LOW FREQUENCY SCATTERING BY A PERFECTLY CONDUCTING BODY

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For a plane electromagnetic wave incident on a perfectly conducting body, the first two sets of terms in the low frequency expansion of the far zone scattered field are derived. The first set are dipole contributions which can be expressed in terms of the electro- and magneto-static potentials, and the next set can be obtained from these same potentials plus one other.

Introduction

When an electromagnetic wave illuminates a body whose dimensions are small compared to the wavelength, the far zone scattered field can be expanded in a series of powers of kL where k is the wavenumber and L is a characteristic dimension of the body. The leading terms are attributable to induced electric and/or magnetic dipoles and are only weakly dependent on the body's shape. Although these are sufficient for many purposes, there are applications such as remote sensing where it is important to obtain more information about the body, and this leads naturally to a consideration of higher order terms in the expansion.

We consider here the problem of a small closed perfectly conducting body, and derive the first two sets of terms in the expansion for the far field. The first terms are dipole contributions and are expressed in terms of polarizability tensors whose elements are weighted integrals of certain potentials. Integral equations are developed from which the potentials can be found. The next terms are defined by these same potentials plus one other, and their contribution to the far field expansion is obtained.

Formulation

A finite closed perfectly conducting body B is illuminated by a plane linearly polarized electromagnetic wave whose electric and magnetic vectors are

$$\bar{E}^{inc} = \hat{a}\, e^{ik\hat{k}.\bar{r}} \quad , \quad \bar{H}^{inc} = Y\hat{b}\, e^{ik\hat{k}\cdot\bar{r}} \quad , \qquad (1)$$

where $\hat{k}$, $\hat{a}$ and $\hat{b}$ are unit vectors specifying the directions of incidence, the electric field (or polarization) and the magnetic field, respectively. The three vectors are mutually perpendicular and, in particular,

$$\hat{b} = \hat{k}_\wedge \hat{a} \quad , \quad \hat{a} \cdot \hat{b} = 0 \ .$$

The propagation constant, permittivity and intrinsic admittance of the surrounding medium are k, ε and Y(=1/Z) respectively, and a time factor $e^{-i\omega t}$ has been assumed and suppressed.

Since the surface can support only an electric current, the scattered field can be expressed in terms of the electric Hertz vector

$$\bar{\Pi}(\bar{r}_0) = \frac{iZ}{4\pi k} \int_B \hat{n}_\wedge \bar{H}^t \frac{e^{ikR}}{R} \, dS \tag{2}$$

where $\hat{n}$ is an outward unit normal to the surface B, the superscript t denotes the total (incident plus scattered) field, and $R = |\bar{r}_0 - \bar{r}|$. The scattered electric field is then

$$\bar{E}(\bar{r}_0) = (\nabla\nabla_\cdot + k^2)\, \bar{\Pi}(\bar{r}_0)$$

and in the far zone

$$\bar{E}(\bar{r}_0) \sim -\frac{e^{ikr_0}}{4\pi r_0} \hat{r}_{0\wedge} \left\{ r_{0\wedge} ikZ \int_B n_\wedge \bar{H}^t \exp(-ik\hat{r}_0 \cdot \bar{r}) \, dS \right\} .$$

At low frequencies the exponential can be expanded in powers of ik, and using the identity

$$\int_B \hat{n}_\wedge \bar{H}\, dS = -ikY \int \hat{n}\cdot\bar{E}\,\bar{r}\, dS$$

(Jones, 1964), we obtain

$$\bar{E}(\bar{r}_o) \sim -\frac{e^{ikr_o}}{4\pi r_o} k^2 \hat{r}_{o\wedge}\hat{r}_{o\wedge} \left\{ \int_B \hat{n}\cdot\bar{E}^t\bar{r}\, dS + Z\int_B \hat{n}_\wedge\bar{H}^t\hat{r}_o\cdot\bar{r}\, dS \right.$$

$$\left. - \frac{ikZ}{2}\int_B \hat{n}_\wedge\bar{H}^t(\hat{r}_o\cdot\bar{r})^2\, dS + \ldots \right\}. \quad (3)$$

The final step is to expand the fields themselves in powers of ik, i.e.,

$$\bar{E} = \sum_{m=o}^{\infty} (ik)^m\, \bar{E}_m \quad , \quad \bar{H} = \sum_{m=o}^{\infty} (ik)^m\, \bar{H}_m \quad .$$

