EFFECT OF PARTICLE SHAPE ON LOW FREQUENCY ABSORPTION

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Abstract: For a homogeneous lossy dielectric spheroid, a simple expression for the absorption cross section at low frequencies is derived. When averaged over all orientations, the cross section is a maximum for a thin oblate spheroid (disk-shaped) particle.

When selecting a material for use in a 'smoke' screen, it is obviously desirable to maximize the absorption of the incident radiation, and if single particle scattering is the only (or dominant) process involved, it is then necessary to maximize the absorption by the individual particles. In addition to the obvious dependence on the material properties of the particle, the absorption depends on the particle shape, and for a particle of given volume and given dielectric constant, it is of interest to consider how the shape should be chosen. A frequency regime where an optimum exists is the low frequency one, i.e., Rayleigh scattering, where the wavelength is much greater than the particle dimensions, and we shall be exclusively concerned with this.

For a homogeneous, non-magnetic lossy dielectric particle illuminated by a plane electromagnetic wave, the far zone scattered field is attributable to an induced electric dipole of moment

$$\bar{p} = \varepsilon_0 \bar{\bar{P}} \cdot \hat{a}$$

where ε_0 is the permittivity of free space, $\hat{a}$ is a unit vector in the direction of the incident electric field and $\bar{\bar{P}}$ is the electric polarizability tensor.[1] $\bar{\bar{P}}$ is a real symmetric tensor whose elements are functions only of the properties of the particle. If, for simplicity, we confine attention to particles with rotational symmetry about the x_3 axis of a Cartesian coordinate system, the tensor diagonalizes with two of its diagonal elements equal, and

$$\bar{p} = \varepsilon_0 \{P_{11}\hat{a} + (P_{33} - P_{11})(\hat{a} \cdot \hat{x}_3)\hat{x}_3 \quad .$$

The scattered electric field in the far zone is then

$$\bar{E}^s = \frac{e^{ikr}}{kr} \bar{S}$$

with

$$\bar{S} = -\frac{k^3}{4\pi} \{P_{11}\hat{r}\wedge(\hat{r}\wedge\hat{a}) + (P_{33} - P_{11})(\hat{r} \cdot \hat{x}_3)\hat{r}\wedge(\hat{r}\wedge\hat{x}_3)\}$$

where k is the (free space) propagation constant, $\hat{r}$ is a unit vector in the direction of the point of observation, and a time factor $e^{-i\omega t}$ has been suppressed. In the forward scattering direction $\hat{r} \cdot \hat{a} = 0$, implying

$$\bar{S} \cdot \hat{a} = \frac{k^3}{4\pi} \{P_{11} + (P_{33} - P_{11})(\hat{a} \cdot \hat{x}_3)^2\} \ , \tag{1}$$

and if we now average over all orientations of the particle in space,

$$\bar{S} \cdot \hat{a} = \frac{k^3}{4\pi}\left\{\frac{1}{3}(2P_{11} + P_{33})\right\} \quad . \qquad (2)$$

We recognize $1/3(2P_{11} + P_{33})$ as the average trace[2] of the polarizability tensor.

The extinction cross section σ_{ext} is the sum of the absorption cross section σ_a and the total (integrated) scattering cross section σ_T, and from the forward scattering theorem[3]

$$\sigma_{ext} = \frac{4\pi}{k^2} \text{ Im. } \bar{S} \cdot \hat{a} = \frac{k}{3} \text{ Im. } (2P_{11} + P_{33}) \quad .$$

At low frequencies, however, σ_T is proportional to k^4 and is therefore negligible compared with the absorption cross section. Hence

$$\sigma_a = \frac{k}{3} \text{ Im. } (2P_{11} + P_{33}) \quad .$$

Studies of Rayleigh scattering in acoustics[4] and electromagnetics[5-7] have shown that for most convex particles, particularly ones having some degree of symmetry, the tensor elements are rather closely approximated by those of an equivalent spheroid for which analytical expressions for P_{11} and P_{33} are available. If V is the volume of the spheroid and ε_r is the complex relative permittivity,[1]

$$P_{11} = 2V\left\{q + \frac{2}{\varepsilon_r - 1}\right\}^{-1}$$

$$P_{33} = V\left\{-q + \frac{\varepsilon_r}{\varepsilon_r - 1}\right\}^{-1} \quad .$$

