LOW FREQUENCY SCATTERING BY A RESISTIVE PLATE

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Abstract

For a plane wave incident on a resistive plate, the first two terms in the low frequency expansion of the far field are determined. Since the magnetic dipole is not excited if the resistivity is non-zero, the leading term is simply the electric dipole contribution, and is identical to that for a perfectly conducting plate. The resistivity appears explicitly in the next term, and it is shown that this can be expressed in terms of potentials analogous to the zeroth-order ones. Some implications of the results are discussed.

Introduction

Resistive sheet materials find many applications, not least for purposes of cross section reduction. To replace a thin metallic plate by a plate made of a composite or other material that can be represented as a resistive sheet can significantly reduce the radar cross section, and analyses and computations for the two-dimensional analogue of a strip have demonstrated [1-3] the advantages at frequencies in the resonance region and above. To see if these advantages persist as the frequency is lowered, we here examine the low frequency behavior of a resistive plate of finite dimensions.

When an electromagnetic wave illuminates a body whose dimensions are small compared to the wavelength, the far zone scattered field can be expanded as a power series in kL where k is the wavenumber and L is a characteristic dimension of the body. The leading (Rayleigh) terms in the expansion are attributable to induced electric and/or magnetic dipoles. For a plate of resistivity $\mathcal{R}$, the boundary conditions imply that the electric dipole is identical to that for a perfectly conducting plate, and that the magnetic dipole contribution is zero. The low frequency expansion is therefore discontinuous in the perfectly conducting limit $\mathcal{R} = 0$, and to see how the scattering depends on $\mathcal{R}$, it is necessary to include higher order terms in the expansion.

The next (first order) contribution is considered and expressed in terms of potentials analogous to the static ones. Some implications of the results are presented.

Formulation

An infinitesimally thin resistive plate B of finite dimensions lies in the plane z = 0 or a Cartesian coordinate system (x,y,z) whose origin is in the vicinity of the plate. It is illuminated by a plane linearly polarized electromagnetic wave whose electric and magnetic vectors are

$$\bar{E}^{inc} = \hat{a} \, e^{ik\hat{k}\cdot\bar{r}} \quad , \quad \bar{H}^{inc} = Y\hat{b} \, e^{ik\hat{k}\cdot\bar{r}} \tag{1}$$

where $\hat{k}$, $\hat{a}$ and $\hat{b}$ are unit vectors specifying the directions of incidence, the electric field (or polarization) and the magnetic field respectively. The three vectors are mutually perpendicular and form a right-handed system. The permittivity and intrinsic admittance of the surrounding free space region are ε and $Y(=1/Z)$, and a time factor $e^{-i\omega t}$ is suppressed.

Since the plate can support only an electric current, the scattered field can be expressed in terms of an electric Hertz vector, and in the far zone [4]

$$\bar{E}(\bar{r}) \sim - \frac{e^{ikr}}{4\pi r} \, k^2 \hat{r}_\wedge \hat{r}_\wedge \left[\int_B \hat{z}\cdot\bar{E}\Big|_-^+ \, \bar{r}' \, dS' + Z \int_B \hat{z}_\wedge \bar{H}\Big|_-^+ \, \hat{r}\cdot\bar{r}' \, dS' \right.$$

$$\left. - \frac{ikZ}{2} \int_B \hat{z}_\wedge \bar{H}\Big|_-^+ \, (\hat{r}\cdot\bar{r}')^2 \, dS' + O(k^2) \right] \tag{2}$$

where a vertical line denotes the discontinuity across the plate. In the near zone (including the surface) the fields can be expanded as

$$\bar{E} = \sum_{m=0}^{\infty} (ik)^m \bar{E}_m \quad , \quad \bar{H} = \sum_{m=0}^{\infty} (ik)^m \bar{H}_m \quad ,$$

and when these are inserted into (2), we have

$$\bar{E}(\bar{r}) \sim - \frac{e^{ikr}}{4\pi r} k^2 \left[\frac{1}{\varepsilon} \hat{r}_\wedge(\hat{r}_\wedge \bar{p}) + Z\hat{r}_\wedge\bar{m} + ik\hat{r}_\wedge[\hat{r}_\wedge\bar{F}(\hat{r})] + O(k^2) \right] \quad , \qquad (3)$$

