BACKSCATTER FROM A RANDOM MEDIUM

by

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ABSTRACT

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The objective of this work was to study the backscatter of waves from a random medium. The model used was that of a random continuum consisting of a slab geometry with a plane wave normally incident from free space. The method of approach decomposed the total field into forward and backscatter components. The Furutsu-Novikov formulation assuming a Markov approximation was extended to include backscatter enabling a closet set of partial differential equations for the first and second order moments of the forward and backscatter feilds to be developed. The first order moment equations were solved exactly to arrive at expressions for the average forward and backscatter field. The second order moment equations were solved numerically. Some analytical results for the second moment were obtained for limiting cases of parameters. It was found that the behavior of the average forward and backscatter field was exponential in form with the amount of backscatter directly dependent on the variance of permittivity fluctuations. The second order moment of coherence equqtions led to expressions for the total energy in the forward and backscatter field dependent on the

variance of permittivity fluctuations and the size of the random slab considered. The second order moment equations were used to study the coherence properties of the forward and backscatter fields. An investigation of the effects of backscatter and forward scatter coherence was performed for varying permittivity fluctuations and slab sizes.

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LIST OF SYMBOLS

$A(\underline{\rho})$	Correlation of the permittivity fluctuations in the $\underline{\rho}$ plane.
$B(\underline{\rho}_d,z)$	Three-dimensional correlation function.
$C(\underline{r}')$	Arbitrary scalar function of a vector coordinate.
$D(\underline{r}')$	Arbitrary scalar function of a vector coordinate.
$E(\underline{r})$	Total scalar electric field intensity in the random medium.
$E^+(\underline{r})$	Forward propagating scalar electric field intensity in the random medium.
$E^-(\underline{r})$	Backward propagation scalar electric field intensity in the random medium.
$f(z)$	Source.
$G(\underline{r},\underline{r}')$	Green's function for elliptic equation in inhomogeneous medium.
$G^{++}(z,z',\underline{\rho}_d)$	Green's function for the second order equation in z for $\psi^{++}(z,\underline{\rho}_d)$.
$G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')$	Green's function for the positive propagating cross coherence ψ^{+-}.
$G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')$	Green's function for the negative propagating cross coherence ψ^{-+}.
$G_i(z',z,\underline{\rho}_d)$, $i = 1,\ldots,4$	Shorthand notation for the kernels of the integral equation for $\psi^{--}(z,\underline{\rho}_d)$.
$\tilde{G}(\underline{r},\underline{r}')$	Free space Green's function of the Helmholtz equation.
$g(\underline{r}')$	Source.
I_i, $i = 1,\ldots,8$	Shorthand notation for single integrals approximated by the method of moments.

k	Free space wave number.
k_{eff}	Effective wave number of a random medium including backscatter.
$K_i(z',z,\underline{\rho}_d)$ $i = 1,\ldots,4$	Shorthand notation for the kernels of the integral equation for $\psi^{++}(z,\underline{\rho}_d)$.
ℓ	Correlation size of permittivity fluctuations.
L	Thickness of a random slab.
L^+	Parabolic operator describing propagation in the pluz z-direction.
L^-	Parabolic operator describing propagation in the minus z-direction.
$s(\underline{r}),\tilde{s}(\underline{r})$	Sources.
$u^+(\underline{r})$	Complex amplitude of a wave propagating in the positive z-direction in the random medium.
$u^-(\underline{r})$	Complex amplitude of a wave propagating in the negative z-direction in the random medium.
$u_o^+(\underline{r})$	Free space forward propagating complex amplitude of the field.
$u_f^+(\underline{r})$	Randomly fluctuating complex amplitude of forward propagating wave.
$u_f^-(\underline{r})$	Randomly fluctuating complex amplitude of backward propagating wave.
$V(z,\underline{\rho}_d)$	Hypothesized dependent variable which obeys a positive going parabolic nonhomogeneous equation.
$\hat{x}$	Unit vector in x-direction.
x	A Cartesian coordinate.
$\hat{y}$	Unit vector in y-direction.
y	A Cartesian coordinate.
$Y = \frac{\pi}{2} \frac{\sigma_\varepsilon^2}{\lambda}^{1/2}$	Shorthand notation for a quantity which occurs in expressions for the average field when $\sigma_\varepsilon^2/\lambda$ is large.
z	A Cartesian coordinate.

$Z = \tau^2 + 4k^2$	Shorthand notation occurring in the integral equations for ψ^{++} and ψ^{--}.
$Z^+(z,z',\underline{\rho},\underline{\rho}')$	Fundamental solution of forward propagating parabolic equations in free space.
$Z^-(z,z',\underline{\rho},\underline{\rho}')$	Fundamental solution of backward propagating parabolic equations in free space.
$Z^+(z,z',\underline{k}_i)$, $Z^-(z,z',\underline{k}_i)$	Two dimensional Fourier transform of $Z^+(z,z',\underline{\rho},\underline{\rho}')$ and $Z^-(z,z',\underline{\rho},\underline{\rho}')$, respectively.
α	Longitudinal position coordinate normalized to slab thickness used as a dummy variable of integration.
β	Ito and Adachi extinction factor.
$\gamma = \frac{k^2}{4}(2A(0)-A(\underline{\rho}_d))$	Coefficient in the first order differential equation in z for ψ^{++} and ψ^{--}.
$\Gamma(z,z',\underline{\rho},\underline{\rho}')$	General notation to represent fundamental solution of parabolic equation propagation in either the plus or minus z-direction.
$\Gamma^+(z,z',\underline{\rho},\underline{\rho}')$	Fundamental solution of forward propagating parabolic equation.
$\Gamma^-(z,z',\underline{\rho},\underline{\rho}')$	Fundamental solution of backward propagating parabolic equation in a random medium.
$\delta(\underline{r})$	Dirac delta function.
$\frac{\delta u^+(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')}$, $\frac{\delta u^-(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')}$	Functional derivative of $u^+(\underline{r})$ (or $u^-(\underline{r})$) with respect to $\tilde{\varepsilon}(\underline{r}')$.
$\varepsilon(\underline{r})$	Relative permittivity.
$\tilde{\varepsilon}(\underline{r})$	Relative permittivity fluctuations.
ζ	Position along the direction of propagation normalized to the slab thickness.
$\eta = \tau \cosh \tau L + \gamma \sinh \tau L$	Denominator of $(z,\underline{\rho}_d)$.

$\eta = (1-\phi_A)^{1/2} \cosh \omega + (1 - \phi_A/2)\sinh \omega$	Normalized version of η.
κ	Coefficient in integral equations for ψ^{++} and ψ^{--} depending on $\sigma_\varepsilon^2/\lambda$ and the normalized spectrum of the permittivity correlation function.
λ	Wavelength.
$\Lambda^+(z,\underline{\rho}_d) = \psi^{--}(z,\underline{\rho}_d) + \psi^{--}(z,\underline{\rho}_d)$	Sum of positive and negative propagating coherences.
$\Lambda^-(z,\underline{\rho}_d) = \psi^{++}(z,\underline{\rho}_d) - \psi^{--}(z,\underline{\rho}_d)$	Difference of positive and negative propagating coherences.
$\underline{\rho} = x\hat{x} + y\hat{y}$	Position vector in transverse plane.
$\underline{\rho}_d$	Difference vector between the two points in a transverse plane.
$\underline{\rho}_c$	Position vector locating a point at the center of the difference vector, $\underline{\rho}_d$, between two points.
σ_ε^2	Variance of permittivity fluctuations.
$\tau = \frac{k^2}{2} (A^2(0) - (A(0)A(\underline{\rho}_d))^{1/2}$	Coefficient in second order differential equation in z for ψ^{++}.
$\phi_1(\underline{\rho},\kappa)$	One dimensional spectrum of the three-dimensional permittivity correlation function.
ϕ_A	Normalized one-dimensional spectrum of permittivity correlation function.

$\chi(z,\underline{\rho}_d) = \frac{\tau\cosh\tau(L-z)+\sinh\tau(L-z)}{\eta}$	Function coefficient appearing in the integral equation for ψ^{++}.
$\hat{\chi}(x,\underline{\rho}_d) = \frac{\tau\cosh\tau_z+\gamma\sinh\tau_z}{\eta}$	Function coefficient appearing in the integral equation for ψ^{--}.
χ	Effective wave number in a random medium including the effects of backscatter normalized to the free space wave number.
$\psi^{++}(\underline{r}_1,\underline{r}_2)$, $\psi^{--}(\underline{r}_1,\underline{r}_2)$	Forward and backward propagating coherence functions of scalar electric field intensities.
$\psi^{+-}(\underline{r}_1,\underline{r}_2)$, $\psi^{-+}(\underline{r}_1,\underline{r}_2)$	Forward and backward propagating cross coherence functions of scalar electric field intensities.
ψ_B^{--}	First Born approximation for backscatter energy.
$\Psi^{++}(\underline{r}_1,\underline{r}_2)$, $\Psi^{--}(\underline{r}_1,\underline{r}_2)$	Forward and backward propagating coherence functions of complex wave amplitudes.
$\Psi^{+-}(\underline{r}_1,\underline{r}_2)$, $\Psi^{-+}(\underline{r}_1,\underline{r}_2)$	Forward and backward propagating cross coherence functions of complex wave amplitudes.
$\omega = \beta(1-\dot{\psi}_A)^{1/2}$	Appears in argument of sinh and cosh and is shorthand notation. Quantity is independent of longitudinal variable.
ω_o	Angular frequency of the time harmonic field variation $\exp(-i\omega_o t)$.
$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$	Laplacian operator.

$\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$	Transverse Laplacian.
$\mid \ \mid^2$	Magnitude squared.
$< \ >$	Ensemble average.
$*$	Complex conjugate.

CHAPTER I. INTRODUCTION

1.1 Background

The objective of this work was to study the backscatter of waves from a random medium, where the medium model includes some of the effects of the multiple scatterings which occur within the random material. Most of the work done to date has concerned itself with the forward scatter problem. This was in response to the need for such models to assist in the design and development of communication systems. The majority of the small amount of work done examining the backscatter problem did not include multiple scattering effects. Due to the increased importance of remote sensing systems the backscatter model was updated in this study to include these multiple scattering effects.

A random medium is a medium in which it is not possible to exactly predict the field at a given position at a given time. the approach is to apply probabilistic techniques to determine such quantities as the average field, the average field intensity, and the variances of the fluctuations of field and field intensity about their respective average values. While a knowledge of the probability density of the field would permit the calculation of the above quantities the complexity of the wave-medium interaction does not make its determination from first principles a tractable approach. It is necessary to determine each quantity of interest by applying probabilistic concepts to the physical laws governing fields.

A random medium may be viewed as discrete or continuous. In the discrete case the uncertainty in the prediction of the field is ascribed to the random location of the particles in the aggregate of particles which models the medium. The principles of scattering theory in conjunction with probability are applied to determine the resultant average field quantities at a point. In the continuous case the medium is modeled using Maxwell's equations incorporating a permittivity which is a random function of position and time. The continuous medium model was used in this work.

The medium of interest was assumed to vary little with time during a cycle of the time harmonically varying field given by $e^{-i\omega_0 t}$. It was modeled charge-free with a permeability given by that of free space. Depolarization effects were neglected. The result was the following scalar wave equation

$$\nabla^2 E(\underline{r}) + k^2 \varepsilon(\underline{r}) E(\underline{r}) = 0 \quad , \qquad (1.1)$$

where $E(\underline{r})$ is the total scalar field and $\varepsilon(\underline{r}) = 1 + \tilde{\varepsilon}(r)$. The relative permittivity fluctuation, $\tilde{\varepsilon}(\underline{r})$, has zero mean and correlation function to be specified in a later section of this thesis.

The solution of the posed problem has two avenues of approach. The first solves the wave equation for E and then uses the expression for E to determine $\langle E \rangle$, $\langle |E|^2 \rangle$ or any statistical quantity of interest. The second approach uses the differential equation for E to determine an equation, either integral, differential or integro-differential, for the mean of interest. The resulting equation for the statistic must then be solved.

The first avenue of approach usually appeals to a body of mathematical techniques known as perturbation theory. This technique studies the dependence of solutions on some parameter. An expression which includes the deviation of a solution from some limiting value of solution is sought. The limiting value of solution is usually for a zero value of parameter and the deviation of the solution is for small perturbation from that value. The solution usually takes the form of an infinite series expansion. This infinite series is truncated at one or, at most, two terms. As applied to the continuous random medium problem, a solution for (1.1) is sought which depends on a small relative permittivity fluctuation. The effect of series truncation is that solutions are valid only over a limited propagation distance. The range limitation of the truncated solution is due to the fact that signal fluctuations are cumulative. The further a wave propagates through a random medium the smaller the average part of the wave becomes while its fluctuating part becomes larger. The larger the fluctuating part of the wave becomes the more terms in the series solution must be considered. Solutions valid for low levels of signal fluctuations and consequently over short propagation distances are said to be of weak fluctuation theory.

Weak fluctuation theory effectively neglects the infinitely many interactions or multiple scatterings which occur within the medium. Several examples of weak fluctuation theory approaches are the Born or single scattering approximation and the geometrical optics techniques discussed by Tatarskii [1] and the Rytov method discussed by Ishimaru [2].

The second avenue of approach, which uses (1.1) to derive an equation for the statistical quantity of interest, usually involves taking the mean of (1.1) or some equivalent form of (1.1) to arrive at an equation for the mean field. Equations for higher order moments can also be developed from equation (1.1). Difficulties arise when the mean of an equation such as (1.1) is taken. Since the random qualities of E are attributed to $\tilde{\varepsilon}(\underline{r})$ the two quantities are correlated. The mean of the product of the two are not easily separable and effectively becomes an additional unknown. This difficulty, commonly referred to as the closure problem, is discussed in detail by Kraichnan [3]. To circumvent the closure problem, simplifying assumptions are made which limit the kind of higher order medium interactions whose effects are accounted for in the final result. The necessity to limit the kind of interactions included restricts the range of validity of results. However, inclusion of higher order medium interactions makes these kinds of approaches valid over greater ranges and for greater permittivity fluctuations than the series truncation approaches described as weak fluctuation theory. These approaches are categorized as strong fluctuation theory approaches. Two examples of this second avenue of solution of equation (1.1) are the multiple scattering approach and the parabolic equation method. Multiple scattering approaches make use of the Dyson and Bethe-Salpeter equations which are developed from equation (1.1) (see Frish [4]). The Dyson equation is an integral equation for the mean Green's function and contains a kernel commonly known as the mass operator. The Bethe-Salpeter equation is an integral equation for the covariance (the covariance or the mutual coherence function describes the correlation of the field between any two points within a random medium) of the Green's function and contains a kernel termed the intensity operator. The mass and intensity operators are composed of an infinite series of terms. Approximations are made which replace the infinite series by finite quantities. The two common approximations are the bilocal and nonlinear approximations. Through the examination of Feynman diagrams, which pictorially describe the wave-medium interactions, one can argue that the approximations constitute summations over selected terms in the infinite term operators and consequently include some of the higher order interactions excluded in the perturbation theory approaches.

The parabolic equation method, of which the backscatter work which is presented here was an extension and because of this will be elaborated on, begins by assuming that scattering occurs primarily in small angles in the forward direction. Mathematically, the substitution

$$E(\underline{r}) = u^{+}(\underline{r})e^{ikz} \qquad (1.2)$$

is made in (1.1) along with the approximation

$$\frac{\partial^2 u^{+}}{\partial z^2} << 2ik\,\frac{\partial u^{+}}{\partial z} \quad , \qquad (1.3)$$

the latter assumption holds for small angles. This yields the equation for $u^{+}(\underline{r})$,

$$2ik\,\frac{\partial u^{+}(\underline{r})}{\partial z} + \nabla_T^2\, u^{+}(\underline{r}) + k^2\tilde{\varepsilon}(\underline{r})u^{+}(\underline{r}) = 0 \qquad (1.4)$$

where

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad .$$

Equation (1.4) is a parabolic partial differential equation with nonconstant coefficients. Taking the mean of (1.4) results in

$$2ik\frac{\partial\langle u^+(\underline{r})\rangle}{\partial z} + \nabla_T^2\langle u^+(\underline{r})\rangle + k^2\langle\tilde{\varepsilon}(\underline{r})u^+(\underline{r})\rangle = 0 \quad . \qquad (1.5)$$

The above equation contains two unknowns $\langle u^+(\underline{r})\rangle$ and $\langle\tilde{\varepsilon}(\underline{r})u^+(\underline{r})\rangle$. The writing of $\langle\tilde{\varepsilon}(\underline{r})u^+(\underline{r})\rangle$ as a term proportional to $\langle u^+(\underline{r})\rangle$ is accomplished by using the Furutsu-Novikov theorem, discussed in detail in Ishimaru [2], for a zero mean Gaussian distributed permittivity function. The theorem states that $\langle\tilde{\varepsilon}(\underline{r})u^+(\underline{r})\rangle$ can be written as

$$\langle\tilde{\varepsilon}(\underline{r})u^+(\underline{r})\rangle = \int d\underline{r}' \quad \langle\tilde{\varepsilon}(\underline{r})\tilde{\varepsilon}(\underline{r}')\rangle\left\langle\frac{\delta u^+(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')}\right\rangle . \qquad (1.6)$$

The first term in brackets in the integrand of (1.6) is the permittivity correlation function. As noted in Tatarskii [1], the correlation in an isotropic, homogeneous medium, is peaked in the direction of propagation. A mathematically equivalent limiting case of this "peakedness" is to say that the medium is delta correlated in the direction of propagation, that is, the permittivity correlations can be written as

$$\langle\tilde{\varepsilon}(\underline{r})\tilde{\varepsilon}(\underline{r}')\rangle = \delta(z-z')A(\underline{\rho} - \underline{\rho}') \qquad (1.7)$$

where $A(\underline{\rho})$ is the permittivity correlation function in a plane transverse to the direction of propagation.

Substituting (1.7) into (1.6) and integrating over z' leads to

$$<\tilde{\varepsilon}(\underline{r})u^{+}(\underline{r})> = \int d\underline{\rho}' \; A(\underline{\rho} - \underline{\rho}')\left\langle \frac{\delta u^{+}(\underline{r})}{\delta\tilde{\varepsilon}(z,\underline{\rho}')} \right\rangle \quad . \qquad (1.8)$$

The bracketed term in the integrand of (1.8) is the functional derivative of $u^{+}(\underline{r})$ with respect to $\tilde{\varepsilon}(\underline{r}')$. The field, $u^{+}(\underline{r})$, is considered to be a functional, that is, a function of a function, $\tilde{\varepsilon}(\underline{r}')$. The functional derivative, $\delta u^{+}(\underline{r})/\delta\tilde{\varepsilon}(\underline{r}')$, describes how the field at a point $\underline{r}$ varies when the permittivity at a point, $\underline{r}'$, is varied. An excellent description of the mechanics of taking functional derivatives is presented in Roman's book on quantum field theory [5]. The quantity $\delta u^{+}(\underline{r})/\delta\tilde{\varepsilon}(z,\underline{\rho}')$ is determined by integrating (1.4) from zero to z, taking the functional derivative of the resulting equation with respect to $\tilde{\varepsilon}(\underline{r}')$ and then taking the limit as $z' \rightarrow z$. Assuming that backscatter is insignificant results in the expression

$$\frac{\delta u^{+}(\underline{r})}{\delta\tilde{\varepsilon}(z,\underline{\rho}')} = \frac{ik}{4} \; \delta(\underline{\rho} - \underline{\rho}')u^{+}(z,\underline{\rho}') \quad . \qquad (1.9)$$

Substituting (1.9) into (1.8) results in

$$<\tilde{\varepsilon}(\underline{r})u^{+}(\underline{r})> = \frac{ik}{4} \; A(o)<u^{+}(\underline{r})> \qquad (1.10)$$

where A(0) is the variance of the permittivity fluctuations. A closed equation for the mean can be written as

$$2ik \frac{\partial \langle u^{+}(\underline{r})\rangle}{\partial z} + \nabla_T^2 \langle u^{+}(\underline{r})\rangle + \frac{ik^3}{4} A(0)\langle u^{+}(\underline{r})\rangle = 0 . \quad (1.11)$$

Now (1.11) is readily solved to result

$$\langle u^{+}(\underline{r})\rangle = u_0^{+}(\underline{r}) \exp\left(-k^2 \frac{A(0)}{8} z\right) , \quad (1.12)$$

where $u_0^{+}(\underline{r})$ is the free space field. The result in (1.12) states that the forward going average field, in a medium where backscatter is negligible, decays in the direction of propagation at a rate dependent on the variance of the permittivity fluctuations. Physically, what is occurring can be seen by imagining that the wave in the random medium is composed of an average and a fluctuating part, that is,

$$u^{+}(\underline{r}) = \langle u^{+}(\underline{r})\rangle + u_f^{+}(\underline{r}) . \quad (1.13)$$

The average field decreases in the direction of propagation while the fluctuating part increases. The further through a random medium a wave propagates the more violently it is perturbed. Note that the only type of wave medium interaction that is neglected is the backscatter event. All other possible forward multiple interactions of which there are an infinite number are included in this formulation. Equation (1.4) can then be used to derive an equation for the mutual coherence function $\langle u^{+}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)\rangle$. The mutual coherence function describes the correlation between the field at any two points in a plane transverse to the direction of propagation. When $\underline{\rho}_1 = \underline{\rho}_2$ it is equal to the average field intensity or energy. Assuming no backscatter and a medium delta correlated in the direction of propagation the mutual coherence function obeys

$$\left\{2ik\frac{\partial}{\partial z} + (\nabla^2_{T_1} - \nabla^2_{T_2}) + \frac{ik^3}{2}[A(0) - A(\underline{\rho}_1 - \underline{\rho}_2)]\right\} \cdot \langle u(z,\underline{\rho}_1)u^{*}(z,\underline{\rho}_2)\rangle = 0 \quad . \tag{1.14}$$

For the case of a plane wave normally incident upon an infinite slab (1.14) can be solved to yield

$$\langle u(z,\underline{\rho}_1)u^*(z,\underline{\rho}_2)\rangle = \exp\left\{-\frac{k^2}{4}[A(0) - A(\underline{\rho}_1 - \underline{\rho}_2)]z\right\} \tag{1.15}$$

The coherence between two fixed $\underline{\rho}$ points decrease as a wave propagates through. However, the situation $\underline{\rho}_1 = \underline{\rho}_2$ is a simple statement of energy conservation, that is, since there is no backscatter all the energy incident upon the medium propagates through it. The details of the parabolic equation method can be found in Ishimaru [2].

1.2 The Problem

The problem addressed in this thesis is the extension of the parabolic equation method to include backscatter. The physical situation is a plane wave normally incident upon a random slab infinite in the plane transverse to the direction of propagation. The slab is statistically homogeneous and isotropic in the transverse plane, isotropic in the longitudinal direction and has a permittivity delta correlated in the direction of propagation. That is, the permittivity correlations are given by $<\tilde{\varepsilon}(z,\underline{\rho})\ \tilde{\varepsilon}(z',\underline{\rho}')>$ $= \delta(z-z')A(\rho_d)$ where $\rho_d = |\underline{\rho}-\underline{\rho}'|$. Note that $A(\rho_d)$ is taken to be either Gaussian or exponential in this work and that $A(0) = \sigma_\varepsilon^2$, the variance of permittivity fluctuations. The total field is resolved into forward and backward components denoted by E^+ and E^- respectively (see Fig. 1.1). The addition of the backward component represents the extension to the conventional parabolic approach. The aim of Chapter II is to derive expressions for the functional derivatives with respect to permittivity fluctuations of the forward and backscatter fields in the random medium. These derivatives are needed in Chapters III and IV to determine partial differential equations for the average fields and the various field coherence functions. Chapter III contains the derivation and solution of the equations governing the average forward and backscatter field quantities. These solutions are obtained analytically. Chapter IV contains the derivation and solution of the field coherence equations. Except for limiting values of various parameters, the coherence equations require numerical treatment. General discussion of the results is given in Chapter V.

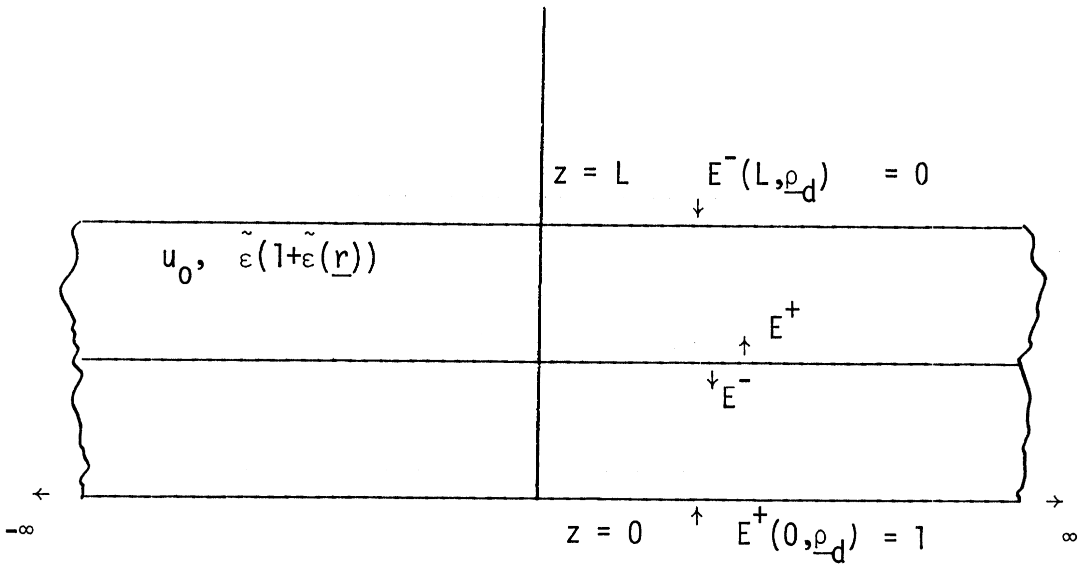


Fig. 1.1 Geometry for Backscatter from a Random Medium.

CHAPTER II. NEW FORMULATION INCLUDING BACKSCATTER

2.1 Field Resolution

The first step in the extension of the parabolic equation method to include backscatter is to reformulate the random medium problem to explicitly account for backscatter. The total field, E, includes backscatter effects but must be resolved in some way to enable those effects to be operated on. The obvious approach is to write the total field as a sum of the forward propagating, E^+, and backscatter, E^-, fields, that is

$$E(\underline{r}) = E^+(\underline{r}) + E^-(\underline{r}) \quad . \tag{2.1}$$

Since there was one equation and one unknown in (1.1) and a second unknown is, by (2.1), introduced, an additional relationship has to be selected between the two unknowns,

$$\frac{\partial E(\underline{r})}{\partial z} = ik[E^+(\underline{r}) - E^-(\underline{r})] \tag{2.2}$$

where k is the free space wave number.

The motivation for the above relationship is the functional form plane waves would take is propagating along the positive and negative z directions. Using (2.1) and (2.2) the following relationships

$$E^{+}(\underline{r}) = \frac{1}{2}\left(E(\underline{r}) + \frac{1}{ik}\frac{\partial E(\underline{r})}{\partial z}\right) \tag{2.3}$$

and

$$E^{-}(\underline{r}) = \frac{1}{2}\left(E(\underline{r}) - \frac{1}{ik}\frac{\partial E(\underline{r})}{\partial z}\right) \tag{2.4}$$

hold. Equations (2.1) and (2.2) in conjunction with the reduced wave equation

$$\nabla^2 E(\underline{r}) + k^2(1 + \tilde{\varepsilon}(\underline{r}))E(\underline{r}) = 0 \tag{2.5}$$

is used to arrive at a set of coupled partial differential equations. The easiest way to derive the coupled set of equations is to begin with the following matrix equation:

$$\frac{\partial}{\partial z}\begin{pmatrix} E(\underline{r}) \\ \frac{\partial E(\underline{r})}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\nabla_T^2 - k^2(1 + \tilde{\varepsilon}(\underline{r})) & 0 \end{pmatrix}\begin{pmatrix} E(\underline{r}) \\ \frac{\partial E(\underline{r})}{\partial z} \end{pmatrix} . \tag{2.6}$$

This matrix formulation was originally described by Corones [6] and McDaniel [7] in separate works. The following

$$\begin{pmatrix} E(\underline{r}) \\ \\ \dfrac{\partial E(\underline{r})}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \\ ik & -ik \end{pmatrix} \begin{pmatrix} E^{+}(\underline{r}) \\ \\ E^{-}(\underline{r}) \end{pmatrix} \tag{2.7}$$

is substituted into (2.6) resulting in

$$\begin{pmatrix} 1 & 1 \\ \\ ik & -ik \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} E^{+}(\underline{r}) \\ \\ E^{-}(\underline{r}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \\ -\nabla_T^2 - k^2(1+\tilde{\varepsilon}(\underline{r})) & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \\ ik & -ik \end{pmatrix} \begin{pmatrix} E^{+}(\underline{r}) \\ \\ E^{-}(\underline{r}) \end{pmatrix} . \tag{2.8}$$

Equation 2.8 can be written as

$$\frac{\partial}{\partial z} \begin{pmatrix} E^{+}(\underline{r}) \\ \\ E^{-}(\underline{r}) \end{pmatrix} = -\frac{1}{2ik} \begin{pmatrix} \nabla_T^2 + k^2(2+\tilde{\varepsilon}(\underline{r})) & \nabla_T^2 + k^2\tilde{\varepsilon}(\underline{r}) \\ \\ -\nabla_T^2 - k^2\tilde{\varepsilon}(\underline{r}) & -\nabla_T^2 - k^2(2+\tilde{\varepsilon})(\underline{r}) \end{pmatrix} \begin{pmatrix} E^{+}(\underline{r}) \\ \\ E^{-}(\underline{r}) \end{pmatrix} . \tag{2.9}$$

The assumption is made that scattering is primarily in the forward and backward directions. The embodiment of the assumption is

$$E^{+}(\underline{r}) = u^{+}(\underline{r})e^{ikz} \tag{2.10}$$

and

$$E^{-}(\underline{r}) = u^{-}(\underline{r})e^{-ikz} \quad . \tag{2.11}$$

Substitution of (2.10) and (2.11) into (2.9) leads to

$$\frac{\partial}{\partial z}\begin{pmatrix} u^+(\underline{r}) \\ \\ u^-(\underline{r}) \end{pmatrix} = -\frac{1}{2ik}\begin{pmatrix} \nabla_T^2+k^2\tilde{\varepsilon}(\underline{r}) & \left(\nabla_T^2+k^2\tilde{\varepsilon}(\underline{r})\right)e^{-2ikz} \\ \\ \left(-\nabla_T^2-k^2\tilde{\varepsilon}(\underline{r})\right)e^{2ikz} & -\nabla_T^2-k^2\tilde{\varepsilon}(\underline{r}) \end{pmatrix}\begin{pmatrix} u^+(\underline{r}) \\ \\ u^-(\underline{r}) \end{pmatrix} \tag{2.12}$$

which is more easily understood in the following form

$$\frac{\partial}{\partial z}\begin{pmatrix} u^+(\underline{r}) \\ \\ u^-(\underline{r}) \end{pmatrix} + \frac{1}{2ik}\begin{pmatrix} \nabla_T^2+k^2\tilde{\varepsilon}(\underline{r}) & 0 \\ \\ 0 & -\nabla_T^2-k^2\tilde{\varepsilon}(\underline{r}) \end{pmatrix}\begin{pmatrix} u^+(\underline{r}) \\ \\ u^-(\underline{r}) \end{pmatrix}$$

$$= -\frac{1}{2ik}\begin{pmatrix} 0 & \left(\nabla_T^2+k^2\tilde{\varepsilon}(\underline{r})\right)e^{-2ikz} \\ \\ \left(-\nabla_T^2-k^2\tilde{\varepsilon}(\underline{r})\right)e^{2ikz} & 0 \end{pmatrix}\begin{pmatrix} u^+(\underline{r}) \\ \\ u^-(\underline{r}) \end{pmatrix}. \tag{2.13}$$

The operator

$$L^+ = \frac{\partial}{\partial z} + \frac{1}{2ik}\left(\nabla_T^2 + k^2\tilde{\varepsilon}(\underline{r})\right)$$

is a parabolic operator and may be interpreted as describing the propagation of a complex wave amplitude through an inhomogeneous medium, where propagation occurs primarily in the positive z-direction.