In particular, for the incident field,

$$\bar{E}_o^{inc} = \hat{a} \qquad \bar{H}_o^{inc} = Y\hat{b}$$

$$\bar{E}_1^{inc} = \hat{a}(\hat{k}\cdot\bar{r}) \qquad \bar{H}_1^{inc} = Y\hat{b}(\hat{k}\cdot\bar{r})$$

To the first two orders in k, (3) then becomes

$$\bar{E}(\bar{r}_0) \sim -\frac{e^{ikr_0}}{4\pi r_0} k^2 \left\{ \frac{1}{\varepsilon} \hat{r}_0 \wedge (\hat{r}_0 \wedge \bar{p}) + Z\hat{r}_0 \wedge \bar{m} \pm ik\hat{r}_0 \wedge \left[\hat{r}_0 \wedge \bar{N}(\hat{r}_0)\right] \right\} , \quad (4)$$

where

$$\bar{N}(\hat{r}_0) = \int_B \hat{n}\cdot\bar{E}_1^t \bar{r}\, dS + Z \int_B \hat{n}\wedge\bar{H}_1^t \hat{r}_0\cdot\bar{r}\, dS - \frac{Z}{2} \int_B \hat{n}\wedge\bar{H}_1^t (\hat{r}_0\cdot\bar{r})^2\, dS \; ; \quad (5)$$

$$\bar{p} = \varepsilon \int_B \hat{n}\cdot\bar{E}_0^t\, \bar{r}\, dS \quad (6)$$

is the electric dipole moment, and

$$\bar{m} = \frac{1}{2} \int_B \bar{r}\wedge(\hat{n}\wedge\bar{H}_0^t)\, dS \quad (7)$$

(Kleinman, 1973) is the magnetic dipole moment. Similarly, for the scattered magnetic field, we have

$$\bar{H}(\bar{r}_0) \sim Y \frac{e^{ikr_0}}{4\pi r_0} k^2 \left\{ \frac{1}{\varepsilon} \hat{r}_0 \wedge \bar{p} - Z\hat{r}_0 \wedge (\hat{r}_0 \wedge \bar{m}) + ik\hat{r}_0 \wedge \bar{N}(\hat{r}_0) \right\} . \quad (8)$$

Dipole Moments

The static fields $\bar{E}_0$ and $\bar{H}_0$ can be obtained from the solutions of certain potential problems. In the case of $\bar{E}_0$ we define $\bar{E}_0 = -\nabla\phi_0$ where ϕ_0 is an exterior potential. The corresponding incident potential is $-\hat{a}\cdot\bar{r}$, and from the boundary condition $\hat{n}\wedge\bar{E}_0 = -\hat{n}\wedge\bar{E}_0^{in}$ on the surface, it follows that

$$\phi_o = \hat{a}\cdot\bar{r} + c \quad \text{on } B \quad , \tag{9}$$

where c is a constant chosen to satisfy the zero induced charge condition

$$\int_B \frac{\partial\phi_o}{\partial n}\, dS = 0$$

for an isolated body.

Because of the form of (9), it is convenient to write

$$\phi_o = \sum_{i=1}^{3} (\hat{a}\cdot\hat{x}_i)\phi_o^i \tag{10}$$

where the x_i, i = 1,2,3, are Cartesian coordinates. Thus

$$\phi_o^i = x_i + c_i \quad \text{on } B \tag{11}$$

and an integral equation from which to determine $\partial\phi_o^i/\partial n$ is

$$-x_i - c_i = \frac{1}{4\pi}\int_B \frac{1}{R}\, \frac{\partial\phi_o^{it}}{\partial n'}\, dS' \quad , \tag{12}$$

where ϕ_o^{it} is the total (incident plus scattered) potential. On substituting (10) into (6) we have

$$\bar{p} = \varepsilon \bar{\bar{P}}\cdot\hat{a} \quad .$$

(Keller et al., 1972), where $\bar{\bar{P}}$ is the electric polarizability tensor with elements

$$P_{ij} = -\int_B \frac{\partial \phi_o^{it}}{\partial n} x_j \, dS$$

which are functions only of the geometry of B. The tensor is real and symmetric, i.e., $P_{ij} = P_{ji}$.