where

$$\bar{F}(\hat{r}) = \int_B \hat{z}\cdot\bar{E}_1\Big|_-^+ \bar{r}' \, dS' + Z \int_B \hat{z}_\wedge\bar{H}_1\Big|_-^+ \hat{r}\cdot\bar{r}' \, dS'$$

$$- \frac{Z}{2} \int_B \hat{z}_\wedge\bar{H}_0\Big|_-^+ (\hat{r}\cdot\bar{r}')^2 \, dS' \quad ; \qquad (4)$$

$$\bar{p} = \varepsilon \int_B \hat{z}\cdot\bar{E}_0\Big|_-^+ \bar{r}' \, dS' \qquad (5)$$

is the electric dipole moment, and [5]

$$\bar{m} = \frac{1}{2} \int_B \bar{r}'_\wedge(\hat{z}_\wedge\bar{H}_0\Big|_-^+) \, dS' \qquad (6)$$

is the magnetic dipole moment. Clearly $\bar{p}$ has components only parallel to the plate and $\bar{m}$ has just the single component normal to the plate.

The next step is to invoke the properties of the plate itself. To appreciate the concept of a resistive sheet, consider a thin layer of conducting material whose permeability is that of free space. If σ is the conductivity and τ is the thickness, a surface resistivity can

be defined as $\mathcal{R} = (\sigma\tau)^{-1}$ ohms, and as τ is decreased we can imagine σ increased in such a manner that $\mathcal{R}$ is finite in the limit $\tau = 0$. The result is an infinitesimally thin sheet whose electromagnetic properties are specified by the quantity $\mathcal{R}$. Thin layers of many types of material can be simulated in this manner, and sheets are commercially available with a wide variety of resistivities. $\mathcal{R}$ is typically measured at dc by applying a voltage between two electrodes on the sheet, and since the resistivity remains constant over a broad range of frequencies, it can be assumed to be independent of k at low frequencies at least.

Zeroth Order Terms

Mathematically a resistive sheet is simply an electric current sheet whose current is proportional to the tangential electric field at its surface [6]. For a sheet lying in the plane z = 0, the boundary conditions on the total (incident plus scattered) fields are

$$\hat{z}_\wedge \bar{E}^t \big|_-^+ = 0 \tag{7}$$

and

$$\hat{z}_\wedge \bar{E}^t = \eta Z \, \hat{z}_\wedge (\hat{z}_\wedge \bar{H} \big|_-^+) \tag{8}$$

where $\eta = \mathcal{R}/Z$, and these are sufficient to ensure a unique solution at all frequencies including, as a limiting case, zero.

To see the implications when k = 0 we have only to recall that the permeability does not differ from that of the surrounding medium.

The sheet is therefore invisible as regards the static magnetic field, i.e., $\bar{H}_0 = 0$, and (8) then implies that for all η $\bar{E}_0$ is the same as if $\eta = 0$. Thus, for a resistive plate, $\bar{m} = 0$ and the electric dipole moment $\bar{p}$ is the same as for a perfectly conducting plate.

The computation of $\bar{p}$ is discussed in [4]. If x_1, x_2, x_3 are Cartesian coordinates with $x_3 = z$,

$$\bar{E}_0 = -\sum_{i=1}^{2} (\hat{a} \cdot \hat{x}_i) \nabla \phi_0^i .$$

ϕ_0^i is an exterior potential satisfying the boundary condition

$$\phi_0^i = x_i + c_i \tag{9}$$

on B, where the constant c_i is chosen to give zero induced charge, i.e.,

$$\int_B \frac{\partial \phi_0^i}{\partial z}\bigg|_-^+ dS = 0 .$$

In terms of ϕ_0^i,

$$\bar{p} = \epsilon \sum_{i=1}^{2} (\hat{a} \cdot \hat{x}_i) \int_B \frac{\partial \phi_0^i}{\partial z'}\bigg|_-^+ \bar{r}' \, dS' \tag{10}$$

and an integral equation from which to determine $\frac{\partial \phi_0^i}{\partial z}\bigg|_-^+$ is

$$-x_i - c_i = \frac{1}{4\pi} \int_B \frac{1}{R} \frac{\partial \phi_0^i}{\partial z'}\bigg|_-^+ dS' \tag{11}$$

where $R = |\bar{r} - \bar{r}'|$. Its solution by the moment method is a relatively simple task [4].

First Order Terms

To see how the low frequency scattered field depends on η, it is necessary to examine the first order terms in the far zone expansion (3).