The operator

$$L^{-} = \frac{\partial}{\partial z} - \frac{1}{2ik} \left(\nabla_T^2 + k^2 \tilde{\varepsilon}(\underline{r})\right)$$

is also parabolic. However, it describes propagation as being primarily in the negative z-direction. That the sense of propagation as discussed above is valid will be obvious when properties of fundamental solutions of parabolic operators are discussed in Section 2.2. The matrix on the right-hand side of (2.13) is a scattering or coupling matrix. It converts waves propagating in the plus z-direction into waves propagating in the minus z-direction and vice versa. As pointed out by Corones [6] and McDaniel [7], the splitting matrix described by (2.7) is only one way, of numerous possibly ways, to resolve the total field into forward and back-scatter fields. Other methods of resolution may be more appealing on physical grounds but lead to intractable equations. Since coupling is caused by permittivity variations it would make sense if when $\tilde{\varepsilon}(\underline{r}) \to 0$ that the coupling matrix vanished. An examination of (2.13) reveals that this clearly is not the case. There are splitting representations for which this kind of limiting process occurs. However, those representations lead to equations for which little in the way of solution can be done. The no backscatter situation results from the field resolution presented in this section if it is assumed that the coupling matrix is zero. Note that the equation for the forward going field is exactly equation (1.4), the no backscatter parabolic equation, from the introduction.

The resolution of the total field as set forth in this section in equation (2.13) is the starting point for the extension of the parabolic equation method to include backscatter and will be used in the subsequent analysis.

2.2 Parabolic Equation Properties

Central to the extension of the parabolic equation method to include backscatter as presented in this thesis is an understanding of the properties of fundamental solutions of parabolic equations. A fundamental solution of a parabolic equation is like a Green's function in elliptic equation theory in that it can be used to write solutions for nonhomogeneous parabolic equations. An excellent treatise on parabolic equations is Friedman's book [8]. Only those properties germane to the backscatter extension problem are outlined here.

The equation for the complex amplitude of a forward propagating wave in the random medium is, as presented in Section 2.1, (2.13) given by

$$\frac{\partial u^+(\underline{r})}{\partial z} + \frac{1}{2ik} \nabla_T^2 u^+(\underline{r}) - \frac{ik}{2} \tilde{\varepsilon}(\underline{r}) u^+(\underline{r}) = -S(\underline{r}) \quad , \qquad (2.14)$$

where the terms on the right-hand side in Section 2.1 are represented by $S(\underline{r})$ and are thought of as source terms. A fundamental solution, $\Gamma^+(z,z',\underline{\rho},\rho')$ obeying

$$\frac{\partial \Gamma^+(z,z',\underline{\rho},\underline{\rho}')}{\partial z} + \frac{i}{2ik} \nabla_T^2 \Gamma^+(z,z',\underline{\rho},\underline{\rho}') - \frac{ik}{2} \tilde{\varepsilon}(\underline{r})\Gamma^+(z,z',\underline{\rho},\underline{\rho}') = 0 \tag{2.15}$$

is sought. Consider the free space equation

$$\frac{\partial Z^+(z,z',\underline{\rho},\underline{\rho}')}{\partial z} + \frac{1}{2ik} \nabla_T^2 Z^+(z,z',\underline{\rho},\underline{\rho}') = -\delta(z-z')\delta(\underline{\rho} - \underline{\rho}') \ . \tag{2.16}$$

Since the random medium is infinite in the plane transverse to the direction of propagation use can be made of the two dimensional Fourier transform over the transverse plane in the solution of (2.16). Writing

$$Z^+(z,z',\underline{\rho},\underline{\rho}') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{Z}^+(z,z',\underline{k_\perp})e^{i\underline{k_\perp}(\underline{\rho}-\underline{\rho}')} \, d\underline{k_\perp} \ , \tag{2.17}$$

where $\hat{Z}^+(z,z',\underline{k_\perp})$ is the two-dimensional Fourier transform of $Z^+(z,z',\underline{\rho},\underline{\rho}')$ relative to the transverse plane and $\int_{-\infty}^{\infty}(\quad)\,d\underline{k_\perp}$ implies a double integration over infinite limits, (2.16) is written as

$$\frac{\partial \hat{Z}^+(z,z',\underline{k_\perp})}{\partial z} - \frac{k_\perp^2}{2ik} \hat{Z}^+(z,z',\underline{k_\perp}) = -\delta(z - z') \ . \tag{2.18}$$

Solving (2.18) subject to the condition

$$\lim_{\varepsilon \to 0} \left(\hat{Z}^+(z,z',\underline{k}_\perp)\Big|_{z=z'+\varepsilon} - \hat{Z}^+(z,z',\underline{k}_\perp)\Big|_{z=z'-\varepsilon} \right) = -1 \qquad (2.19)$$

gives

$$\left.\begin{aligned} \hat{Z}^+(z,z',\underline{k}_\perp) &= -\exp \frac{k_\perp^2}{2ik}(z - z') \qquad && z \geq z' \\ &= 0 && z < z' \end{aligned}\right\} \qquad (2.20)$$

The inverse Fourier transform of (2.30) yields

$$\left.\begin{aligned} Z^+(z,z'\underline{\rho},\underline{\rho}') &= \frac{1}{2\pi} \frac{ik}{z - z'} \exp \frac{ik}{2} \frac{|\underline{\rho} - \underline{\rho}'|^2}{z - z'} \qquad && z \geq z' \\ &= 0 && z < z' \end{aligned}\right\} \qquad (2.21)$$

Substituting (2.20) into (2.17) the following observation is made

$$\lim_{z' \to z} Z^+(z,z',\underline{\rho},\underline{\rho}') = \frac{-1}{(2\pi)^2} \lim_{z' \to z} \int_{-\infty}^{\infty} \exp\left(i\underline{k}_\perp(\underline{\rho} - \underline{\rho}') + \frac{k_\perp^2}{2ik}(z - z') \right) d\underline{k}_\perp \qquad (2.22)$$

or

$$Z^+(z,z,\underline{\rho},\underline{\rho}') = -\delta(\underline{\rho} - \underline{\rho}') \quad . \qquad (2.23)$$

The above property will be crucial in what follows. Friedman [8] has shown that a fundamental solution $\Gamma^+(z,z',\underline{\rho},\underline{\rho}')$ exists, is unique and can be written in the following form

$$\Gamma^+(z,z',\underline{\rho},\underline{\rho}') = Z^+(z,z',\underline{\rho},\underline{\rho}') - \frac{ik}{2}\int_{z'}^{z} d\beta \int_{-\infty} d\underline{y} Z^+(z,\beta,\underline{\rho},\underline{y})$$

$$\cdot\ \tilde{\varepsilon}(\beta,\underline{y})\Gamma^+(\beta,z',\underline{y},\underline{\rho}') \quad \text{for} \quad z > z' \ . \tag{2.24}$$

That (2.24) is a solution of (2.15) can be easily demonstrated by operating on (2.24) with

$$L^+ = \frac{\partial}{\partial z} + \frac{1}{2ik}\nabla_T^2 - \frac{ik}{2}\tilde{\varepsilon}(\underline{r})$$

and making use of (2.23). Since (2.24) is true for position z forward of the excitation point, z', L^+ can be interpreted as describing forward propagation. Note that the property in (2.23) is also true for (2.24) since the integral in (2.24) goes to zero as $z' \to z$. That is,

$$\lim_{z' \to z} \Gamma^+(z,z',\underline{\rho},\underline{\rho}') = \lim_{z' \to z} Z^+(z,z',\underline{\rho},\underline{\rho}') = -\delta(\underline{\rho} - \underline{\rho}') \tag{2.25}$$

holds. The complex amplitude of a wave propagating in the negative z-direction obeys

$$\frac{\partial u^-(\underline{r})}{\partial z} - \frac{1}{2ik}\nabla_T^2 u^-(\underline{r}) + \frac{ik}{2}\tilde{\varepsilon}(\underline{r})u^-(\underline{r}) = -\tilde{S}(\underline{r}) \qquad (2.26)$$

while its corresponding fundamental solution obeys

$$\frac{\partial \Gamma^-(z,z',\underline{\rho},\underline{\rho}')}{\partial z} - \frac{1}{2ik}\nabla_T^2 \Gamma^-(z,z',\underline{\rho},\underline{\rho}') + \frac{ik}{2}\tilde{\varepsilon}(\underline{r})\Gamma^-(z,z',\underline{\rho},\underline{\rho}') = 0 \ . \qquad (2.27)$$

The fundamental solution of the backward propagating wave in free space obeys

$$\frac{\partial Z^-(z,z',\underline{\rho},\underline{\rho}')}{\partial z} - \frac{1}{2ik}\nabla_T^2 Z^-(z,z',\underline{\rho},\underline{\rho}') = -\delta(z-z')\delta(\underline{\rho}-\underline{\rho}') \ . \qquad (2.28)$$

The solution of (2.28) can be shown to be

$$\left.\begin{aligned} Z^-(z,z',\underline{\rho},\underline{\rho}') &= -\frac{1}{2\pi}\,\frac{ik}{z-z'}\exp -\frac{ik}{2}\,\frac{|\underline{\rho}-\underline{\rho}'|^2}{z-z'} \qquad & z \le z' \\ &= 0 & z > z' \end{aligned}\right\} \qquad (2.29)$$

by the same approach as used for $Z^+(z,z',\underline{\rho},\underline{\rho}')$. As for $Z^+(z,z',\underline{\rho},\underline{\rho}')$ (2.29) can be written as

$$Z^-(z,z,\underline{\rho},\underline{\rho}') = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\underline{k}_\perp \exp\left(i\underline{k}_\perp(\underline{\rho}-\underline{\rho}') - \frac{k_\perp^2}{2ik}(z-z')\right)$$

implying that

$$\lim_{z' \to z} Z^-(z,z',\underline{\rho},\underline{\rho}') = -\delta(\underline{\rho} - \underline{\rho}') \quad . \tag{2.31}$$

Now a solution for (2.27) can be constructed which obeys

$$\Gamma^-(z,z',\underline{\rho},\underline{\rho}') = Z^-(z,z',\underline{\rho},\underline{\rho}') - \frac{ik}{2} \int_z^{z'} d\beta \int_{-\infty}^{\infty} d\underline{y}\, Z^-(z,\beta,\underline{\rho},\underline{y})$$

$$\cdot\ \tilde{\varepsilon}(\beta,\underline{y})\Gamma^-(\beta,z',\underline{y},\underline{\rho}') \quad , \tag{2.32}$$

for $z' > z$. The above can easily be shown to satisfy (2.27) by operating on (2.32) with

$$L^- = \frac{\partial}{\partial z} - \frac{1}{2ik} \nabla_T^2 + \frac{ik}{2} \tilde{\varepsilon}(\underline{r}) \quad .$$

Since (2.32) is true for observation point z in the backward direction from the source, z', L^- can be intepreted as describing a backward propagating wave. An additional property of parabolic equations relating $\Gamma^+(z,z',\underline{\rho},\underline{\rho}')$ and $\Gamma^-(z,z',\underline{\rho},\underline{\rho}')$, which is also presented in Friedman's book [8], may be interpreted as a reciprocity condition in wave propagation. The property is given by

$$\Gamma^+(z,z',\underline{\rho},\underline{\rho}') = \Gamma^-(z',z,\underline{\rho}',\underline{\rho}) \quad . \tag{2.33}$$

Physically (2.33) states that in an inhomogeneous medium the source and observation point may be interchanged. The final property of the fundamental solutions of parabolic equations is that they can be used to write solutions for quantities that obey the nonhomogeneous parabolic equations such as Eqs. (2.14) and (2.26). It is easy to see that

$$u^+(\underline{r}) = -\int_{-\infty}^{\infty} d\underline{\rho}'\Gamma^+(z,0,\underline{\rho},\underline{\rho}')u^+(0,\underline{\rho}') + \int_0^z dz'\int_{-\infty}^{\infty} d\underline{\rho}' \cdot \Gamma^+(z,z',\underline{\rho},\underline{\rho}')S(z',\underline{\rho}') \quad , \qquad (2.34)$$

satisfies equation (2.14) where $u^+(0,\underline{\rho}')$ is $u^+(\underline{r})$ at the boundary at $z = 0$. Equation (2.34) is written for the slab geometry considered in this thesis. Note that $u^+(0,\underline{\rho}')$ is an arbitrary excitation on that slab boundary. Similarly,

$$u^-(\underline{r}) = -\int_z^L dz'\int_{-\infty}^{\infty} d\underline{\rho}' \ \Gamma^-(z,z',\underline{\rho},\underline{\rho}')\hat{S}(z',\underline{\rho}') \quad , \qquad (2.35)$$

where use has been made of the fact that there is no backscatter at the far side of the slab at $z = L$, that is $u^-(L,\underline{\rho}) = 0$.

The above parabolic equation properties are all that is necessary to extend the parabolic equation method to include backscatter.

2.3 Elliptic/Parabolic Correspondence

The correspondence between elliptic and parabolic equations is examined in this section. The relationship between the two is examined initially for the no backscatter case and then the natural extension of this correspondence to include backscatter is presented.

The introductory chapter in this work presents the total field, E, as possessing a behavior as described by the elliptic equation

$$\nabla^2 E + k^2(1 + \tilde{\varepsilon}(\underline{r}))E = 0 \quad . \tag{2.36}$$

Assuming scattering primarily in the forward direction, the substitution $E = u^+(\underline{r})e^{ikz}$ results in, approximately,

$$2ik \frac{\partial u^+(\underline{r})}{\partial z} + \nabla_T^2 u^+(\underline{r}) + k^2\tilde{\varepsilon}(\underline{r})u^+(\underline{r}) = 0 \quad . \tag{2.37}$$

As discussed earlier, the above equation is parabolic. Section 2.2 describes fundamental solutions, $\Gamma^+(z,z',\underline{\rho},\underline{\rho}')$, for equations containing parabolic operators of the form

$$L^+ = \frac{\partial}{\partial z} + \frac{1}{2ik}\nabla_T^2 - \frac{ik}{2}\tilde{\varepsilon}(\underline{r})$$

in terms of which solutions for nonhomogeneous parabolic equations can be written. The work of John [9] describes a method of constructing fundamental solutions or Green's functions for elliptic differential equations such as (2.36). Since (2.37) and (2.36) are related by a transformation it is reasonable to expect that the fundamental solution of (2.37) and the Green's function for (2.36) are related in some way. The nature of the relationship is clear if a comparison is made of the free space Green's function for the elliptic equation to the free space fundamental solution for the positive propagating parabolic equation. The Green's function for (2.36) in free space is given by

$$\tilde{G}(\underline{r},\underline{r}') = \frac{1}{4\pi} \frac{1}{\sqrt{(z-z')^2 + |\underline{\rho} - \underline{\rho}'|^2}} \exp\left(ik\sqrt{(z-z')^2 + |\underline{\rho} - \underline{\rho}'|^2}\right) \tag{2.38}$$

while the fundamental solution for (2.37) in free space is

$$\left.\begin{aligned} Z^{+}(z,z',\underline{\rho},\underline{\rho}') &= \frac{1}{2\pi}\frac{ik}{z-z'} \exp \frac{ik}{2}\frac{|\underline{\rho} - \underline{\rho}'|^2}{z-z'} & z > z' \\ &= 0 & z < z' \end{aligned}\right\} \tag{2.39}$$

Expression (2.38) describes propagation away from a source at $(z,\underline{\rho}')$ as occurring on spherical wavefronts in all directions. Now expression (2.39) confines propagation to the positive z-direction and states that it occurs on parabolic wavefronts. Approximately when $(z-z') \gg |\underline{\rho}-\underline{\rho}'|$

$$\tilde{G}(\underline{r},\underline{r}') \simeq \frac{1}{4\pi} \frac{1}{|z-z'|} e^{ik|z-z'| + ik\frac{|\underline{\rho} - \underline{\rho}'|^2}{|z-z'|}} \; . \tag{2.40}$$

Now for $z > z'$ the following correspondence

$$\tilde{G}(\underline{r},\underline{r}') \approx \frac{e^{ik(z-z')}}{2ik} Z^{+}(z,z',\underline{\rho},\underline{\rho}') \qquad (2.41))$$

is seen to exist. Analogously, it can be argued that in the homogeneous medium the Green's function, $G(\underline{r},\underline{r}')$, for equation (2.36) is approximately related to the fundamental solution, $\Gamma^{+}(z,z',\underline{\rho},\underline{\rho}')$, for equation (2.37) in the following manner

$$G(\underline{r},\underline{r}') = \frac{e^{ik(z-z')}}{\partial ik} \Gamma^{+}(z,z',\underline{\rho},\underline{\rho}') \qquad (2.42)$$

for $z > z'$. This approximation completely ignores backscatter, i.e., the $z < z'$ situation. Mathematically from eq. (2.40) it is seen that in free space

$$\tilde{G}(\underline{r},\underline{r}') \approx \begin{cases} \dfrac{e^{ik(z-z')}}{2ik} z+(z,z',\underline{\rho},\underline{\rho}') & z > z' \\ \dfrac{e^{-ik(z-z')}}{2ik} z^{-}(z,z',\underline{\rho},\underline{\rho}') & z < z' \end{cases} . \qquad (2.43)$$

Section 2.1 describes a resolution of the total field $E(\underline{r})$ into forward and backward propagating parts $E^{+}(\underline{r})$ and $E^{-}(\underline{r})$, respectively. Let the Green's function $G(\underline{r},\underline{r}')$ be written as

$$G(\underline{r},\underline{r}') = G^{+}(\underline{r},\underline{r}') + G^{-}(\underline{r},\underline{r}') \quad , \qquad (2.44)$$

where $G^{+}(\underline{r},\underline{r}')$ describes forward propagation and $G^{-}(\underline{r},\underline{r}')$ backward propagation. Since, from Section 2.1, the substitutions $E^{+}(\underline{r}) = u^{+}(\underline{r})e^{ikz}$ and $E^{-}(\underline{r}) = u^{-}(\underline{r})e^{-ikz}$ lead to a pair of coupled parabolic differential equations whose solutions can be written in terms of the fundamental solutions $\Gamma^{+}(z,z',\underline{\rho},\underline{\rho}')$ and $\Gamma^{-}(z,z',\underline{\rho},\underline{\rho}')$, the implication of equation (2.43) and the material leading to equation (2.43) in this section is that it is possible to write

$$G^{+}(\underline{r},\underline{r}') \approx \frac{e^{ik(z-z')}}{2ik} \Gamma^{+}(z,z',\underline{\rho},\underline{\rho}') \qquad z > z' \tag{2.45}$$

and

$$G^{-}(\underline{r},\underline{r}') \approx \frac{e^{-ik(z-z')}}{2ik} \Gamma^{-}(z,z',\underline{\rho},\underline{\rho}') \qquad z < z' \tag{2.46}$$

The respresentation is complete if

$$G^{+}(\underline{r},\underline{r}') \approx 0 \qquad z < z' \tag{2.47}$$

and

$$G^{-}(\underline{r},\underline{r}') \approx 0 \qquad z > z' \tag{2.48}$$

is also assumed. The physical picture is that a spherical wave front is resolved into a parabolic wave front propagating in the positive z-direction and a parabolic wave front propagating in the minus z-direction. Note that equations (2.3) and (2.4) in Section 2.1 imply that the relationships

$$\begin{aligned} G^{+}(\underline{r},\underline{r}') &= \frac{1}{2}\left(G(\underline{r},\underline{r}') + \frac{1}{ik} \frac{\partial G(\underline{r},\underline{r}')}{\partial z} \right) \\ &= \frac{e^{ik(z-z')}}{2ik} \Gamma^{+}(z,z',\underline{\rho},\underline{\rho}') \qquad z > z' \\ &= 0 \qquad z < z' \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} G^{-}(\underline{r},\underline{r}') &= \frac{1}{2}\left(G(\underline{r},\underline{r}') - \frac{1}{ik} \frac{\partial G(\underline{r},\underline{r}')}{\partial z} \right) \\ &= \frac{e^{-ik(z-z')}}{2ik} \Gamma^{-}(z,z',\underline{\rho},\underline{\rho}') \qquad z < z' \\ &= 0 \qquad z > z' \end{aligned} \tag{2.50}$$

hold. The reciprocity condition states that

$$\Gamma^{+}(z',z,\underline{\rho},\underline{\rho}') = \Gamma^{-}(z,z',\underline{\rho},\underline{\rho}') \tag{2.51}$$

so that

$$G^-(\underline{r},\underline{r}') \approx \frac{1}{2}\left(G(\underline{r},\underline{r}') - \frac{1}{2ik}\frac{\partial G(\underline{r},\underline{r}')}{\partial z}\right)$$

$$= \frac{e^{-ik(z-z')}}{2ik}\ \Gamma^+(z',z,\underline{\rho},\underline{\rho}') \qquad z < z' \qquad (2.52)$$

can be written.

The statement of the elliptic/parabolic correspondence extended to include backscatter is now complete. The above relationships will be used in the following section.

2.4 Functional Derivatives

The introductory chapter of this thesis describes the parabolic equation method for the no-backscatter situation. The point was made that following currently established mathematical procedure, in order to calculate the functional derivative of the field with respect to permittivity fluctuations it was assumed that backscatter is negligible. The objective of this section is to show that using the material in Section 2.1 through Section 2.3 it is possible to lift the restriction; that is, it is possible to develop approximate expressions for the functional derivatives of interest while including the effects of backscatter.

The total field in a region containing a source $g(\underline{r}')$ obeys

$$\nabla'^2 E(\underline{r}') + k^2(1 + \tilde{\varepsilon}(\underline{r}'))E(\underline{r}') = -g(\underline{r}') \quad , \qquad (2.53)$$

where primed coordinates were used so that unprimed coordinates will designate the observation point. Equation (2.53) is assumed to hold in a random slab of finite thickness L, in z, the direction of propagation of the incident wave. The slab is immersed in free space. The plane parallel surfaces transverse to the z-direction are infinite in extent. The incident radiation is a plane wave normally incident from free space and given by e^{ikz}. The total field $E(\underline{r})$ can be resolved into the sum of $E^+(\underline{r})$ and $E^-(\underline{r})$. $E^+(\underline{r})$ at z = 0 is given by 1 while $E^-(\underline{r})$ at z = L is 0. The above boundary conditions result since if $\tilde{\varepsilon}(\underline{r}') \to 0$ only the incident free space wave is supported as a propagating mode. Just into the medium at z = 0 the forward propagating field is the incident field and is deterministic. The transition from random medium to free space supports no backscatter since it is the inhomogeneities that give rise to backscatter. The Green's function $G(\underline{r}',\underline{r})$ corresponding to $E(\underline{r}')$ is a solution of

$$\nabla'^2 G(\underline{r}',\underline{r}) + k^2(1 + \tilde{\varepsilon}(\underline{r}'))G(\underline{r}',\underline{r}) = -\delta(\underline{r}'-\underline{r}) \ . \quad (2.54)$$

Multiplying (2.53) by $G(\underline{r}',r)$ and (2.54) by $E(\underline{r}')$ and subtracting the two expressions results in

$$\nabla' \cdot [G(\underline{r}',\underline{r})\nabla' E(\underline{r}') - E(\underline{r}')\nabla' G(\underline{r}',\underline{r})] = -g(\underline{r}')G(\underline{r}',\underline{r}) + \delta(\underline{r}',-\underline{r})E(\underline{r}') \quad (2.55)$$

where the simple identity

$$C(\underline{r}')\nabla'^2 D(\underline{r}') = \nabla' \cdot [C(\underline{r}')\nabla' D(\underline{r}')] - \nabla' C(\underline{r}') \cdot \nabla' D(\underline{r}') \quad (2.56)$$

was used. Integrating (2.55) over V', the volume of the slab, results in

$$\oint_{S'} [G(\underline{r}',\underline{r})\, \nabla' E(\underline{r}') - E(\underline{r}')\nabla' G(\underline{r}'.\underline{r})] \cdot d\underline{S}'$$

$$= - \int_{V'} d\underline{r}'\, g(\underline{r}')G(\underline{r}',\underline{r}) + E(\underline{r}) \quad , \quad (2.57)$$

where S' is the surface area of the scattering body. The divergence theorem was used in writing (2.57). Explicitly writing the volume and area integral in (2.57)

$$E(\underline{r}) = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\; g(\underline{r}')G(\underline{r}',\underline{r}) + \int_{-\infty}^{\infty} d\underline{\rho}' \left(-G(0,\underline{\rho}',z,\underline{\rho}) \cdot \left.\frac{\partial E(\underline{r}')}{\partial z'}\right|_{z'=0} + E(0,\underline{\rho}') \left.\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\right|_{z'=0} \right) + \int_{-\infty}^{\infty} d\underline{\rho}'$$

$$\cdot \left(G(L,\underline{\rho}',z,\underline{\rho}) \left.\frac{\partial E(\underline{r}')}{\partial z'}\right|_{z'=L} - E(L,\underline{\rho}') \left.\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\right|_{z'=L} \right) \quad . \quad (2.58)$$

The second integral in (2.58) can be written as

$$\int_{-\infty}^{\infty} d\underline{\rho}' \left(-ikG(0,\underline{\rho}',z,\underline{\rho})E(0,\underline{\rho}') - G(0,\underline{\rho}',z,\underline{\rho}) \left.\frac{\partial E(\underline{r}')}{\partial z'}\right|_{z'=0} + ikG(0,\underline{\rho}',z,\underline{\rho})E(0,\underline{\rho}') + E(0,\underline{\rho}') \left.\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\right|_{z'=0} \right) \quad (2.59)$$

or

$$-2ik\int_{-\infty}^{\infty} d\underline{\rho}' \left(G(0,\underline{\rho}',z,\underline{\rho}) \left[\frac{1}{2}E(0,\underline{\rho}') + \frac{1}{2ik}\left.\frac{\partial E(\underline{r}')}{\partial z'}\right|_{z'=o}\right]\right.$$

$$\left. - E(0,\underline{\rho}')\left[\frac{1}{2}G(0,\underline{\rho}',z,\underline{\rho}) + \frac{1}{2ik}\left.\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\right|_{z'=o}\right]\right) . \qquad (2.60)$$

The third integral in (2.58) is written as

$$\int_{-\infty}^{\infty} d\underline{\rho}' \left(ikG(L,\underline{\rho}',z,\underline{\rho})E(L,\underline{\rho}') + G(L,\underline{\rho}',z,\underline{\rho}) \left.\frac{\partial E(\underline{r}')}{\partial z'}\right|_{z'=L}\right.$$

$$\left. - ikG(L,\underline{\rho}',z,\underline{\rho})E(L,\underline{\rho}') - E(L,\underline{\rho}') \left.\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\right|_{z'=L}\right) \qquad (2.61)$$

or

$$-2ik\int_{-\infty}^{\infty} d\underline{\rho}' \left(G(L,\underline{\rho}',z,\underline{\rho}) \left[\frac{1}{2}E(L,\underline{\rho}') - \frac{1}{2ik}\left.\frac{\partial E(\underline{r}')}{\partial z'}\right|_{z'=L}\right]\right.$$

$$\left. - E(L,\underline{\rho}')\left[\frac{1}{2}G(L,\underline{\rho}',z,\underline{\rho}) - \frac{1}{2ik}\left.\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\right|_{z'=L}\right]\right) . \qquad (2.62)$$

The relationships established for $E^{+}(\underline{r}')$ and $E^{-}(\underline{r}')$ in Section 2.1 are

$$E^{+}(\underline{r}') = \frac{1}{2}E(\underline{r}') + \frac{1}{2ik}\frac{\partial E(\underline{r}')}{\partial z'} \qquad (2.63)$$

$$E^{-}(\underline{r}') = \frac{1}{2}E(\underline{r}') - \frac{1}{2ik}\frac{\partial E(\underline{r}')}{\partial z'} . \qquad (2.64)$$

Equations (2.63) and (2.64) can be used in (2.60) and (2.61). Writing equation (2.58) incorporating (2.60), (2.62), (2.63) and (2.64) results in

$$E(\underline{r}) = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; g(\underline{r}')G(\underline{r}',\underline{r})$$
$$-2ik\int_{-\infty}^{\infty} d\underline{\rho}' \left(G(0,\underline{\rho}',z,\underline{\rho})E^{+}(\underline{r}')\Big|_{z'=0} - E(0,\underline{\rho}')\left[\frac{1}{2}G(0,\underline{\rho}',z,\underline{\rho})\right.\right.$$
$$\left.\left. + \frac{1}{2ik}\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\Big|_{z'=0}\right]\right) - 2ik\int_{-\infty}^{\infty} d\underline{\rho}' \left(G(L,\underline{\rho}',z,\underline{\rho})E^{-}(\underline{r}')\Big|_{z'=L}\right.$$
$$\left. - E(L,\underline{\rho}')\left[\frac{1}{2}G(L,\underline{\rho}',z,\underline{\rho}) - \frac{1}{2ik}\frac{\partial G(\underline{r}',\underline{r})}{\partial z'}\Big|_{z'=L}\right]\right). \qquad (2.65)$$

The symmetry property discussed in Roach's book on Green's functions [10], $G(\underline{r}',\underline{r}) = G(\underline{r},\underline{r}')$ is used to write (2.65) as

$$E(\underline{r}) = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; g(\underline{r}')G(\underline{r},\underline{r}')$$
$$-2ik\int_{-\infty}^{\infty} d\underline{\rho}' \left(G(z,\underline{\rho},0,\underline{\rho}')E^{+}(\underline{r}')\Big|_{z'=0} - E(0,\underline{\rho}')\left[\frac{1}{2}G(z,\underline{\rho},0,\underline{\rho}')\right.\right.$$
$$\left.\left. + \frac{1}{2ik}\frac{\partial G(\underline{r},\underline{r}')}{\partial z'}\Big|_{z'=0}\right]\right) - 2ik\int_{-\infty}^{\infty} d\underline{\rho}' \left(G(z,\underline{\rho},L,\underline{\rho}')E^{-}(\underline{r}')\Big|_{z'=L}\right.$$
$$\left. - E(L,\underline{\rho}')\left[\frac{1}{2}G(z,\underline{\rho},L,\underline{\rho}') - \frac{1}{2ik}\frac{\partial G(\underline{r},\underline{r}')}{\partial z'}\Big|_{z'=L}\right]\right). \qquad (2.66)$$

Within the context of the problem considered here it can be argued that

$$\frac{\partial G(\underline{r},\underline{r}')}{\partial z} = -\frac{\partial G(\underline{r},\underline{r}')}{\partial z'} \quad . \tag{2.67}$$

Now (2.66) is rewritten as

$$E(\underline{r}) = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; g(\underline{r}')G(\underline{r},\underline{r}')$$
$$-2ik \int_{-\infty}^{\infty} d\underline{\rho}' \left(G(z,\underline{\rho},0,\underline{\rho}')E^{+}(\underline{r}')\Big|_{z'=0} - E(0,\underline{\rho}') \left[\frac{1}{2} G(z,\underline{\rho},0,\underline{\rho}') \right.\right.$$
$$\left.\left. - \frac{1}{2ik} \frac{\partial G(\underline{r},\underline{r}')}{\partial z}\Big|_{z'=0} \right] \right) - 2ik \int_{-\infty}^{\infty} d\underline{\rho}' \left(G(z,\underline{\rho},L,\underline{\rho}')E^{-}(\underline{r}')\Big|_{z'=L} \right.$$
$$\left. - E(L,\underline{\rho}') \left[\frac{1}{2} G(z,\underline{\rho},L,\underline{\rho}') + \frac{1}{2ik} \frac{\partial G(\underline{r},\underline{r}')}{\partial z}\Big|_{z'=L} \right] \right) \quad . \tag{2.68}$$

A direct comparison can be made between the terms in square braces in (2.68) and (2.49) and (2.50) in Section 2.3. Since

$$\frac{1}{2} G(\underline{r},\underline{r}') + \frac{1}{2ik} \frac{\partial G(\underline{r},\underline{r}')}{\partial z} = 0 \qquad z < z' \tag{2.69a}$$

and

$$\frac{1}{2}\,G(\underline{r},\underline{r}') - \frac{1}{2ik}\,\frac{\partial G(\underline{r},\underline{r}')}{\partial z} = 0 \qquad z > z' \tag{2.69b}$$

for 0 < z < L (2.68) can be written as

$$E(\underline{r}) = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\; g(\underline{r}')G(\underline{r},\underline{r}') - 2ik \int_{-\infty}^{\infty} d\underline{\rho}'$$

$$\cdot\; G(z,\underline{\rho},0,\underline{\rho}')E^{+}(\underline{r}')\Big|_{z'=o} - 2ik \int_{-\infty}^{\infty} d\underline{\rho}'\; G(z,\underline{\rho},L,\underline{\rho}')E^{-}(\underline{r}')\Big|_{z'=L} \;. \tag{2.70}$$

