THE CURRENT INDUCED IN A RESISTIVE HALF PLANE

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Abstract

For a resistive half plane illuminated by an E-polarized plane wave at edge-on incidence, an exact expression is available for the total induced current as a function of the electrical distance kx from the edge. The expression is cast in a form that is amenable to computation for any complex 'resistivity'. The results of such a computation are presented, and it is shown how the data can be used to predict the backscattered field of a strip.

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1. Introduction

In a recent study (Senior, 1979a) of scattering by resistive and impedance strips, it was shown that for a strip of large electrical width kw illuminated by a plane electromagnetic wave, the bistatic far field can be expressed in terms of the current induced in the corresponding half plane. The required values of the current are those at locations appropriate to the front and rear edges of the strip, and by using the known expressions for these currents, the resulting formulas for the far field of the strip are uniform in angle. In the particular case of a plane wave at edge-on incidence on a strip of uniform (real) resistivity R, the current is that supported by a half plane with surface impedance $\eta Z = 2R$ when illuminated by an E-polarized plane wave incident edge-on. An exact expression for this is available and some values have been computed. It is found that if these are used in preference to asymptotic expressions for the current, the values obtained for the backscattered far field of a strip are accurate even for kw as small as unity. The ability to compute the half plane current is therefore helpful in the selection of a strip material for low radar cross section.

To produce a viable structure it is necessary to rigidize the resistive sheet material, for example, by encasing it in plastic or fiberglass, thereby producing a complex 'resistivity'. The computation of the half plane current is more difficult if η is complex, and the purpose of this paper is to develop a formulation which is valid for all η, $|\arg \eta| \leq \pi/2$, and is amenable to computation. Some numerical results are presented, and their application is described.

2. Expression for the Current

A half plane having resistivity $R = \eta Z/2$, where Z is the intrinsic impedance of free space, occupies the region $x \geq 0$, $-\infty < z < \infty$ of the plane $y = 0$ of the Cartesian coordinate system (x,y,z), and is illuminated by an E-polarized plane wave at edge-on incidence. If $\underline{E}^i = \hat{z}\exp(ikx)$ where a time factor $\exp(-i\omega t)$ is assumed and suppressed, the total induced electric current is in the z direction and is

$$ZJ(x) = \frac{2i}{\pi} \int_C g(\beta) \frac{\cos\frac{\beta}{2}}{1 + \eta \sin\beta} e^{ikx\cos\beta} \alpha\beta$$

(Senior, 1979a). The path C (see Fig. 1) consists of the straight line segments $\beta = i\infty + \varepsilon$ to ε, ε to $\pi-\varepsilon$ and $\pi-\varepsilon$ to $\pi-\varepsilon-i\infty$ with $\varepsilon > 0$. The function $g(\beta)$ is expressible in terms of the split functions resulting from the solution of the diffraction problem by the Wiener-Hopf technique, and as shown by Senior (1952),

$$g(\beta) = \cos\frac{\beta}{2} \frac{K_+(k)}{K_+(k\cos\beta)} \tag{1}$$

where $K_+(\xi)$ is defined in (21) of that reference. Writing $\xi = k\cos\beta$ we have, after some manipulation,

$$g(\beta) = \left[\frac{\cos^2\chi - \sin^2\beta}{\cos^2\chi}\right]^{1/4}\left[\frac{\cos\beta + \sin\chi}{\cos\beta - \sin\chi}\,\frac{1-\sin\chi}{1+\sin\chi}\right]^{\chi/(2\pi)}$$

$$\exp\left[\frac{\cos\chi}{\pi}\int_0^{\sin\beta}\frac{\sin^{-1} y}{\cos^2\chi - y^2}\,dy\right]$$

which is more conveniently written as

$$g(\beta) = \left[\frac{\cos\beta + \sin\chi}{1+\sin\chi}\right]^{\chi/\pi} \exp$$

$$\left\{-\frac{i}{\pi}\int_0^{\sin\beta} \frac{\cos\chi\,\ln(\sqrt{1-y^2}+iy) - iy(\pi/2-\chi)}{\cos^2\chi - y^2}\right\} dy \qquad (2)$$

where $\cos\chi = 1/\eta$. Equations with which to compute χ knowing η are given in the Appendix. As evident from (2), $g(\beta)$ is an even function of β and is finite at $\beta = \pm(\pi/2 - \chi)$.

To evaluate the integral in (2) it is advantageous to deform C into the steepest descent path S(0) through $\beta = 0$ (see Fig. 1). The path is such that

$$\cos\beta = 1 + it^2 \qquad (3)$$

implying

$$\sin\frac{\beta}{2} = \frac{t}{\sqrt{2}}\,e^{-i\pi/4}$$

where t runs from $-\infty$ to ∞, and in the complex β plane

$$\text{Im}\,\beta = -2\tanh^{-1}\left\{\tan\left(\frac{1}{2}\,\text{Re}\,\beta\right)\right\}. \tag{4}$$