The procedure for the magnetic field is similar. We define $\bar{H}_o = -Y\nabla\psi_o$ where ψ_o is again an exterior potential, and since the corresponding incident field potential is $-\hat{b}\cdot\bar{r}$, the boundary condition $\hat{n}\cdot\bar{H}_o = -\hat{n}\cdot\bar{H}_o^{inc}$ on the surface implies

$$\frac{\partial \psi_o}{\partial n} = \hat{n}\cdot\hat{b} \quad \text{on B} \quad . \tag{13}$$

If

$$\psi_o = \sum_{i=1}^{3} (\hat{b}\cdot\hat{x}_i)\psi_o^i \quad , \tag{14}$$

the condition (13) becomes

$$\frac{\partial \psi_o^i}{\partial n} = \hat{n}\cdot\hat{x}_i \quad \text{on B} \quad , \tag{15}$$

and an integral equation with which to compute the total potential ψ_o^{it} is

$$\psi_o^{it}(\bar{r}) = -2x_i + \frac{1}{2\pi}\int_B \psi_o^{it}(\bar{r}') \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) dS' \quad . \tag{16}$$

In terms of this potential

$$\bar{m} = -Y\bar{\bar{M}}\cdot\hat{b} \quad ,$$

(Keller et al., 1972), where $\bar{\bar{M}}$ is the magnetic polarizability tensor with elements

$$M_{ij} = -\int_B \hat{n}\cdot\hat{x}_j \psi_0^{it} \, dS \quad .$$

These also are functions only of the geometry of B, and the tensor is real and symmetric, i.e., $M_{ij} = M_{ji}$.

Programs have been written to solve the integral equations (12) and (16) by the moment method and, hence, compute the tensor elements for any rotationally symmetric body whose profile is made up of straight line and circular arc segments (Senior and Ahlgren, 1973). Computed data for some aerospace configurations have been given by Kleinman and Senior (1975). In addition, a variety of isoperimetric and other bounds on the tensor elements have been developed (Payne, 1967; Kleinman and Senior, 1972), and these could serve to adequately approximate the elements.

First Order Terms

The first order contribution to the far zone scattered field is given by the function $\bar{N}(\hat{r}_0)$, and the third term in the expression (15) is easily evaluated.

$$\int_B \hat{n}_\wedge \bar{H}_0^t(\hat{r}_0\cdot\bar{r})^2\,dS = Y\int_B \hat{n}_\wedge\hat{b}(\hat{r}_0\cdot\bar{r})^2\,dS - Y\int_B \hat{n}_\wedge\nabla\psi_0(\hat{r}_0\cdot\bar{r})^2\,dS$$

$$= Y\int_B \hat{n}_\wedge\hat{b}(\hat{r}_0\cdot\bar{r})^2\,dS - Y\int_B \hat{n}_\wedge\left[\nabla\left\{(\hat{r}_0\cdot\bar{r})^2\,\psi_0\right\} - 2(\hat{r}_0\cdot\bar{r})\hat{r}_0\psi_0\right]dS$$

$$= Y\int_B \hat{n}_\wedge\hat{b}(\hat{r}_0\cdot\bar{r})^2\,dS - 2Y\hat{r}_{0\wedge}\int_B \hat{n}(\hat{r}_0\cdot\bar{r})\psi_0\,dS$$

from Stokes' theorem. Also

$$\int_B \hat{n}\cdot\bar{E}_1^{inc}\,\bar{r}\,dS = \int_B \hat{n}\cdot\hat{a}(\hat{k}\cdot\bar{r})\bar{r}\,dS$$

$$= -\int_B \hat{n}\cdot(\hat{k}_\wedge\hat{b})(\hat{k}\cdot\bar{r})\bar{r}\,dS$$

$$= -\frac{1}{2}\int_B \hat{n}_\wedge\hat{b}(\hat{k}\cdot\bar{r})^2\,dS$$

on using the lemma in Kleinman (1967a), and

$$\int_B \hat{n}_\wedge\bar{H}_1^{inc}\,(\hat{r}_0\cdot\bar{r})\,dS = Y\int_B \hat{n}_\wedge\hat{b}(\hat{k}\cdot\bar{r})(\hat{r}_0\cdot\bar{r})dS\ .$$

Hence

$$\bar{N}(\hat{r}_0) = \frac{1}{2}\hat{b}_\wedge \int_B \hat{n}\left\{(\hat{k} - \hat{r}_0)\cdot\bar{r}\right\}^2 dS + \hat{r}_{0\wedge}\int_B \hat{n}(\hat{r}_0\cdot\bar{r})\psi_0 \, dS$$

$$+ \int_B \hat{n}\cdot\bar{E}_1 \, \bar{r} \, dS + Z\int_B \hat{n}_\wedge\bar{H}_1(\hat{r}_0\cdot\bar{r}) \, dS \quad . \tag{17}$$

We observe that the first term is known and vanishes in the forward scattering direction $\hat{r}_0 = \hat{k}$, and that the second term is known once the zeroth order potential ψ_0 has been determined. Unfortunately, the two remaining terms are more complex.