Equation (2.70) inserted into equation (2.63) results in

$$E^{+}(\underline{r}) = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\; g(\underline{r}')\left(\frac{1}{2}\,G(\underline{r},\underline{r}') + \frac{1}{2ik}\,\frac{\partial G(\underline{r},\underline{r}')}{\partial z}\right)$$

$$- 2ik \int_{-\infty}^{\infty} d\underline{\rho}' \left(\frac{1}{2}\,G(z,\underline{\rho},0,\underline{\rho}') + \frac{1}{2ik}\,\frac{\partial G(z,\underline{\rho},0,\underline{\rho}')}{\partial z}\right) E^{+}(\underline{r}')\Big|_{z'=o}$$

$$- 2ik \int_{-\infty}^{\infty} d\underline{\rho}' \left(\frac{1}{2}\,G(z,\underline{\rho},L,\underline{\rho}') + \frac{1}{2ik}\,\frac{\partial G(z,\underline{\rho},L,\underline{\rho}')}{\partial z}\right) E^{-}(\underline{r}')\Big|_{z'=L}$$

(2.71)

while (2.70) in (2.64) leads to

$$E^-(r) = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; g(\underline{r}') \left(\frac{1}{2} G(\underline{r},\underline{r}') - \frac{1}{2ik} \frac{\partial G(\underline{r},\underline{r}')}{\partial z} \right)$$

$$- 2ik \int_{-\infty}^{\infty} d\underline{\rho}' \left(\frac{1}{2} G(z\underline{\rho},0,\underline{\rho}') - \frac{1}{2ik} \frac{\partial G(z,\underline{\rho},0,\underline{\rho}')}{\partial z} \right) E^+(\underline{r}') \Bigg|_{z'=0}$$

$$- 2ik \int_{-\infty}^{\infty} d\underline{\rho}' \left(\frac{1}{2} G(z,\underline{\rho},L,\underline{\rho}') - \frac{1}{2ik} \frac{\partial G(z,\underline{\rho},L,\underline{\rho}')}{\partial z} \right) E^-(\underline{r}') \Bigg|_{z'=L} \tag{2.72}$$

for $0 < z < L$. The task is to find the functional derivative of $E(\underline{r})$ with respect to $\tilde{\varepsilon}(\underline{r}')$. The equation now of interest is

$$\nabla^2 E(\underline{r}) + k^2(1 + \tilde{\varepsilon}(\underline{r}))E(\underline{r}) = 0 \; . \tag{2.73}$$

The functional derivative of (2.73) with respect to $\tilde{\varepsilon}(\underline{r}')$ results in

$$\left(\frac{\delta E(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} \right) + k\,(1 + \tilde{\varepsilon}(r)) \left(\frac{\delta E(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} \right) = -k^2 \delta(\underline{r} - \underline{r}')E(\underline{r}) \; . \tag{2.74}$$

Notice that (2.74) is of the same form as (2.53) in this section, therefore (2.70) through (2.72) apply to (2.74) with $E(\underline{r})$ replaced by $\delta E(\underline{r})/\delta\tilde{\varepsilon}(\underline{r}')$, $E^+(\underline{r})$ replaced by $\delta E^+(\underline{r})/\delta\tilde{\varepsilon}(\underline{r}')$, $E^-(\underline{r})$ replaced by $\delta E^-(\underline{r})/\delta\tilde{\varepsilon}(\underline{r}')$ and $g(\underline{r}')$ by $-k^2\delta(\underline{r} - \underline{r}')E(\underline{r})$. The functional derivatives of interest are

$$\frac{\delta E^+(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')} = -k^2\int_0^L dz''\int_{-\infty}^{\infty} d\underline{\rho}''\,\delta\,(\underline{r}'' - \underline{r}')E(\underline{r}'')$$

$$\cdot\left(\frac{1}{2}\,G(\underline{r},\underline{r}'') + \frac{1}{2ik}\,\frac{\partial G(\underline{r},\underline{r}'')}{\partial z}\right). \tag{2.75}$$

and

$$\frac{\delta E^-(r)}{\delta\tilde{\varepsilon}(\underline{r}')} = -k^2\int_0^L dz''\int_{-\infty}^{\infty} d\underline{\rho}''\,\delta\,(\underline{r}'' - \underline{r})E(\underline{r}'')$$

$$\cdot\left(\frac{1}{2}\,G(\underline{r},\underline{r}'') - \frac{1}{2ik}\,\frac{\partial G(\underline{r},\underline{r}'')}{\partial z}\right). \tag{2.76}$$

Since $E^+(r)\Big|_{z=0} = 1$ and $E^-(\underline{r})\Big|_{z=L} = 0$

$$\left.\frac{\delta E^+(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')}\right|_{z=0} = 0 \tag{2.77}$$

and

$$\left.\frac{\delta E^-(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')}\right|_{z=L} = 0 \quad , \tag{2.78}$$

where (2.69a) and (2.69b) were used in (2.75) and (2.76). Now (2.49) and (2.50) from Section 2.3 can be used to write (2.75) and (2.76) as

$$\frac{\delta E^+(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\int_0^z dz'' \int_{-\infty}^{\infty} d\underline{\rho}''\, \delta(\underline{r}'' - \underline{r}')E(\underline{r}'')e^{ik(z-z'')}$$

$$\cdot\, \Gamma^+(z,z'',\underline{\rho},\underline{\rho}'') \qquad (2.79)$$

and

$$\frac{\delta E^-(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\int_z^L dz'' \int_{-\infty}^{\infty} d\underline{\rho}''\, \delta(\underline{r}'' - \underline{r}')E(\underline{r}'')e^{-ik(z-z'')}$$

$$\cdot\, \Gamma^-(z,z'',\underline{\rho},\underline{\rho}'') \quad . \qquad (2.80)$$

Anticipating that z and z' will be involved in a limit, care must be taken when integrating over the delta functions. The introduction of the following step function

$$\left.\begin{aligned} \theta(z) &= 1 \qquad z > 0 \\ &= 1/2 \qquad z = 0 \\ &= 0 \qquad z < 0 \end{aligned}\right\} \qquad (2.81)$$

leads to

$$\frac{\delta E^+(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\int_0^L dz'' \int_{-\infty}^{\infty} d\underline{\rho}''\; \theta(z - z'')\delta(\underline{r}'' - \underline{r}')$$

$$\cdot\, E(\underline{r}'')\, e^{ik(z-z'')}\, \Gamma^+(z,z'',\underline{\rho},\underline{\rho}'') \qquad (2.82)$$

and

$$\frac{\delta E^-(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\int_0^L dz'' \int_{-\infty}^{\infty} d\underline{\rho}''\, \theta(z'' - z)\delta(\underline{r}'' - \underline{r}')E(\underline{r}'')$$

$$\cdot\; e^{-ik(z - z'')}\; \Gamma^-(z,z'',\underline{\rho},\underline{\rho}'') \quad . \tag{2.83}$$

Integrating over z'' and $\underline{\rho}''$ results in

$$\frac{\delta E^+(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\; \theta(z - z')E(\underline{r}')\; e^{ik(z-z')}\, \Gamma^+(z,z',\underline{\rho},\underline{\rho}') \tag{2.84}$$

and

$$\frac{\delta E^-(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\; \theta(z' - z)E(\underline{r}')\; e^{-ik(z-z')}\, \Gamma^-(z,z',\underline{\rho},\underline{\rho}') \quad . \tag{2.85}$$

Realizing that $E(\underline{r}') = E^+(\underline{r}') + E^-(\underline{r}')$ and making the substitution $E^+(\underline{r}) = u^+(\underline{r})e^{ikz}$ and $E^-(\underline{r}) = u^-(\underline{r})e^{-ikz}$ (2.84) and (2.85) are written as

$$\frac{\delta u^+(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\, \theta(z - z')\; (u^+(\underline{r}') + u^-(\underline{r}')e^{-2ikz'})\; \Gamma^+(z,z',\underline{\rho},\underline{\rho}') \tag{2.86}$$

and

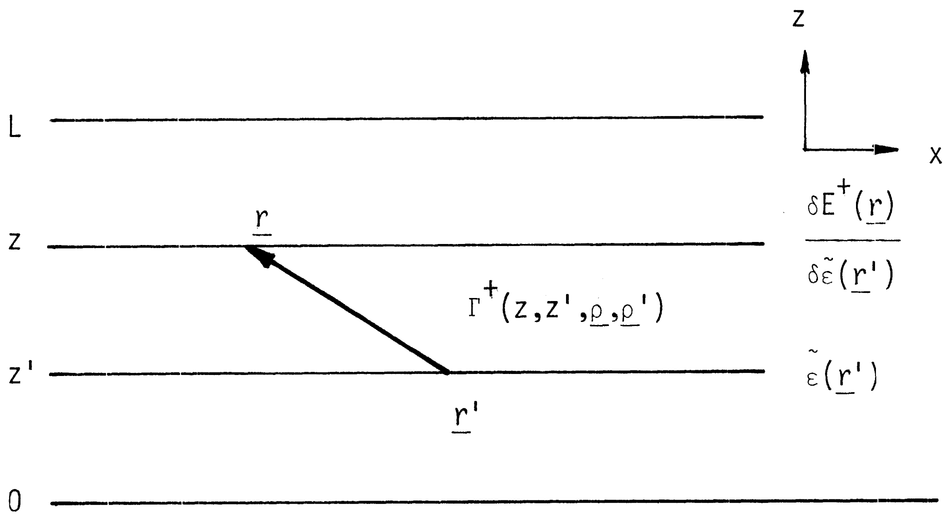
$$\frac{\delta u^-(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} = \frac{ik}{2}\; \theta(z' - z)(u^+(\underline{r}')e^{2ikz'} + u^-(\underline{r}'))\, \Gamma^-(z,z',\underline{\rho},\underline{\rho}') \quad . \tag{2.87}$$

The limit of z' → z of (2.86) and (2.87) takes place when (2.86) and (2.87) are used in the Furutsu-Novikov relation to obtain closure expressions. Discussion of these limits will be delayed until the closure expressions are required. The functional derivatives have a fairly simple physical interpretation. Equation (2.84) leads to the conclusion that the forward propagating field perturbation due to varying the permittivity fluctuation occurs in some plane z forward of plane z', where the permittivity is perturbed, and is proportional to the total field at the permittivity perturbation point as modified by the forward inhomogeneous propagation path between perturbation and observation point. Figure 2.1(a) illustrates the relationship implied by (2.84). Equation (2.85) implies that the backward propagating field perturbation due to varying the permittivity fluctuation occurs in some plane z backward of the permittivity perturbation plane, z', and is proportional to the total field at the permittivity perturbation point as modified by the backward inhomogeneous propagation path between perturbation and observation point. Figure 2.1(b) illustrates the relationship implied by (2.85). Note that the implication is that

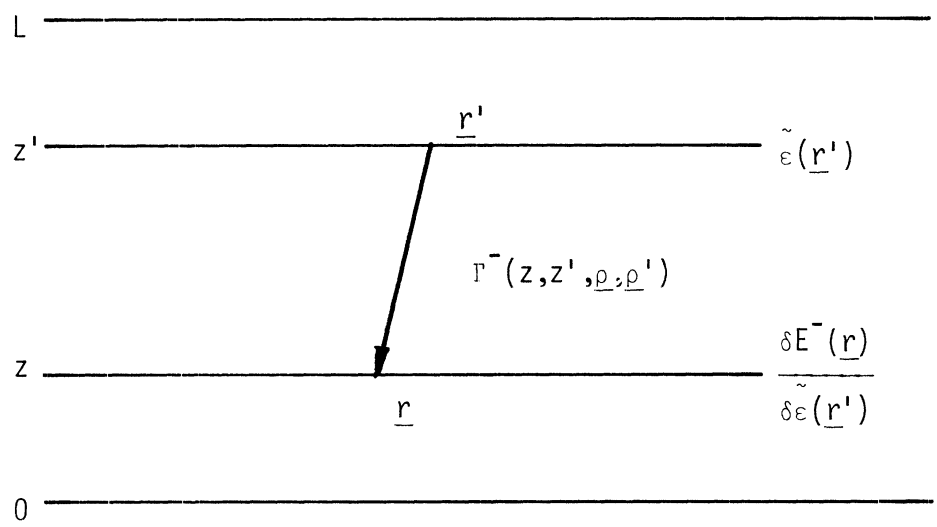
$$\frac{\delta u^{+}(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} = 0 \qquad z < z' \tag{2.88}$$

and

$$\frac{\delta u^{-}(\underline{r})}{\delta \tilde{\varepsilon}(\underline{r}')} = 0 \qquad z > z' \quad . \tag{2.89}$$



(a) Forward functional derivative $\delta E^{+}(\underline{r})/\delta\tilde{\varepsilon}(\underline{r}') = ik/2\ \theta(z - z')E(\underline{r}')$
$\cdot\ e^{ik(z-z')}\ \Gamma^{+}(z,z',\underline{\rho},\underline{\rho}')$



(b) Backward functional derivative $\delta E^{-}(\underline{r})/\delta\tilde{\varepsilon}(\underline{r}') = ik/2\ \ \theta(z'-z)E(\underline{r}')$
$\cdot\ e^{-ik(z-z')}\ \Gamma^{-}(z,z',\underline{\rho},\underline{\rho}')$.

Fig. 2.1 Field Functional Derivatives.

The effect described by (2.86) and (2.87) is a first-order effect. Higher order effects are lost since the elliptic Green's function is split into a purely forward propagating part and a purely backward propagating part, as a consequence of the approximation

$$G^{+}(\underline{r},\underline{r}') \approx 0 \qquad z < z' \tag{2.90}$$

and

$$G^{-}(\underline{r},\underline{r}') \approx 0 \qquad z > z' \ . \tag{2.91}$$

The results of this section can be used to determine closure relations which enable partial differential equations for the various moments of the forward and backward propagating fields in the random medium to be written.

CHAPTER III. MEAN FIELD

The representation of a field in a random medium as consisting of the sum of an average field and a fluctuating field can be extended to the medium in which backscatter is signficiant. The forward and backward propagating fields are written as

$$u^+(\underline{r}) = \langle u^+(\underline{r})\rangle + u_f^+(\underline{r}) \tag{3.1}$$

and

$$u^-(\underline{r}) = \langle u^-(\underline{r})\rangle + u_f^-(\underline{r}) \tag{3.2}$$

where $\langle u^+(\underline{r})\rangle$ and $\langle u^-(\underline{r})\rangle$ are the average and $u_f^+(\underline{r})$ and $u_f^-(\underline{r})$ are the fluctuating parts. The equivalent representation for the no backscatter situation, as discussed in Chapter I, has a fairly simple physical interpretation. The fluctuating part of the field increases with propagation distance at the expense of the average field. When backscatter is significant the physical interpretation of the representation is not obvious because of the infinitely many interactions which occur within the medium. Study of the average forward and backward propagating fields as described by the Bremmer series expansion lends some understanding to the process which occurs within a random body and gives physical meaning to the formulation presented in (3.1) and (3.2).

3.1 Mean Field Equations

A set of coupled partial differential equations for the average forward and average backward propagating waves in a random medium can be determined by using the material of Chapter II of this thesis. The no backscatter situation required a statement for correlations of the form $\langle u^+(\underline{r})\tilde{\varepsilon}(\underline{r})\rangle$ proportional to $\langle u^+(\underline{r})\rangle$ to realize a model consisting of a single partial differential equation and a single unknown. The backscatter case requires that $\langle u^+(\underline{r})\tilde{\varepsilon}(\underline{r})\rangle$ be written in a form proportional to $\langle u^+(\underline{r})\rangle$ and $\langle u^-(\underline{r})\rangle$ and in addition that an expression for $\langle u^-(\underline{r})\tilde{\varepsilon}(\underline{r})\rangle$ be determined proportional to $\langle u^+(\underline{r})\rangle$ and $\langle u^-(\underline{r})\rangle$ to arrive at a model with two equations and two unknowns. The Furutsu-Novikov expression will now be used to derive the required correlations. When $\tilde{\varepsilon}(\underline{r})$ is zero mean Gaussian distributed

$$\langle\tilde{\varepsilon}(\underline{r})u^+(\underline{r})\rangle = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \, \langle\tilde{\varepsilon}(\underline{r})\tilde{\varepsilon}(\underline{r}')\rangle \left\langle \frac{\delta u^+(\underline{r})}{\delta\tilde{\varepsilon}(\underline{r}')} \right\rangle . \quad (3.3)$$

The delta correlated medium assumption is made,

$$\langle\tilde{\varepsilon}(\underline{r})\tilde{\varepsilon}(\underline{r}')\rangle = \delta(z - z')A(\underline{\rho} - \underline{\rho}') \quad , \quad (3.4)$$

in what follows, where $A(\underline{\rho} - \underline{\rho}')$ is the correlation in the transverse plane.

Substituting equation (2.86) from Section 2.4 and (3.4) from above into (3.3) leads to

$$<\tilde{\varepsilon}(\underline{r})u^+(\underline{r})> \;=\; \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \;\delta(z - z')A(\underline{\rho} - \underline{\rho}')\,\frac{ik}{2}\,\theta(z - z')$$

$$<(u^+(\underline{r}') + u^-(\underline{r}')e^{-2ikz'})\,\Gamma^+(z,z',\underline{\rho},\underline{\rho}')> \;. \qquad (3.5)$$

Performing the integrations in (3.5)

$$<\tilde{\varepsilon}(\underline{r})u^+(\underline{r})> \;=\; \frac{ik}{4}\int_{-\infty}^{\infty} d\underline{\rho}' \;A(\underline{\rho} - \underline{\rho}')\;<(u^+(z,\underline{\rho}') + u^-(z,\underline{\rho}')e^{-2ikz}$$

$$\cdot\;\Gamma^+(z,z,\underline{\rho},\underline{\rho}')> \qquad (3.6)$$

and making use of the property of fundamental solutions of parabolic equations (Eq. 2.25)

$$\lim_{z' \to z} \;\Gamma^+(z,z',\underline{\rho},\underline{\rho}') \;=\; \delta(\underline{\rho} - \underline{\rho}') \qquad (3.7)$$

derived in Section 2.2, results in

$$<\tilde{\varepsilon}(\underline{r})u^+(\underline{r})> \;=\; \frac{ik}{4}\,A(0)\;(<u^+(\underline{r})> + <u^-(\underline{r})>\,e^{-2ikz}) \;. \qquad (3.8)$$

In a completely analogous manner

$$<\tilde{\varepsilon}(\underline{r})u^-(\underline{r})> \;=\; \int_{-\infty}^{\infty} d\underline{\rho}' \;A(\underline{\rho} - \underline{\rho}')\left\langle \frac{\delta u^-(\underline{r})}{\delta\tilde{\varepsilon}(z,\underline{\rho}')} \right\rangle \qquad (3.9)$$

where

$$\frac{\delta u^-(\underline{r})}{\delta\tilde{\varepsilon}(z,\underline{\rho}')} = \frac{ik}{4}\left(u^+(z,\underline{\rho}')e^{2ikz} + u^-(z,\underline{\rho}')\right)\Gamma^-(z,z',\underline{\rho},\underline{\rho}'). \quad (3.10)$$

Using the property

$$\lim_{z'\to z} \Gamma^-(z,z',\underline{\rho},\underline{\rho}') = \delta(\underline{\rho} - \underline{\rho}') \quad (3.11)$$

in (3.9) leads to

$$<\tilde{\varepsilon}(\underline{r})u^-(\underline{r})> = \frac{ik}{4} A(0) \left(<u^+(\underline{r})> e^{2ikz} + <u^-(\underline{r})>\right) . \quad (3.12)$$

These correlations can be used to write closed equations for $<u^+(\underline{r})>$ and $<u^-(\underline{r})>$.

3.2 Solution of the Mean Field Equations

A Bremmer series representation of the fields in a random medium is given by

$$\frac{\partial u^+(\underline{r})}{\partial z} + \frac{1}{2ik}\nabla_T^2 u^+(\underline{r}) - \frac{ik}{2}\tilde{\varepsilon}(\underline{r})u^+(\underline{r}) = \left(-\frac{1}{2ik}\nabla_T^2 u^-(\underline{r}) + \frac{ik}{2}\tilde{\varepsilon}(\underline{r})u^-(\underline{r})\right) e^{-2ikz} \quad (3.13)$$

and

$$\frac{\partial u^-(\underline{r})}{\partial z} - \frac{1}{2ik} \nabla_T^2 u^-(\underline{r}) + \frac{ik}{2} \tilde{\varepsilon}(\underline{r}) u^-(\underline{r}) = \left(\frac{1}{2ik} \nabla_T^2 u^+(\underline{r}) - \frac{ik}{2} \tilde{\varepsilon}(\underline{r}) u^+(\underline{r}) \right) e^{2ikz} \quad . \qquad (3.14)$$

Taking the mean of (3.13) and (3.14) and using the approximations

$$<\tilde{\varepsilon}(\underline{r}) u^+(\underline{r})> = \frac{ik}{4} A(0) \left(<u^+(\underline{r})> + <u^-(r)> e^{-2ikz} \right) \qquad (3.15)$$

and

$$<\tilde{\varepsilon}(\underline{r}) u^-(\underline{r})> = \frac{ik}{4} A(0) \left(<u^+(\underline{r})> e^{2ikz} + <u^-(\underline{r})> \right) \qquad (3.16)$$

results in

$$\frac{\partial <u^+(\underline{r})>}{\partial z} + \frac{1}{2ik} \nabla_T^2 <u^+(\underline{r})> + \frac{k^2}{4} A(0) <u^+(\underline{r})> = \left(- \frac{1}{2ik} \nabla_T^2 <u^-(\underline{r})> - \frac{k^2}{4} A(0) <u^-(\underline{r})> \right) e^{-2ikz} \qquad (3.17)$$

and

$$\frac{\partial <u^-(\underline{r})>}{\partial z} - \frac{1}{2ik} \nabla_T^2 <u^-(\underline{r})> - \frac{k^2}{4} A(0) <u^-(\underline{r})> = \left(\frac{1}{2ik} \nabla_T^2 <u^+(\underline{r})> + \frac{k^2}{4} A(0) <u^+(\underline{r})> \right) e^{+2ikz} \quad . \qquad (3.18)$$

Equations (3.17) and (3.18) can be easily solved for $<u^+(\underline{r})>$ and $<u^-(\underline{r})>$. The Fourier transform of $<u^+(\underline{r})>$ and $<u^-(\underline{r})>$ with respect to the plane transverse to the direction of propagation is given by

$$<u^+(\underline{r})> = \int_{-\infty}^{\infty} <\hat{u}^+(z,\underline{k}_\perp)> e^{i\underline{k}_\perp\cdot\underline{\rho}}\, d\underline{k}_\perp \qquad (3.19)$$

and

$$<u^-(\underline{r})> = \int_{-\infty}^{\infty} <\hat{u}^-(z,\underline{k}_\perp)> e^{i\underline{k}_\perp\cdot\underline{\rho}}\, d\underline{k}_\perp \qquad (3.20)$$

where the notation $\int_{-\infty}^{\infty} d\underline{k}_\perp$ represents a double integration. Using (3.19) and (3.20) in (3.13) and (3.14) leads to

$$\frac{\partial<\hat{u}^+>}{\partial z} - \frac{k_\perp^2}{2ik} <\hat{u}^+> + \frac{k^2}{4} A(0) <\hat{u}^+> = \left(\frac{k_\perp^2}{2ik} - \frac{k^2}{4} A(0)\right) <\hat{u}^-> e^{-2ikz} \qquad (3.21)$$

and

$$\frac{\partial<\hat{u}^->}{\partial z} + \frac{k_\perp^2}{2ik} <\hat{u}^-> - \frac{k^2}{4} A(0) <\hat{u}^-> = \left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right) <\hat{u}^+> e^{+2ikz} \quad . \qquad (3.22)$$

Solving (3.21) for $<\hat{u}^->$ and writing

$$\hat{L}^+ = \frac{\partial}{\partial z} - \frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)$$

$$<\hat{u}^-(z,k_\perp)> = \frac{e^{2ikz}}{\left(\frac{k_\perp^2}{2ik} - \frac{k^2}{4} A(0)\right)} \hat{L}^+ <\hat{u}^+> \quad . \qquad (3.23)$$

Differentiating (3.23) with respect to z gives

$$\frac{\partial\langle\hat{u}^-\rangle}{\partial z} = \frac{2ik\, e^{2ikz}}{\left(\frac{k_\perp^2}{2ik} - \frac{k^2}{4} A(0)\right)} \hat{L}^+\langle\hat{u}^+\rangle + \frac{e^{2ikz}}{\left(\frac{k_\perp^2}{2ik} - \frac{k^2}{4} A(0)\right)}$$

$$\cdot \left(\frac{\partial^2\langle\hat{u}^+\rangle}{\partial z^2} - \frac{k_\perp^2}{2ik} \frac{\partial\langle\hat{u}^+\rangle}{\partial z} + \frac{k^2}{4} A(0) \frac{\partial\langle\hat{u}^+\rangle}{\partial z}\right) \quad . \qquad (3.24)$$

Substituting (3.23) and (3.24) into (3.22) results in

$$\frac{2ik\, e^{2ikz}}{\left(\frac{k_\perp^2}{2ik} - \frac{k^2}{4} A(0)\right)} \hat{L}^+\langle\hat{u}^+\rangle + \frac{e^{2ikz}}{\left(\frac{k_\perp^2}{2ik} - \frac{k^2}{4} A(0)\right)}$$

$$\cdot \left(\frac{\partial^2\langle\hat{u}^+\rangle}{\partial z^2} - \frac{k_\perp^2}{2ik} \frac{\partial\langle\hat{u}^+\rangle}{\partial z} + \frac{k^2}{4} A(0) \frac{\partial\langle\hat{u}^+\rangle}{\partial z}\right) + e^{2ikz}\, L^+\langle\hat{u}^+\rangle$$

$$= \left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right) \langle\hat{u}^+\rangle\, e^{2ikz} \qquad (3.25)$$

which can be reduced to

$$\frac{\partial\langle\hat{u}^+\rangle}{\partial z^2} + 2ik \frac{\partial\langle\hat{u}^+\rangle}{\partial z} - k_\perp^2\langle\hat{u}^+\rangle + \frac{ik^3}{2} A(0)\, \langle\hat{u}^+\rangle = 0 \quad . \qquad (3.26)$$

Solving (3.22) for $\langle\hat{u}^+\rangle$ and writing

$$\hat{L}^- = \frac{\partial}{\partial z} + \frac{k_\perp^2}{2ik} - \frac{k^2}{4} A(0)$$

results in

$$\langle\hat{u}^+\rangle = \frac{e^{-2ikz}}{\left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right)} \hat{L}^- \langle\hat{u}^-\rangle \ . \qquad (3.27)$$

Differentiating (3.27) with respect to z gives

$$\frac{\partial\langle\hat{u}^+\rangle}{\partial z} = \frac{-2ik\, e^{-2ikz}}{\left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right)} \hat{L}^- \hat{u}^- + \frac{e^{-2ikz}}{\left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right)}$$

$$\cdot \left(\frac{\partial^2\langle\hat{u}^-\rangle}{\partial z^2} + \frac{k_\perp^2}{2ik} \frac{\partial\langle\hat{u}^-\rangle}{\partial z} - \frac{k^2}{4} A(0) \frac{\partial\langle\hat{u}^-\rangle}{\partial z}\right) \ . \qquad (3.28)$$

Substituting (3.27) and (3.28) into (3.21) leads to

$$\frac{-2ik\, e^{-2ikz}}{\left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right)} \hat{L}^- \langle\hat{u}^-\rangle + \frac{e^{-2ikz}}{\left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right)}$$

$$\cdot \left(\frac{\partial^2\langle\hat{u}^-\rangle}{\partial z^2} + \frac{k_\perp^2}{2ik} \frac{\partial\langle\hat{u}^-\rangle}{\partial z} - \frac{k^2}{4} A(0) \frac{\partial\langle\hat{u}^-\rangle}{\partial z}\right) + e^{-2ikz} \hat{L}^- \langle\hat{u}^-\rangle$$

$$= - \left(-\frac{k_\perp^2}{2ik} + \frac{k^2}{4} A(0)\right) \langle\hat{u}^-\rangle \ e^{-2ikz} \qquad (3.29)$$

which can be reduced to

$$\frac{\partial^2\langle\hat{u}^-\rangle}{\partial z^2} - 2ik\,\frac{\partial\langle\hat{u}^-\rangle}{\partial z} - k_\perp^2\langle\hat{u}^-\rangle + \frac{ik^3}{2}\,A(0)\langle\hat{u}^-\rangle \;=\; 0 \quad . \qquad (3.30)$$

Now taking the inverse Fourier transform of (3.26) and (3.30) two uncoupled partial differential equations for $\langle u^+(\underline{r})\rangle$ and $\langle u^-(\underline{r})\rangle$ result which are given by

$$\frac{\partial^2\langle u^+\rangle}{\partial z^2} + 2ik\,\frac{\partial\langle u^+\rangle}{\partial z} + \nabla_T^2\langle u^+\rangle + \frac{ik^3}{2}\,A(0)\,\langle u^+\rangle \;=\; 0 \qquad (3.31)$$

and

$$\frac{\partial^2\langle u^-\rangle}{\partial z^2} - 2ik\,\frac{\partial\langle u^-\rangle}{\partial z} + \nabla_T^2\,\langle u^-\rangle + \frac{ik^3}{2}\,A(0)\,\langle u^-\rangle \;=\; 0 \quad . \qquad (3.32)$$

Note that (3.31) and (3.32) can be written as

$$\nabla^2\langle u^+\rangle + 2ik\,\frac{\partial\langle u^+\rangle}{\partial z} + \frac{ik^3}{2}\,A(0)\,\langle u^+\rangle \;=\; 0 \qquad (3.33)$$

and

$$\nabla^2\langle u^-\rangle - 2ik\,\frac{\partial\langle u^-\rangle}{\partial z} + \frac{ik^3}{2}\,A(0)\,\langle u^-\rangle \;=\; 0 \quad . \qquad (3.34)$$

Substituting $\langle u^+\rangle = \langle E^+\rangle e^{-ikz}$ and $\langle u^-\rangle = \langle E^-\rangle\, e^{ikz}$ into (3.33) and (3.34) leads to

$$\nabla^2\langle E^+\rangle + k^2\left(1 + \frac{ik}{2}\,A(0)\right)\langle E^+\rangle \;=\; 0 \qquad (3.35)$$

and

$$\nabla^2\langle E^-\rangle + k^2\left(1 + \frac{ik}{2} A(0)\right) \langle E^-\rangle = 0 \ . \tag{3.36}$$

Now

$$\nabla^2\langle E\rangle + k^2_{eff}\langle E\rangle = 0 \ , \tag{3.37}$$

where

$$k^2_{eff} = k^2 \left(1 + \frac{ik}{2} A(0)\right)$$

can be solved by separation of variables. Writing (3.37) in cylindrical coordinates

$$\frac{\partial^2\langle E\rangle}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial\langle E\rangle}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\langle E\rangle}{\partial\phi^2} + k^2_{eff}\langle E\rangle = 0 \ . \tag{3.38}$$

Substituting $\langle E\rangle = R(\rho)\Phi(\phi)Z(z)$ into (3.38) results in

$$\frac{1}{\rho R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{1}{\rho^2\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} + k^2_{eff} = 0 \ . \tag{3.39}$$

Since the physical situation studied is a plane wave normally incident upon a slab infinite in the plane transverse to the direction of propagation and finite in the direction of propagation the average fields are independent of ρ and ϕ. Then (3.39) becomes

$$\frac{\partial^2 Z(z)}{\partial z^2} + k_{eff}^2 Z(z) = 0 \tag{3.40}$$

which has the solution

$$\langle E^+\rangle = Z(z) = A_1^+ \exp(ik_{eff}z) + A_2^+ \exp(-ik_{eff}z) \tag{3.41}$$

or

$$\langle u^+\rangle = A_1^+ \exp[-i(k-k_{eff})z] + A_2^+ \exp[-i(k+k_{eff})z] \tag{3.42}$$

and similarly

$$\langle u^-\rangle = A_1^- \exp[i(k+k_{eff})z] + A_2^- \exp[i(k-k_{eff})z] \quad . \tag{3.43}$$