For a prolate spheroid

$$q = -\frac{1}{2}\xi(\xi^2 - 1)\ln\frac{\xi + 1}{\xi - 1} + \xi^2$$

where ξ is the radial variable defining the surface. In terms of the ratio of the length (ℓ) to the width (w) of the body

$$\xi = \frac{\ell/w}{\{(\ell/w)^2 - 1\}^{1/2}}$$

and varies from ∞ for a sphere to 1 for an infinitesimally thin spheroid (or needle) having w = 0. Over the entire range of ξ, q varies only from 1 for a needle down to 2/3 for a sphere. In the case of an oblate spheroid

$$q = \xi(\xi^2 + 1)\tan^{-1}\frac{1}{\xi} - \xi^2$$

where now

$$\xi = \frac{\ell/w}{\{1 - (\ell/w)^2\}^{1/2}}$$

and varies from ∞ for a sphere to 0 for an infinitesimally thin disk having ℓ = 0. The corresponding variation of q is from 2/3 for a sphere to 0 for a disk. Figure 1 shows a plot of q vs ℓ/w.

If $\varepsilon_r = 1 + a + ib$ where a,b are real with $a,b \geq 0$, it can easily be shown that

$$\text{Im. } P_{11} = \frac{4bV}{(2 + aq)^2 + (bq)^2}$$

$$\text{Im. } P_{33} = \frac{bV}{\{1 + a(1 - q)\}^2 + \{b(1 - q)\}^2}$$

and hence

$$\sigma_a = \frac{1}{3} kbV \Gamma \tag{4}$$

with

$$\Gamma = 8[(2 + aq)^2 + (bq)^2]^{-1} + [\{1 + a(1 - q)\}^2 + \{b(1 - q)\}^2]^{-1} \ . \tag{5}$$

For a disk (q = 0)

$$\Gamma = 2 + [(1 + a)^2 + b^2]^{-1} \tag{6}$$

whereas for a needle (q = 1)

$$\Gamma = 1 + 2[(1 + a/2)^2 + (b/2)^2]^{-1} \ . \tag{7}$$

These are approximately 2 and 1 respectively if a and/or b is large, and for almost all a and b (7) is less than (6). For a sphere (q = 2/3)

$$\Gamma = 27[(3 + a)^2 + b^2]^{-1} \tag{8}$$

which tends to zero as a and/or b $\rightarrow \infty$. Unless b lies in a narrow range of values with the range existing only if a $<$ 0.3,

$$\Gamma_{disk} > \Gamma_{needle} > \Gamma_{sphere} \ , \tag{9}$$

and Γ is actually a minimum for a sphere.

To illustrate the behavior as a function of q, Γ is plotted in Figure 2 for two contrasting values of the refractive index n: $n = 1.29 + i0.32$ implying $a = 0.56$ and $b = 0.83$, and $n = 2.12 + i0.71$, implying $a = b = 3.0$. In both cases Γ is a maximum for a disk. With the larger refractive index Γ decreases rapidly with increasing q, reaching a minimum for $q = 2/3$. It then rises to a value about one half that for a disk as q approaches unity. With the smaller refractive index the variation is less pronounced and the value of Γ for $q = 0$ is only about 10 percent greater than the value for $q = 1$.

Given a particular lossy dielectric we can therefore maximize the absorption cross section per unit volume of the material by using disk or plate-like particles. The resulting value of σ_a is significantly larger than for a sphere or similar-shaped particle, most particularly when the real and/or imaginary parts of the refractive index are large, but if the disk shape is difficult to obtain, a needle or rod-like particle is an alternative though less effective choice. Neither conclusion is surprising from a physical point of view. For maximum absorption we obviously require the maximum field concentration within the material, and provided the particles considered all lie in the Rayleigh region, the maximum can be achieved by minimizing one (or more) of the particle's dimensions. The fact that the tensor element P_{11}(associated with the transverse component of the dipole moment) appears twice in the expression for the average tensor trace, leads to a disk rather than a needle as the optimum particle shape.