Since $\bar{H}_0 = 0$, the third term in the expression (4) for $\bar{F}(\hat{r})$ is zero. To evaluate the second term, we remark that on the plate

$$\bar{H}_z^t = \frac{Y}{ik} \left(\frac{\partial E_y^t}{\partial x} - \frac{\partial E_x^t}{\partial y} \right)$$

$$= \frac{\eta}{ik} \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) \Big|_-^+$$

$$= - \frac{\eta}{ik} \frac{\partial H_z}{\partial z} \Big|_-^+$$

and hence

$$\frac{\partial H_{1z}}{\partial z} \Big|_-^+ = - \frac{1}{\eta} H_{0z}^t = - \frac{Y}{\eta} \hat{b} \cdot \hat{z}$$

with $H_{1z}\Big|_-^+ = 0$. From Maxwell's equations we also have

$$\frac{\partial}{\partial z} (\hat{z} \wedge \bar{H}_1) = \hat{z} \wedge \nabla H_{1z} + Y\hat{z} \wedge (\hat{z} \wedge \bar{E}_0) \quad , \tag{12}$$

showing that the quantity on the left is continuous across the plate.

The Cartesian components of $\bar{H}_1$ are themselves exterior potentials, and therefore

$$H_{1z} = -\frac{1}{4\pi} \int_B \frac{1}{R} \left.\frac{\partial H_{1z}}{\partial z'}\right|_-^+ dS' = \frac{\gamma}{4\pi\eta} \hat{b}\cdot\hat{z} \int_B \frac{1}{R} dS' \quad . \tag{13}$$

Also,

$$\hat{z}_\wedge \bar{H}_1 = \frac{1}{4\pi} \int_B \left.\hat{z}_\wedge \bar{H}_1\right|_-^+ \frac{\partial}{\partial z'} \left(\frac{1}{R}\right) dS'$$

$$= -\frac{1}{4\pi} \frac{\partial}{\partial z} \int_B \left.\hat{z}_\wedge \bar{H}_1\right|_-^+ \frac{1}{R} dS'$$

and hence, on the plate,

$$\frac{\partial}{\partial z} \left(\hat{z}_\wedge \bar{H}_1\right) = -\frac{1}{4\pi} \frac{\partial^2}{\partial z^2} \int_B \left.\hat{z}_\wedge \bar{H}_1\right|_-^+ \frac{1}{R} dS' \quad . \tag{14}$$

Using (12), (13) and the fact that $\hat{z}_\wedge \bar{E}_0 = -\hat{z}_\wedge \hat{a}$ on B, the left-hand side of (14) can be evaluated, and the solution of (14) can then be written as

$$\left.\hat{z}_\wedge \bar{H}_1\right|_-^+ = -\gamma \hat{z}_\wedge (\hat{z}_\wedge \hat{a}) \left.\psi_0^3\right|_-^+ + \frac{\gamma}{\eta} \hat{b}\cdot\hat{z} \, \bar{u} \tag{15}$$

where $\bar{u}$ is a tangential vector satisfying the integral equation

$$\frac{1}{4\pi} \oint_C \frac{1}{R} \overline{d\ell}' = \frac{1}{4\pi} \frac{\partial^2}{\partial z^2} \int_B \bar{u}(\bar{r}') \frac{1}{R} dS' \tag{16}$$

with the line integral taken around the perimeter C of the plate, and $\psi_0^3\big|_-^+$ satisfies

$$-1 = \frac{1}{4\pi} \frac{\partial^2}{\partial z^2} \int_B \psi_0^3\Big|_-^+ \frac{1}{R} \, dS' \quad . \tag{17}$$

ψ_0^3 is proportional to the magnetostatic potential for a perfectly conducting plate [4], and a program has been written to solve (17) by the moment method. Only a simple modification is necessary to solve (16), and knowing $\psi_0^3\big|_-^+$ and $\bar{u}$, we have

$$Z \int_B \hat{z}_\wedge \bar{H}_1\Big|_-^+ \hat{r}\cdot\bar{r}' \, dS' = -\hat{z}_\wedge(\hat{z}_\wedge\hat{a}) \int_B \psi_0^3\Big|_-^+ \hat{r}\cdot\bar{r}' \, dS' + \frac{\hat{b}\cdot\hat{z}}{\eta} \int_B \bar{u}(\bar{r}')\hat{r}\cdot\bar{r}' \, dS' \quad .$$