In the deformation of the path the pole at $\beta = -\pi/2 + \chi$ may be captured. The manner in which Re χ and Im χ vary with $|\eta|$ and arg η is illustrated in Figs. 2 and 3. Since $|\arg\eta| \leq \pi/2$ corresponding to a passive impedance, $0 \leq \text{Re}\,\chi \leq \pi/2$, and capture cannot occur unless $\arg\eta \geq 0$. In fact, from an examination of the pole location relative to the path defined by (4), the condition for capture is found to be

$$\text{Re}\left\{e^{-i\pi/4}\cos\frac{1}{2}\left(\frac{\pi}{2}+\chi\right)\right\} \leq 0 \quad , \tag{5}$$

and the resulting values of $|\eta|$ and arg η are shown in Fig. 4. Deformation of C into S(0) then gives

$$ZJ(x) = 8\Gamma A\, e^{ikx\sin\chi} + \frac{2i}{\pi}\int_{S(0)} g(\beta)\,\frac{\cos\frac{\beta}{2}}{1+\eta\sin\beta}\,e^{ikx\cos\beta}\,d\beta \tag{6}$$

where

$$A = \frac{1}{2}\,g\left(-\frac{\pi}{2}+\chi\right)\cot\chi\cos\frac{1}{2}\left(\frac{\pi}{2}-\chi\right) \tag{7}$$

and $\Gamma = 1$ if the pole is captured, but zero otherwise.

The first term on the right-hand side of (6) is the surface wave contribution. The second (integral) term must be evaluated numerically and to avoid the complication caused by the pole, we write it as

$$\int_{S(0)} \left\{ g(\beta) \frac{\cos \frac{\beta}{2}}{1 + \eta \sin \beta} - \frac{A}{\cos \frac{1}{2}(\beta - \frac{\pi}{2} - \chi)} \right\} e^{ikx \cos \beta} d\beta$$

$$+ A \int_{S(0)} \frac{e^{ikx \cos \beta}}{\cos \frac{1}{2} (\beta - \frac{\pi}{2} - \chi)} d\beta \quad . \tag{8}$$

The integrand of the first integral in (8) is now finite at $\beta = -\pi/2 + \chi$ and the second integral can be evaluated exactly as

$$\mp 4A\sqrt{\pi} \quad e^{i(kx \sin \chi + \pi/4)} \; F[\pm\sqrt{2kx} \cos \frac{1}{2} (\frac{\pi}{2} + \chi)] \tag{9}$$

where

$$F(\tau) = \int_{\tau}^{\infty} e^{iu^2} du \tag{10}$$

is the Fresnel integral. The upper (lower) signs in (9) must be chosen according as the condition (5) is violated (satisfied). When (9) and (8) are used in (6), the residue contribution can be absorbed, and using also the symmetry of the path S(0) about $\beta = 0$, we have

$$ZJ(x) = \frac{8A}{\sqrt{\pi}} e^{i(kx \sin \chi - \pi/4)} F[\sqrt{2kx} \cos \frac{1}{2}(\frac{\pi}{2} + \chi)]$$

$$+ \frac{2i}{\pi} \int_{S(0)} \left\{ g(\beta) \frac{\cos^2 \chi}{\cos^2 \beta - \sin^2 \chi} - \frac{2A \cos \frac{1}{2} (\frac{\pi}{2} + \chi)}{\cos \beta - \cos \chi} \right\}$$

$$\cdot \cos \frac{\beta}{2} e^{ikx \cos \beta} d\beta \quad . \tag{11}$$

The final step is to change the variable of integration from β to t as shown in (3). The expression for the current then becomes

$$ZJ(x) = \frac{8A}{\sqrt{\pi}} e^{i(kx \sin\chi - \pi/4)} F[\sqrt{2kx} \cos \tfrac{1}{2} (\tfrac{\pi}{2} + \chi)]$$

$$+ \frac{4\sqrt{2}}{\pi}(1 - \sin\chi)\, e^{i(kx + \pi/4)} \int_0^\infty \left\{ h(t) - A \sec \tfrac{1}{2} (\tfrac{\pi}{2} + \chi) \right\} \cdot \frac{e^{-kxt^2}}{1 - \sin\chi + it^2}\, dt \qquad (12)$$

where

$$h(t) = \frac{1 + \sin\chi}{1 + \sin\chi + it^2}\, g(\beta) ,$$

and from (2) with $y = \{1 - (1 + iv^2)^2\}^{1/2}$,

$$h(t) = \left[\frac{1 + \sin\chi + it^2}{1 + \sin\chi} \right]^{\chi/\pi - 1} \exp \left\{ \frac{2 \cos\chi}{\pi} \int_0^t \left[\ln \left(1 + iv^2 + i \left\{ 1 - (1 + iv^2)^2 \right\}^{1/2} \right) - i \left(\tfrac{\pi}{2} - \chi \right) \sec\chi \left\{ 1 - (1 + iv^2)^2 \right\}^{1/2} \right] \left[\sin^2\chi - (1 + iv^2)^2 \right]^{-1} \cdot \left[1 - (1 + iv^2)^2 \right]^{-1/2} (1 + iv^2) v\, dv \right\} , \qquad (13)$$

Clearly, $h(0) = 1$. Also, from (7) and (2) with the substitution $y = -v \cos \chi$,

$$A = \cos \frac{1}{2} \left(\frac{\pi}{2} + \chi \right) \left[\frac{2 \sin \chi}{1 + \sin \chi} \right]^{\chi/\pi - 1}$$

$$\cdot \exp \left\{ \frac{i}{\pi} \int_0^1 \left[\ln \left(\left\{ 1 - v^2 \cos^2 \chi \right\}^{1/2} - iv \cos \chi \right) + i \left(\frac{\pi}{2} - \chi \right) v \right] (1 - v^2)^{-1} \, dv \right\} \qquad (14)$$

valid for all $\chi \neq 0$, i.e., $\eta \neq 1$. If $\eta = 1$ the pole is neither captured nor lies close to the path of integration, and A can be put equal to zero.