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References

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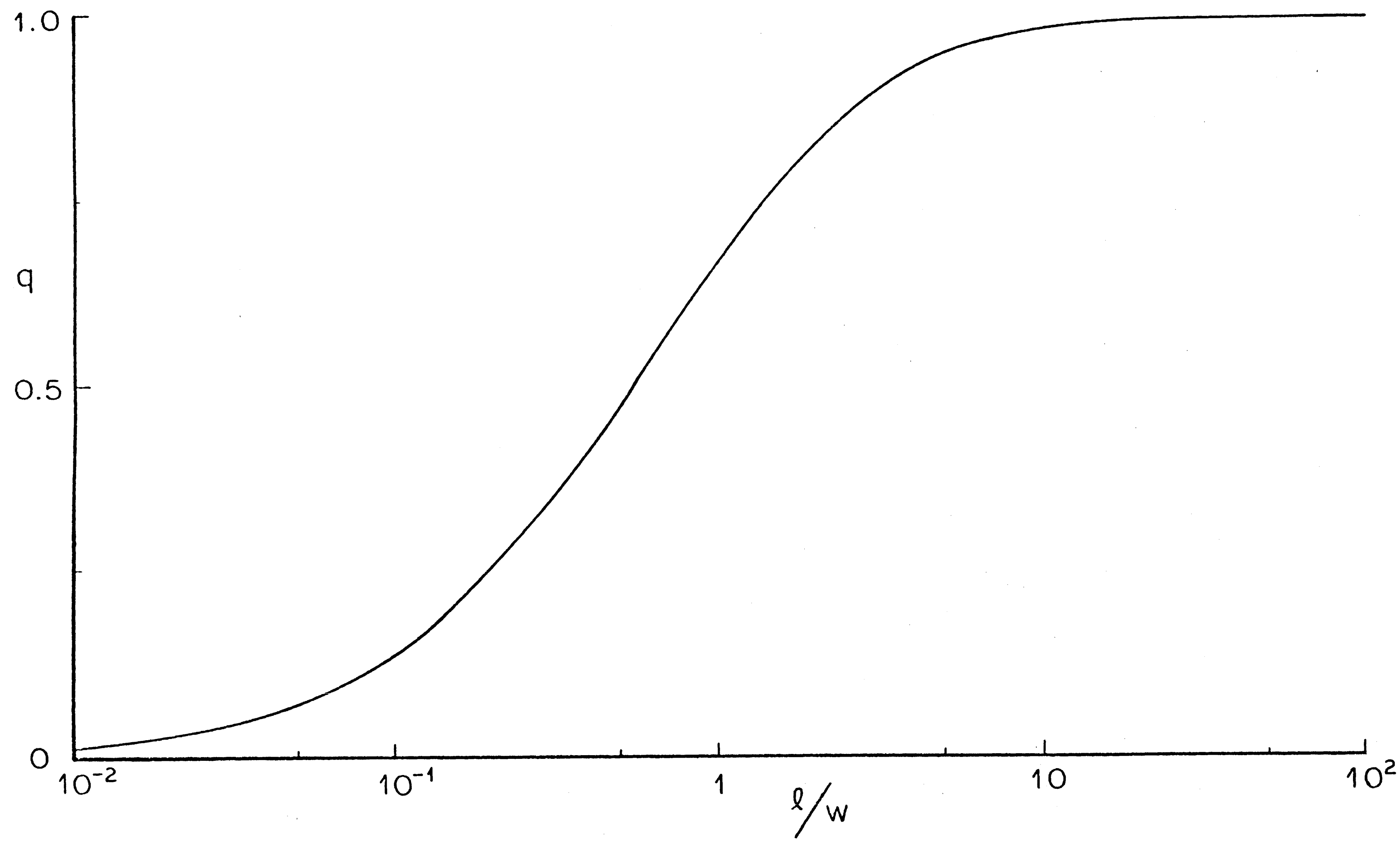
Legends for Figures

Fig. 1 q as a function of the length to width ratio ℓ/w of a spheroid.