$$\tag{18}$$

The next step is to compute the first term on the right-hand side of (4). Since $\bar{H}_0 = 0$, $\bar{E}_1$ is the gradient of an exterior potential [7], i.e., $\bar{E}_1 = -\nabla\phi_1$ with

$$\int_B \frac{\partial\phi_1}{\partial z}\Big|_-^+ \, dS = 0 \quad .$$

Hence

$$\int_B \hat{z}\cdot\bar{E}_1\Big|_-^+ \bar{r}' \, dS' = -\sum_{i=1}^{2} \hat{x}_i \int_B \frac{\partial\phi_1}{\partial z'}\Big|_-^+ (x_i' + c_i) \, dS'$$

and from (9) and the reciprocity theorem for exterior potentials,

$$\int_B \hat{z} \cdot \bar{E}_1 \Big|_-^+ \bar{r}' \, dS' = -\sum_{i=1}^2 \hat{x}_i \int_B \phi_1 \hat{z} \cdot \nabla' \phi_0^i \Big|_-^+ \, dS' \quad . \quad (19)$$

The zeroth order potential ϕ_0^i is associated with an incident field having $\hat{a} = \hat{x}_i$. If the corresponding first order scattered magnetic field is $\bar{H}_1^i$, Maxwell's equations imply

$$\hat{z} \cdot \nabla \phi_0^i \Big|_-^+ = Z \hat{z} \cdot \nabla_\wedge \bar{H}_1^i \Big|_-^+ \quad ,$$

and when this is substituted into (19), the vector relation in [8, p. 531] can be used to give

$$\int_B \hat{z} \cdot \bar{E}_1 \Big|_-^+ \bar{r}' \, dS' = Z \sum_{i=1}^2 \hat{x}_i \int_B \hat{z}_\wedge \nabla' \phi_1 \cdot \bar{H}_1^i \Big|_-^+ \, dS' \quad .$$

From the boundary condition (8)

$$\hat{z}_\wedge \nabla \phi_1 = \hat{z}_\wedge \bar{E}_1^{inc} - \eta Z \hat{z}_\wedge (\hat{z}_\wedge \bar{H}_1 \Big|_-^+)$$

$$= \hat{z}_\wedge \left\{ \hat{a}(\hat{k} \cdot \bar{r}) - \hat{a} \eta \psi_0^3 \Big|_-^+ - \bar{u}(\hat{b} \cdot \hat{z}) \right\} \quad ,$$

and hence

$$\int_B \hat{z} \cdot \bar{E}_1 \Big|_-^+ \bar{r}' \, dS' = -Z \sum_{i=1}^2 \hat{x}_i \int_B \left\{ (\hat{k} \cdot \bar{r}') \, \hat{a} - \eta \psi_0^3 \Big|_-^+ \hat{a} - (\hat{b} \cdot \hat{z}) \bar{u} \right\} \cdot \hat{z}_\wedge \bar{H}_1^i \Big|_-^+ \, dS' \quad .$$

Finally, from (15)

$$\hat{z}_\wedge \bar{H}^i_1 \Big|^+_- = -\gamma\hat{z}_\wedge(\hat{z}_\wedge\hat{x}_i)\psi^3_0\Big|^+_- + \frac{\gamma}{\eta}\hat{z}\cdot(\hat{k}_\wedge\hat{x}_i)\bar{u} \quad,$$

and when the vectors are combined to eliminate the summation,

$$\int_B \hat{z}\cdot\bar{E}_1\Big|^+_-\ \bar{r}'\ dS' = \int_B \Bigg[\Bigg[\Big\{(\hat{k}\cdot\bar{r}')\hat{z}_\wedge(\hat{z}_\wedge\hat{a}) - (\hat{b}\cdot\hat{z})\hat{z}_\wedge(\hat{z}_\wedge\bar{u})$$

$$+ (\hat{a}\cdot\bar{u})(\hat{z}_\wedge\hat{k})\Big\}\ \psi^3_0\Big|^+_- - \eta\hat{z}_\wedge(\hat{z}_\wedge\hat{a})\psi^3_0\Big|^+_-\ \psi^3_0\Big|^+_-$$

$$- \frac{1}{\eta}(\hat{z}_\wedge\hat{k})\Big\{(\hat{k}\cdot\bar{r}')\hat{a}\cdot\bar{u} - (\hat{b}\cdot\hat{z})\bar{u}\cdot\bar{u}\Big\}\Bigg]\Bigg]dS' \quad. \quad (20)$$