The above results are exact and (12) through (14) enable $ZJ(x)$ to be computed as a function of kx for any η. The Fresnel integral was introduced to handle the special case when the pole lies on the path. For other values of η it is mathematically permissible to put $A = 0$ in the integral in (12) and to replace the Fresnel integral by $\sqrt{\pi}\, e^{i\pi/4}$ or 0 depending on whether the pole is or is not captured in the deformation of the path. Numerically, however, it is desirable to retain the more complicated form (12) whenever the pole lies close to the path. The pole contribution represents a surface wave of amplitude 8A and its presence has a major effect on the current, particularly for small values of kx. The only case in which the surface wave is unattenuated is when the sheet is purely reactive ($\arg \eta = \pi/2$), and (14) then implies

$$|A| = \left|\frac{\cos\chi}{2\sqrt{\sin\chi}}\right| . \tag{15}$$

3. The Program and Its Application

A program designated SURFCOM has been written to compute ZJ(x) as a function of kx for any given η and is available from the author. The numerical integration required to compute A, h(t) and, finally, ZJ(x) from (12) is performed by a fifth-order Runge-Kutta method as described, for example, by Lambert (1973). The Fresnel integral is evaluated using series expansions, and, for simplicity, is retained regardless of η. The input parameters are η in modulus and argument, and the initial, incremental and final values of kx. As the program is presently written the maximum value of t used in evaluating the integral in (12) is $(18/kx)^{1/2} \leq 30$, which therefore requires that $kx \geq 0.02$. We remark that

$$ZJ(0) = 2\eta^{-1/2} K_+(k) \tag{16}$$

Its computation has been discussed by Senior (1975, 1979b), and asymptotic approximations are available which are valid for small and large $|\eta|$. In particular, ZJ(0) = 0.4645 and 0.4968 exp $(-0.475\pi i)$ for $\eta = 4$ and $4i$ respectively.

To illustrate the results obtained from the program, the amplitude and phase of the current ZJ(x) as functions of kx are plotted in Figs. 5 and 6 for $\eta = 4\exp(i\theta)$ with $\theta = 0,30,45,60$ and $90°$.

As evident from Fig. 4, $\theta = 45°$ is a case when the pole lies almost precisely on the path and the inclusion of the Fresnel integral is then vital. For $\theta \geq 45°$ the pole is captured and the surface wave contribution is responsible for the initial increase in $ZJ(x)$ with x. When $\theta = 90°$ the surface wave is unattenuated and $|ZJ(x)|$ approaches the surface wave amplitude $8A(=1.016)$ as x increases. For $\theta < 0$ the current amplitude is similar to that for $\theta = 0$, but slightly less except within a fraction of a wavelength of the edge. For all values of θ the dominant part of the phase is that of the incident field, viz. kx.

Senior (1979a) showed that for a resistive strip of width w illuminated by an E-polarized plane wave at edge-on incidence, the backscattered far field can be written as

$$E^s = \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-\pi/4)}\, P$$

with

$$P = P^f + P^r \quad , \tag{17}$$

where

$$P^f = -\frac{i\eta}{16}\{ZJ(0)\}^2 \tag{18}$$

and

$$P^r = \frac{i}{4\eta}\left\{\frac{ZJ(w)}{ZJ(0)}\right\}^2 \tag{19}$$

are the front and rear edge contributions respectively, and the current is that on a half plane of the same resistivity. Although this is a high frequency approximation, the resulting values of P are remarkably accurate even for small values of w. This is illustrated in Fig. 7 where the values of $|P|$ computed using the half plane currents for $\eta = 4$ and 4i are compared with those obtained from a numerical solution of the integral equation for a strip (Senior, 1979b). The agreement is excellent even for kw as small as 0.3 ($w/\lambda \simeq 0.05$), though for smaller values of $|\eta|$ discrepancies are evident for $kw \lesssim 1$.

4. Concluding Remarks

A problem of some practical importance is to reduce the radar scattering cross section of a planar structure such as the wing or fin of an aircraft. The scattering can often be approximated by that of a strip or ribbon, and this leads to a consideration of a strip composed of a composite and, perhaps, multilayer material. In many instances a material of this type can be simulated by a resistive sheet of possibly complex resistivity. If the resistivity is reasonably large, the reduction in the backscattering cross section for edge-on incidence is typical of that achieved at all angles (Senior, 1979b), and the ability to compute the half plane current for any resistivity then constitutes a valuable design tool.

5. Acknowledgements

The author is indebted to C. J. Roussi and T. M. Willis for their assistance with the computer program, and to Dr. R. M. Bevensee of Lawrence Livermore Laboratory for providing the complex error function code used to compute the Fresnel integral.

This work was supported by the Air Force Office of Scientific Research under Grant 77-3188.