Since

$$\nabla_\wedge\bar{H}_1 = -Y\bar{E}_0 \neq 0 \quad ,$$

$\bar{H}_1$ is <u>not</u> the gradient of a potential, but we can write $\bar{H}_1$ as

$$\bar{H}_1 = Y\bar{G} - Y\nabla\psi_1 \quad , \tag{18}$$

(Kleinman, 1967b) where ψ_1 is an exterior potential. Clearly

$$\nabla_\wedge\bar{G} = \nabla\phi_0 . \tag{19}$$

and as shown by Stevenson (1954),

$$\bar{G} = \frac{1}{4\pi}\nabla_\wedge\int\frac{1}{R}\nabla'\phi_0 \, dV'$$

where the volume integration is carried out over all space. It is therefore necessary to introduce an interior potential Φ_o corresponding to ϕ_o and such that $\hat{n}\cdot\nabla\phi_o = \hat{n}\cdot\nabla\Phi_o$ on B. Since

$$\frac{1}{R}\nabla'\phi_o = \nabla'\left(\frac{\phi_o}{R}\right) + \phi_o\nabla\frac{1}{R}$$

and the second term vanishes under the operation $\nabla_\wedge$, it follows that

$$\bar{G} = -\frac{1}{4\pi}\nabla_\wedge\int_B \hat{n}'(\phi_o - \Phi_o)\frac{dS'}{R} \tag{20}$$

with (of course) $\phi_o = \hat{a}\cdot\bar{r}' + c$ on B. By analogy with (10) it is convenient to write

$$\bar{G} = \sum_{i=1}^{3}(\hat{a}\cdot\hat{x}_i)\bar{G}^i \quad , \tag{21}$$

where

$$\bar{G}^i = -\frac{1}{4\pi}\nabla_\wedge\int_B \hat{n}'(\phi_o^i - \Phi_o^i)\frac{dS'}{R} \tag{22}$$

and Φ_o^i is the interior potential appropriate to ϕ_o^i. The determination of Φ_o^i is a simple Neumann problem in potential theory, and an integral equation from which to obtain $\phi_o^i - \Phi_o^i$ is

$$-\left\{\phi_o^i(\bar{r}) - \Phi_o^i(\bar{r})\right\} = -2(x_i + c_i) + \frac{1}{2\pi}\int_B (\phi_o^i - \Phi_o^i)\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) dS' \quad . \tag{23}$$

The evaluation of the term involving $\bar{H}_1$ in (17) is now straightforward. We have

$$Z\int_B \hat{n}_\wedge \bar{H}_1 (\hat{r}_0\cdot\bar{r})dS = \int_B \hat{n}_\wedge \bar{G}(\hat{r}_0\cdot\bar{r})dS - \int_B \hat{n}_\wedge \nabla\psi_1 (\hat{r}_0\cdot\bar{r})dS$$

and

$$\begin{aligned}\int_B \hat{n}_\wedge \nabla\psi_1 (\hat{r}_0\cdot\bar{r})dS &= \hat{r}_{0\wedge}\int_B \hat{n}\psi_1 \, dS \\ &= \hat{r}_{0\wedge} \sum_{i=1}^{3} x_i \int_B \psi_1 \hat{n}\cdot\hat{x}_i \, dS \\ &= \hat{r}_{0\wedge} \sum_{i=1}^{3} \hat{x}_i \int_B \psi_1 \hat{n}\cdot\nabla\psi_0^i \, dS \\ &= \hat{r}_{0\wedge} \sum_{i=1}^{3} \hat{x}_i \int_B \psi_0^i \, \hat{n}\cdot\nabla\psi_1 \, dS ,\end{aligned}$$