Now from (3.17) it must be true that

$$\frac{\partial\langle u^+\rangle}{\partial z} + \frac{k^2}{4} A(0) \langle u^+\rangle = - \frac{k^2}{4} A(0) \langle u^-\rangle \; e^{-2ikz} \quad . \tag{3.44}$$

Using (3.42), (3.43) and (3.44) the following two equations result

$$-i(k - k_{eff})A_1^+ + \frac{k^2}{4} A(0) A_1^+ = - \frac{k^2}{4} A(0) A_1^- \tag{3.45}$$

and

$$-i(k + k_{eff})A_2^+ + \frac{k^2}{4} A(0) A_2^+ = - \frac{k^2}{4} A(0) A_2^- \quad . \tag{3.46}$$

Now from (3.18)

$$\frac{\partial \langle u^-\rangle}{\partial z} - \frac{k^2}{4} A(0) \langle u^-\rangle = \frac{k^2}{4} A(0) \langle u^+\rangle e^{2ikz} \tag{3.47}$$

results. Substituting (3.42) and (3.43) into (3.47) produces two equations, that is

$$i(k + k_{eff})A_1^- - \frac{k^2}{4} A(0) A_1^- = \frac{k^2}{4} A(0) A_1^+ \tag{3.48}$$

and

$$i(k - k_{eff})A_2^- - \frac{k^2}{4} A(0) A_2^- = \frac{k^2}{4} A(0) A_2^+ . \tag{3.49}$$

Now, since a plane wave is normally incident upon a slab extending from z = 0 to z = L in the direction of propagation, at z = 0

$$\langle u^+(0,\underline{\rho}_d)\rangle = 1 . \tag{3.50}$$

Assume that at z = L, the interface between random medium and free space, there is no backscatter, that is

$$\langle u^-(L,\underline{\rho}_d)\rangle = 0 . \tag{3.51}$$

The following equations result from (3.50) and (3.51)

$$A_1^+ + A_2^+ = 1 \tag{3.52}$$

and

$$A_1^- \exp[i(k+k_{eff})L] + A_2^- \exp[i(k-k_{eff})L] = 0 \quad .(3.53)$$

Careful examination reveals that (3.45) and (3.48) and (3.46) and (3.49) are equivalent. Solving (3.45) for A_1^- and (3.46) for A_2^- results in

$$A_1^- = \left(\frac{i(k-k_{eff}) - \frac{k^2}{4} A(0)}{\frac{k^2}{4} A(0)} \right) A_1^+ \tag{3.54}$$

and

$$A_2^- = \left(\frac{i(k+k_{eff}) - \frac{k^2}{4} A(0)}{\frac{k^2}{4} A(0)} \right) A_2^+ \quad . \tag{3.55}$$

Now solving (3.53), (3.54), (3.55) and (3.56) simultaneously results in the following expressions for $<u^+>$ and $<u^->$

$$<u^+> = \frac{\left\{ [-i(k - k_{eff}) + \frac{k^2}{4} A(0)] \exp[i(k-k_{eff})(L-z)] - [-i(k - k_{eff}) + \frac{k^2}{4} A(0)] \exp[i(k+k_{eff})(L-z)] \right\}}{\left\{ [-i(k + k_{eff}) + \frac{k^2}{4} A(0)]\exp[i(k-k_{eff})L] - [-i(k - k_{eff}) + \frac{k^2}{4} A(0)]\exp[i(k+k_{eff})L] \right\}} \tag{3.56}$$

and

$$\langle u^- \rangle = \frac{k^2}{4} A(0) \frac{\left\{ -\exp[i[(k + k_{eff})z + (k - k_{eff})L]] + \exp[i[(k - k_{eff})z + (k + k_{eff})L]] \right\}}{\left\{ [-i(k + k_{eff}) + \frac{k^2}{4} A(0)] \exp[i(k - k_{eff})L] - [-i(k - k_{eff}) + \frac{k^2}{4} A(0)] \exp[i(k + k_{eff})L] \right\}} . \quad (3.57)$$

The average forward propagating and backscatter fields depend upon a complex effective wave number, k_{eff}. Noting that A(0) is σ_ε^2, the variance of permittivity fluctuations, and utilizing a normalized spatial variable

$$\zeta = \frac{z}{L} \quad (3.58)$$

equations (3.56) and (3.57) can be written as

$$\langle u^+ \rangle = \frac{\left\{ \left[\frac{\pi}{2} \frac{\sigma_\varepsilon^2}{\lambda} - i(1 + x) \right] e^{ikL(1-x)(1-\zeta)} - \left[\frac{\pi}{2} \frac{\sigma_\varepsilon^2}{\lambda} - i(1 - x) \right] e^{ikL(1+x)(1-\zeta)} \right\}}{\left\{ \left[\frac{\pi}{2} \frac{\sigma_\varepsilon^2}{\lambda} - i(1 + x) \right] e^{ikL(1-x)} - \left[\frac{\pi}{2} \frac{\sigma_\varepsilon^2}{\lambda} - i(1 - x) \right] e^{ikL(1+x)} \right\}} \quad (3.59)$$

and

$$\langle u^- \rangle = \frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda} \frac{\left\{ e^{ikL[(1-x)\zeta + (1+x)]} - e^{ikL[(1+x)\zeta + (1-x)]} \right\}}{\left\{ \left[\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda} - i(1+x)\right] e^{ikL(1-x)} - \left[\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda} - i(1-x)\right] e^{ikL(1+x)} \right\}} , \quad (3.60)$$

where

$$x = \left(1 + i\pi \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2} . \quad (3.61)$$

Examination of (3.59) and (3.60) reveal that $\langle u^+ \rangle$ and $\langle u^- \rangle$ depend on the parameters $\sigma_\varepsilon^2/\lambda$ and L/λ. The first parameter describes how violently perturbed the medium is while the second describes its thickness in terms of wavelengths. The implications of (3.59) and (3.60) will be examined in the next section.

3.3 Mean Field Results

One of the difficulties in discussing this work is the almost complete lack of previous work to which comparisons can be made. When the variance of the permittivity fluctuations is small the average field including backscatter is best compared to the average field excluding backscatter. When $\sigma_\varepsilon^2/\lambda$ is very small

$$x \simeq \left(1 + i\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right) \quad (3.62)$$

which when substituted in (3.59) leads to

$$\langle u^+ \rangle \approx \exp(-k^2 L \sigma_\varepsilon^2 \zeta/4) \quad . \qquad (3.63)$$

Equation (3.60) for small $\sigma_\varepsilon^2/\lambda$ implies that

$$\langle u^- \rangle \approx 0 \quad . \qquad (3.64)$$

The average forward propagating field decays with distance through a random medium. Weak permittivity fluctuations result in an average backscatter field negligible in comparison to the average forward propagating field. Very small $\sigma_\varepsilon^2/\lambda$ means that the permittivity varies little from its mean which is free space. A medium which is almost free space will have little reflected energy as indicated by (3.64). If σ_ε^2 is allowed to go to zero then $\langle u^+ \rangle \to 1$, the free space propagating wave. The formulation which begins with the initial assumption that backscatter is negligible, discussed in Chapter I, results in

$$\langle u^+ \rangle = \exp(-k^2 L \sigma \varepsilon^2/8) \ \zeta \quad . \qquad (3.65$$

Note that (3.63) decays twice as fast as (3.65). Even though (3.64) indicates that the average backscatter field is negligible the effect of backscatter interactions within the medium upon the average forward scattered field is significant. The difference between (3.63) and (3.65) is due to the conversion of coherent energy or energy in the average forward propagating field in the no backscatter situation to backward and forward propagating fluctuating energy in the case where backscatter is included. In other words, the difference in $\langle u^+(\underline{r})\rangle$ between the backscatter and no backscatter cases is due to an increase in $u_f^+(\underline{r})$ and $u_f^-(\underline{r})$ when backscatter is taken into account. The above statement will be examined in greater detail in the following chapter. Now if L becomes increasingly large relative to a wavelength it is seen from equation (3.63) that $\langle u^+\rangle \to 0$. Physically what is occurring is that more and more of the positive going coherent energy is being transformed into backward and forward propagating fluctuating energy until there is no deterministic components of propagating signal.

When $\sigma_\varepsilon^2/\lambda$ is very large and L/λ is fixed it is true that

$$x \simeq \left(\frac{\pi\sigma_\varepsilon^2}{2\lambda}\right)^{1/2}(1+i) \quad . \tag{3.66}$$

When (3.66) is substituted in (3.59) and (3.60) the result is

$$\langle u^+\rangle \simeq \frac{\left\{\left[\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}+1\right]\exp\left[-ikL\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)(1-\zeta)\right] - \left[\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}-1\right]\exp\left[ikL\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)(1-\zeta)\right]\right\}}{\left\{\left[\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}+1\right]\exp\left[-ikL\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)\right] - \left[\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}-1\right]\exp\left[ikL\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)\right]\right\}} \tag{3.67}$$

and

$$\langle u^- \rangle \approx \left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2} \frac{\left\{\exp\left[ikL\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)(1-\zeta)\right] - \exp\left[-ikL\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)(1-\zeta)\right]\right\}}{\left\{\left[\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}+1\right]\exp\left[-ikL\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)\right] - \left[\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}-1\right]\exp\left[ikL\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}(1+i)\right]\right\}} \tag{3.68}$$

when L/λ is very small from (3.67) and (3.68)

$$\langle u^+ \rangle \to 1 \tag{3.69}$$

and

$$\langle u^- \rangle \to 0\ . \tag{3.70}$$

A very thin layer of random material with very large permittivity fluctuation has little effect on the free space field. When $\sigma_\varepsilon^2/\lambda$ is large and $L/\lambda \gtrsim 1$ then from (3.67) and (3.68)

$$\langle u^+ \rangle \approx \exp\left[-kL\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}\zeta\right]\exp\left[ikL\left(\frac{\pi}{2}\ \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}\zeta\right] \tag{3.71}$$

and

and

$$\langle u^- \rangle \approx -\left[\frac{\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}}{\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}+1}\right]\exp\left[-kL\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}\zeta\right]\exp\left[ikL\left(\frac{\pi}{2}\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2}\zeta\right] \tag{3.72}$$

where (3.72) is true for all $\zeta \neq 1$. Note that $kL[(\pi/2)(\sigma_\varepsilon^2/\lambda)]^{1/2}$ must be such that $\langle u^- \rangle$ is small near z = 1. Compare (3.71) and (3.72) with (3.63) and (3.64). Since (3.71) and (3.72) are valid for large permittivity fluctuations a significant amount of coherent energy can be reflected whereas (3.62) and (3.63) are true for small deviations from free space conditions resulting in little reflected coherent energy. The coefficient of (3.72) indicates that the maximum amount of reflected coherent energy is determined by the size of permittivity fluctuations. When $\sigma_\varepsilon^2/\lambda$ is held at a fixed value no increase in L/λ or layer size will cause an increase in reflected coherent energy. However, since at z = 1 $\langle u^+ \rangle$ decreases for increasing L/λ it must be concluded that the energy in u_f^+ and u_f^- is increasing. That is, for large $\sigma_\varepsilon^2/\lambda$ the average backscatter remains constant but the transmitted average signal becomes increasingly random as slab thickness is increased. Analogous to the skin depth phenomenon of lossy media, $\langle u^+ \rangle$ and $\langle u^- \rangle$ may be thought of as existing only to some finite depth from z = 0, within the random medium. Whereas for conducting media beyond the skin depth essentially no fields exist, in a random medium fluctuating fields exist. Finally, from (3.71) and (3.72) note that as $\sigma_\varepsilon^2/\lambda \to \infty$ $\langle u^+ \rangle$ and $\langle u^- \rangle$ approach zero everywhere but at z = 0. What occurs in the limit is that the medium becomes impenetrable. All energy is reflected at the z = 0 interface. This will be born out by results in the following chapter.

CHAPTER IV. COHERENCE FUNCTION

The coherence function describes how the field at two points in space, $\underline{r}_1$, and $\underline{r}_2$, is correlated. The coherence function determines the signal to noise ratio in communication systems as discussed by Fried [11] and the resolution in synthetic aperture radar systems as discussed by Chu [12]. Writing the forward propagating field as an average forward propagating field and a fluctuating forward propagating field

$$u^+(\underline{r}) = \langle u^+(\underline{r})\rangle + u_f^+(\underline{r}) \tag{4.1}$$

and the backscatter field as an average and fluctuating part

$$u^-(\underline{r}) = \langle u^-(\underline{r})\rangle + u_f^-(\underline{r}) \tag{4.2}$$

the forward and backward coherence functions are given by

$$\langle u^+(\underline{r}_1)u^{+*}(\underline{r}_2)\rangle = \langle u^+(\underline{r}_1)\rangle \langle u^{+*}(\underline{r}_2)\rangle + \langle u_f^+(\underline{r}_1)u_f^{+*}(\underline{r}_2)\rangle \tag{4.3}$$

and

$$\langle u^-(\underline{r}_1)u^{-*}(\underline{r}_2)\rangle = \langle u^-(\underline{r}_1)\rangle \langle u^{-*}(\underline{r}_2)\rangle + \langle u_f^-(\underline{r}_1)u_f^{-*}(\underline{r}_2)\rangle \tag{4.4}$$

respectively. When $\underline{r}_1 = \underline{r}_2$ (4.3) and (4.4) are the total forward and backward intensities. The total forward intensity is the sum of the coherent intensity, $|\langle u^+(\underline{r})\rangle|^2$, and $\langle|u_f^+(\underline{r})|^2\rangle$ the incoherent intensity. The total backscatter intensity also consists of a coherent and incoherent part. In a statistically homogeneous medium

$$\langle u^+(\underline{r}_1)u^{+*}(\underline{r}_2)\rangle = \langle u^+(\underline{r}_2)u^{+*}(\underline{r}_1)\rangle \qquad (4.5)$$

and

$$\langle u^-(\underline{r}_1)u^{-*}(\underline{r}_2)\rangle = \langle u^-(\underline{r}_2)u^{-*}(\underline{r}_1)\rangle \quad . \qquad (4.6)$$

The following shorthand notation is used throughout this thesis

$$\Psi^{++}(\underline{r}_1,\underline{r}_2) = \langle u^+(\underline{r}_1)u^{+*}(\underline{r}_2)\rangle \qquad (4.7)$$

and

$$\Psi^{--}(\underline{r}_1,\underline{r}_2) = \langle u^-(\underline{r}_1)u^{-*}(\underline{r}_2)\rangle \quad . \qquad (4.8)$$

Note that since $E^+(\underline{r}_1) = u^+(\underline{r}_1)\exp(ikz_1)$ and $E^-(\underline{r}_1) = u^-(\underline{r}_1)\exp(-ikz_1)$ it is also true that

$$\Psi^{++}(\underline{r}_1,\underline{r}_2) = \langle E^+(\underline{r}_1)E^{+*}(\underline{r}_2)\rangle \exp[ik(z_2-z_1)]$$

$$= \psi^{++}(\underline{r}_1,\underline{r}_2) \exp[ik(z_2-z_1)] \qquad (4.9)$$

and

$$\Psi^{--}(\underline{r}_1,\underline{r}_2) = \langle E^-(\underline{r}_1)E^{-*}(\underline{r}_2)\rangle \exp[-ik(z_2-z_1)$$

$$= \psi^{--}(\underline{r}_1,\underline{r}_2) \exp[-ik(z_2-z_1)] \ . \qquad (4.10)$$

It is also possible to define the forward going cross coherence

$$\Psi^{+-}(\underline{r}_1,\underline{r}_2) = \langle u^{+}(\underline{r}_1)u^{-*}(\underline{r}_2)\rangle = \langle E^{+}(\underline{r}_1)E^{-*}(\underline{r}_2)\rangle \exp[-ik(z_2+z_1)]$$

$$= \psi^{+-}(\underline{r}_1,\underline{r}_2)\exp[-ik(z_2+z_1)] \quad (4.11)$$

and the backward going cross coherence

$$\Psi^{-+}(\underline{r}_1,\underline{r}_2) = \langle u^{-}(\underline{r}_1)u^{+*}(\underline{r}_2)\rangle = \langle E^{-}(\underline{r}_1)E^{+*}(\underline{r}_2)\rangle \exp[ik(z_2+z_1)]$$

$$= \psi^{-+}(\underline{r}_1,\underline{r}_2) \exp[ik(z_2+z_1)] \quad (4.12)$$

In a statistically homogeneous medium $\Psi^{+-}(\underline{r}_1,\underline{r}_2) = \Psi^{-+*}(\underline{r}_1,\underline{r}_2)$. The propagation directions of (4.11) and (4.12) will be clear in a later section of this chapter. Equations will now be developed which govern the behavior of the four coherence functions.

4.1 Derivation of Coherence Equations

The development of a set of equations that describes the behavior of the coherence functions in a random medium which includes the effects of backscatter begins with equation (2.13) in Section 2.1. Since the physical situation considered in this work is a finite slab whose thickness is aligned longitudinal to the direction of propagation of incident radiation, the z direction, the field correlations are examined for fixed z between the points in the transverse plane. Writing $\underline{r} = z\hat{z} + \rho\hat{\rho}$ the equation parabolic in $u^{+}(z,\underline{\rho})$ is considered at $\underline{\rho}_1$

$$\frac{\partial u^{+}(z,\underline{\rho}_1)}{\partial z} + \frac{1}{2ik}\nabla^2_{T_1} u^{+}(z,\underline{\rho}_1) - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)u^{+}(z,\underline{\rho}_1)$$

$$= \left(-\frac{1}{2ik}\nabla^2_{T_1} u^{-}(z,\underline{\rho}_1) + \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)u^{-}(z,\underline{\rho}_1)\right) e^{-2ikz} \qquad (4.13)$$

and at $\underline{\rho}_2$

$$\frac{\partial u^{+}(z,\underline{\rho}_2)}{\partial z} + \frac{1}{2ik}\nabla^2_{T_2} u^{+}(z,\underline{\rho}_2) - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)u^{+}(z,\underline{\rho}_2)$$

$$= \left(-\frac{1}{2ik}\nabla^2_{T_2} u^{-}(z,\underline{\rho}_2) + \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)u^{-}(z,\underline{\rho}_2)\right) e^{-2ikz} \ . \qquad (4.14)$$

Multiplying (4.13) by $u^{+*}(z,\underline{\rho}_2)$ where * designates complex conjugate, taking the conjugate of (4.14) and multiplying it by $u^{+}(z,\underline{\rho}_1)$ and then adding the two products results in

$$\frac{\partial}{\partial z}\left(u^{+}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)\right) + \frac{1}{2ik}\left(\nabla^2_{T_1} - \nabla^2_{T_2}\right) u^{+}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)$$

$$- \frac{ik}{2}[\tilde{\varepsilon}(z,\underline{\rho}_1) - \tilde{\varepsilon}(z,\underline{\rho}_2)]u^{+}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2) = \left(-\frac{1}{2ik}\nabla^2_{T_1} + \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)\right)$$

$$\cdot\, u^{-}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)e^{-2ikz} + \left(\frac{1}{2ik}\nabla^2_{T_2} - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)\right)$$

$$\cdot\, u^{-*}(z,\underline{\rho}_2)u^{+}(z,\underline{\rho}_2)e^{2ikz} \qquad . \quad (4.15)$$

Now writing the equation parabolic in $u^-(z,\underline{\rho})$ at $\underline{\rho}_1$ gives

$$\frac{\partial u^-(z,\underline{\rho}_1)}{\partial z} - \frac{1}{2ik}\nabla^2_{T_1} u^-(z,\underline{\rho}_1) + \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)u^-(z,\underline{\rho}_1)$$

$$= \left(\frac{1}{2ik}\nabla^2_{T_1} u^+(z,\underline{\rho}_1) - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)u^+(z,\underline{\rho}_1)\right)e^{2ikz} \qquad (4.16)$$

and at $\underline{\rho}_2$ gives

$$\frac{\partial u^-(z,\underline{\rho}_2)}{\partial z} - \frac{1}{2ik}\nabla^2_{T_2} u^-(z,\underline{\rho}_2) + \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)u^-(z,\underline{\rho}_2)$$

$$= \left(\frac{1}{2ik}\nabla^2_{T_2} u^+(z,\underline{\rho}_2) - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)u^+(z,\underline{\rho}_2)\right)e^{2ikz} \ . \qquad (4.17)$$

Multiplying (4.16) by $u^{-*}(z,\underline{\rho}_2)$, conjugating (4.17) and multiplying the resulting equation by $u^-(z,\underline{\rho}_2)$ and then adding the two products results in

$$\frac{\partial}{\partial z}\left(u^-(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)\right) - \frac{1}{2ik}\left(\nabla^2_{T_1} - \nabla^2_{T_2}\right)u^-(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)$$

$$+ \frac{ik}{2}\,[\tilde{\varepsilon}(z,\underline{\rho}_1) - \tilde{\varepsilon}(z,\underline{\rho}_2)]u^-(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2) = \left(\frac{1}{2ik}\nabla^2_{T_1} - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)\right)$$

$$\cdot\ u^+(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)e^{2ikz} - \left(-\frac{1}{2ik}\nabla^2_{T_2} - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)\right)$$

$$\cdot\ u^-(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)e^{-2ikz} \ . \qquad (4.18)$$

Multiplying the conjugate of (4.17) by $u^{+}(z,\underline{\rho}_1)$ and (4.13) by $u^{-*}(z,\underline{\rho}_2)$ and then adding the products leads to

$$\frac{\partial}{\partial z}\left(u^{+}(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)\right) + \frac{1}{2ik}\left(\nabla^2_{T_1} + \nabla^2_{T_2}\right)u^{+}(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)$$

$$- \frac{ik}{2}\left[\tilde{\varepsilon}(z,\underline{\rho}_1) + \tilde{\varepsilon}(z,\underline{\rho}_2)\right]u^{+}(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2) = \left(-\frac{1}{2ik}\nabla^2_{T_1} + \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)\right)$$

$$\cdot\, u^{-}(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)e^{-2ikz} + \left(-\frac{1}{2ik}\nabla^2_{T_2} + \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)\right)$$

$$\cdot\, u^{+}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)e^{-2ikz} \quad . \qquad (4.19)$$

The fourth and final equation is obtained by multiplying (4.16) by $u^{+*}(z,\underline{\rho}_2)$, conjugating (4.14) and multiplying the result by $u^{-}(z,\underline{\rho}_1)$ and then adding the products leading to

$$\frac{\partial\left(u^{-}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)\right)}{\partial z} - \frac{1}{2ik}\left(\nabla^2_{T_1} + \nabla^2_{T_2}\right)u^{-}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)$$

$$+ \frac{ik}{2}\left[\tilde{\varepsilon}(z,\underline{\rho}_1) + \tilde{\varepsilon}(z,\underline{\rho}_2)\right]u^{-}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2) = \left(\frac{1}{2ik}\nabla^2_{T_1} - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_1)\right)$$

$$\cdot\, u^{+}(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)e^{2ikz} + \left(\frac{1}{2ik}\nabla^2_{T_2} - \frac{ik}{2}\tilde{\varepsilon}(z,\underline{\rho}_2)\right)$$

$$\cdot\, u^{-}(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)e^{2ikz} \quad . \qquad (4.20)$$

Equations (4.15), (4.18), (4.19) and (4.20) are four coupled partial differential equations with random coefficients. These four equations are now used to relate the two coherence functions, $\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)$ and $\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)$, and two cross coherences, $\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)$ and $\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)$, as defined in the introductory section in this chapter. The desired relationship is established by taking the mean of (4.15), (4.18), (4.19) and (4.20). Since $u^+(z,\underline{\rho})$ and $u^-(z,\underline{\rho})$ derive their random nature from $\tilde{\varepsilon}(z,\underline{\rho})$ the various correlations of $\tilde{\varepsilon}(z,\underline{\rho})$, $u^+(z,\underline{\rho})$ and $u^-(z,\underline{\rho})$ must be expressed in terms of $\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)$, $\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)$, $\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)$ and $\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)$. As in the derivation of the mean field equations this objective is accomplished by using the Furutsu-Novikov relationship, the delta correlated medium assumption and the functional derivative expressions accounting for backscatter as determined in Section 2.4. Taking the mean of (4.15) necessitates that an expression for $\langle\tilde{\varepsilon}(z,\underline{\rho}_1)u^+(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)\rangle$ be determined in terms of the four coherences. Using the Furutsu-Novikov relationship, valid for $\tilde{\varepsilon}(z,\underline{\rho}_1)$ zero mean Gaussian distributed, results in

$$\langle\tilde{\varepsilon}(z,\underline{\rho}_1)u^+(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)\rangle = \int_0^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\ \delta(z - z')A(\underline{\rho}_1 - \underline{\rho}')$$

$$\cdot \left(\left\langle \frac{\delta u^+(z,\underline{\rho}_1)}{\delta\tilde{\varepsilon}(\underline{r}')} u^{+*}(z,\underline{\rho}_2) \right\rangle + \left\langle u^+(z,\underline{\rho}_1) \frac{\delta u^{+*}(z,\underline{\rho}_2)}{\delta\tilde{\varepsilon}(\underline{r}')} \right\rangle \right) \quad . \quad (4.21)$$

Substituting (2.86) into (4.21) leads to

$$<\tilde{\varepsilon}(z,\underline{\rho}_1)u^+(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)> \approx \frac{ik}{4}(A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))\,\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)$$

$$+ \frac{ik}{4}A(0)\,\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} - \frac{ik}{4}A(\underline{\rho}_1 - \underline{\rho}_2)\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} \;. \quad (4.22)$$

In addition to (4.22) taking the means of (4.15), (4.18), (4.19) and (4.20) requires the knowledge of seven other correlations. Similar to the approach used in the determination of (4.22) the following expressions can be determined

$$<\tilde{\varepsilon}(z,\underline{\rho}_2)u^+(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)> \approx \frac{ik}{4}(A(\underline{\rho}_2 - \underline{\rho}_1) - A(0))\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)$$

$$+ \frac{ik}{4}A(\underline{\rho}_2 - \underline{\rho}_1)\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} - \frac{ik}{4}A(0)\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} \quad (4.23)$$

$$<\tilde{\varepsilon}(z,\underline{\rho}_1)u^-(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)> \approx \frac{ik}{4}A(0)\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz}$$

$$- \frac{ik}{4}A(\underline{\rho}_1 - \underline{\rho}_2)\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} + \frac{ik}{4}(A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)$$

(4.24)

$$<\tilde{\varepsilon}(z,\underline{\rho}_2)u^{-*}(z,\underline{\rho}_2)u^+(z,\underline{\rho}_1)> \approx -\frac{ik}{4}A(0)\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz}$$

$$+ \frac{ik}{4}A(\underline{\rho}_2 - \underline{\rho}_1)\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} - \frac{ik}{4}(A(0) - A(\underline{\rho}_2 - \underline{\rho}_1))\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)$$

(4.25)

$$<\tilde{\varepsilon}(z,\underline{\rho}_1)u^-(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)> \;\approx\; \frac{ik}{4}\,(A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))\Psi^{--}(\underline{\rho}_1,\underline{\rho}_2)$$

$$+ \frac{ik}{4}\,A(0)\,\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} - \frac{ik}{4}\,A(\underline{\rho}_1 - \underline{\rho}_2)\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} \quad (4.26)$$

$$<\tilde{\varepsilon}(z,\underline{\rho}_2)u^-(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)> \;\approx\; -\frac{ik}{4}\,(A(0) - A(\underline{\rho}_2 - \underline{\rho}_1))\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)$$

$$- \frac{ik}{4}\,A(0)\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} + \frac{ik}{4}\,A(\underline{\rho}_2 - \underline{\rho}_1)\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} \quad (4.27)$$

$$<\tilde{\varepsilon}(z,\underline{\rho}_1)u^+(z,\underline{\rho}_1)u^{-*}(z,\underline{\rho}_2)> \;\approx\; \frac{ik}{4}\,(A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)$$

$$+ \frac{ik}{4}\,A(0)\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} - \frac{ik}{4}\,A(\underline{\rho}_1 - \underline{\rho}_2)\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} \quad (4.28)$$

and

$$<\tilde{\varepsilon}(z,\underline{\rho}_2)u^-(z,\underline{\rho}_1)u^{+*}(z,\underline{\rho}_2)> \;\approx\; -\frac{ik}{4}\,(A(0) - A(\underline{\rho}_2 - \underline{\rho}_1))\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)$$

$$- \frac{ik}{4}\,A(0)\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} + \frac{ik}{4}\,A(\underline{\rho}_2 - \underline{\rho}_1)\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} \quad . \quad (4.29)$$

Now taking the mean of (4.15), (4.18), (4.19) and (4.20) and using (4.22) through (4.29) results in

$$\frac{\partial\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)}{\partial z} + \frac{1}{2ik}[\nabla^2_{T_1} - \nabla^2_{T_2}]\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2) + \frac{k^2}{4}(2A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))$$

$$\cdot\,\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2) = \frac{k^2}{4}A(\underline{\rho}_1 - \underline{\rho}_2)\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2) + \left(-\frac{1}{2ik}\nabla^2_{T_1} - \frac{k^2}{4}\right.$$

$$\left.\cdot\,(A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))\right)\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} + \left(\frac{1}{2ik}\nabla^2_{T_2} - \frac{k^2}{4}\right.$$

$$\left.\cdot\,(A(0) - A(\underline{\rho}_2 - \underline{\rho}_1))\right)\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} \qquad (4.30)$$

$$\frac{\partial\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)}{\partial z} - \frac{1}{2ik}[\nabla^2_{T_1} - \nabla^2_{T_2}]\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2) - \frac{k^2}{4}(2A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))$$

$$\cdot\,\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2) = -\frac{k^2}{4}A(\underline{\rho}_1 - \underline{\rho}_2)\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2) + \left(\frac{1}{2ik}\nabla^2_{T_1} + \frac{k^2}{4}\right.$$

$$\left.\cdot\,(A(0) - A(\underline{\rho}_1 - \underline{\rho}_2))\right)\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} + \left(-\frac{1}{2ik}\nabla^2_{T_2} + \frac{k^2}{4}\right.$$

$$\left.\cdot\,(A(0) - A(\underline{\rho}_2 - \underline{\rho}_1))\right)\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} \qquad (4.31)$$

$$\frac{\partial\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2)}{\partial z} + \frac{1}{2ik}[\nabla^2_{T_1} + \nabla^2_{T_2}]\Psi^{+-}(z,\underline{\rho}_1,\underline{\rho}_2) = \left(-\frac{1}{2ik}\nabla^2_{T_1} - \frac{k^2}{4}A(0)\right)$$

$$\cdot\,\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz} + \left(-\frac{1}{2ik}\nabla^2_{T_2} + \frac{k^2}{4}A(0)\right)\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)e^{-2ikz}$$

$$(4.32)$$

and finally

$$\frac{\partial\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2)}{\partial z} - \frac{1}{2ik}\left[\nabla^2_{T_1} + \nabla^2_{T_2}\right]\Psi^{-+}(z,\underline{\rho}_1,\underline{\rho}_2) = \left(\frac{1}{2ik}\nabla^2_{T_2} - \frac{k^2}{4}A(0)\right)$$

$$\cdot\,\Psi^{--}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} + \left(-\frac{1}{2ik}\nabla^2_{T_1} + \frac{k^2}{4}A(0)\right)\Psi^{++}(z,\underline{\rho}_1,\underline{\rho}_2)e^{2ikz} \quad . \tag{4.33}$$

Equations (4.30) through (4.33) are formidable. Some simplification can be realized through the symmetry of the problem. Making the following substitution

$$\underline{\rho}_c = \frac{\underline{\rho}_1 + \underline{\rho}_2}{2} = x_c\hat{x} + y_c\hat{y} \tag{4.34}$$

and

$$\underline{\rho}_d = \underline{\rho}_1 - \underline{\rho}_2 = x_d\hat{x} + y_d\hat{y} \tag{4.35}$$

leads to

$$\nabla^2_{T_1} = \frac{1}{4}\nabla^2_{T_c} + \nabla_{T_c}\cdot\nabla_{T_d} + \nabla_{T_d}^{\;2} \tag{4.36}$$

and

$$\nabla^2_{T_2} = \frac{1}{4}\nabla^2_{T_c} - \nabla_{T_c}\cdot\nabla_{T_d} + \nabla^2_{T_d} \tag{4.37}$$

where

$$\nabla^2_{T_c} = \frac{\partial^2}{\partial x_c^2} + \frac{\partial^2}{y_c^2} \tag{4.38}$$

and

$$\nabla^2_{T_d} = \frac{\partial^2}{\partial x_d^2} + \frac{\partial^2}{\partial y_d^2} \ . \qquad (4.39)$$