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Appendix A

By definition $\cos\chi = 1/\eta$. Let $\eta = |\eta|e^{i\theta}$ where $\theta = \arg\eta$ and $|\theta| \leq \pi/2$. Then if $\chi = \chi_r + i\chi_i$ with $0 \leq \chi_r \leq \pi/2$, we have

$$\chi_r = \sin^{-1}\left\{\frac{1}{2|\eta|^2}\left[|\eta|^2 - 1 + \left(\left\{|\eta|^2 - 1\right\}^2 + 4|\eta|^2\sin^2\theta\right)^{1/2}\right]\right\}^{1/2} \tag{A.1}$$

and

$$\chi_i = \sinh^{-1}\left(\frac{\sin\theta}{|\eta|\sin\chi_r}\right) = \cosh^{-1}\left(\frac{\cos\theta}{|\eta|\cos\chi_i}\right) . \tag{A.2}$$

Clearly

$$\chi_r(-\theta) = \chi_r(\theta) \quad , \quad \chi_i(-\theta) = -\chi_i(\theta) \quad .$$

Values of χ_r and χ_i as functions of η, $0.1 \leq |\eta| \leq 10$ and $0 \leq \theta \leq \pi/2$, are shown in Figs. 2 and 3 respectively.

Legends for Figures

Fig. 1: The paths of integration in the complex β plane. The pole at $\beta = \chi - \pi/2$ is captured if it lies in the shaded region.

Fig. 2: Curves of constant χ_r computed from (A.1).

Fig. 3: Curves of constant χ_i computed from (A.2).

Fig. 4: Arg η as a function of $|\eta|$ for which the pole at $\beta = \chi - \pi/2$ lies on the path S(0). For all larger values of arg η the pole is captured.

Fig. 5: Current amplitude for $|\eta| = 4$ and arg η = 0,30,45,60 and 90 degrees.

Fig. 6: Current phase for $|\eta| = 4$ and arg η = 0,30,45,60 and 90 degrees.

Fig. 7: $|P|$ for a strip of width w having $\eta = 4$ (——) and 4i (---) computed using (17) through (19) and data for the half plane currents. The circled points are values obtained by numerical solution of the integral equation for a strip.

