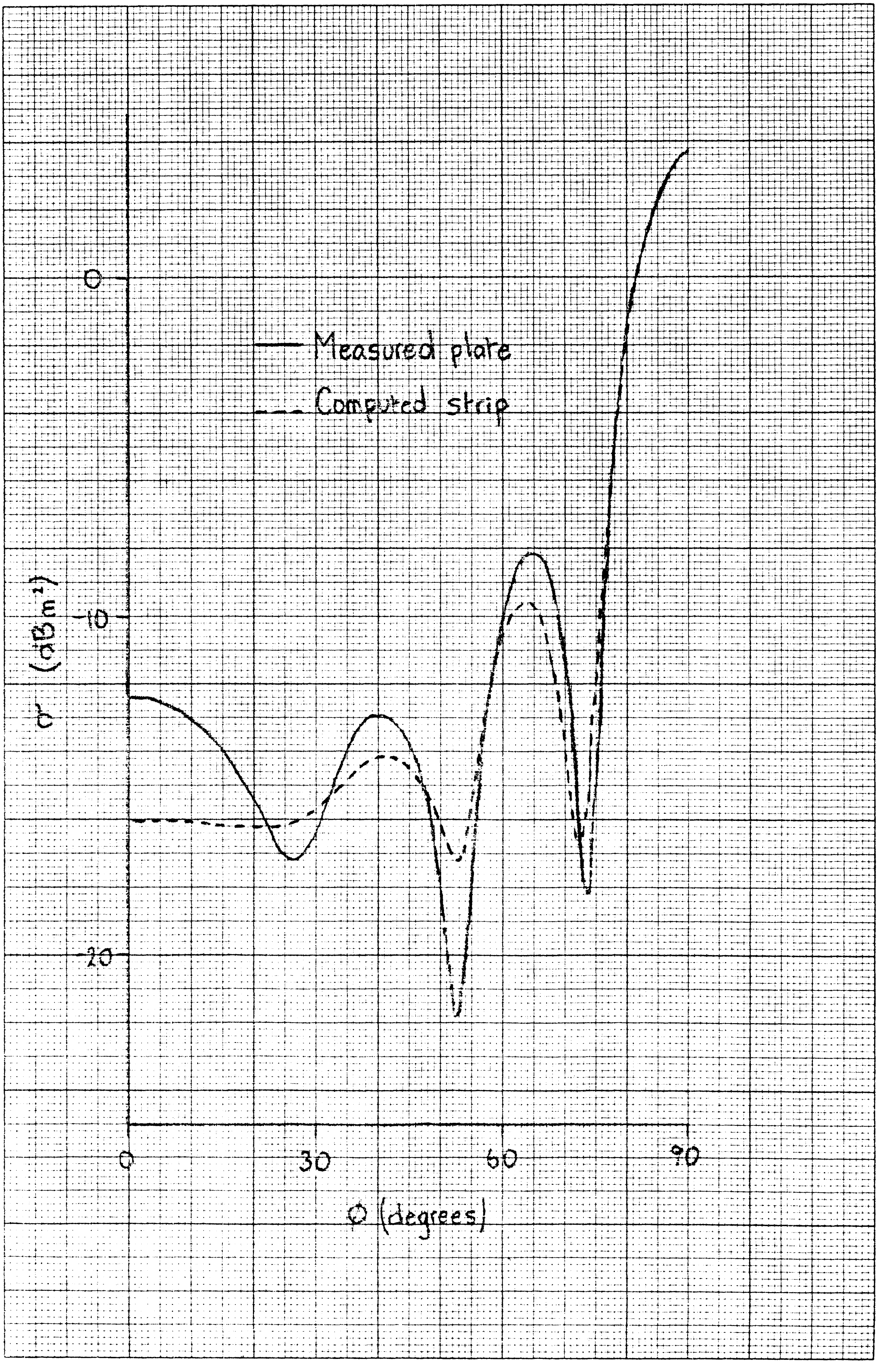
$$1 = \frac{1}{4\pi} \iint \psi\Big|_{-}^{+} \frac{\partial^2}{\partial z^2} \left(\frac{1}{|\underline{r}-\underline{r}'|}\right) dx'dy' \quad .$$

Using a simple segmentation procedure, programs have been written for solving these integral equations for a number of plate geometries, and from the results obtained there seems little doubt that the procedures are adequate and effective. However, I'll talk about these results at a later time.