Since a plane wave is normally incident upon a slab of random medium assumed to be statistically homogeneous in the x-y plane it is true that

$$\nabla^2_{T_c} = 0 \ . \qquad (4.40)$$

Equation (4.40) implies

$$\nabla^2_{T_1} = \nabla^2_{T_2} = \nabla^2_{T_d} \ . \qquad (4.41)$$

$\Psi^{++}, \Psi^{--}, \Psi^{+-}$ and Ψ^{-+} depend only on the difference of two points in the x-y plane of a statistically homogeneous slab with a normally incident plane wave and not where in the x-y plane those two points are selected. Explicitly writing the $\underline{\rho}_d$ dependence of Ψ as $\Psi(z,\underline{\rho}_d)$ equations (4.30) through (4.33) take the form

$$\frac{\partial \Psi^{++}(z,\underline{\rho}_d)}{\partial z} + \frac{k^2}{4}\left(2A(0) - A(\underline{\rho}_d)\right)\Psi^{++}(z,\underline{\rho}_d) = \frac{k^2}{4} A(\underline{\rho}_d)\Psi^{--}(z,\underline{\rho}_d)$$

$$+ \left(-\frac{1}{2ik}\nabla^2_{T_d} - \frac{k^2}{4}\left(A(0) - A(\underline{\rho}_d)\right)\right)\Psi^{-+}(z,\underline{\rho}_d)e^{-2ikz}$$

$$+ \left(\frac{1}{2ik}\nabla^2_{T_d} - \frac{k^2}{4}\left(A(0) - A(\underline{\rho}_d)\right)\right)\Psi^{+-}(z,\underline{\rho}_d)e^{+2ikz} \qquad (4.42)$$

$$\frac{\partial\Psi^{--}(z,\underline{\rho}_d)}{\partial z} - \frac{k^2}{4}(2A(0) - A(\underline{\rho}_d))\Psi^{--}(z,\underline{\rho}_d) = -\frac{k^2}{4}A(\underline{\rho}_d)\,\Psi^{++}(z,\underline{\rho}_d)$$

$$+\left(\frac{1}{2ik}\nabla^2_{T_d} + \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\right)\Psi^{+-}(z,\underline{\rho}_d)e^{2ikz}$$

$$+\left(-\frac{1}{2ik}\nabla^2_{T_d} + \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\right)\Psi^{-+}(z,\underline{\rho}_d)e^{-2ikz} \qquad (4.43)$$

$$\frac{\partial\Psi^{+-}(z,\underline{\rho}_d)}{\partial z} + \frac{1}{ik}\nabla^2_{T_d}\,\Psi^{+-}(z,\underline{\rho}_d) = \left(-\frac{1}{2ik}\nabla^2_{T_d} - \frac{k^2}{4}A(0)\right)\Psi^{--}(z,\underline{\rho}_d)e^{-2ikz}$$

$$+\left(-\frac{1}{2ik}\nabla^2_{T_d} + \frac{k^2}{4}A(0)\right)\Psi^{++}(z,\underline{\rho}_d)e^{-2ikz} \qquad (4.44)$$

$$\frac{\partial\Psi^{-+}(z,\underline{\rho}_d)}{\partial z} - \frac{1}{ik}\nabla^2_{T_d}\,\Psi^{-+}(z,\underline{\rho}_d) = \left(\frac{1}{2ik}\nabla^2_{T_d} - \frac{k^2}{4}A(0)\right)\Psi^{--}(z,\underline{\rho}_d)e^{2ikz}$$

$$+\left(\frac{1}{2ik}\nabla^2_{T_d} + \frac{k^2}{4}A(0)\right)\Psi^{++}(z,\underline{\rho}_d)e^{2ikz}\ . \qquad (4.45)$$

Equations (4.42) through (4.45) are the coherence relationships of interest. Manipulations in subsequent sections are simplified if e^{2ikz} and e^{-2ikz} are eliminated from the above equations. Substituting (4.9) through (4.12) for $z_1 = z_2$ in (4.42) through (4.45) realizes this elimination resulting in

$$\frac{\partial\psi^{++}(z,\underline{\rho}_d)}{\partial z} + \frac{k^2}{4}\,(2A(0) - A(\underline{\rho}_d))\psi^{++}(z,\underline{\rho}_d) \;=\; \frac{k^2}{4}\,A(\underline{\rho}_d)\psi^{--}(z,\underline{\rho}_d)$$

$$+ \left(-\,\frac{1}{2ik}\;\nabla_T^2 - \frac{k^2}{4}\,(A(0) - A(\underline{\rho}_d))\right)\psi^{-+}(z,\underline{\rho}_d)$$

$$+ \left(\frac{1}{2ik}\,\nabla_T^2 - \frac{k^2}{4}\,(A(0) - A(\underline{\rho}_d))\right)\psi^{+-} \qquad (4.46)$$

$$\frac{\partial\psi^{--}(z,\underline{\rho}_d)}{\partial z} - \frac{k^2}{4}\,(2A(0) - A(\underline{\rho}_d))\psi^{--}(z,\underline{\rho}_d) \;=\; -\,\frac{k^2}{4}\,A(\underline{\rho}_d)\psi^{++}(z,\underline{\rho}_d)$$

$$+ \left(-\,\frac{1}{2ik}\,\nabla_T^2 + \frac{k^2}{4}\,(A(0) - A(\underline{\rho}_d))\right)\psi^{-+}(z,\underline{\rho}_d)$$

$$+ \left(\frac{1}{2ik}\,\nabla_T^2 + \frac{k^2}{4}\,(A(0) - A(\underline{\rho}_d))\right)\psi^{+-}(z,\underline{\rho}_d) \qquad (4.47)$$

$$\frac{\partial\psi^{+-}(z,\underline{\rho}_d)}{\partial z} + \frac{\nabla_T^2}{ik}\,\psi^{+-}(z,\underline{\rho}_d) - 2ik\psi^{+-}(z,\underline{\rho}_d) \;=\; \left(-\,\frac{1}{2ik}\,\nabla_T^2 - \frac{k^2}{4}\,A(0)\right)$$

$$\cdot\,\psi^{--}(z,\underline{\rho}_d) + \left(-\,\frac{1}{2ik}\,\nabla_T^2 + \frac{k^2}{4}\,A(0)\right)\psi^{++}(z,\underline{\rho}_d) \qquad (4.48)$$

$$\frac{\partial\psi^{-+}(z,\underline{\rho}_d)}{\partial z} - \frac{\nabla_T^2}{ik}\,\psi^{-+}(z,\underline{\rho}_d) + 2ik\psi^{-+}(z,\underline{\rho}_d) \;=\; \left(\frac{1}{2ik}\,\nabla_T^2 - \frac{k^2}{4}\,A(0)\right)$$

$$\cdot\,\psi^{--}(z,\underline{\rho}_d) + \left(\frac{1}{2ik}\,\nabla_T^2 + \frac{k^2}{4}\,A(0)\right)\psi^{++}(z,\underline{\rho}_d)\quad . \qquad (4.49)$$

Note that to simplify the notation somewhat $\nabla^2_{T_d}$ has become ∇^2_T in the above. The task in the rest of this chapter is to extract as much information as possible concerning the behavior of ψ^{++} and ψ^{--} from equations (4.46) through (4.49).

4.2 Integral Equation Representation of the Coherence Function

The coherence functions were seen to obey a system of four simultaneous partial differential equations with variable coefficients. No approach is known to exist for handling such a system analytically. The strategy is to represent the quantities of interest, $\psi^{++}(z,\underline{\rho}_d)$ and $\psi^{--}(z,\underline{\rho}_d)$, as a pair of coupled integral equations which can then be tackled numerically. Starting with

$$\frac{\partial\psi^{++}}{\partial z} + \frac{k^2}{4}(2A(0) - A(\underline{\rho}_d))\psi^{++} = \frac{k^2}{4} A(\underline{\rho}_d)\psi^{--} - \frac{1}{2ik} \nabla^2_T\psi^{-+}$$

$$- \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\psi^{-+} + \frac{1}{2ik} \nabla^2_T\psi^{+-} - \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\psi^{+-} \qquad (4.50)$$

$$\frac{\partial\psi^{--}}{\partial z} - \frac{k^2}{4} (2A(0) - A(\underline{\rho}_d))\psi^{--} = - \frac{k^2}{4} A(\underline{\rho}_d)\psi^{++} - \frac{1}{2ik} \nabla^2_T\psi^{-+}$$

$$+ \frac{k^2}{4} (A(0) - A(\underline{\rho}_d))\psi^{-+} + \frac{1}{2ik} \nabla^2_T\psi^{+-} + \frac{k^2}{4} (A(0) - A(\underline{\rho}_d))\psi^{+-} \qquad (4.51)$$

$$\frac{\partial\psi^{+-}}{\partial z} + \frac{\nabla^2_T}{ik} \psi^{+-} - 2ik\psi^{+-} = - \frac{1}{2ik} \nabla^2_T\psi^{--} - \frac{k^2}{4} A(0)\psi^{--}$$

$$- \frac{1}{2ik} \nabla^2_T\psi^{++} + \frac{k^2}{4} A(0)\psi^{++} \qquad (4.52)$$

$$\frac{\partial\psi^{-+}}{\partial z} - \frac{\nabla_T^2}{ik}\psi^{-+} + 2ik\psi^{-+} = \frac{1}{2ik}\nabla_T^2\psi^{--} - \frac{k^2}{4}A(0)\psi^{--}$$

$$+ \frac{1}{2ik}\nabla_T^2\,\psi^{++} + \frac{k^2}{4}A(0)\psi^{++} \qquad (4.53)$$

equation (4.50) is solved for ψ^{--}

$$\psi^{--}(z,\underline{\rho}_d) = \frac{4}{k^2}\frac{1}{A(\underline{\rho}_d)}\left(\frac{\partial\psi^{++}}{\partial z} + \frac{k^2}{4}(2A(0) - A(\underline{\rho}_d))\psi^{++} + \frac{1}{2ik}\nabla_T^2\psi^{-+}\right.$$

$$\left.+ \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\psi^{-+} - \frac{1}{2ik}\nabla_T^2\psi^{+-} + \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\psi^{+-}\right) . \qquad (4.54)$$

Now, differentiating (4.54) with respect to z,

$$\frac{\partial\psi^{--}(z,\underline{\rho}_d)}{\partial z} = \frac{4}{k^2}\frac{1}{A(\underline{\rho}_d)}\left(\frac{\partial^2\psi^{++}}{\partial z^2} + \frac{k^2}{4}(2A(0) - A(\underline{\rho}_d))\frac{\partial\psi^{++}}{\partial z} + \frac{1}{2ik}\nabla_T^2\frac{\partial\psi^{-+}}{\partial z}\right.$$

$$\left.+ \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\frac{\partial\psi^{-+}}{\partial z} - \frac{1}{2ik}\nabla_T^2\frac{\partial\psi^{+-}}{\partial z} + \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\frac{\partial\psi^{+-}}{\partial z}\right). \qquad (4.55)$$

Substituting (4.54) and (4.55) into (4.51)

$$\frac{\partial^2 \psi^{++}}{\partial z^2} - \frac{k^4}{4} A(0)(A(0) - A(\underline{\rho}_d))\psi^{++} = -\frac{1}{2ik} \nabla_T^2 \left(\frac{\partial \psi^{-+}}{\partial z} - \frac{\partial \psi^{+-}}{\partial z} \right)$$

$$- \frac{k^2}{4} (A(0) - A(\underline{\rho}_d)) \left(\frac{\partial \psi^{-+}}{\partial z} + \frac{\partial \psi^{+-}}{\partial z} \right) + \frac{k^2}{2} (A(0) - A(\underline{\rho}_d)) \frac{\nabla_T^2}{2ik} (\psi^{-+} - \psi^{+-})$$

$$+ \frac{k^4}{8} A(0) (A(0) - A(\underline{\rho}_d))(\psi^{-+} + \psi^{+-}) \quad (4.56)$$

is obtained after some manipulation. Now writing (4.56) using a compact notation

$$\frac{\partial^2 \psi^{++}}{\partial z^2} - \tau^2 \psi^{++} = f(z) \quad (4.57)$$

where

$$\tau = \frac{k^2}{2} (A^2(0) - A(0)A(\underline{\rho}_d))^{1/2} \quad (4.58)$$

and $f(z)$ is the right-hand side of (4.56), assume the existence of a Green's function $G^{++}(z,z',\underline{\rho}_d)$ corresponding to $\psi^{++}(z,\underline{\rho}_d)$ such that

$$\frac{\partial^2 G^{++}}{\partial z^2} - \tau^2 G^{++} = \delta(z - z') \quad . \quad (4.59)$$

The difference of the product of ψ^{++} and (4.59) from the product of G^{++} and (4.57) can be written as

$$\frac{\partial}{\partial z}\left(G^{++}(z,z',\underline{\rho}_d)\,\frac{\partial\psi^{++}(z,\underline{\rho}_d)}{\partial z} - \psi^{++}(z,\underline{\rho}_d)\,\frac{\partial G^{++}(z,z',\underline{\rho}_d)}{\partial z}\right)$$

$$= G^{++}(z,z'\underline{\rho}_d)f(z) - \psi^{++}(z,\underline{\rho}_d)\delta(z - z') \quad . \qquad (4.60)$$

Integrating (4.60) over z from 0 to L, where L is the thickness of the random slab, results in

$$G^{++}(L,z',\underline{\rho}_d)\,\frac{\partial\psi^{++}(z,\underline{\rho}_d)}{\partial z}\bigg|_{z=L} - \psi^{++}(L,\underline{\rho}_d)\frac{G^{++}(z,z',\underline{\rho}_d)}{\partial z}\bigg|_{z=L}$$

$$- G^{++}(0,z',\underline{\rho}_d)\,\frac{\partial\psi^{++}(z,\underline{\rho}_d)}{\partial z}\bigg|_{z=0} + \psi^{++}(0,\underline{\rho}_d)\,\frac{\partial G^{++}(z,z',\underline{\rho}_d)}{\partial z}\bigg|_{z=0}$$

$$= \int_0^L G^{++}(z,z',\underline{\rho}_d)f(z)dz - \psi^{++}(z',\underline{\rho}_d) \quad . \quad (4.61)$$

Using the fact that $\psi^{--} = \psi^{-+} = \psi^{+-} = 0$ at $z = L$ in conjunction with equation (4.50) results in

$$\left.\frac{\partial\psi^{++}}{\partial z}\right|_{z=L} + \gamma\psi^{++}(L,\underline{\rho}_d) = 0 \qquad (4.62)$$

where

$$\gamma = \frac{k^2}{4}\,(2A(0) - A(\underline{\rho}_d)) \quad .$$

Substituting (4.62) into (4.61) results in

$$-\gamma G^{++}(L,z',\underline{\rho}_d)\psi^{++}(L,\underline{\rho}_d) - \psi^{++}(L,\underline{\rho}_d)\left.\frac{\partial G^{++}(z,z',\underline{\rho}_d)}{\partial z}\right|_{z=L}$$

$$- G^{++}(0,z',\underline{\rho}_d)\left.\frac{\partial\psi^{++}(z,\underline{\rho}_d)}{\partial z}\right|_{z=o} + \psi^{++}(0,\underline{\rho}_d)\left.\frac{\partial G^{++}(z,z',\underline{\rho}_d)}{\partial z}\right|_{z=o}$$

$$= \int_0^L G^{++}(z,z',\underline{\rho}_d)f(z)dz - \psi^{++}(z',\underline{\rho}_d) \quad . \qquad (4.63)$$

Now if

$$\left.\frac{\partial G^{++}(z,z',\underline{\rho}_d)}{\partial z}\right|_{z=L} + \gamma G^{++}(L,z',\underline{\rho}_d) = 0 \qquad (4.64)$$

then (4.63) can be written as

$$\psi^{++}(z',\underline{\rho}_d) = G^{++}(0,z',\underline{\rho}_d) \left.\frac{\partial\psi^{++}(z,\underline{\rho}_d)}{\partial z}\right|_{z=0} - \psi^{++}(0,\underline{\rho}_d) \left.\frac{\partial G^{++}(z,z',\underline{\rho}_d)}{\partial z}\right|_{z=0}$$

$$+ \int_0^L G^{++}(z,z',\underline{\rho}_d) f(z) dz \quad . \qquad (4.65)$$

Taking $G^{++}(0,z',\underline{\rho}_d) = 0$, $\psi^{++}(z',\underline{\rho}_d)$ is written as

$$\psi^{++}(z',\underline{\rho}_d) = \int_0^L G^{++}(z,z',\underline{\rho}_d) f(z) dz - \psi^{++}(0,\underline{\rho}_d) \left.\frac{\partial G^{++}(z,z',\underline{\rho}_d)}{\partial z}\right|_{z=0} \quad .$$

(4.66)

Solving (4.59) subject to $G^{++}(0,z',\underline{\rho}_d) = 0$ and (4.64) leads to

$$G^{++}(z,z',\underline{\rho}_d) = -\frac{\sin \tau z}{\tau} \left(\frac{\tau \cosh \tau(L-z') + \gamma \sinh \tau(L-z')}{\tau \cosh \tau L + \gamma \sinh \tau L}\right) \quad z \leq z'$$

$$= -\frac{\sinh \tau z'}{\tau} \left(\frac{\tau \cosh \tau(L-z) + \gamma \sinh \tau(L-z)}{\tau \cosh \tau L + \gamma \sinh \tau L}\right) \quad z \geq z' \quad .$$

(4.67)

Interchanging the role of z and z' and substituting in expressions for $f(z')$ and $G^{++}(z',z,\underline{\rho}_d)$ in (4.66) results in

$$\psi^{++}(z,\underline{\rho}_d) = -\left(\frac{\tau\cosh\tau(L-z)+\gamma\sinh\tau(L-z)}{\tau\cosh\tau L+\gamma\sinh\tau L}\right)\int_0^z dz'\left(\frac{\sinh\tau z'}{\tau}\right)$$

$$\left[-\frac{\nabla_T^2}{2ik}\left(\frac{\partial\psi^{-+}}{\partial z'}-\frac{\partial\psi^{+-}}{\partial z'}\right)-\frac{k^2}{4}(A(0)-A(\underline{\rho}_d))\left(\frac{\partial\psi^{-+}}{\partial z'}+\frac{\partial\psi^{+-}}{\partial z'}\right)\right.$$

$$\left.+\frac{k^2}{2}(A(0)-A(\underline{\rho}_d))\frac{\nabla_T^2}{2ik}(\psi^{-+}-\psi^{+-})+\frac{k^4}{8}A(0)(A(0)-A(\underline{\rho}_d))(\psi^{-+}+\psi^{+-})\right]$$

$$-\left(\frac{\sinh\tau z}{\tau}\right)\int_z^L dz'\left(\frac{\tau\cosh\tau(L-z')+\gamma\sinh\tau(L-z')}{\tau\cosh\tau L+\gamma\sinh\tau L}\right)$$

$$\left[-\frac{\nabla_T^2}{2ik}\left(\frac{\partial\psi^{-+}}{\partial z'}-\frac{\partial\psi^{+-}}{\partial z'}\right)-\frac{k^2}{4}(A(0)-A(\underline{\rho}_d))\left(\frac{\partial\psi^{-+}}{\partial z'}+\frac{\partial\psi^{+-}}{\partial z'}\right)\right.$$

$$\left.+\frac{k^2}{2}(A(0)-A(\underline{\rho}_d))\frac{\nabla_T^2}{2ik}(\psi^{-+}-\psi^{+-})+\frac{k^4}{8}A(0)(A(0)-A(\underline{\rho}_d))(\psi^{-+}+\psi^{+-})\right]$$

$$+\psi^{++}(0,\underline{\rho}_d)\left(\frac{\tau\cosh\tau(L-z)+\gamma\sinh\tau(L-z)}{\tau\cosh\tau L+\gamma\sinh\tau L}\right). \quad (4.68)$$

Integrating (4.68) by parts to eliminate z' differentiation within the integrands results in

$$\psi^{++}(z,\underline{\rho}_d) = \chi(z,\underline{\rho}_d)\int_0^z dz' \frac{\nabla_T^2}{2ik}(\psi^{-+} - \psi^{+-})\left(\cosh \tau z' + \frac{k^2}{2}(A(0) - A(\underline{\rho}_d)\right)$$

$$\cdot \left.\frac{\sinh \tau z'}{\tau}\right) - \frac{k^2}{4}(A(0) - A(\underline{\rho}_d))\chi(z,\underline{\rho}_d)\int_0^z dz' (\psi^{-+} + \psi^{+-})\left(\cosh \tau z'\right.$$

$$\left. + \frac{k^2}{2} A(0) \frac{\sinh \tau z'}{\tau}\right) + \frac{k^2}{4}\frac{A(\underline{\rho}_d)}{\eta} \frac{\sinh \tau z}{\tau}\int_z^L dz' \frac{\nabla_T^2}{2ik}(\psi^{-+} - \psi^{+-})$$

$$\cdot \left(\frac{k^2}{2}(A(0) - A(\underline{\rho}_d))\sinh \tau(L-z') + \tau \cosh \tau(L-z')\right) - \frac{k^4}{16}\frac{A(\underline{\rho}_d)}{\eta}$$

$$\cdot (A(0) - A(\underline{\rho}_d))\frac{\sinh \tau z}{\tau}\int_z^L dz' (\psi^{-+} + \psi^{+-})\left(\frac{k^2}{2} A(0) \sinh \tau(L-z')\right.$$

$$\left. + \tau \cosh \tau(L-z')\right) + \chi(z,\underline{\rho}_d)\psi^{++}(0,\underline{\rho}_d) \qquad (4.69)$$

where

$$\chi(z,\underline{\rho}_d) = \frac{\tau \cosh \tau(L-z) + \gamma \sinh \tau(L-z)}{\tau \cosh \tau L + \gamma \sinh \tau L} \tag{4.70}$$

and

$$\eta = \tau \cosh \tau L + \gamma \sinh \tau L \quad . \tag{4.71}$$

The notation

$$K_1(z',z,\underline{\rho}_d) = -\chi(z,\underline{\rho}_d) \left(\cosh \tau z' + \frac{k^2}{2} \left(A(0) - A(\underline{\rho}_d)\right) \frac{\sinh \tau z'}{\tau}\right) \tag{4.72}$$

$$K_2(z',z,\underline{\rho}_d) = -\frac{\chi(z,\underline{\rho}_d)}{2} \left(\cosh \tau z' + \frac{k^2}{2} A(0) \frac{\sinh \tau z'}{\tau}\right) \tag{4.73}$$

$$K_3(z',z,\underline{\rho}_d) = \frac{k^2}{4} \frac{A(\underline{\rho}_d)}{\eta} \frac{\sinh \tau z}{\tau} \left(\frac{k^2}{2} \left(A(0) - A(\underline{\rho}_d)\right) \cdot \sinh \tau(L-z') + \tau \cosh \tau(L-z')\right) \tag{4.74}$$

$$K_4(z',z,\underline{\rho}_d) = -\frac{k^2}{8} \frac{A(\underline{\rho}_d)}{\eta} \frac{\sinh \tau z}{\tau} \left(\frac{k^2}{2} A(0) \sinh \tau(L-z') + \tau \cosh \tau(L-z')\right) \tag{4.75}$$

is used to write (4.69) as

$$\psi^{++}(z,\underline{\rho}_d) = \int_0^z dz' \frac{\nabla_T^2}{2ik} (\psi^{-+} - \psi^{+-}) K_1(z',z,\underline{\rho}_d) + \frac{k^2}{2}\left(A(0) - A(\underline{\rho}_d)\right)$$

$$\cdot \int_0^z dz' \, (\psi^{-+} + \psi^{+-}) K_2(z',z\underline{\rho}_d) + \int_z^L dz' \frac{\nabla_T^2}{2ik} (\psi^{-+} - \psi^{+-}) K_3(z',z,\underline{\rho}_d)$$

$$+ \frac{k^2}{2}\left(A(0) - A(\underline{\rho}_d)\right) \int_z^L dz' (\psi^{-+} + \psi^{+-}) K_4(z',z,\underline{\rho}_d) + \chi(z,\underline{\rho}_d)\psi^{++}(0,\underline{\rho}_d) \quad . \tag{4.76}$$

A corresponding expression for $\psi^{--}(z,\underline{\rho}_d)$ is determined by differentiating (4.76) with respect to z and substituting the result and (4.76) into (4.54) . The expression that is obtained is given by

$$\psi^{--}(z,\underline{\rho}_d) = \int_0^z dz' \frac{\nabla_T^2}{2ik} (\psi^{-+} - \psi^{+-}) G_1(z',z,\underline{\rho}_d) + \frac{k^2}{2} \left(A(0) - A(\underline{\rho}_d)\right)$$

$$\cdot \int_0^z dz' (\psi^{-+} + \psi^{+-}) G_2(z',z,\underline{\rho}_d) + \int_z^L dz' \frac{\nabla_T^2}{2ik} (\psi^{-+} - \psi^{+-}) G_3(z',z,\underline{\rho}_d)$$

$$+ \frac{k^2}{2}\left(A(0) - A(\underline{\rho}_d)\right) \int_z^L dz' (\psi^{-+} + \psi^{+-}) G_4(z',z,\underline{\rho}_d)$$

$$+ \frac{k^2}{4} A(\underline{\rho}_d) \frac{\sinh \tau(L - z)}{\eta} \psi^{++}(0,\underline{\rho}_d) \quad . \tag{4.77}$$

where

$$G_1(z',z,\underline{\rho}_d) = -\frac{k^2 A(\underline{\rho}_d)}{4} \frac{\sinh \tau(L - z)}{\eta}$$

$$\left(\cosh \tau z' + \frac{k^2}{2} \left(A(0) - A(\underline{\rho}_d)\right) \frac{\sinh \tau z'}{\tau}\right) \quad (4.78)$$

$$G_2(z',z,\underline{\rho}_d) = -\frac{k^2 A(\underline{\rho}_d)}{8} \frac{\sinh \tau(L - z)}{\eta}$$

$$\left(\cosh \tau z' + \frac{k^2 A(0)}{2} \frac{\sinh \tau z'}{\tau}\right) \quad (4.79)$$

$$G_3(z',z,\underline{\rho}_d) = \left[\frac{\tau \cosh \tau z + \gamma \sinh \tau z}{\eta}\right]\left[\cosh \tau(L - z')\right.$$

$$\left. + \frac{k^2}{2} \left(A(0) - A(\underline{\rho}_d)\right) \frac{\sinh \tau(L - z')}{\tau}\right] \quad (4.80)$$

$$G_4(z',z,\underline{\rho}_d) = -\frac{1}{2} \left(\frac{\tau \cosh \tau z + \gamma \sinh \tau z}{\eta}\right)$$

$$\left(\cosh \tau(L - z') + \frac{k^2 A(0)}{2} \frac{\sinh \tau(L - z')}{\tau}\right) \quad (4.81)$$

It is necessary to determine $\nabla_T^2(\psi^{-+} - \psi^{+-})$ and $\psi^{-+} + \psi^{+-}$ in terms of ψ^{++} and ψ^{--}. Note that the left-hand side of equations (4.52) and (4.53) are parabolic operators. Some of the properties of such operators were outlined in Section 2.2. Treating the right-hand side as a source term equation (4.52) is written as

$$\frac{\partial\psi^{+-}}{\partial z} + \frac{\nabla_T^2}{ik}\psi^{+-} - 2ik\psi^{+-} = -S(z,\underline{\rho}_d) \quad . \qquad (4.82)$$

A Green's function $G^{+-}(z-z',\underline{\rho}_d - \underline{\rho})$ corresponding to $\psi^{+-}(z,\underline{\rho}_d)$ satisfying

$$\frac{\partial G^{+-}(z-z',\underline{\rho}_d-\underline{\rho})}{dz} + \frac{\nabla_T^2}{ik} G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}) - 2ikG^{+-}(z-z',\underline{\rho}_d-\underline{\rho})$$

$$= -\delta(z-z')\delta(\underline{\rho}_d-\underline{\rho}) \qquad (4.83)$$

is assumed to exist. Since the geometry of the problem considered consists of a random slab whose dimensions are finite in the z-direction but infinite in the plane transverse to the direction of propagation, use can be made of the Fourier transform of (4.83) with respect to the transverse coordinate. Using exactly the same Fourier transform technique as in Section 2.2 G^{+-} is shown to be given by

$$G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}) = \frac{ik}{4\pi}\frac{1}{z-z'}\exp\left(2ik(z-z') + \frac{ik}{4}\frac{|\underline{\rho}_d-\underline{\rho}|^2}{z-z'}\right) \quad z \geq z'$$

$$= 0 \qquad z < z' \qquad (4.84)$$

Similarly, the Green's function for (4.53) can be shown to be

$$G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}) = 0 \qquad z > z'$$

$$= -\frac{ik}{4\pi}\frac{1}{z-z'}\exp\left(-2ik(z-z') - \frac{ik}{4}\frac{|\underline{\rho}_d-\underline{\rho}|^2}{z-z'}\right) \quad z \leq z' .$$

(4.85)

Friedman [8] has demonstrated that a solution for (4.82) can be written in terms of $G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')$ in the following way

$$\psi^{+-}(z,\underline{\rho}_d) = \int_{-\infty}^{\infty} G^{+-}(z-z_0,\underline{\rho}_d-\underline{\rho})\psi^{+-}(z_0.\underline{\rho}')d\underline{\rho}'$$

$$+ \int_{z_0}^{z} dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')S(z',\underline{\rho}') \quad z \geq z_0 \qquad (4.86)$$

where z_0 is some z plane where $\psi^{+-}(z,\underline{\rho}_d)$ is known. Note that (4.86) differs from (2.34) in that (2.34) is valid for a parabolic equation with variable coefficients. In a similar manner if (4.53) is written as

$$\frac{\partial\psi^{-+}}{\partial z} - \frac{\nabla_T^2}{ik}\psi^{-+} + 2ik\psi^{-+} = -\tilde{S}(z,\underline{\rho}_d) \qquad (4.87)$$

then

$$\psi^{-+}(z,\underline{\rho}_d) = \int_{-\infty}^{\infty} G^{-+}(z-z_0,\underline{\rho}_d-\underline{\rho}')\psi^{-+}(z_0,\underline{\rho}')d\underline{\rho}'$$

$$-\int_{z}^{z_0} dz' \int_{-\infty}^{\infty} d\underline{\rho}' G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')\tilde{S}(z',\underline{\rho}') \quad z \leq z_0 \quad . \qquad (4.88)$$

An expression for $\psi^{+-}(z,\underline{\rho}_d)$ valid for $z \leq z_0$ is needed to avoid boundary condition difficulties in the development of the integral equation solution for ψ^{++} and ψ^{--}. The conjugate of (4.82) is given by

$$\frac{\partial\psi^{+-*}}{\partial z} - \frac{\nabla_T^2}{ik}\psi^{+-*} + 2ik\psi^{+-*} = -S^*(z,\underline{\rho}_d) \ . \quad (4.89)$$

Note that the form of equation (4.89) is the same as (4.87). The solution of (4.89) can be written by making use of (4.88), that is

$$\psi^{+-*}(z,\underline{\rho}_d) = \int_{-\infty}^{\infty} G^{-+}(z-z_0,\underline{\rho}_d-\underline{\rho}')\psi^{+-*}(z_0,\underline{\rho}')$$

$$- \int_{z}^{z_0} dz' \int_{-\infty}^{\infty} d\underline{\rho}' \ G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')S^*(z',\underline{\rho}') \ z \leq z_0 \ . \quad (4.90)$$

The conjugate of (4.90) gives the desired expression

$$\psi^{+-}(z,\underline{\rho}_d) = \int_{-\infty}^{\infty} G^{-+*}(z-z_0,\underline{\rho}_d-\underline{\rho}')\psi^{+-}(z_0,\underline{\rho}')$$

$$- \int_{z}^{z_0} dz' \int_{-\infty}^{\infty} d\underline{\rho}' \ G^{-+*}(z-z',\underline{\rho}_d-\underline{\rho}')S(z',\underline{\rho}') \quad z \leq z_0 \ . \quad (4.91)$$

Now if $z_0 = L$, where $\psi^{+-}(L,\underline{\rho}_d) = \psi^{-+}(L,\underline{\rho}_d) = 0$, (4.88) and (4.91) are

$$\psi^{-+}(z,\underline{\rho}_d) = +\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \left[\frac{ik}{4\pi} \frac{1}{z-z'} \exp\left(-2ik(z-z') - \frac{ik}{4} \cdot \frac{|\underline{\rho}_d-\underline{\rho}'|^2}{z-z'}\right)\right] \tilde{S}(z',\underline{\rho}') \qquad z \leq L \tag{4.92}$$

$$\psi^{+-}(z,\underline{\rho}_d) = -\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \left[\frac{ik}{4\pi} \frac{1}{z-z'} \exp\left(-2ik(z-z') - \frac{ik}{4} \cdot \frac{|\underline{\rho}_d-\underline{\rho}'|^2}{z-z'}\right)\right] \tilde{S}(z',\underline{\rho}') \qquad z \leq L \ . \tag{4.93}$$

The above are the desired expressions for ψ^{+-} and ψ^{-+} in terms of the values of ψ^{+-} and ψ^{-+} at $z = L$.