Fig. 2 Γ as a function of q for the refractive indices n = 1.29 + i0.32 (solid line) and 2.12 + i0.71 (broken line).



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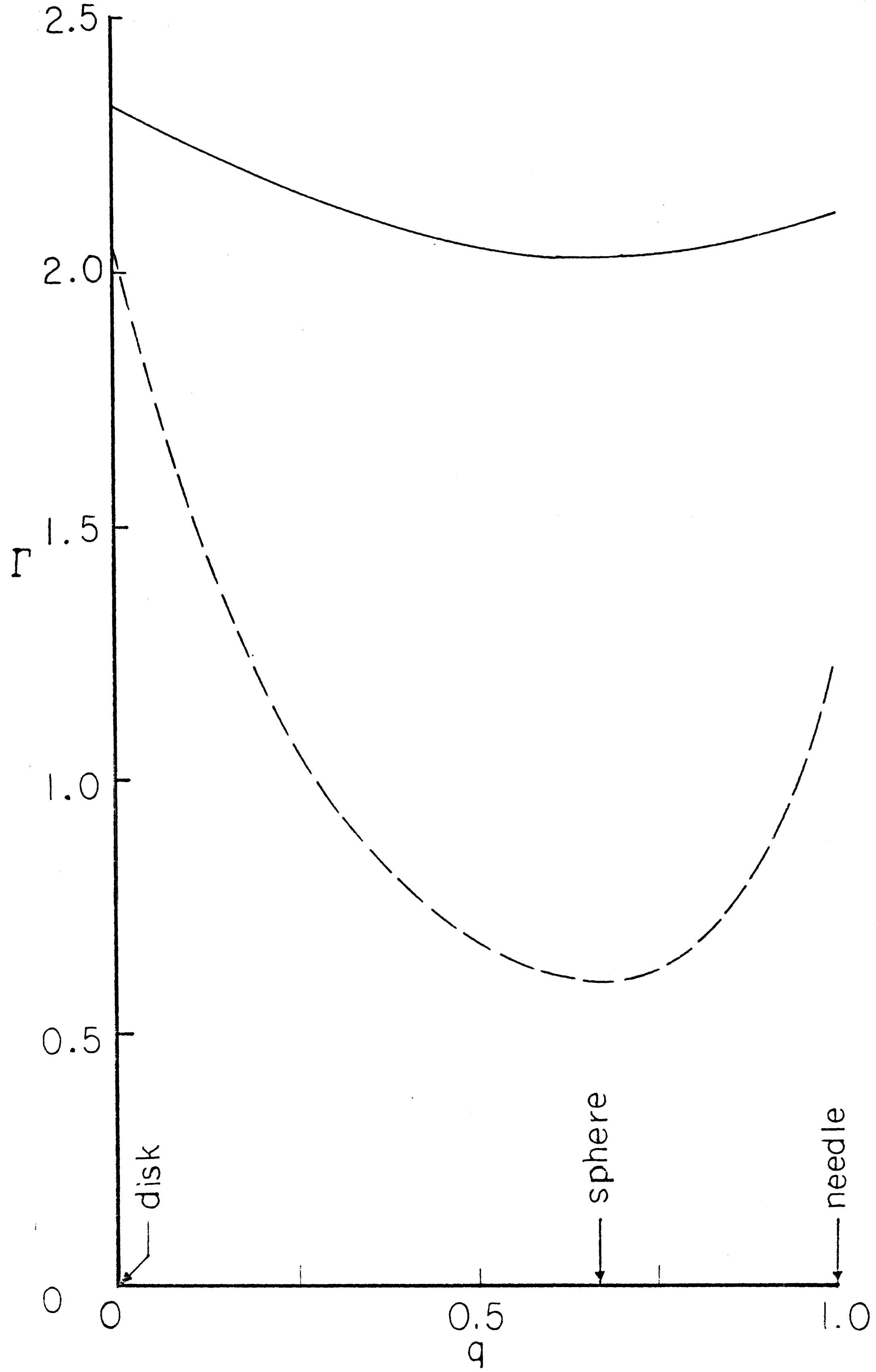