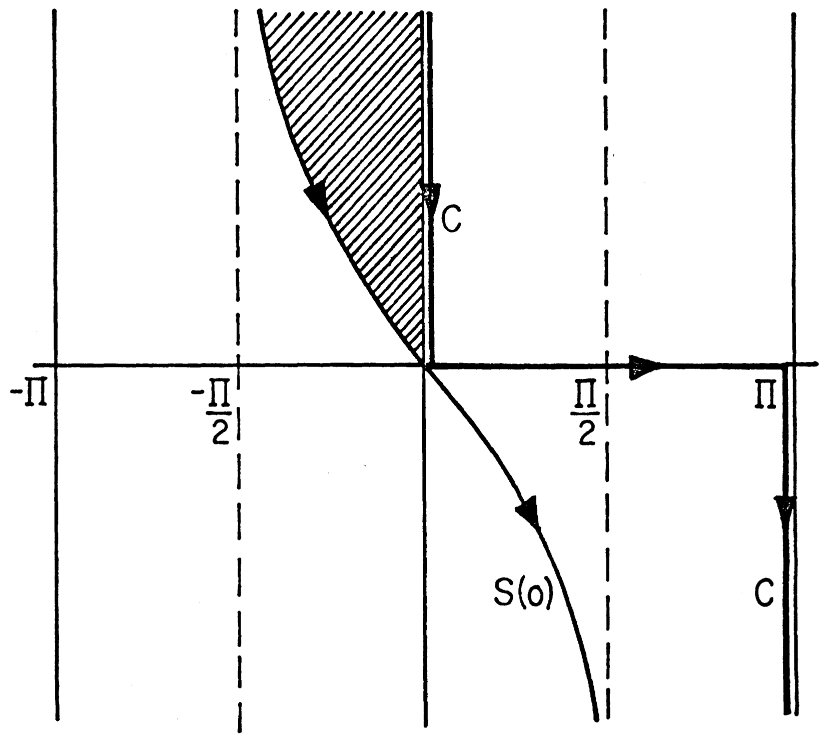
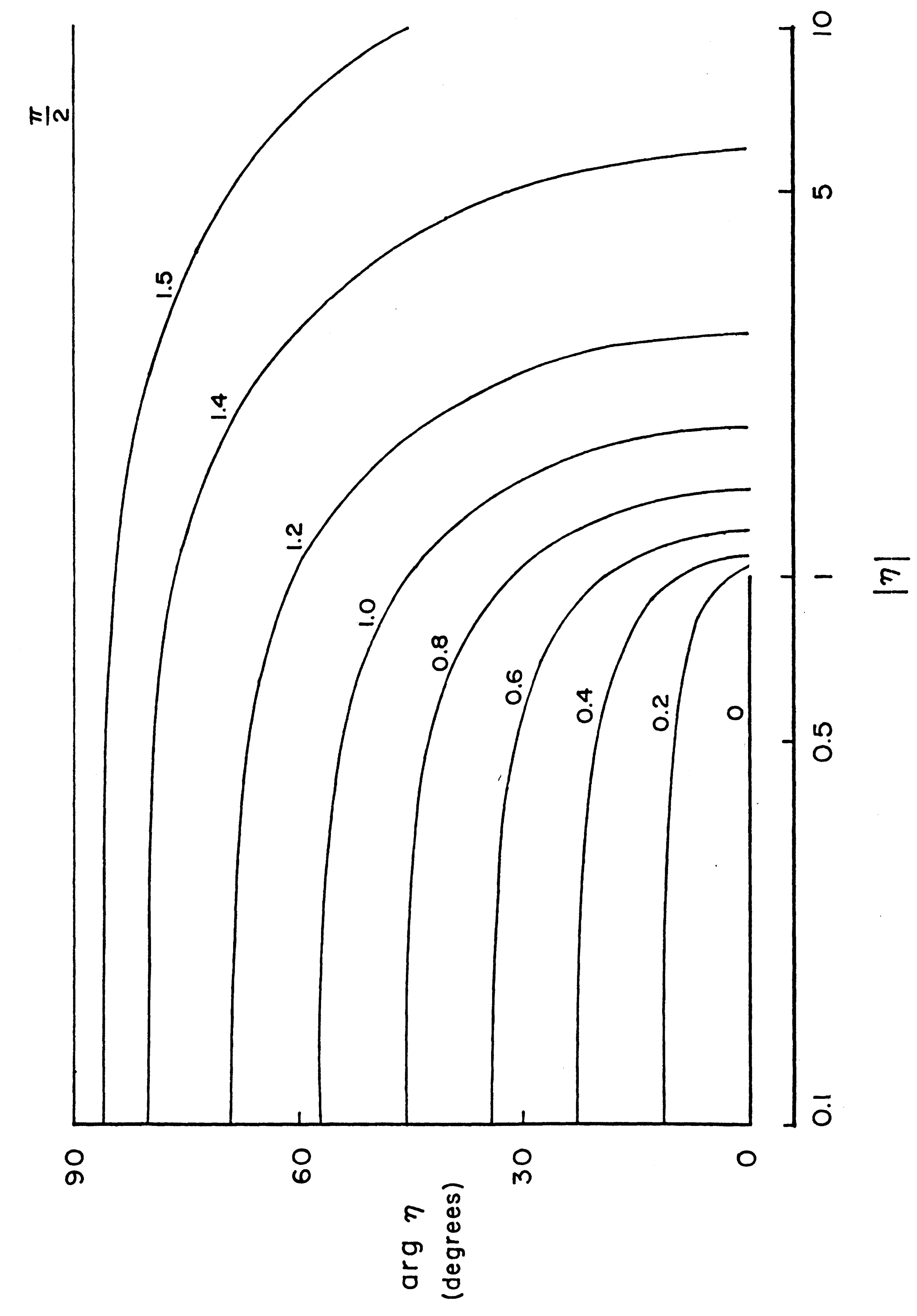
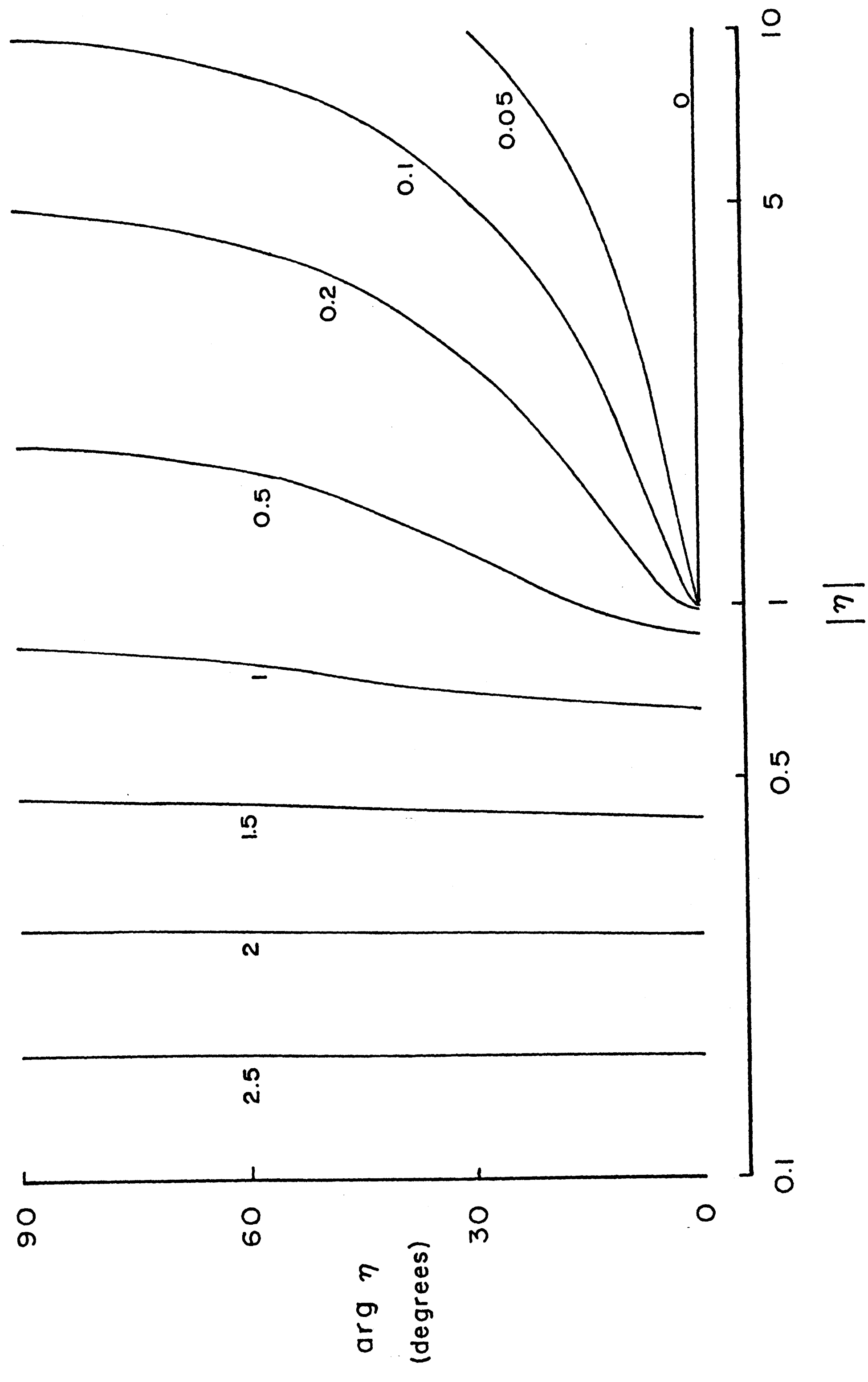