One more relationship must be determined before the desired expressions for $\nabla_T^2(\psi^{-+} - \psi^{+-})$ and $\psi^{-+} + \psi^{+-}$ in terms of ψ^{++} and ψ^{--} can be written. Writing $|\underline{\rho}_d - \underline{\rho}'| = \tilde{\rho}$ and $z - z' = \tilde{z}$ it is desired to simplify an expression of the type

$$\int_{-\infty}^{\infty} \exp\left(- \frac{ik}{4} \frac{\tilde{\rho}^2}{\tilde{z}}\right) \tilde{\nabla}_T^2 \, F(\tilde{\rho}) \, d\tilde{\underline{\rho}} \ . \tag{4.94}$$

Consider the following expression:

$$\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\tilde{\nabla}_T^2 F(\tilde{\rho}) - F(\tilde{\rho})\,\tilde{\nabla}_T^2 \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right) = \nabla_T\cdot\left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right.$$

$$\left.\cdot\,\nabla_T F(\tilde{\rho}) - F(\tilde{\rho})\,\tilde{\nabla}_T \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right] \quad . \qquad (4.95)$$

Integrating (4.95) over a plane transverse to the direction of propagation apply the two dimensional version of Gauss' law, discussed in C. C. Johnson's book [13],

$$\int_S \left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\tilde{\nabla}_T^2 F(\tilde{\rho}) - F(\tilde{\rho})\,\tilde{\nabla}_T^2 \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right] d\tilde{S}$$

$$= \int_S \tilde{\nabla}_T\cdot\left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}}{\tilde{z}}\right)\tilde{\nabla}_T F(\tilde{\rho}) - F(\tilde{\rho})\,\tilde{\nabla}_T \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right] d\tilde{S}$$

$$= \oint_\ell \left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\tilde{\nabla}_T F(\tilde{\rho}) - F(\tilde{\rho})\,\tilde{\nabla}_T \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right]\cdot\hat{n}\,d\ell \qquad (4.96)$$

or

$$\int_S \left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\tilde{\nabla}_T^2 F(\tilde{\rho}) - F(\tilde{\rho})\,\tilde{\nabla}_T^2 \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right] d\tilde{S}$$

$$= \int_0^{2\pi} \tilde{\rho}\left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\frac{\partial F(\tilde{\rho})}{\partial\tilde{\rho}} - F(\tilde{\rho})\frac{\partial \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)}{\partial\tilde{\rho}}\right] d\phi$$

$$= 2\pi\tilde{\rho}\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\frac{\partial F(\tilde{\rho})}{\partial\tilde{\rho}} + 2\pi F(\tilde{\rho})\frac{ik}{2}\frac{\tilde{\rho}}{\tilde{z}}\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right) \quad . \tag{4.97}$$

Using a standard technique in which k is assumed to have a small negative imaginary part, it is seen that for sufficiently well-behaved $F(\tilde{\rho})$, that is as long as $F(\tilde{\rho})$ does not decrease more slowly than $\exp[(ik/4)(\tilde{\rho}^2/\tilde{z})]/\tilde{\rho}^n$ where $n > 2$, the right-hand side of (4.97) goes to zero as $\tilde{\rho}$ approaches infinity. Then (4.97) is written as

$$\int_S \left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\tilde{\nabla}_T^2 F(\tilde{\rho}) - F(\tilde{\rho})\,\tilde{\nabla}_T^2 \exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right] dS = 0 \quad . \tag{4.98}$$

Now the small imaginary part of k is allowed to approach zero resulting in

$$\int_S \left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\tilde{\nabla}_T^2 F(\tilde{\rho})\right]d\tilde{S} = \int_S F(\tilde{\rho})\,\tilde{\nabla}_T^2\left[\exp\left(-\frac{ik}{4}\frac{\tilde{\rho}^2}{\tilde{z}}\right)\right] d\tilde{S} \ . \tag{4.99}$$

Equation (4.99) is useful in the determination of $\psi^{-+} + \psi^{+-}$.

Now an expression for ψ^{+-} can be determined. Substituting the expression for $S(z',\underline{\rho}')$ in (4.93)

$$\psi^{+-}(z,\underline{\rho}_d) = +\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \left[\frac{ik}{4\pi}\frac{1}{z-z'} \exp\left(2ik(z-z') + \frac{ik}{4} \cdot \frac{|\underline{\rho}_d - \underline{\rho}'|^2}{z-z'}\right)\right]\left[-\frac{1}{2ik}\nabla_T'^2(\psi^{--}+\psi^{++}) + \frac{k^2}{4}A(0)(\psi^{++}-\psi^{--})\right] . \tag{4.100}$$

Next, use is made of expression (4.99) resulting in

$$\psi^{+-}(z,\underline{\rho}_d) = \int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \ \nabla_T'^2\left[\frac{ik}{4\pi}\frac{1}{z-z'} \exp\left(2ik(z-z') + \frac{ik}{4} \cdot \frac{|\underline{\rho}_d - \underline{\rho}'|^2}{z-z'}\right)\right]\left(-\frac{1}{2ik}(\psi^{--}+\psi^{++})\right) + \frac{k^2 A(0)}{4}$$

$$\cdot \int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \left[\frac{ik}{4\pi}\frac{1}{z-z'} \exp\left(2ik(z-z') + \frac{ik}{4}\frac{|\underline{\rho}_d - \underline{\rho}'|^2}{z-z'}\right)\right] \cdot (\psi^{++}-\psi^{--}) \ . \tag{4.101}$$

Since G^{+-} is an even function of $|\underline{\rho}_d - \underline{\rho}'|$ and (4.101) contains, $\nabla_T'^2$, a second order derivative then $\nabla_T'^2$ can be replaced by ∇_T^2 and (4.101) is written as

$$\psi^{+-}(z,\underline{\rho}_d) = \nabla_T^2 \int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; G^{+-}(z - z',\underline{\rho}_d - \underline{\rho}')$$

$$\cdot \left(-\frac{1}{2ik} \; (\psi^{++} + \psi^{--})\right) + \frac{k^2A(0)}{4} \int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'$$

$$\cdot \; G^{+-}(z-z',\underline{\rho}_d - \underline{\rho}')(\psi^{++} - \psi^{--}) . \quad (4.102)$$

Consider the system

$$\frac{\partial V(z\underline{\rho}_d)}{\partial z} + \frac{\nabla_T^2}{ik} \; V(z,\underline{\rho}_d) - 2ikV(z,\underline{\rho}_d) = S(z,\underline{\rho}_d) \qquad (4.103)$$

where $V(L,\underline{\rho}_d) = 0$. As presented earlier the solution to (4.103) can be written as

$$V(z,\underline{\rho}_d) = \int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; S(z',\underline{\rho}')G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}') . \quad (4.104)$$

Assume that $S(z',\underline{\rho}')$ in (4.104) is given by $-(1/2ik)(\psi^{--}(z',\underline{\rho}') + \psi^{++}(z',\underline{\rho}'))$. Comparing the first term in (4.102) and (4.104) it is seen that the first term in (4.102) is actually $\nabla_T^2 V(z,\underline{\rho}_d)$. Solving (4.103) for $\nabla_T^2 V(z,\underline{\rho}_d)$, (4.102) can be written as

$$\psi^{+-}(z,\underline{\rho}_d) = \frac{1}{2}\frac{\partial}{\partial z}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\; G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')(\psi^{--}+\psi^{++})$$

$$- ik\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\; G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')(\psi^{--}+\psi^{++})$$

$$+ \frac{k^2A(0)}{4}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\; G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')(\psi^{++}-\psi^{--})$$

$$-\frac{1}{2}\left(\psi^{--}(z,\underline{\rho}_d)+\psi^{++}(z,\underline{\rho}_d)\right) \quad . \qquad (4.105)$$

Now an expression for ψ^{-+} can be determined. Since the medium considered is statistically homogeneous in the transverse plane, $\psi^{+-}(z,\underline{\rho}_d) = \psi^{-+*}(z,\underline{\rho}_d)$ and $\psi^{-+}(z,\underline{\rho}_d)$ can be immediately written down as

$$\psi^{-+}(z,\underline{\rho}_d) = \frac{1}{2}\frac{\partial}{\partial z}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')(\psi^{--}+\psi^{++})$$

$$+ ik\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')(\psi^{--}+\psi^{++})$$

$$+ \frac{k^2A(0)}{4}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')(\psi^{++}-\psi^{--}) - \frac{1}{2}(\psi^{++}+\psi^{--})$$

(4.106)

where (4.5) and (4.6) were used. Now (4.105) and (4.106) can be used to write an expression for $\psi^{+-} + \psi^{-+}$,

$$\psi^{+-} + \psi^{-+} = \frac{1}{2}\frac{\partial}{\partial z}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\Big(G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}') + G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')\Big)$$

$$\cdot\,(\psi^{--}+\psi^{++}) - ik\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \Big(G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}') - G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')\Big)$$

$$\cdot\,(\psi^{--}+\psi^{++}) + \frac{k^2A(0)}{4}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \Big(G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')$$

$$+ G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')\Big)(\psi^{++}-\psi^{--}) - (\psi^{++}+\psi^{--}) \quad . \qquad (4.107)$$

The term $\psi^{+-} + \psi^{-+}$ as given by (4.107) is one of the sought after quantities to be used in equation (4.76) for ψ^{++} and in (4.77) for ψ^{--}. It is desirable to eliminate $\underline{\rho}'$ integrations from the above equation. The method of stationary phase can be used to eliminate these integrations. Examine

$$\int_{-\infty}^{\infty} d\underline{\rho}' \; G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')\Lambda^{+}(z',\underline{\rho}') = \int_{-\infty}^{\infty} d\underline{\rho}' \left(\frac{ik}{4\pi} \frac{1}{z - z'}\right.$$

$$\cdot \; \exp \; 2ik(z-z') + \frac{ik}{4} \frac{|\underline{\rho}_d - \underline{\rho}'|^2}{z - z'} \Big) \Lambda^{+}(z',\underline{\rho}') \quad . \qquad (4.108)$$

where

$$\Lambda^{+}(z,\underline{\rho}_d) = \psi^{++}(z,\underline{\rho}_d) + \psi^{--}(z,\underline{\rho}_d) \quad . \qquad (4.109)$$

Also allow

$$\Lambda^{-}(z,\underline{\rho}_d) = \psi^{++}(z,\underline{\rho}_d) - \psi^{--}(z,\underline{\rho}_d) \quad . \qquad (4.110)$$

Stationary phase is an asymptotic method used for the approximate evaluation of integrals whose integrands are oscillatory. It is assumed in (4.108) that the $\exp(ik/4)(|\underline{\rho}_d - \underline{\rho}'|^2)/z-z'$ term in the integrand varies so rapidly relative to $\Lambda^{+}(z',\underline{\rho}')$ that for $\underline{\rho}' \neq \underline{\rho}_d$ contributions to the integral cancel. Equation (4.108) then depends primarily on its value near $\underline{\rho}' = \underline{\rho}_d$ and can be approximately written as

$$\int_{-\infty}^{\infty} d\underline{\rho}' \; G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')\Lambda^{+}(z',\underline{\rho}') \;=\; \Lambda^{+}(z',\underline{\rho}_d)\, \frac{ik}{4} \, \frac{1}{z-z'}$$

$$\cdot \exp \; 2ik(z-z') \int_{-\infty}^{\infty} d\underline{\rho}' \;\; \exp \; \frac{ik}{4} \, \frac{|\underline{\rho}_d - \underline{\rho}'|^2}{z-z'} \quad . \qquad (4.111)$$

Integrating over $\underline{\rho}'$ (4.111) results in

$$\int_{-\infty}^{\infty} d\underline{\rho}' \; G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')\Lambda^{+}(z',\underline{\rho}') \;\approx\; -\Lambda^{+}(z',\underline{\rho}_d)\exp \; 2ik(z-z') \quad . \qquad (4.112)$$

The physical implication of the above approximation is that $\psi^{++}(z,\underline{\rho}_d)$ and $\psi^{--}(z,\underline{\rho}_d)$ must vary slowly in the $\underline{\rho}_d$ plane. When a plane wave is normally incident upon a slab infinite in extent in the transverse plane, geometry makes the idea of little variation in the $\underline{\rho}_d$ plane intuitively appealing. Similarly, $\underline{\rho}'$ integrations of $G^{-+}\Lambda^{+}$ can be approximated as

$$\int_{-\infty}^{\infty} d\underline{\rho}' \; G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}') \; \Lambda^{+}(z,\underline{\rho}') \;\approx\; -\Lambda^{+}(z',\underline{\rho}_d) \; e^{-2ik(z-z')} \; . \qquad (4.113)$$

Now eliminating all $\underline{\rho}'$ integrations in (4.107) results in

$$\psi^{+-} + \psi^{-+} = -\frac{\partial}{\partial z}\int_z^L dz'\ \Lambda^+(z',\underline{\rho}_d)\cos 2k(z - z')$$

$$- 2k\int_z^L dz'\ \Lambda^+(z',\underline{\rho}_d)\sin 2k(z - z') - \frac{k^2A(0)}{2}\int_z^L dz'$$

$$\cdot\ \Lambda^-(z',\underline{\rho}_d)\cos 2k(z - z') - \Lambda^+(z,\underline{\rho}_d)\quad . \qquad (4.114)$$

Carrying out the differentiation in (4.114) results in

$$\psi^{+-} + \psi^{-+} = -\frac{k^2A(0)}{2}\int_z^L dz'\ \Lambda^-(z',\underline{\rho}_d)\cos 2k(z-z') \qquad (4.115)$$

Similarly writing an expression for the difference of ψ^{+-} and ψ^{-+}

$$\psi^{+-} - \bar{\psi}^{-+} = \frac{1}{2}\frac{\partial}{\partial z}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\Big(G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}') - G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')\Big)$$

$$\cdot\ \Lambda^+(z',\underline{\rho}') - ik\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\ \Big(G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}') + G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')\Big)$$

$$\cdot\ \Lambda^-(z',\underline{\rho}') + \frac{k^2A(0)}{4}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}'\ \Big(G^{+-}(z-z',\underline{\rho}_d-\underline{\rho}')$$

$$- G^{-+}(z-z',\underline{\rho}_d-\underline{\rho}')\Big)\ \Lambda^-(z',\underline{\rho}')\quad . \qquad (4.116)$$

Stationary phase is applied to (4.116) resulting in

$$\psi^{+-} - \psi^{-+} = -i \frac{\partial}{\partial z} \int_z^L dz' \, \Lambda^+(z',\underline{\rho_d}) \sin 2k(z - z') + 2ik \int_z^L dz'$$

$$\cdot \, \Lambda^+(z',\underline{\rho_d}) \cos 2k(z - z') - i \frac{k^2 A(0)}{2} \int_z^L dz' \, \Lambda^-(z',\underline{\rho_d}) \sin 2k(z - z') \tag{4.117}$$

Now differentiating in (4.117)

$$\psi^{+-} - \psi^{-+} = -i \frac{k^2 A(0)}{2} \int_z^L dz' \, \Lambda^-(z',\underline{\rho_d}) \sin 2k(z - z') \quad . \tag{4.118}$$

The above equations can be used to determine an expression for $\nabla_T^2(\psi^{-+} - \psi^{+-})$. Adding equations (4.52) and (4.53) results in

$$\frac{\partial}{\partial z} (\psi^{+-} + \psi^{-+}) + \frac{\nabla_T^2}{ik} (\psi^{+-} - \psi^{-+}) - 2ik(\psi^{+-} - \psi^{-+})$$

$$= \frac{k^2 A(0)}{2} \Lambda^-(z,\underline{\rho_d}) \quad . \tag{4.119}$$

Substituting (4.115) and (4.118) in (4.119) to determine $\nabla_T^2(\psi^{+-} - \psi^{-+})$

$$- \frac{k^2A(0)}{2} \frac{\partial}{\partial z} \int_z^L dz' \, \Lambda^-(z',\underline{\rho}_d) \cos 2k(z - z') + \frac{\nabla_T^2}{ik} (\psi^{+-} - \psi^{+-})$$

$$- k^3A(0) \int_z^L dz' \, \Lambda^-(z',\underline{\rho}_d) \sin 2k(z - z') \;=\; \frac{k^2A(0)}{2} \Lambda^-(z,\underline{\rho}_d) \quad (4.120)$$

or

$$\nabla_T^2(\psi^{+-} - \psi^{+-}) \;=\; 0 \quad (4.121)$$

results. Now (4.115) and (4.121) can be used in (4.76) and (4.77) resulting in

$$\psi^{++}(z,_{-d}) \;=\; -\tau^2 \int_0^z dz' \; K_2(z',z,\underline{\rho}_d) \int_{z'}^L dz'' \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z'-z'')$$

$$-\tau^2 \int_z^L dz' \, K_4(z',z,\underline{\rho}_d) \int_{z'}^L dz'' \, \Lambda^-(z'',\underline{\rho}_d) \cosh 2k(z'-z'')$$

$$+ \; \chi(z,\underline{\rho}_d)\psi^{++}(0,P_d) \quad (4.122)$$

and

$$\psi^{--}(z,\underline{\rho}_d) = -\tau^2 \int_0^z dz' \, G_2(z',z,\underline{\rho}_d) \int_{z'}^L dz'' \, \Lambda^-(z',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$-\tau^2 \int_z^L dz' \quad G_4(z,z,\underline{\rho}_d) \int_{z'}^L dz'' \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$+ \frac{k^2}{4} A(\underline{\rho}_d) \, \frac{\sinh \tau(L - z)}{\eta} \quad \psi^{++}(0,\underline{\rho}_d) \quad . \tag{4.123}$$

It is now desirable to eliminate the double integrations in (4.122) and (4.123). Reversing the order of integration, the first term in (4.122) is written as

$$\int_0^z dz' \quad K_2(z',z,\underline{\rho}_d) \int_{z'}^L dz'' \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$= \int_0^z dz'' \int_0^{z''} dz' \, K_2(z',z,\underline{\rho}_d) \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$+ \int_z^L dz'' \int_0^z dz' \, K_2(z',z,\underline{\rho}_d) \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'') \, . \tag{4.124}$$

Substituting in the expression for K_2 from (4.73)

$$\int_0^z dz' \, K_2(z',z,\underline{\rho}_d) \int_{z'}^{L} dz'' \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$= - \frac{\chi(z,\underline{\rho}_d)}{2} \int_0^z dz'' \, \Lambda^-(z'',\underline{\rho}_d) \int_0^{z''} dz' \, (\cosh \tau z'$$

$$+ \frac{k^2 A(0)}{2} \frac{\sin \tau z'}{\tau}) \cos 2k(z' - z'') - \frac{\chi(z,\underline{\rho}_d)}{2} \int_z^L dz''$$

$$\cdot \Lambda^- (z'',\underline{\rho}_d) \int_0^z dz' \, (\cosh \tau z' + \frac{k^2 A(0)}{2} \frac{\sinh \tau z'}{\tau}) \cos 2k(z'-z'')$$

(4.125)

and integrating

$$\int_0^z dz' \, K_2(z',z,\underline{\rho}_d) \int_{z'}^{L} dz'' \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'') = - \frac{\chi(z,\underline{\rho}_d)}{2}$$

$$\cdot \left[\int_0^z dz'' \, \frac{\Lambda^-(z'',\underline{\rho}_d)}{Z} (\tau \sinh \tau z'' + 2k \sin 2kz'' + \frac{k^2 A(0)}{2} (\cosh \tau z'' \right.$$

$$- \cos 2kz'' + \int_z^L dz'' \, \frac{\Lambda^-(z'',\underline{\rho}_d)}{Z} (\tau \sinh \tau z \cos 2k(z-z'')$$

$$+ 2k(\cosh \tau z \sin 2k(z-z'') + \sin 2kz'') + \frac{k^2 A(0)}{2} (\cosh \tau z \cosh 2k(z-z'')$$

$$\left. - \cos 2kz'') + \frac{k^2 A(0)}{\tau} \sinh \tau z \sin 2k(z - z'') \right] \qquad (4.126)$$

where $Z = \tau^2 + 4k^2$. The second term in (4.122) can be written as

$$\int_z^L dz' \int_{z'}^L dz'' \; K_4(z',z,\underline{\rho}_d)\, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$= \int_z^L dz'' \int_z^{z''} dz' \; K_4(z',z,\underline{\rho}_d)\, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'') \quad . \qquad (4.127)$$

Substituting the expression for K_4 into (4.127)

$$\int_z^L dz' \int_{z'}^L dz'' \; K_4(z',z,\underline{\rho}_d)\, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$= -\frac{k^2}{8} \frac{A(\underline{\rho}_d)}{\eta} \frac{\sinh \tau z}{\tau} \int_z^L dz'' \, \Lambda^-(z'',\underline{\rho}_d) \int_z^{z''} dz''$$

$$\cdot \left(\frac{k^2 A(0)}{2} \sinh \tau(L - z') + \tau \cosh \tau(L - z') \cos 2k(L - z') \right) \qquad (4.128)$$

and integrating

$$\int_z^L dz' \int_z^L dz'' \, K_4(z',z,\underline{\rho}_d) \, \Lambda^-(z'',\underline{\rho}_d) \cos 2k(z' - z'')$$

$$= \frac{k^2}{8} \frac{A(\underline{\rho}_d)}{\eta} \frac{\sinh \tau z}{Z^\tau} \int_z^L dz'' \, \Lambda^-(z'',\underline{\rho}_d) \left[\frac{\tau k^2 A(0)}{2} (\cosh \tau(L - z'') \right.$$

$$- \cosh \tau(L - z) \cosh 2k(z - z'') + k^3 A(0) \sinh \tau(L - z) \sin 2k(z - z'')$$

$$+ \tau^2 (\sinh \tau(L - z'') - \sinh \tau(L - z) \cos 2k(z - z''))$$

$$\left. + 2k\tau \cosh \tau(L - z) \sin 2k(z - z'') \right] \quad . \qquad (4.129)$$

Now using (4.126) and (4.128) in (4.122) and simplifying results in

$$\psi^{++}(z,\underline{\rho}_d) = \tau^2 \left[\frac{\chi(z,\underline{\rho}_d)}{2Z} \int_0^z dz'' \, \Lambda^-(z'',\underline{\rho}_d)(\tau \sinh \tau z'' + 2k \sin 2kz'' \right.$$

$$+ \frac{k^2 A(0)}{2} (\cosh \tau z'' - \cos 2kz'')) - \int_z^L dz'' \, \Lambda^-(z'',\underline{\rho}_d) \left(- \frac{k}{Z} (\chi(z,\underline{\rho}_d) \right.$$

$$\left. \cdot \sin 2kz'' + \sin 2k(z-z'') \right) + \frac{k^2 A(0)}{4Z} (\chi(z,\underline{\rho}_d) \cos 2kz'' - \cos 2k(z - z'')$$

$$\left. + \frac{k^2 A(\underline{\rho}_d)}{8} \frac{\sinh \tau z}{2Z} \left(\frac{k^2 A(0)}{2} \cosh \tau(L - z'') + \sinh \tau(L - z'') \right) \right] \qquad (4.130)$$

or writing in $\Lambda^{-}(z'',\underline{\rho}_d)$

$$\psi^{++}(z,\underline{\rho}_d) = \frac{\tau^2\chi(z,\underline{\rho}_d)}{2Z} \int_0^z \psi^{++}(z'',\underline{\rho}_d)\Big(\tau \sinh \tau z'' + 2k \sin 2kz''$$

$$+ \frac{k^2 A(0)}{2} (\cosh \tau z'' - \cos 2kz'')\Big) - \tau^2 \int_z^L dz'' \, \psi^{++}(z'',\underline{\rho}_d)$$

$$\cdot \Big(- \frac{k^2}{Z} (\chi(z,\underline{\rho}_d)\sin 2kz'' + \sin 2k(z - z'')\Big) + \frac{k^2 A(0)}{4Z} (\chi(z,\underline{\rho}_d)$$

$$\cdot \cos 2kz'' - \cos 2k(z - z'')) + \frac{k^2 A(\underline{\rho}_d)}{8} \frac{\sinh \tau z}{2Z} \Big(\frac{k^2 A(0)}{2}$$

$$\cdot \cosh \tau(L - z'') + \tau \sinh \tau(L - z'')\Big)\Big) - \frac{\tau^2\chi(z,\underline{\ }_d)}{2Z} \int_0^z \psi^{--}(z'',\underline{\rho}_d)$$

$$\cdot \Big(\tau \sinh \ z'' + 2k \sin 2kz'' + \frac{k^2 A(0)}{2}(\cosh \tau z'' - \cos 2kz'')\Big)$$

$$+ \tau^2 \int_z^L dz'' \ \psi^{--}(z'',\underline{\rho}_d) \Big(- \frac{k}{Z} (\chi(z,\underline{\rho}_d)\sin 2kz'' + \sin 2k(z - z'')\Big)$$

$$+ \frac{k^2 A(0)}{4Z} (\chi(z,\underline{\rho}_d)\cos 2kz'' - \cos 2k(z - z'')\Big) + \frac{k^2 A(\underline{\rho}_d)}{8} \frac{\sinh \tau z}{\eta Z}$$

$$\cdot \left(\frac{k^2 A(0)}{2} \cosh \tau(L-z'') + \sinh \tau(L-z'')\right)\Bigg) + \chi(z,\underline{\rho}_d)\psi^{++}(0,\underline{\rho}_d) \ . \quad (4.131)$$

which is an integral equation with only single integrals. Now examining (4.123) it is possible, in a manner completely analogous to that for the ψ^{++} equation to demonstrate that

$$\psi^{--}(z,\underline{\rho}_d) = \frac{\tau^2 k^2 A(\underline{\rho}_d)}{8Z} \frac{\sinh \tau(L - z)}{\eta} \int_0^z dz'' \, \psi^{++}(z'',\underline{\rho}_d)(\tau \sinh \tau z''$$

$$+ 2k \sin 2kz'' + \frac{k^2 A(0)}{2} (\cosh \tau z'' - \cos 2kz'')\Big) - \tau^2 \int_z^L dz'' \, \psi^{++}(z'',\underline{\rho}_d)$$

$$\cdot \left(\frac{k}{Z} (\sin 2k(z - z'') - \frac{k^2 A(\underline{\rho}_d)}{4\eta} \sinh \tau(L - z) \sin 2kz''\right) + \frac{k^2 A(0)}{4Z}$$

$$\cdot (-\cos 2k(z - z'') + \frac{k^2 A(\underline{\rho}_d)}{4\eta} \sinh \tau (L - z) \cos 2kz'') + \frac{\hat{\chi}(z,\underline{\rho}_d)}{2Z}$$

$$\cdot \left(\frac{k^2 A(0)}{2} \cosh \tau(L - z'') + \tau \sinh \tau(L - z'')\right)\Big) - \frac{\tau^2 k^2 A(\underline{\rho}_d)}{8Z} \frac{\sinh \tau(L - z)}{\eta}$$

$$\cdot \int_0^z dz'' \, \psi^{--}(z'',\underline{\rho}_d)\Big(\tau \sinh \tau z'' + 2k \sin 2kz'' + \frac{k^2 A(0)}{2} (\cosh \tau z''$$

$$- \cos 2kz'')\Big) + \tau^2 \int_z^L dz'' \, \psi^{--}(z'',\underline{\rho}_d) \left(\frac{k}{Z} (\sin 2k(z - z'')\right.$$

$$- \frac{k^2 A(\underline{\rho}_d)}{4} \sin \tau(L - z) \sinh 2kz''\Big) + \frac{k^2 A(0)}{4Z} (-\cos 2k(z - z'')$$

$$+ \frac{k^2 A(\underline{\rho}_d)}{4} \sinh \tau (L - z) \cos 2kz''\Big) + \frac{\hat{\chi}(z,\underline{\rho}_d)}{2Z} \Big(\frac{k^2 A(0)}{2} \cosh (L - z'')$$

$$+ \tau \sinh \tau(L - z'')\Big) + \frac{k^2 A(\underline{\rho}_d)}{4} \frac{\sinh \tau (L - z)}{\eta} \psi^{++}(0,\underline{\rho}_d) \qquad (4.132)$$

where

$$\hat{\chi}(z,\underline{\rho}_d) = \frac{\tau \cosh \tau z + \gamma \sinh \tau z}{\tau \cosh \tau L + \gamma \sinh \tau L} . \qquad (4.133)$$

The above integral equations are formidable in appearance. A measure of confidence in their validity can be established by noting that $\psi^{++}(0,\underline{\rho}_d) = 1$ from (4.131) and $\psi^{--}(L,\underline{\rho}_d) = 0$ from (4.132). The pair of integral equations can be categorized as a Fredholm equation of the second kind. The remainder of this chapter extracts information from (4.131) and (4.132) in an attempt to understand the behavior of $\psi^{++}(z,\underline{\rho}_d)$ and $\psi^{--}(z,\underline{\rho}_d)$.