where we have used in succession the vector relations in Kleinman (1973), the boundary condition (15), and the reciprocity theorem for exterior potentials. The boundary condition $\hat{n}\cdot\bar{H}_1^t = 0$ at the surface implies

$$\hat{n}\cdot\nabla\psi_1 = \hat{n}\cdot(Z\bar{H}_1^{inc} + \bar{G}) \qquad (24)$$

on B, and hence

$$Z \int_B \hat{n}\wedge\bar{H}_1(\hat{r}_0\cdot\bar{r})dS = \int_B \hat{n}\wedge\bar{G}(\hat{r}_0\cdot\bar{r})dS - \hat{r}_0\wedge \sum_{i=1}^{3} \hat{x}_i \int_B \psi_0^i\hat{n}\cdot\left\{\hat{b}(\hat{k}\cdot\bar{r}) + \bar{G}\right\} dS \ . \tag{25}$$

The only remaining task is to evaluate the term involving $\bar{E}_1$ in (17). As shown by Kleinman (1967b),

$$\bar{E}_1 = \bar{F} - \nabla\phi_1 \quad ,$$

where ϕ_1 is an exterior potential and

$$\bar{F} = -\nabla\wedge \frac{1}{4\pi} \int_B \hat{n}'\wedge\bar{E}_1^{inc} \frac{dS'}{R} + \frac{Z}{4\pi} \int_B \hat{n}'\wedge\bar{H}_0 \frac{dS'}{R} \quad ,$$

$$= \nabla\wedge \frac{1}{4\pi} \int_B \left\{\hat{n}'\psi_o - \hat{n}'\wedge\hat{a}\,(\hat{k}\cdot\bar{r}')\right\} \frac{dS'}{R} \ . \tag{26}$$

Hence

$$\int_B \hat{n}\cdot\bar{E}_1\,\bar{r}\,dS = \int_B \hat{n}\cdot\bar{F}\,\bar{r}\,dS - \int_B \frac{\partial\phi_1}{\partial n}\,\bar{r}\,dS \tag{27}$$

and by using the boundary condition (11) on ϕ_0^i and the reciprocity theorem, the second term on the right-hand side of (27) can be written as

$$\sum_{i=1}^{3} \hat{x}_i \int_B \phi_0^i\hat{n}\cdot\nabla\phi_1\,dS = \sum_{i=1}^{3} \hat{x}_i \int_B \phi_1\hat{n}\cdot\nabla\phi_0^i\,dS \ . \tag{28}$$

It might seem that we are now home, but unfortunately the boundary condition $\hat{n}_\wedge \bar{E}^t_1 = 0$ implies

$$\hat{n}_\wedge \nabla\phi_1 = \hat{n}_\wedge(\bar{E}^{inc}_1 + \bar{F}) \tag{29}$$

and specifies the tangential derivatives of ϕ_1 on B rather than ϕ_1 itself. This difficulty can be overcome at the expense of introducing the function $\bar{G}^i$. From (19) and (21)

$$\int_B \phi_1 \hat{n}\cdot\nabla\phi^i_o \, dS = \int_B \phi_1 \hat{n}\cdot\nabla_\wedge\bar{G}^i \, dS$$

$$= -\int_B \bar{G}^i\cdot\hat{n}_\wedge\nabla\phi_1 \, dS$$

from Stokes' theorem, and the boundary condition (29) can now be inserted to give

$$\int_B \phi_1 \hat{n}\cdot\nabla\phi^i_o \, dS = -\int_B \bar{G}^i \cdot(\hat{n}_\wedge\hat{a})(\hat{k}\cdot\bar{r})dS - \int_B \bar{G}^i \cdot \hat{n}_\wedge\bar{F} \, dS \quad . \tag{30}$$

The last term can be simplified as follows:

$$\int_B \bar{G}^i \cdot \hat{n}_\wedge\bar{F} \, dS = \int_B \hat{n} \cdot (\bar{F}_\wedge \bar{G}^i) \, dS$$

$$= -\int_\nu \left\{\bar{G}^i \cdot \nabla_\wedge\bar{F} - \bar{F} \cdot \nabla_\wedge\bar{G}^i\right\} dV \quad ,$$

where ν is the volume exterior to B, and

$$\nabla_\wedge \bar{F} = -\nabla\psi_0 \quad , \quad \nabla_\wedge \bar{G}^i = \nabla\phi_0^i \quad .$$