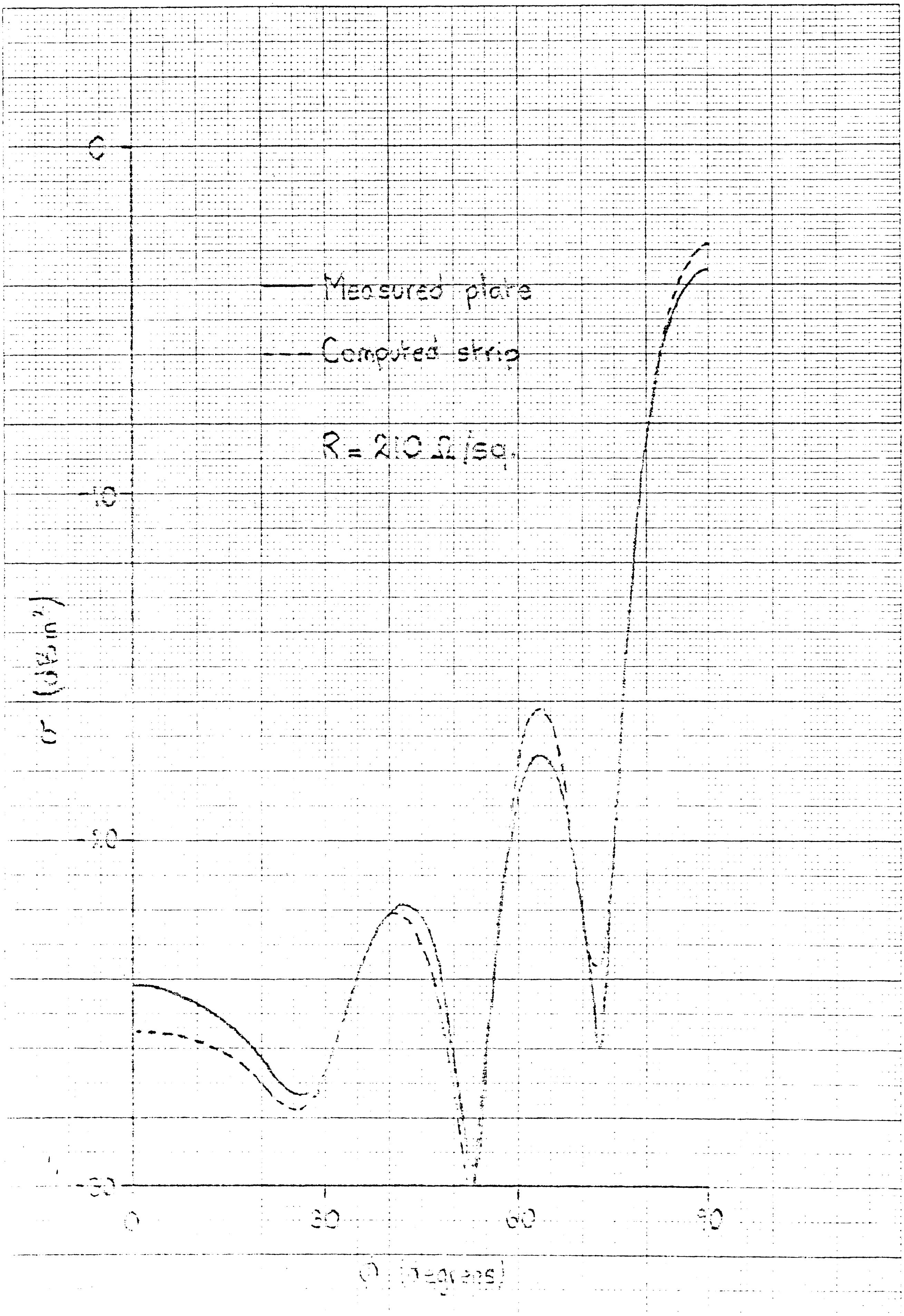
We have not yet tackled the non-static problem, but it is of interest to note that the static problem leads naturally to the singularity possessed by the second pair of integral equations. In fact, we have not yet found any straightforward approach to the potential problem producing kernels having second tangential derivatives, and for this reason out intent now is to focus on the second pair of integral equations rather than the first.

Reference

Y. Rahmat-Samii and R. Mittra, Integral equation solution and RCS computation of a thin rectangular plate, IEEE Trans. Antennas and Propagat. AP-22 (608-610) 1974.



Measured plate
Computed strip
0
-10
-20
σ (dBm²)
0
30
60
90
φ (degrees)



Measured plate
Computed strip
R = 210 Ω/sq.
σ (dBm²)
0
-10
-20
-30
0
30
60
90
θ (degrees)