4.3 Coherence Function Results

The integral equations describing the behavior of $\psi^{++}(z,\underline{\rho}_d)$ and $\psi^{--}(z,\underline{\rho}_d)$ can be made somewhat more palatable by normalization. A normalization scheme is employed similar to that used by Ito and Adachi [14]. Given a permittivity function delta correlated in the direction of propagation it is true that

$$A(\underline{\rho}_d) = \int_{-\infty}^{\infty} B(\underline{\rho}_d,z)\, dz \qquad (4.134)$$

where $B(\underline{\rho}_d,z) = A(\underline{\rho}_d)\delta(z) = \langle\tilde{\varepsilon}(\underline{\rho}_1,z_1)\tilde{\varepsilon}(\underline{\rho}_2,z_2)\rangle$. Defining a one dimensional spectrum in the following way

$$\phi_1(\underline{\rho}_d,k_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \;\; B(\underline{\rho}_d,z)\, e^{-ik_3 z} \qquad (4.135)$$

it is seen that

$$A(\underline{\rho}_d) = 2\pi\phi_1(\underline{\rho}_d,0) \tag{4.136}$$

and

$$A(0) = 2\pi\phi_1(0,0) = \sigma_\varepsilon^2 \ . \tag{4.137}$$

A normalized spectrum is written as

$$\phi_A = \frac{\phi_1(\underline{\rho}_d,0)}{\phi_1(0,0)} \ . \tag{4.138}$$

Note that ϕ_A ranges from one to zero as $\underline{\rho}_d$ ranges from zero to infinity for $A(\underline{\rho}_d)$ Gaussian or exponential. Ito and Adachi [14] define an extinction coefficient β as

$$\beta = \pi k^2 L\phi_1(0,0) = 2\pi^2 \frac{L}{\lambda} \frac{\sigma_\varepsilon^2}{\lambda} \ .$$

The extinction coefficient given above was the only independent parameter which appeared in the work of Ito and Adachi [14]. Increases in β, in the referenced work, can be interpreted as increases in the size of the slab or as increases in the variance of permittivity fluctuations. The present work has independent occurrences of β and L/λ in the defining integral equations. An increase in permittivity fluctuations is realized by increasing β with L/λ held fixed. An increase in slab thickness requires increases in β and L/λ. All z variations are normalized to L, the slab thickness. Using these quantities the ψ^{++} equation can be written

$$\psi^{++}(\zeta,\underline{\rho}_d) = \frac{1}{2}\kappa\chi(\zeta,\underline{\rho}_d)\int_0^{\zeta} d\alpha\,\psi^{++}(\alpha,\underline{\rho}_d)\Big(\beta(1-\phi_A)^{1/2}$$

$$\cdot \sinh\omega\alpha + 2kL\sin 2kL\alpha + \beta(\cosh\omega\alpha - \cos 2kL\alpha)\Big)$$

$$-\frac{1}{2}\kappa\int_{\zeta}^{1} d\alpha\,\psi^{++}(\alpha,\underline{\rho}_d)\Big(-(\beta\sin 2kL\zeta + 2kL(\chi(\zeta,\underline{\rho}_d) - \cos 2kL\zeta)\Big)$$

$$\cdot \sin 2kL\alpha + \Big(\beta(\chi(\zeta,\underline{\rho}_d) - \cos 2kL\zeta) - 2kL\sin 2kL\zeta\Big)\cos 2kL\alpha$$

$$+\frac{\beta\phi_A}{2}\frac{\sinh\omega\zeta}{\eta'}[\cosh\omega(1-\alpha) + (1-\phi_A)^{1/2}\sinh\omega(1-\alpha)]\Big)$$

$$-\frac{1}{2}\kappa\chi(\zeta,\underline{\rho}_d)\int_0^{\zeta}\psi^{--}(\alpha,\underline{\rho}_d)\Big(\beta(1-\phi_A)^{1/2}\sinh\omega\alpha + 2kL\sin 2kL\alpha$$

$$+\beta(\cosh\omega\alpha - \cos 2kL\alpha)\Big) + \frac{1}{2}\kappa\int_{\zeta}^{1} d\alpha\,\psi^{--}(\alpha,\underline{\rho}_d)$$

$$\cdot\Big(-(\beta\sin 2kL\zeta + 2kL(\chi(\zeta,\underline{\rho}_d) - \cos 2kL\zeta)\Big)\sin 2kL\zeta$$

$$+\Big(\beta(\chi(\zeta,\underline{\rho}_d) - \cos 2kL\zeta) - 2kL\sin 2kL\zeta\Big)\cos 2kL\alpha + \frac{\beta\phi_A}{2}\frac{\sinh\omega\zeta}{\eta'}$$

$$\cdot\Big(\cos\omega(1-\alpha) + (1-\phi_A)^{1/2}\sinh\omega(1-\alpha)\Big) + \chi(\zeta,\underline{\rho}_d)\psi^{++}(0,\underline{\rho}_d)$$

(4.139)

where

$$\kappa = \frac{\left(\frac{\sigma_\varepsilon^2}{\lambda}\right)^2}{\left(\frac{\sigma_\varepsilon^2}{\lambda}\right) + \frac{4}{\pi^2} \frac{1}{(1 - \phi_A)}} \tag{4.140}$$

$$\chi(\zeta,\underline{\rho}_d) = \frac{(1 - \phi_A)^{1/2} \cosh \omega(1 - \zeta) + (1 - \frac{\phi_A}{2}) \sinh \omega(1 - \zeta)}{\eta'} \tag{4.141}$$

$$\eta' = (1 - \phi_A)^{1/2} \cosh \omega + (1 - \frac{\phi_A}{2}) \sinh \omega \tag{4.142}$$

$$\omega = \beta(1 - \phi_A)^{1/2} \tag{4.143}$$

and ζ and α range from zero to one. The normalized version of the ψ^{--} integral equation is given by

$$
\begin{aligned}
\psi^{--}(\zeta,\underline{\rho}_d) = {} & \frac{1}{2}\kappa \; \frac{\phi_A}{2} \; \frac{\sinh \omega(1-\zeta)}{\eta'} \int_0^{\zeta} d\alpha \; \psi^{++}(\alpha,\underline{\rho}_d)(\beta(1-\phi_A)^{1/2} \\
& \cdot \sinh \omega\alpha + 2kL \sin 2kL\alpha + \beta(\cosh \omega\alpha - \cos 2kL\alpha)) - \frac{1}{2}\kappa \int_{\zeta}^{1} d\alpha \\
& \cdot \psi^{++}(\alpha,\underline{\rho}_d) \; \left(-2kL\left(\frac{\phi_A}{2} \; \frac{\sinh \omega(1-\zeta)}{\eta'} + \cos 2kL\zeta\right) - \beta \sin 2kL\zeta\right) \\
& \cdot \sin 2kL\alpha + \left(2kL \sin 2kL\zeta + \beta\left(\frac{\phi_A}{2} \; \frac{\sinh \omega(1-\zeta)}{\eta'} - \cos 2kL\zeta\right)\right) \\
& \cdot \cos 2kL\zeta + \beta\hat{\chi}(\zeta,\underline{\rho}_d)(\cosh \omega(1-\alpha) + (1-\phi_A)^{1/2} \sinh \omega(1-\alpha))) \\
& - \frac{1}{2}\kappa \, \frac{\phi_A}{2} \; \frac{\sinh \omega(1-\zeta)}{\eta'} \int_0^{\zeta} d\alpha \; \psi^{--}(\alpha,\underline{\rho}_d)\Big(\beta(1-\phi_A)^{1/2} \sinh \omega\alpha \\
& + 2kL \sin 2kL\alpha + \beta(\cosh \omega\alpha - \cos 2kL\alpha)\Big) + \frac{1}{2}\kappa \int_{\zeta}^{1} d\alpha \; \psi^{--}(\alpha,\underline{\rho}_d) \\
& \cdot \left(\left(-2kL\left(\frac{\phi_A}{2} \; \frac{\sinh \omega(1-\zeta)}{\eta'} + \cos 2kL\zeta\right) - \beta \sin 2kL\zeta\right) \sin 2kL\alpha \right. \\
& + \left(2kL \sin 2kL\zeta + \beta\left(\frac{\phi_A}{2} \; \frac{\sinh \omega(1-\zeta)}{\eta'} - \cos 2kL\zeta\right)\right) \cos 2kL\alpha \\
& \left. + \beta\hat{\chi}(\zeta,\underline{\rho}_d) \; (\cosh \omega(1-\zeta) + (1-\phi_A)^{1/2} \sinh \omega(1-\alpha))\right) \\
& + \frac{\phi_A}{2} \; \frac{\sinh \omega(1-\zeta)}{\eta'} \; \psi^{++}(0,\underline{\rho}_d) \qquad (4.144)
\end{aligned}
$$

where

$$\hat{\chi}(\zeta,\underline{\rho}_d) = \frac{(1-\phi_A)^{1/2}\cosh \omega\zeta + 1 - \frac{\phi_A}{2}\ \sinh \omega\zeta}{\eta'} \cdot \quad (4.145)$$

The complicated nature of equations (4.139) and (4.144) requires a numerical solution. A numerical approach is described in Appendix A. Much information can be gained from the equations themselves by considering limiting values of parameters β and $\sigma_\varepsilon^2/\lambda$ and the independent variable ρ_d. There are two types of information which can be garnered from $\psi^{++}(\zeta,\underline{\rho}_d)$ and $\psi^{--}(\zeta,\underline{\rho}_d)$. When $\rho_d = 0$ energy information is obtained. When $\rho_d \neq 0$ the correlation properties for the field at any two points in any z plane for forward propagating or backscatter fields can be determined.

An expression for the total energy profile for the forward propagating wave is obtained by taking the limit as $\rho_d \to 0$ of (4.139) resulting in

$$\psi^{++}(\zeta,0) = \frac{2+\beta(1-\zeta)}{2+\beta} \quad (4.145)$$

where $\psi^{++}(0,\underline{\rho}_d) = \psi^{++}(0,0) = 1$ for plane wave incidence. An expression for the total energy profile for the backward propagating wave is obtained by taking the limit of $\rho_d \to 0$ of (4.144) resulting in

$$\psi^{--}(\zeta,0) = \frac{\beta(1 - \zeta)}{2 + \beta} . \tag{4.146}$$

The forward scatter energy out of the random medium is given by

$$\psi^{++}(1,0) = \frac{2}{2 + \beta} \tag{4.147}$$

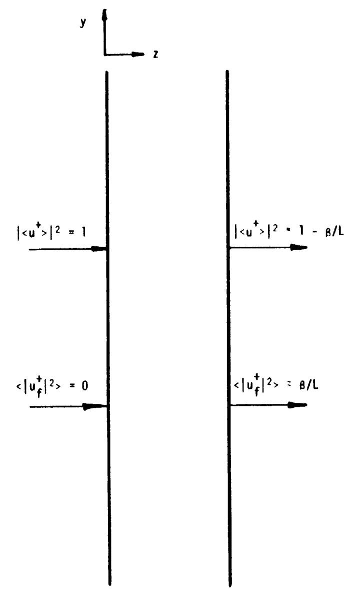
while the backscatter energy out of the slab is given by

$$\psi^{--}(0,0) = \frac{\beta}{2 + \beta} . \tag{4.148}$$

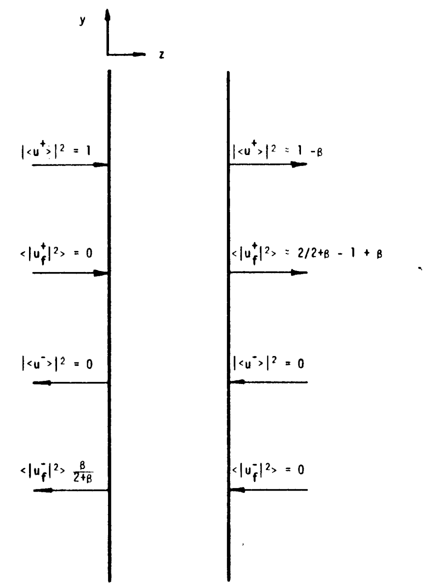
Since the medium is not lossy the total energy into the medium must equal the total energy out of the medium or

$$\psi^{++}(1,0) + \psi^{--}(0,0) = 1 , \tag{4.149}$$

which by (4.147) and (4.148) is the case. Note that at $\zeta = 1$, $\psi^{+-} = \psi^{-+} = 0$ and at $\zeta = 0$ Re ψ^{+-} = Re ψ^{-+} and consequently the cross energy terms do not enter into a balance consideration. The contention was made that the difference between the mean field results for the no backscatter and backscatter situation for small $\sigma_\varepsilon^2/\lambda$ was due to an increase in energy in the fluctuating forward and backward propagating fields when backscatter is taken into account. That this is the situation can be easily seen by examining Fig. 4.1. Note that for each situation illustrated in Fig. 4.1 the total energy into the slab is equal to the total



(a) Coherent and noncoherent energy balance no backscatter situation.



(b) Coherent and noncoherent energy balance with backscatter.

Fig. 4.1 Random Slab Energy Balance.

energy out of the slab. The notation $|\langle u \rangle|^2$ designates the energy in the coherent field while the notation $\langle |u_f|^2 \rangle$ designates the energy in the fluctuating field. The difference in coherent energy out of the slab between a and b is given by $\beta/2$ while the difference in fluctuating energy out of the slab between a and b is given by

$$\frac{2}{2+\beta} - 1 + \beta + \frac{\beta}{2+\beta} = \frac{\beta}{2} \quad .$$

which supports the previous contention.

Earlier in this thesis it was mentioned that there is little work to which the results presented here can be compared. Ito and Adachi [14] did, however, consider a cumulative forward scatter single backscatter approach to the backscatter problem. A principle result of their work was contained in a plot of backscatter energy normalized to the Born or single scatter approximation for backscatter versus the extinction factor, β, for various permittivity correlation functions. A similar plot will be presented here. In Appendix B an expression was obtained from first prinipl\es for the Born approximation which agrees up to a multiplicative constant with the result found in many texts (see, for example, Ishimaru [2]). Development from first principles was used to make available a result compatible within the framework of the present formulation. The Born approximation to backscatter is given by

$$\psi_B^{--} = \frac{\beta}{2} . \tag{4.150}$$

Normalized to (4.150) the multiple scattering factor for backscatter is given by

$$\frac{\psi^{--}(0,0)}{\psi_B^{--}} = \frac{2}{2 + \beta} . \tag{4.151}$$

Equation (4.151) is plotted in Fig. 4.2 versus β. Increasing β implies a decrease in the average forward scatter field (see Section 3.3) arriving at some point internal to the random medium. Since the coherent wave medium interaction is confined to the $z = 0$ boundary with increasing β the amount of multiple scattering activity decreases with increasing β. Consequently, Fig. 4.2 behaves as illustrated. Comparing Fig. 4.2 to the corresponding figure in the work of Ito and Adachi [14], it is seen that the behavior of multiple scattering factor with increasing β is the same except that the result in this work falls off more rapidly with with increasing β. The reason for this more rapid decrease in multiple scattering is the more rapid decline in average forward scatter field as discussed in Section 3.3. Note that (4.151) depends on the variance of permittivity fluctuations and not on the shape of correlation function used. This is in contrast to the Ito and Adachi [14] results which were correlation dependent.

Now leaving the energy interpretation of ψ^{++} and ψ^{--} and considering the correlation interpretation the question of when

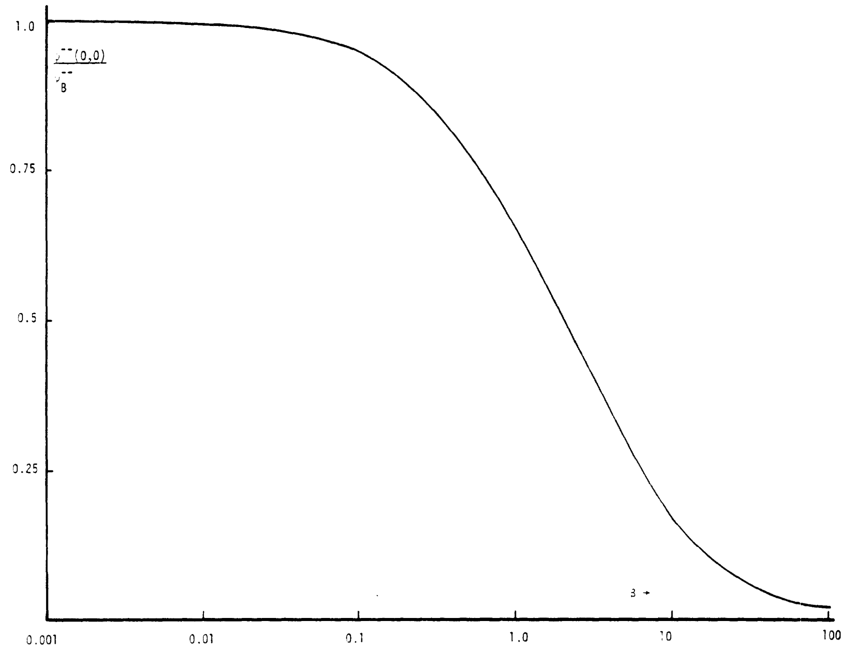


Fig. 4.2 Multiple Scattering Factor for Backscatter.

the integral terms in equations (4.139) and (4.144) are important must be considered. Each integral term in equations (4.139) and and (4.144) is multiplied by κ. The ψ^{++} equation contains a driving force term given by $\chi(\zeta,\rho_d)$ while the ψ^{--} equation contains $(\phi_A/2)(\sinh \omega(1 - \zeta)/\eta')$ as a driving force. The integral terms in each equation must be compared to the driving force terms. The relative sizes of the integral terms versus the driving functions are a fairly complex function of $\rho_d,\sigma_\varepsilon^2/\lambda$ and β. A graphical presentation of the pertinent coefficients makes it more clear when the integral terms in the equations must be considered, that is when a numerical solution is required, and when they are unimportant. Figures 4.3 through 4.5 display κ as the coefficient of the integral terms, $\chi(1,\rho_d)$ the ψ^{++} equation driving force, $\phi_A/2$ $(\sinh \omega)/\eta'$ the ψ^{--} equation driving force, respectively. Note that by definition ψ^{++} and ψ^{--} must fall between zero and one. When κ is much greater than $\chi(1,\rho_d)$ or $\phi_A/2$ $(\sinh \omega)/\eta'$ the integral terms in the corresponding equation should be considered. For example, a comparison of Fig. 4.3 and Fig. 4.4 indicates that for small $\sigma_\varepsilon^2/\lambda$ and small β the integral terms contributing to ψ^{++} are negligible for all values of ρ_d/ℓ , however a comparison of Figs. 4.3 and 4.5 indicate that the integral terms are important contributors to ψ^{--} for most values of β and $\sigma_\varepsilon^2/\lambda$ at high ρ_d/ℓ values. These figures are a useful aid in what follows.

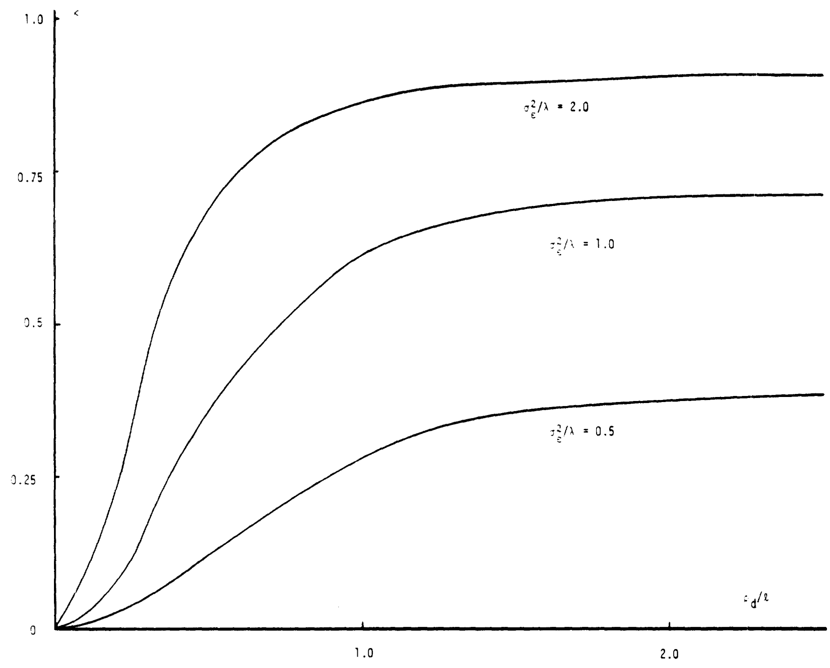


Fig. 4.3 Integral Coefficient.

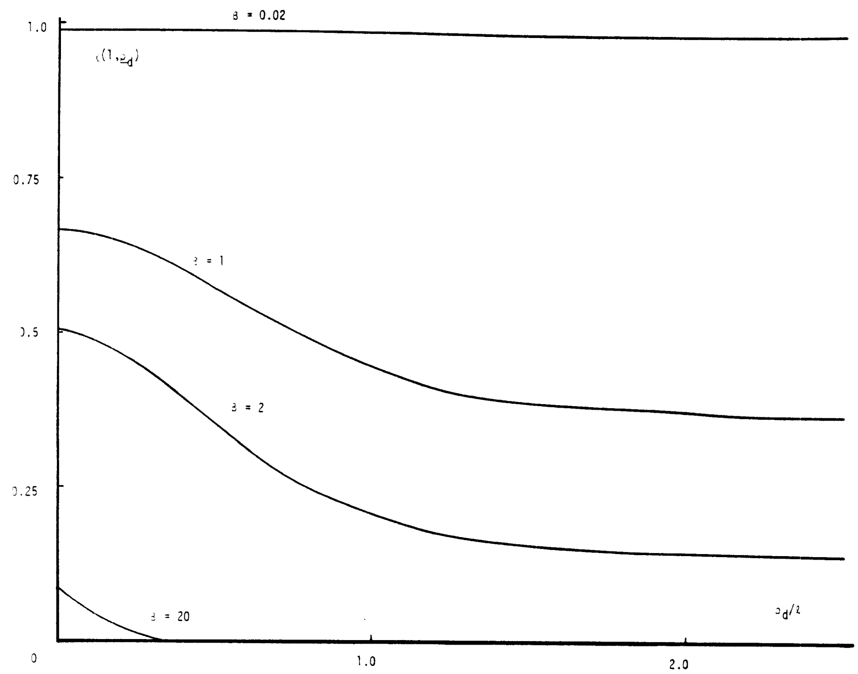


Fig. 4.4 ψ^{++} Driving Force.

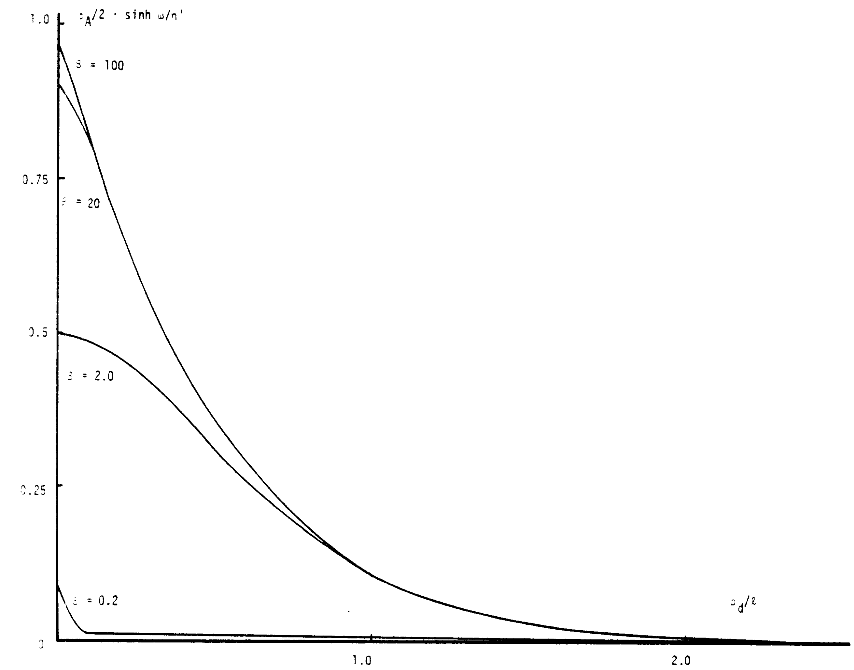


Fig. 4.5 ψ^{++} Driving Force.

Consider $\sigma_\varepsilon^2/\lambda$ very small, specifically

$$\sigma_\varepsilon^2/\lambda \ll \frac{2}{\pi} \quad , \tag{4.152}$$

then

$$\kappa \approx \frac{\pi^2}{4}(1 - \phi_A)\left(\frac{\sigma_\varepsilon^2}{\lambda}\right)^2 \tag{4.153}$$

resulting in κ much less than one. In addition to (4.153) when L/λ is very small

$$\chi(\zeta,\underline{\rho}_d) \approx \frac{1 + \left(1 - \frac{\phi_A}{2}\right)\beta(1 - \zeta)}{1 + \left(1 - \frac{\phi_A}{2}\right)\beta} \quad . \tag{4.154}$$

Equation (4.154) is less than or equal to one for all possible values of ρ_d and ζ. Since L/λ and $\sigma_\varepsilon^2/\lambda$ are small β is small. An examination of Figs. 4.3 and 4.4 and equation (4.139) leads to the expression

$$\psi^{++}(\zeta,\underline{\rho}_d) \approx \frac{1 + \left(1 - \frac{\phi_A}{2}\right)\beta(1 - \zeta)}{1 + \left(1 - \frac{\phi_A}{2}\right)\beta} \quad . \tag{4.155}$$

The integral terms in (4.139) are negligible. Comparing Figs. 4.3 and 4.5 for the same relative parameter size as discussed above, since the ψ^{--} equation driving term, $\phi_A/2\ (\sinh\omega)/\eta'$, decreases to zero as $\rho_d \to \infty$ at some ρ_d value it is possible that the integral terms in the equation become significant. The integral terms are not important for ρ_d of the order of at least several correlation lengths and within that range of ρ_d values

$$\psi^{--}(\zeta,\underline{\rho}_d) \approx \frac{\phi_A}{2} \frac{(1-\beta)}{1+\left(1-\frac{\phi_A}{2}\right)\beta} . \tag{4.156}$$

Note that as $\sigma_\varepsilon^2/\lambda$ or L/λ approaches zero that $\psi^{++}(\zeta,\underline{\rho}_d) \to 1$ and $\psi^{--}(\zeta,\underline{\rho}_d) \to 0$, a return to free space conditions is exhibited. The situation concerning ψ^{--} which occurs at higher ρ_d values is completely analogous to that which occurs for large $\sigma_\varepsilon^2/\lambda$ at lower ρ_d values. Comparisons between coherence functions can be made for various parameters if they are normalized to one at $\rho_d = 0$. When ρ_d becomes infinite the normalized forward propagating coherence function is given from (4.155) by

$$\frac{\psi^{++}(1,\infty)}{\psi^{++}(1,0)} = \frac{1}{2} \frac{2+\beta}{1+\beta} . \tag{4.157}$$

Since β is small (4.157) is almost one indicating that the forward scatter field is almost completely coherent. Since $\sigma_\varepsilon^2/\lambda$ and L/λ are small most of the signal incident on the random medium is passing through unperturbed and will always be correlated with itself. Figures 4.6 through 4.13 are for a Gaussian permittivity correlation while Fig. 4.14 and 4.15 consider exponential correlation. Figure 4.6 is a plot of normalized forward scatter coherence for a small $\sigma_\varepsilon^2/\lambda$. The effects close to ρ_d equals zero where the plots

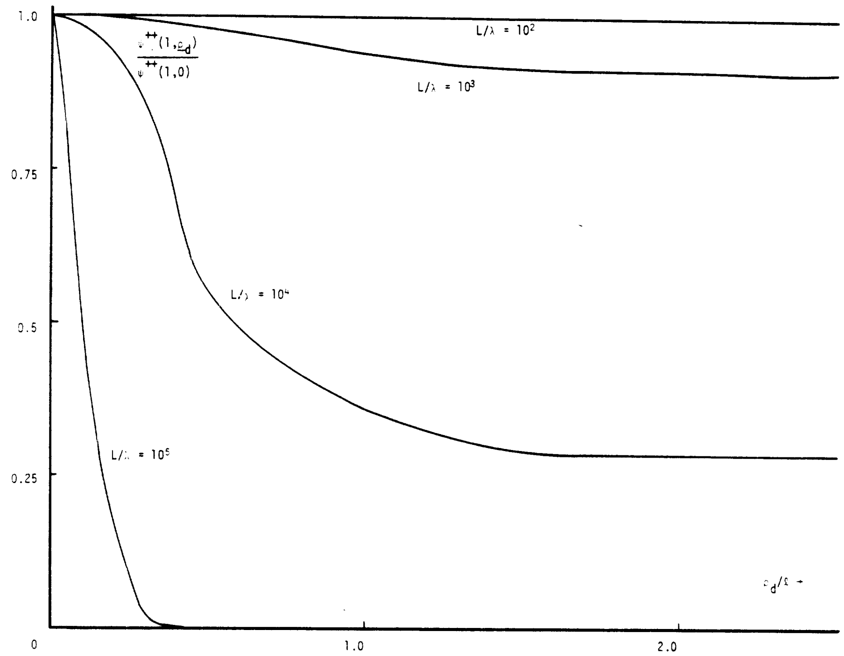


Fig. 4.6 Forward Scatter Coherence Function: Gaussian Permittivity Correlation. ($\sigma_\varepsilon^2/\lambda = 10^{-5}$)

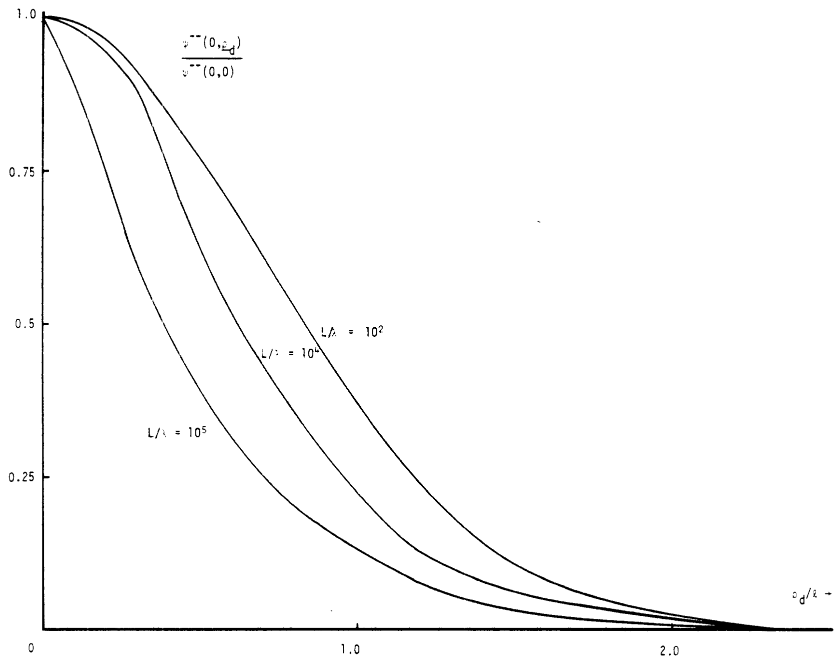


Fig. 4.7 Backscatter Coherence Function: Gaussian Permittivity Correlation. ($\sigma_\varepsilon^2/\lambda = 10^{-5}$)

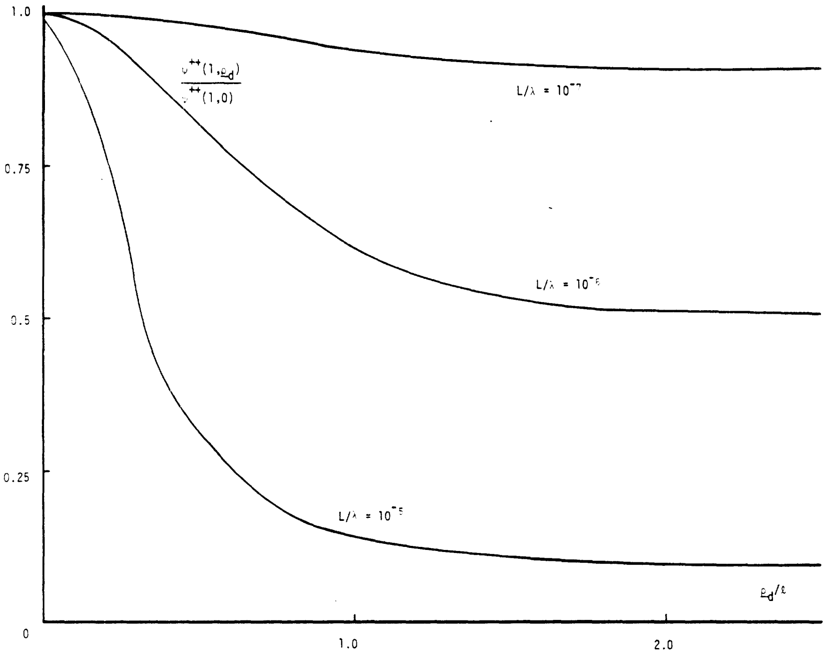


Fig. 4.8 Forward Scatter Coherence Function: Gaussian Permittivity Correlation. $(\sigma_{\varepsilon}^2/\lambda = 10^5)$

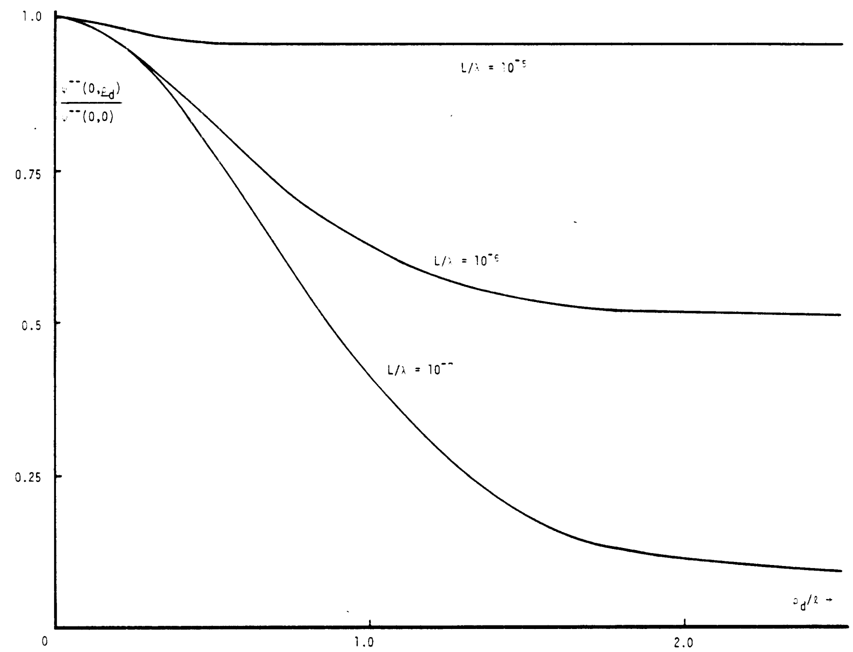


Fig. 4.9 Backscatter Coherence Function: Gaussian Permittivity Correlation. ($\sigma_\varepsilon^2/\lambda = 10^5$)