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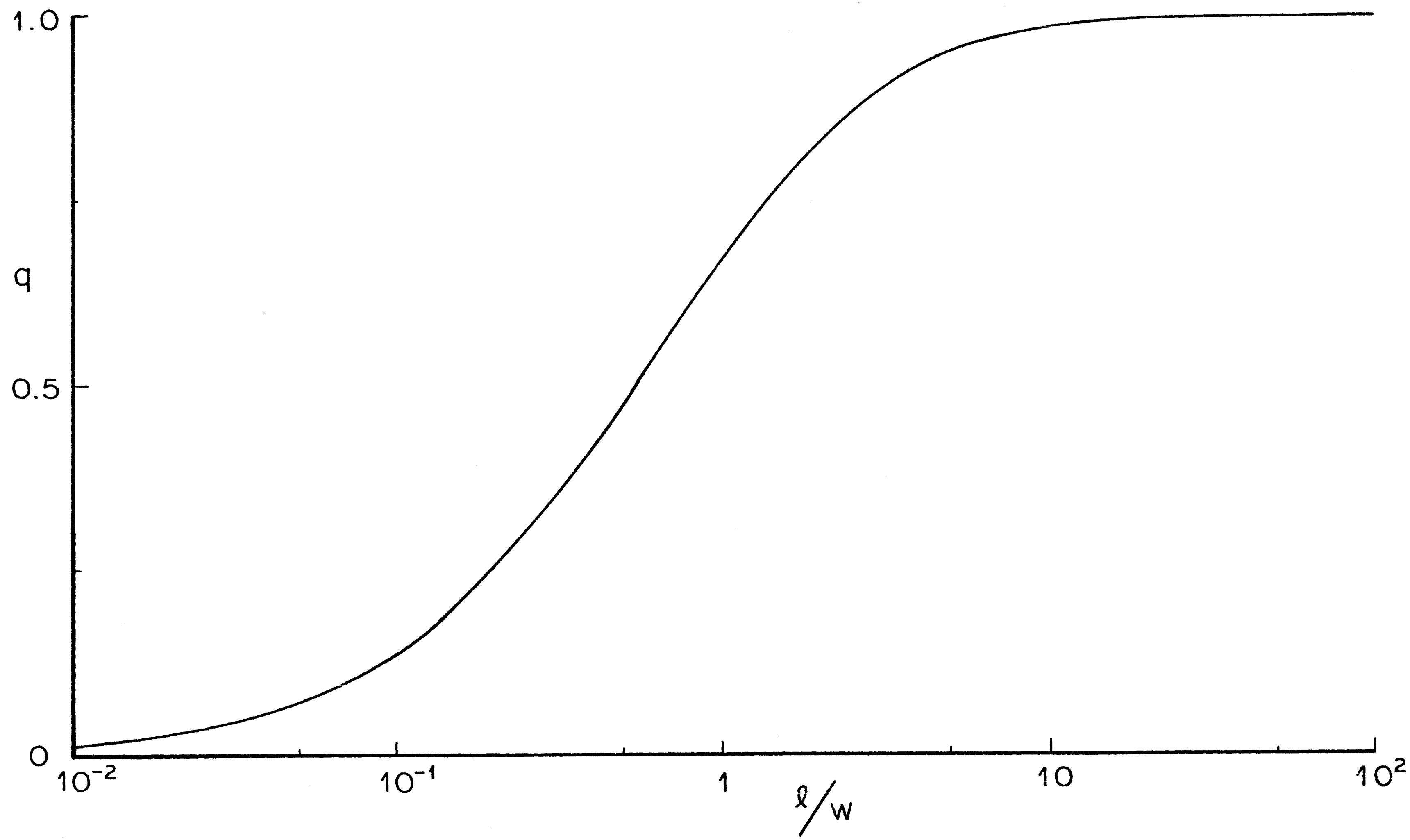