Hence

$$\int_B \bar{G}^i \cdot \hat{n}_\wedge \bar{F} \, dS = \int_\nu \left\{ \nabla\cdot(\bar{F}\phi_0^i + \bar{G}^i\psi_0) - \phi_0^i \nabla\cdot\bar{F} - \psi_0\nabla\cdot\bar{G}^i \right\} dV \quad ,$$

and since both vectors have zero divergence,

$$\int_B \bar{G}^i \cdot \hat{n}_\wedge \bar{F} \, dS = - \int_B \hat{n} \cdot (\bar{F}\phi_0^i + \bar{G}^i\psi_0) \, dS \quad . \qquad (31)$$

When this is substituted into (30) and thence into (28) and (27), application of the boundary condition on ϕ_0^i shows that the first term on the right-hand side of (31) cancels the corresponding term in (27), and thus

$$\int_B \hat{n}\cdot\bar{E}_1 \, \bar{r} \, dS = - \sum_{i=1}^{3} \hat{x}_i \int_B \left\{ \hat{a}\cdot\hat{n}_\wedge\bar{G}^i(\hat{k}\cdot\bar{r}) + \hat{n}\cdot\bar{G}^i\psi_0 \right\} dS \quad . \qquad (32)$$

From (31), (25) and (17) the resulting expression for the first order contribution is

$$\bar{N}(\hat{r}_0) = \frac{1}{2}\hat{b}\wedge\int_B \hat{n}\left\{(\hat{k}-\hat{r}_0)\cdot\bar{r}\right\}^2 dS + \hat{r}_0\wedge\int_B \hat{n}(\hat{r}_0\cdot\bar{r})\psi_0\, dS$$
$$+\int_B \hat{n}\wedge\bar{G}(\hat{r}_0\cdot\bar{r})dS - \sum_{i=1}^{3}\hat{x}_i\int_B \hat{a}\cdot(\hat{n}\wedge\bar{G}^i)(\hat{k}\cdot\bar{r})\, dS$$
$$-\hat{r}_0\wedge\sum_{i=1}^{3}\hat{x}_i\int_B \psi_0^i\,\hat{n}\cdot\left\{\bar{G}+\hat{b}(\hat{k}\cdot\bar{r})\right\} dS - \sum_{i=1}^{3}\hat{x}_i\int_B \hat{n}\cdot\bar{G}^i\psi_0\, dS \quad . \tag{33}$$

This involves the zeroth order magnetostatic potential ψ_0 and, through $\bar{G}$, the zeroth order potentials ϕ_0 and Φ_0 as well. It is easily verified that $\hat{a}\cdot\bar{N}(\hat{k}) = 0$ as required by the forward scattering theorem for a lossless body.

Concluding Remarks

The fact that $\bar{N}(\hat{r}_0)$ can be expressed in terms of zeroth order potentials is not surprising. Van Bladel (1968) showed that this is true in acoustic scattering by hard and soft bodies and Jones (1980) obtained a similar result for a dielectric body illuminated by an electromagnetic wave. In each instance the expression involves double integrals over the surface, and we remark that Jones' result fails in the limiting case of perfect conductivity.

The scattering problem for a perfectly conducting body is a true boundary value problem and we had hoped that its simpler nature would lead to a simpler expression for the first order contribution. In fact, (33) is comparable to the result for a dielectric body, and in one respect it is less desirable. Whereas for a dielectric body

ϕ_0 specifies immediately the boundary values of the interior potential Φ_0, for perfect conductivity the determination of Φ_0 constitutes a potential problem additional to those for ϕ_0 and ψ_0. The potential Φ_0 enters via the function $\bar{G}$, and though it seems illogical that the solution of (19) in the region outside and on B should entail the solution of an interior problem, we have found no satisfactory way to avoid it except in the special case of a flat plate of infinitesimal thickness. Alternatively, if (29) could be integrated to yield the value of ϕ_1 on the surface (a process which is certainly feasible in the context of a numerical solution), (28) could be evaluated directly to provide the weighted integral (27) of $\hat{n}\cdot\bar{E}_1$. It would then be possible to evaluate the left-hand side of (25) without introducing $\bar{G}$ by using the counterpart of the procedure which we have here applied to the electric field. Unfortunately, integration of (29) is tantamount to the expression of a gradient as a curl, which brings us back full circle. This topic will be addressed in a future article.

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