The complete first order contribution is therefore

$$\bar{F}(\hat{r}) = \int_B \Bigg[\Bigg[\Big\{(\hat{k} - \hat{r})\cdot\bar{r}'\ \hat{z}_\wedge(\hat{z}_\wedge\hat{a}) - (\hat{b}\cdot\hat{z})\hat{z}_\wedge(\hat{z}_\wedge\bar{u})$$

$$+ (\hat{a}\cdot\bar{u})(\hat{z}_\wedge\hat{k})\Big\}\ \psi^3_0\Big|^+_- - \eta\hat{z}_\wedge(\hat{z}_\wedge\hat{a})\psi^3_0\Big|^+_-\ \psi^3_0\Big|^+_-$$

$$+ \frac{1}{\eta}\Big\{(\hat{r}\cdot\bar{r}')(\hat{b}\cdot\hat{z})\bar{u} - (\hat{z}_\wedge\hat{k})\Big((\hat{k}\cdot\bar{r}')\hat{a}\cdot\bar{u} - (\hat{b}\cdot\hat{z})\bar{u}\cdot\bar{u}\Big)\Big\}\Bigg]\Bigg]\ dS' \quad (21)$$

and we observe that this has components only parallel to the plate. As expected, $\bar{F}$ is a function of η, and the presence of terms inversely proportional to η makes explicit the discontinuous behavior as $\eta \to 0$. For any given $\eta \neq 0$, there is a frequency (which depends on the plate dimensions) below which $\bar{F}$ no longer represents a valid correction to the zeroth order contribution.

A case of some interest is that in which the incident magnetic vector is parallel to the plate, so that $\hat{b}\cdot\hat{z} = 0$. Then $\hat{z}\wedge\hat{k} = -(\hat{a}\cdot\hat{z})\hat{b}$ and

$$\bar{F}(\hat{r}) = \hat{z}\wedge(\hat{z}\wedge\hat{a}) \int_B \left\{ (\hat{k} - \hat{r})\cdot\bar{r}' - \eta\psi_o^3\big|_-^+ \right\} \psi_o^3\big|_-^+ \, dS'$$

$$+ \frac{1}{\eta} \hat{b}(\hat{a}\cdot\hat{z}) \int_B \left\{ \hat{k}\cdot\bar{r}' - \eta\psi_o^3\big|_-^+ \right\} \hat{a}\cdot\bar{u} \, dS' \quad . \quad (22)$$

If, in addition, $\hat{a}\cdot\hat{z} = 0$ implying normal incidence on the plate, the second integral in (22) vanishes. The first two order terms in the low frequency expansion then go over smoothly into the result for a perfectly conducting plate [4] as $\eta \to 0$. On the other hand, if $\hat{k}\cdot\hat{z} = 0$ corresponding to grazing (edge-on) incidence, $\hat{a}$ is in the z direction (and perpendicular to $\bar{u}$) and $\bar{F}(\hat{r}) = 0$. This is consistent with the fact that any electric current sheet is invisible to a plane wave at grazing incidence for perpendicular polarization.

Concluding Remarks

The fact that $\bar{F}(\hat{r})$ can be expressed in terms of potentials analogous to the zeroth order ones is not surprising. Similar results have been obtained for acoustically soft and hard bodies [9] and for a dielectric body of non-zero volume [10], but all involve double integrals over the surface of the body. The simpler expression (21) is made possible by our ability to determine the first order magnetic field in terms of zeroth order potentials, but the result is still 'heavier' than for a perfectly conducting plate [4], not least because of the presence of the function $\bar{u}$.

If $\hat{b}\cdot\hat{z} = 0$, the scattered field approaches that of a perfectly conducting plate as the frequency decreases, and the cross reduction which the resistivity provides progressively diminishes. If $\hat{b}\cdot\hat{z} \neq 0$, however, the limiting values differ, and we can expect that, in general, the resistive plate will have a lower cross section at all frequencies. At any given frequency, the solution goes over smoothly to the perfectly conducting one as $\eta \rightarrow 0$ only if $\hat{b}\cdot\hat{z} = \hat{a}\cdot\hat{z} = 0$, i.e., for normal incidence on the plate.

The results obtained are also applicable to the scattering of mm and infrared radiation by the atmosphere, where many species of particles resemble dielectric platelets.

Acknowledgement

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