figure 1



$\frac{\pi}{2}$
1.5
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
90
60
30
0
arg η
(degrees)
0.1
0.5
1
5
10
$|\eta|$

figure 2



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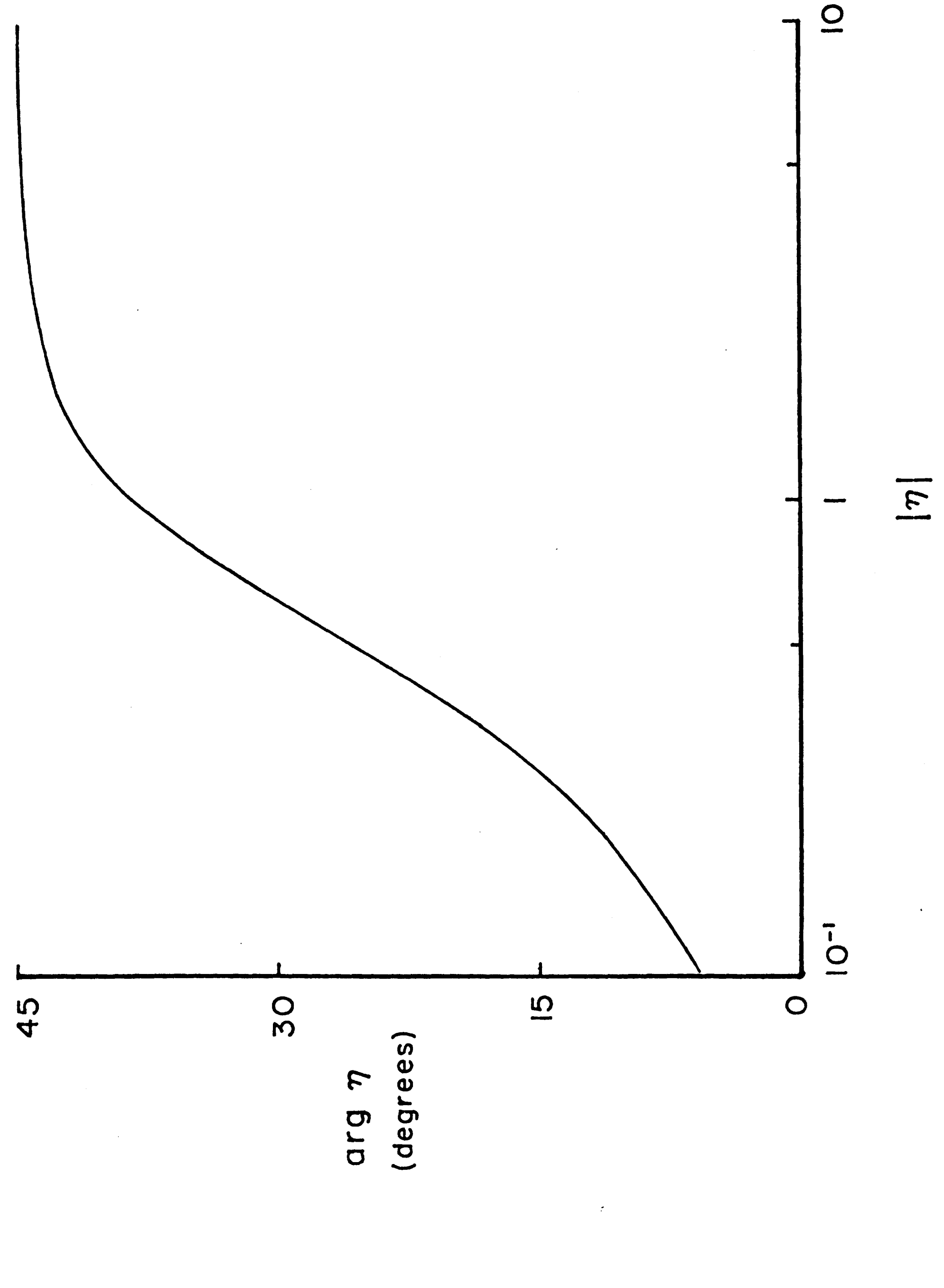