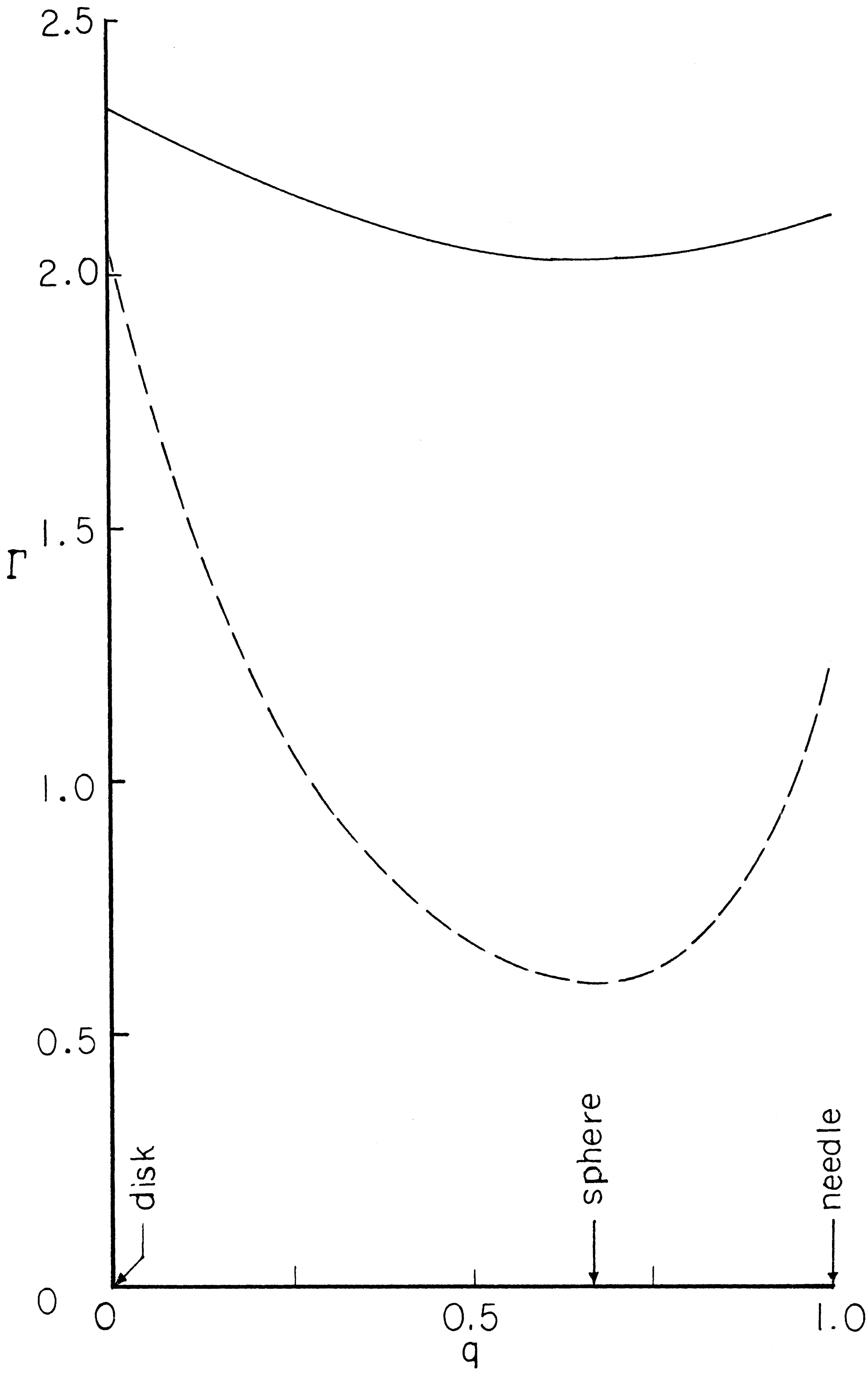


1.0
q
0.5
0
10⁻²
10⁻¹
1
10
10²
ℓ/w



2.5
2.0
1.5
Γ
1.0
0.5
0
0
0.5
1.0
q
disk
sphere
needle