figure 4

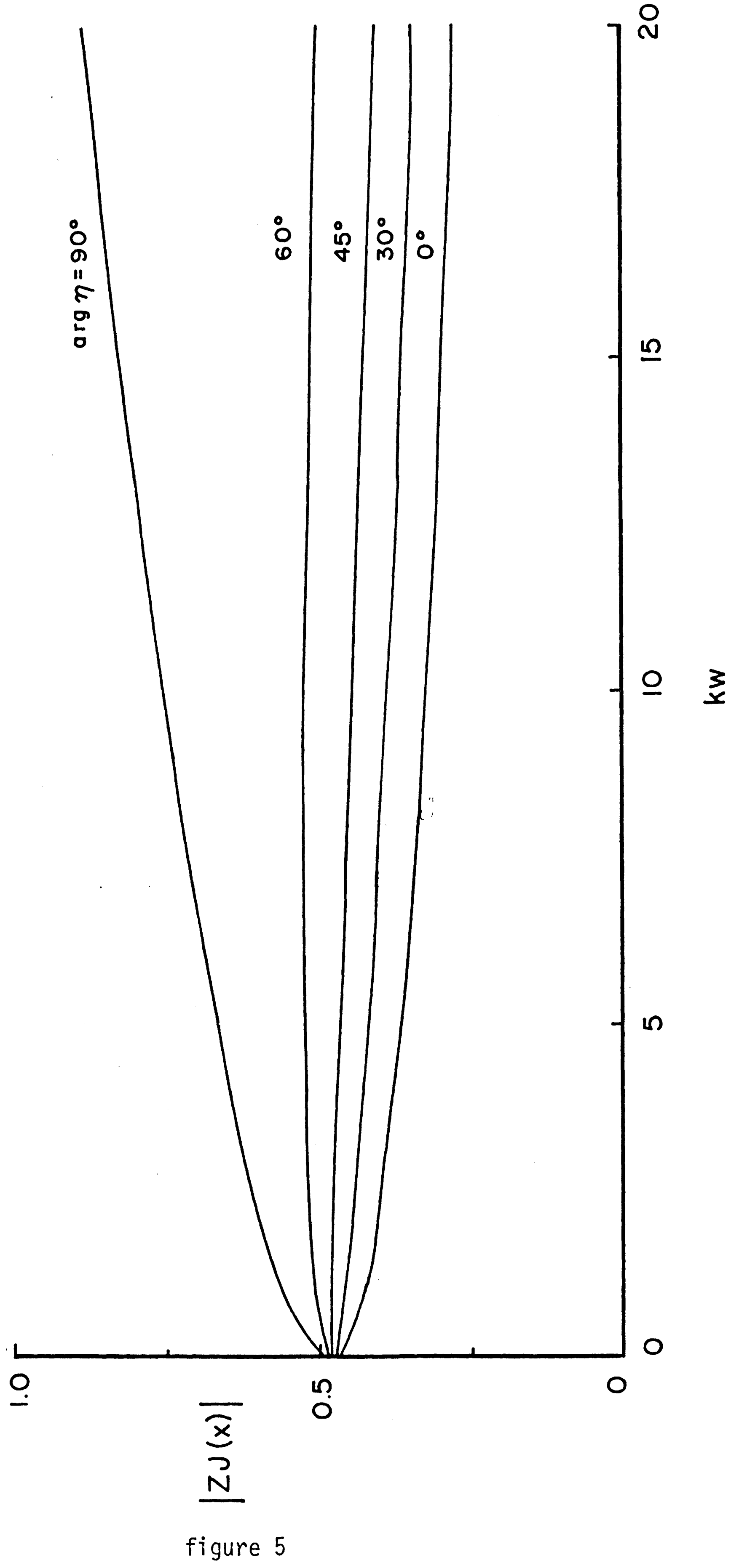
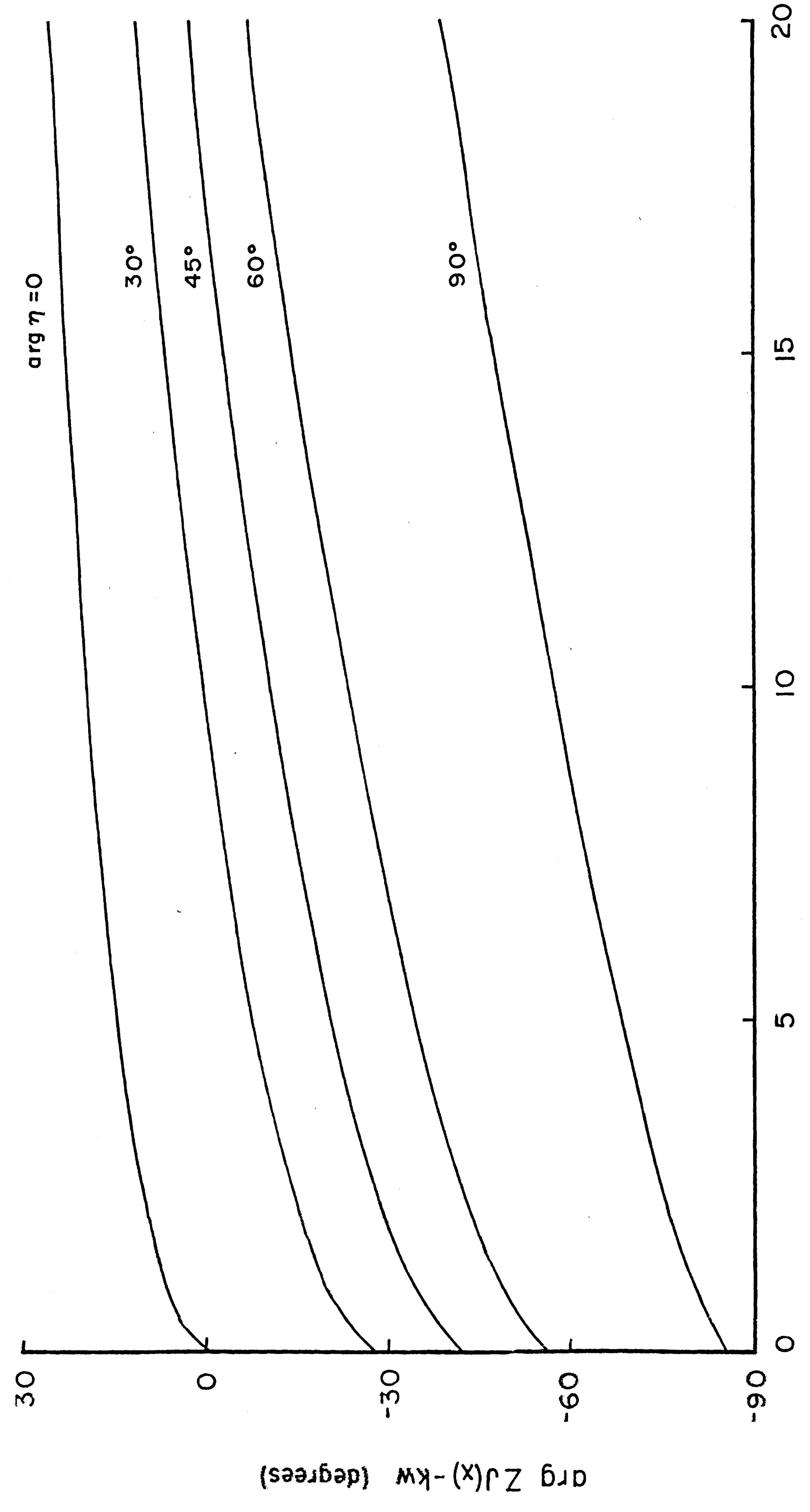


figure 5



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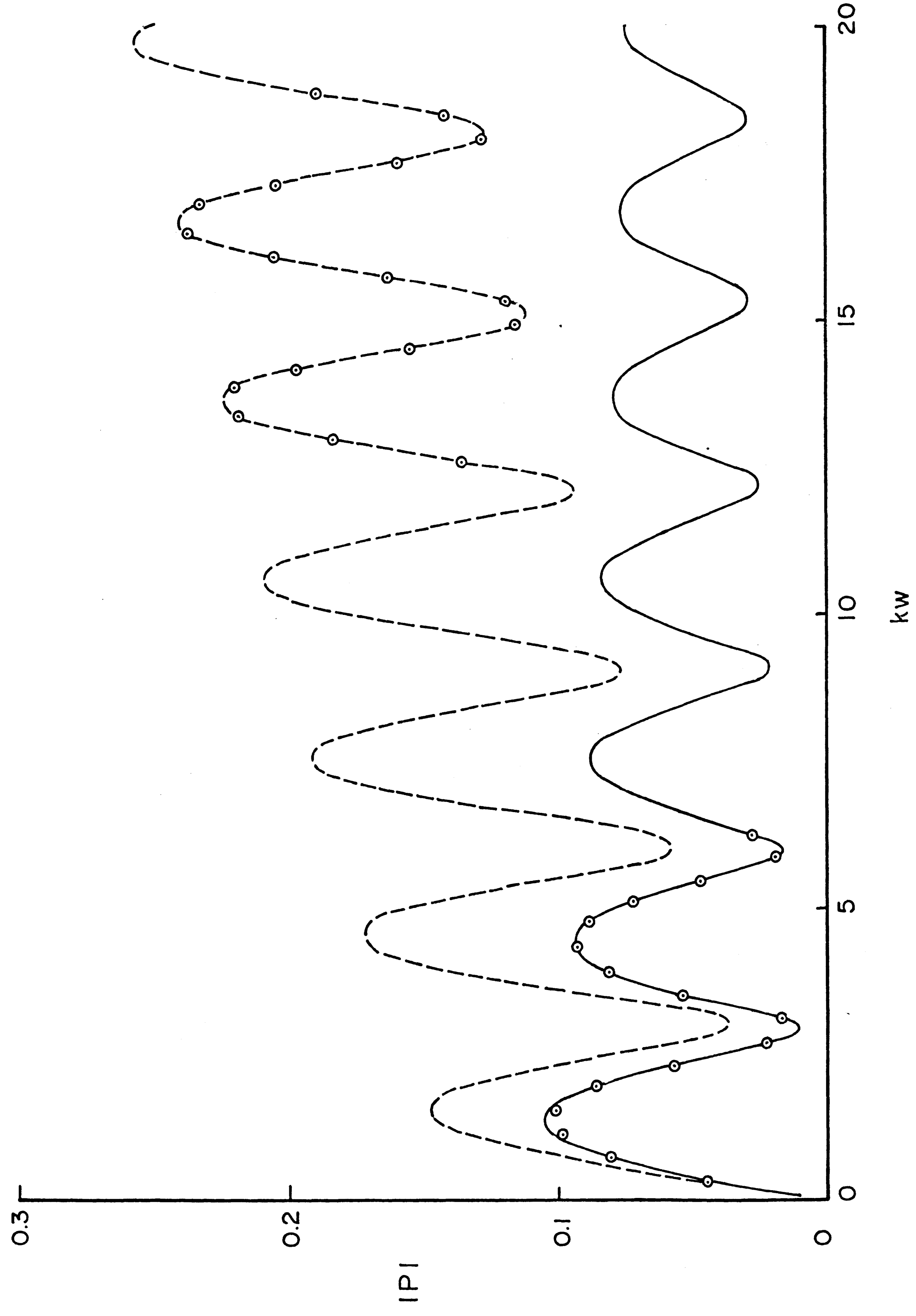

figure 7