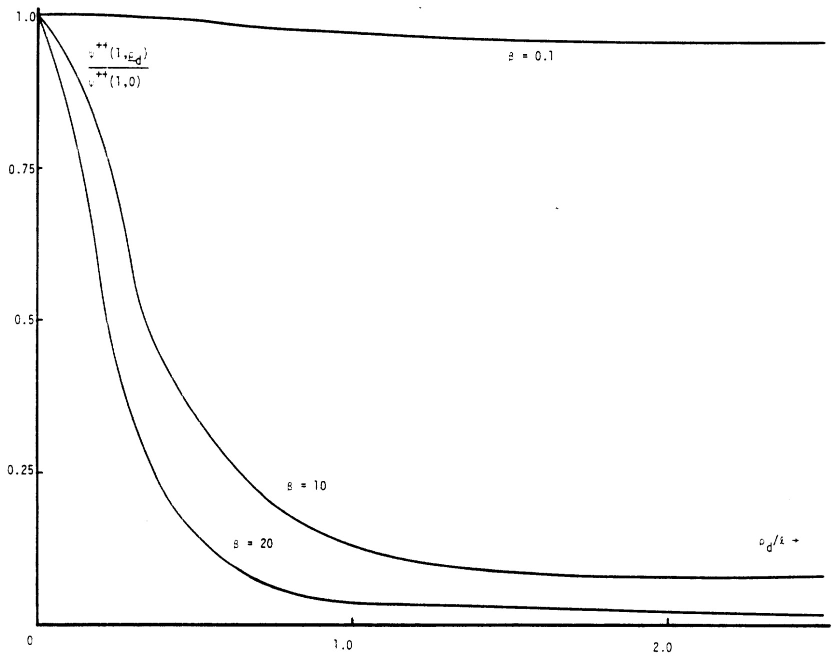


Fig. 4.10 Forward Scatter Coherence Function: Gaussian Permittivity Correlation. ($L/\lambda = 0.125$)

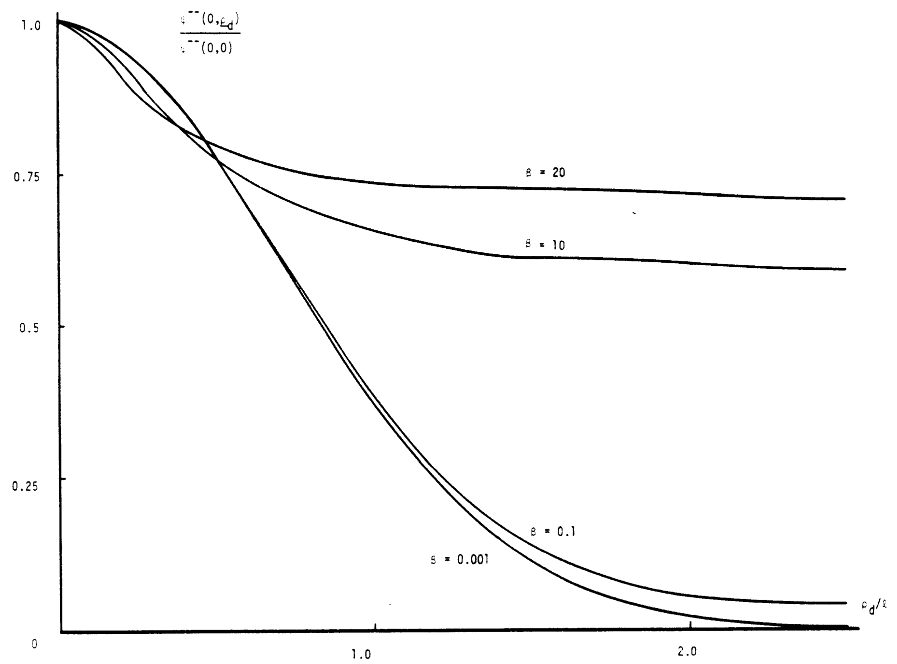


Fig. 4.11 Backscatter Coherence Function: Gaussian Permittivity Correlation. (L/λ = 0.125)

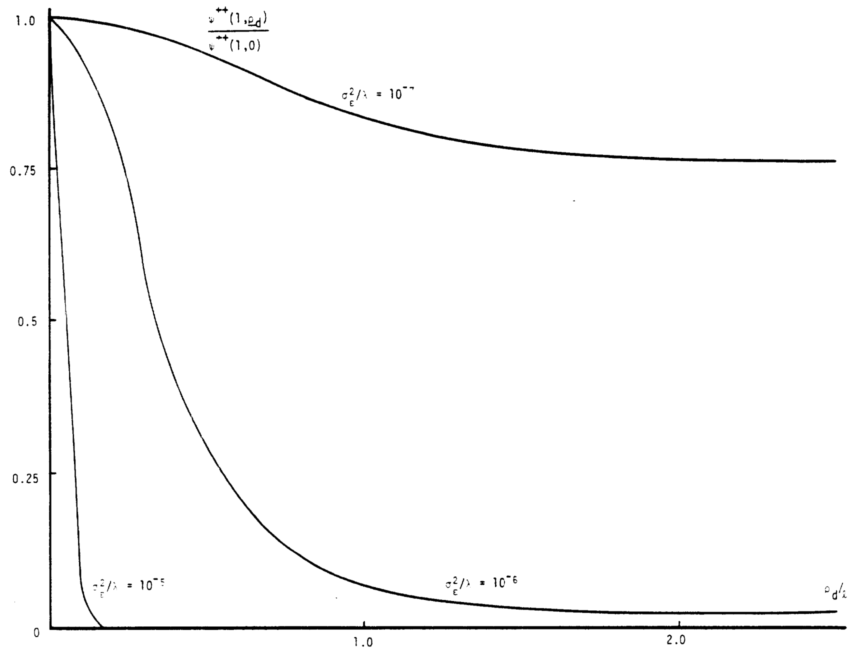


Fig. 4.12 Forward Scatter Coherence Function: Gaussian Permittivity Correlation. (L/λ = 250,000)

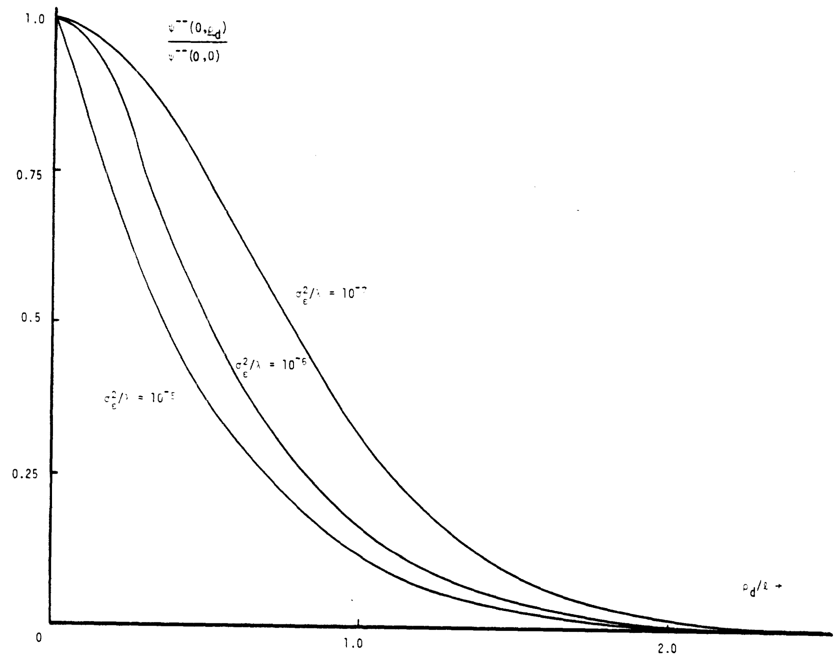


Fig. 4.13 Backscatter Coherence Function: Gaussian Permittivity Correlation. (L/λ = 250,000)

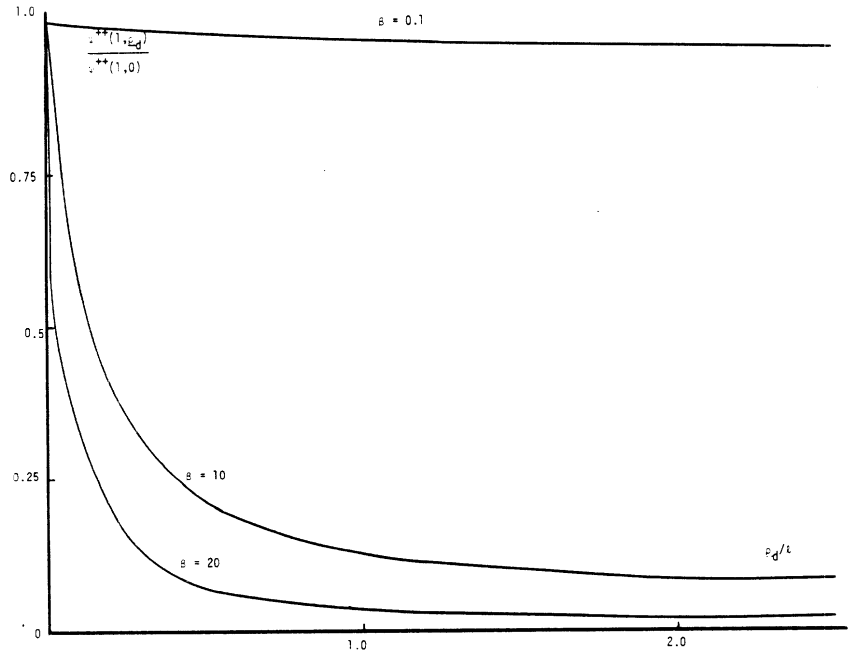


Fig. 4.14 Forward Scatter Coherence Function: Exponential Permittivity Correlation. (L/λ = 0.125)

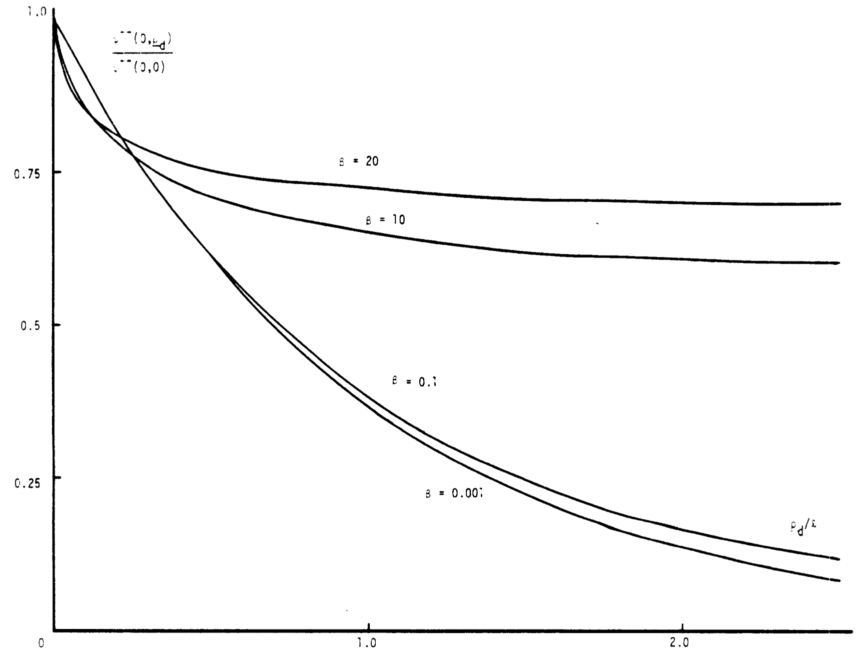


Fig. 4.15 Backscatter Coherence Function: Exponential Permittivity Correlation. (L/λ = 0.125)

change most significantly with ρ_d are effects due to the correlation size of the permittivity fluctuations and can be attributed to a decrease in signal coherence with increasing ρ_d of the fluctuating signal component. The asymptotic portions of the curves occur above some transverse separation distance, ρ_d, beyond which fluctuating field components have little in common with each other and is due to the average field component. The asymptote of the curves is equal to the portion of the total forward scatter signal energy contained in the average forward scatter field. Using equation (3.63) the normalized energy in the average forward scatter signal component is given by

$$\left(\frac{2+\beta}{2}\right) e^{-\beta} \quad . \tag{4.158}$$

The above equation compares with the asymptotic portions of Fig. 4.6. The portion of Fig. 4.6 where effects of the fluctuating signal components are most pronounced indicates that coherence decreases with increasing slab thicknes at a fixed ρ_d. This is due to the cumulative effect of random inhomogeneities in traversing from one z plane to the next. The decrease in asymptotic portions of the curve with increasing L/λ indicates that increasing amounts of average transmitted field is being backscattered. Figure 4.7 is a plot of the backscatter coherence function where the dominant effect illustrated is that of the decorrelation of the fluctuating backscatter field with increasing ρ_d.

At some very large value of ρ_d there is a small but nonzero level of coherence due to the extremely small average component of backscatter field. Backscatter coherence at some fixed ρ_d decreases with increasing slab thickness due to the increased randomness of contributions to the fluctuating backscatter field as these contributions travel over larger distances through the random medium. When the slab thickness, L/λ, is increased without limit for some small fixed value of $\sigma_\varepsilon^2/\lambda$ the backscatter coherence approaches a lower limit below which the coherence does not fall for further increases in L/λ. The limit is given by

$$\frac{\phi_A}{2} \frac{1}{(1 - \phi_A)^{1/2} + (1 - \phi_A)} \tag{4.159}$$

which can be demonstrated by writing ψ^{--} and ψ^{++} in a series expansion in terms of $1/2kL$. What is occurring is that only those fluctuating signal components relatively close to the plane of incidence of $z = 0$ contribute to the backscatter coherence while fluctuating backscatter signal components generated at some distance from the slab interface become completely uncorrelated by virtue of their propagation through the random slab. The demonstration of equation (4.159) has an intermediate result that

$$\lim_{kL \to \infty} \psi^{++}(\zeta,\underline{\rho}_d) = \psi^{--}(\zeta,\underline{\rho}_d) \to 0$$

for all $\zeta \neq 0$ consistent with the above interpretation.

When the variance of permittivity fluctuations is increased substantially the forward and backscatter coherence relationships plotted in Fig. 4.8 and 4.10 result. Figure 4.8 depicts the same behavior as Fig. 4.6, in other words, transmitted signal coherence decreases at a fixed ρ_d when slab thickness increases. The reason for the substantially different ranges of L/λ plotted in Figs. 4.6 and 4.8 is that to plot Fig. 4.6 for $10^{-7} \leq L/\lambda \leq 10^{-5}$ would result in a plot equal to unity for all ρ_d and to plot Fig. 4.8 for $10^3 \leq L/\lambda \leq 10^5$ is impossible because of computational limitations. The great disparity in $\sigma_\varepsilon^2/\lambda$ between Fig. 4.6 and 4.8, was selected to emphasize the propagation difference between small and large σ_ε^2. When $L/\lambda = 10^{-7}$ in Fig. 4.8, the transmitted field has a large coherent component. Increasing slab thickness decreases the transmitted coherent field. The portion of the transmitted energy in the transmitted coherent field is given by

$$\left(\frac{\beta + 2}{2}\right) \frac{8Y^2}{[(Y^2 + Y)^2 + (1 + Y)^2]\, e^{2kLY} + [(Y^2 - Y)^2 + (1 - Y)^2]e^{-2kLY} \begin{array}{l} -2[(Y^2 - Y)(Y^2 + Y) + (1 - Y)(1 + Y)\cos 2kLY \\ + 2L^{-}(1 + Y)(Y^2 - Y) - (Y^2 + Y)(1 - Y)]\sin 2kLY \end{array}} \tag{4.160}$$

where

$$Y = \left(\frac{\pi}{2} \frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2} . \tag{4.161}$$

Equation (4.160) is the magnitude squared of (3.59) at $\zeta = 1$ with

$$\left(1 + \frac{i\pi\sigma_\varepsilon^2}{\lambda}\right)^{1/2} \approx \left(\frac{\pi}{2}\,\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2} (1 + i)$$

normalized to $2/(2 + \beta)$. Calculation of equation (4.160) for $\sigma_\varepsilon^2/\lambda = 10^5$ and $L/\lambda = 10^{-7}$, 10^{-6} and 10^{-5} yields excellent agreement with Fig. 4.8 at $\rho_d = 2.5$. Figure 4.9 is a plot of backscatter coherence for large $\sigma_\varepsilon^2/\lambda$. When $L/\lambda = 10^{-7}$ there is a small component of coherent backscatter. The portion of the total backscatter energy contained in the coherent backscatter field is given by

$$\left(\frac{2 + \beta}{\beta}\right) Y^4$$

$$\cdot \frac{e^{-2kLY} + e^{2kLY} - 2\cos 2kLY}{\begin{aligned}&[(Y^2+Y) + (1+Y)^2]e^{2kLY} + [(Y^2-Y)^2 + (1-Y)^2]e^{-2kLY} \\ &- 2[(Y^2-Y)(Y^2+Y) + (1-Y)(1+Y)]\cos 2kLY \\ &+ 2[(L + Y)(Y - Y) - (Y + Y)(1 - Y)]\sin 2kLY\end{aligned}} \quad , \qquad (4.162)$$

where Y is as given in equation (4.161). Equation (4.162) is the magnitude squared of (3.60) at $\zeta = 0$ with

$$\left(1 + \frac{\pi\sigma_\varepsilon^2}{\lambda}\right)^{1/2} \approx \left(\frac{\pi}{2}\,\frac{\sigma_\varepsilon^2}{\lambda}\right) (1 + i) \quad .$$

Calculation of (4.162) for $L/\lambda = 10^{-7}$ and $\sigma_\varepsilon^2/\lambda = 10^5$ provides excellent agreement with Fig. 4.9 at $\rho_d = 2.5$. Figure 4.9 demonstrates that as slab thickness increases when $\sigma_\varepsilon^2/\lambda$ is large the backscatter coherence increases. The reason for this behavior is that the backscatter of incident energy resulting in average backscatter field occurs over some small but finite depth into the medium. The range of slab thicknesses plotted is within this depth of penetration resulting in increasing average backscatter with increasing L/λ and consequently increasing coherence. The increase in coherence with increasing slab thickness over a range of thin slabs also occurs for weak $\sigma_\varepsilon^2/\lambda$. When $\sigma_\varepsilon^2/\lambda$ is large and $L/\lambda \gtrsim 1$ (4.162) is given approximately by

$$\frac{\frac{\pi}{2}\,\frac{\sigma_\varepsilon^2}{\lambda}}{\left[\left(\frac{\pi}{2}\,\frac{\sigma_\varepsilon^2}{\lambda}\right)^{1/2} + 1\right]^2} \quad . \tag{4.163}$$

Note that for large $\sigma_\varepsilon^2/\lambda$ (4.163) is almost one. There is a very strong component of average backscatter field which almost completely dominates any fluctuating signal component. Equation (4.163) demonstrates that when L/λ is some finite value and $\sigma_\varepsilon^2/\lambda$ is large increasing L/λ further has no effect on the upper limit of backscatter coherence. Most of the incident energy is reflected as average backscatter field. Figure 4.10 depicts the normalized forward scatter coherence for a thin layer when β or $\sigma_\varepsilon^2/\lambda$ is increased. When $\beta = 0.1$ there is a strong coherent component of transmitted field since most of the energy incident upon the medium is passing through unperturbed. When β increases the discontinuity in free space provided by the presence of the medium increases resulting in a greater average component of backscatter field. The result is a decrease in forward scatter coherence. The value of the curve at $\rho_d/\ell = 2.5$ for $\beta = 0.1$ is given by (4.158). $\beta = 10$ and 20 require a more difficult computation using (3.59) with (3.61). Figure 4.11 plots backscatter coherence for a thin slab over a range of β. When the $\sigma_\varepsilon^2/\lambda$ of a thin slab is increased an increasing amount of the average forward propagating field is reflected at the discontinuity formed by the interface of the random medium. Since less coherent field penetrates the slab less fluctuating backscatter energy is produced resulting in a decrease in the fluctuating portion of the curve coincident with an increase in the asymptotic portion due to an increase in $\sigma_\varepsilon^2/\lambda$ or β. At ρ_d close to zero effects due to a decrease in fluctuating field predominate while at large ρ_d effects due to increases in coherent backscatter field predominate. Note that at some intermediate value of ρ_d there is a transition region where with increasing β fluctuating effects first predominate then coherent backscatter effects predominate. Holding ρ_d constant at any value and sweeping through the range of $\sigma_\varepsilon^2/\lambda$ will at first exhibit a predominance of fluctuating effects transitioning into a situation where average backscatter effects are more important. When $\sigma_\varepsilon^2/\lambda \to \infty$ it is seen from (3.71) and (3.72) that all the energy incident upon the slab is reflected as coherent energy implying that $(\psi^{--}(0,\underline{\rho}_d)/\psi^{--}(0,0)) \to 1$ for all ρ_d. Figure 4.12 plots the forward scatter coherence for a thick slab for a range of $\sigma_\varepsilon^2/\lambda$ values. The same relationship is observed in Fig. 4.12 as in Fig. 4.10, increasing $\sigma_\varepsilon^2/\lambda$ results in decreasing coherence. The plot depicts coherence effects related to the correlation of permittivity fluctuations and due to the average transmitted field. Since when $\sigma_\varepsilon^2/\lambda$ increases the average backscatter field increases less average signal is transmitted resulting in a decreased forward scatter coherence. The curves in Fig. 4.12 are asymptotic to the normalized energy in the average forward scatter field as can be determined by comparing the values of the curves at $\rho_d/\ell = 2.5$ to the values calculated by using (4.158). Figure 4.13 is a plot of backscatter coherence for a thick slab for a range of $\sigma_\varepsilon^2/\lambda$. Since average backscatter field increases with increasing $\sigma_\varepsilon^2/\lambda$ the fluctuating part of the backscatter field decreases. The fact that the average backscatter field increases for $\sigma_\varepsilon^2/\lambda$ increasing over a range of small values can be seen by examining

$$|\langle u^-\rangle|^2 = \frac{1}{4}\,\frac{Y^4}{Y^4+1}\left(1 + e^{-4kLY^2} - 2\,e^{-2kLY^2}\cos 2kL\right) \qquad (4.164)$$

for increasing $\sigma_\varepsilon^2/\lambda$. Note that, as discussed earlier for small $\sigma_\varepsilon^2/\lambda$ with increases in L/λ, there is a lower limit to the backscatter coherence given by equation (4.159). This limit also holds for increases of $\sigma_\varepsilon^2/\lambda$ over a range of small values.

Increases in $\sigma_\varepsilon^2/\lambda$ will produce no further changes in backscatter coherence until the average backscatter field becomes significant with respect to the fluctuating field, at which point the coherence will increase with further $\sigma_\varepsilon^2/\lambda$ increases. When $\sigma_\varepsilon^2/\lambda \to \infty$, as seen before, all the incident energy is reflected as coherent backscatter field at the discontinuity at $z = 0$. The reason for the lower limit to the decrease in coherence is that when $\sigma_\varepsilon^2/\lambda$ is increased within a range of small values the average forward propagating field is at some $\sigma_\varepsilon^2/\lambda$ value confined primarily to the interface at $z = 0$. However, forward propagating fluctuating fields exist within the random volume. Only those forward propagating fluctuating fields reflected within a finite distance of the interface will add coherently. Those fields a greater distance away are statistically independent and contribute nothing to backscatter coherence. Figures 4.14 and 4.15 plot the forward and backscatter coherence for a thin slab for increasing $\sigma_\varepsilon^2/\lambda$ for an exponential permittivity correlation function. A comparison of Fig.4.14 and 4.15 with Fig. 4.10 and 4.11 shows that except for slight differences in shape the relationships identified in Fig. 4.10 and 4.11 are preserved in Fig. 4.14 and 4.15. Note that the Gaussian and exponential correlation functions are equal at three points $\rho_d = 0, \ell$ and ∞. At these three points ψ^{++} and ψ^{--} must be equal for either function. Since the ψ^{++} and ψ^{--} integral equations presented in this work do not strongly depend on the slope of the correlation function, in other words, there are no transverse derivatives, the two correlation functions will provide the same results. No further computations using exponential correlation functions will be performed.

CHAPTER V. SUMMARY AND RECOMMENDATIONS

A study of backscatter from a random medium based on the random continuum model and using functional equation techniques as presented in this thesis leads to the conclusion that two phenomena contribute to backscatter effects. The first phenomenon is due to the occurrence of the discontinuity in free space as presented by the random volume. The discontinuity gives rise to an average backscatter field component whose magnitude is proportional to the variance of permittivity fluctuations of the random material. The greater the permittivity variance the greater the average backscatter field. The second phenomenon is the generation of a fluctuating backscatter field component due to propagation of average fields over some distance through the medium.

Small permittivity fluctuations give rise to negligible average backscatter field components and fluctuating backscatter field dominates. As the slab thickness is increased backscatter fluctuating energy increases until at $kL \to \infty$ all the incident energy is returned as a fluctuating signal. As the slab thickness is increased the backscatter coherence decreases. Only those backscatter fluctuating signals within some finite distance of the surface of a very thick random slab will contribute to the backscatter cohrence. Fluctuating signal components arising from deep within the volume become independent by virtue of their propagation to the surface.

Increasing permittivity fluctuations increases average backscatter field to a point where it is no longer dominated by the fluctuating part of the backscatter field. The energy in the backscatter field now consists of the energy in the fluctuating and average backscatter fields. As slab thickness increases the energy in the average and fluctuating backscatter fields increases. The average backscatter field is the average forward scatter field that has reflected within some distance from the medium boundary into the random slab dependent on the variance of permittivity fluctuations. When the slab is thin reflection occurs throughout the entire slab, when the slab is thick this reflection is confined to be close to the slab interface when compared with the entire slab thickness. The latter situation results in an average backscatter field independent of slab thickness, proportional only to the variance of permittivity fluctuations. The energy in the fluctuating backscatter field increases with increasing slab thickness but the coherence of the backscatter field as observed at the surface of the random slab decreases. As discussed in the previous paragraph only those backscatter fluctuating signals within some finite distance of the surface of a very thick random slab will contribute to backscatter coherence and consequently there is a lower limit to the decrease in coherence with increasing slab thickness.

Very large permittivity fluctuations give rise to a situation in which there is very little fluctuating backscatter field and the average backscatter field dominates. Nearly all of the backscatter energy is contained in the average backscatter field. The average forward propagating field is reflected within a thin layer of the surface of the random slab and consequently the average backscatter field is independent of slab thickness for slabs several layers thick. Since the backscatter energy is essentially all contained in the average backscatter field the backscatter energy is almost completely coherent.

A comment concerning the interpretation of the work presented in this thesis is in order. It is not possible to say what order of multiple interaction is included within the formulation used in this work. It can only be stated that backscatter is included and that some level of multiple interaction is included. Previous work using functional techniques to derive closed equations for the moment of interest began with the assumption that backscatter is negligible. The interpretation then is that the infinite number of possible forward interactions are included in the formulation. The term applied to such a formulation is the cumulative forward scatter approach. The work of Ito and Adachi [14] which included backscatter but was not based on functional techniques made use of Green's functions which included the cumulative forward and cumulative backscatter interactions but found it necessary to iterate and neglect higher order terms. The result was readily interpretable as a cumulative forward scatter single backscatter approach. The work presented here started with an assumption on the form of the functional derivative which made it possible to include backscatter. The form of the functional derivative used can be shown to be the lowest order terms in an infinite series expansion. Consequently, it cannot be stated that all higher order forward and backscatter interactions are accounted for in the results presented here. Understanding the nature of the higher order terms in the expression for the functional derivative can lend further insight into types of interactions included here. The study of these higher order terms and the extension of this theory to include them provides a possible avenue for future efforts along these lines.

APPENDIX A. NUMERICAL COMPUTATION OF COHERENCE EQUATIONS

The complicated nature of the integral equations for the forward going and backscattered coherence necessitate a numerical approach to their solution. The approach used is a version of the method of moments where the basis functions are pulses and the weighting functions are delta functions. The objective of the technique is to reduce a linear integral equation to a set of simultaneous equations which can be solved for the desired unknowns. Many texts are available which describe the method and consequently little background information will be provided here. The book by Jawson and Symm provides a particularly clear presentation of the method of moments with many useful examples.

The equations for $\psi^{++}(y,\underline{\rho}_d)$ and $\psi^{--}(y,\underline{\rho}_d)$, as given by (4.139) and (4.144), valid for a fixed ρ_d, contain single integral terms which are approximated by segmenting their domains of integration into unitary intervals. The intervals are taken to be sufficiently small such that the value of the unknown dependent variable over the region covered by the cell is essentially that at the cell center. The integral terms in the integral equations for ψ^{++} and ψ^{--} are approximated by adding up the integrals over the cells with the unknown dependent variable constant over the cells. An integral term from 0 to ζ is approximated as

$$\int_0^{\zeta} d\alpha\ \psi^{++}(\alpha,\underline{\rho}_d)\sinh\omega\alpha \approx \sum_{j=1}^{m-1} \psi^{++}(j\Delta)\int_{(j-1/2)\Delta}^{(j+1/2)\Delta} d\alpha\ \sinh\omega\alpha$$

$$+\ \psi^{++}(0)\int_0^{\Delta/z} d\alpha\ \sinh\omega\alpha\ +\ \psi^{++}(m\Delta)\int_{(m-1/2)\Delta}^{m\Delta} d\alpha\ \sinh\omega\alpha \qquad \text{(A.1)}$$

where $m\Delta$ is the sampled version of ζ. N1, the number of parallel slabs into which the random slab is divided is related to Δ by $m = 1/\Delta$. The notation $\psi(\ )$ in (A.1) and in what follows omits the $\underline{\rho}_d$. $\psi(\ )$, however, does depend on $\underline{\rho}_d$. The integral is approximated such that the unknown dependent variable values at the endpoints of the domain of integration appear explicitly in the solution. The last two terms in (A.1) result since the value of the unknown dependent variable is taken at the cell center. The integrals in (A.1) are easily performed resulting in

$$\int_0^{\zeta} d\alpha\ \psi^{++}(\alpha,\underline{\rho}_d)\ \sinh\omega\alpha \approx \frac{1}{\omega}\sum_{j=1}^{m-1} \psi^{++}(j\Delta)\Big(\cosh\omega(j+1/2)\Delta$$

$$-\cosh\omega(j-1/2)\Delta\Big) + \frac{1}{\omega}\ \psi^{++}(0)\ \left(\cosh\frac{\omega\Delta}{2} - 1\right) + \frac{1}{\omega}\ \psi^{++}(m\Delta)$$

$$\cdot\left(\cosh\omega m\Delta - \cosh\omega\left(m-\frac{1}{2}\right)\Delta\right). \qquad \text{(A.2)}$$

The approximation of integrals with domains ζ to 1 are essentially the same as those for 0 to ζ.

Examination of the integral equations for ψ^{++} and ψ^{--} reveals that there are eight single integrals to be approximated

$$I_1 = \int_0^z d\alpha\ \psi(\alpha,\underline{\rho}_d)\sinh \omega\alpha \tag{A.3}$$

$$I_2 = \int_0^z d\alpha\ \psi(\alpha,\underline{\rho}_d)\sin 2kL\alpha \tag{A.4}$$

$$I_3 = \int_0^z d\alpha\ \psi(\alpha,\underline{\rho}_d)\cosh \omega\alpha \tag{A.5}$$

$$I_4 = \int_0^z d\alpha\ \psi(\alpha,\underline{\rho}_d)\cos 2kL\alpha \tag{A.6}$$

$$I_5 = \int_z^1 d\alpha\ \psi(\alpha,\underline{\rho}_d)\sin 2kL\alpha \tag{A.7}$$

$$I_6 = \int_z^1 d\alpha\ \psi(\alpha,\underline{\rho}_d)\cosh 2kL\alpha \tag{A.8}$$

$$I_7 = \int_z^1 d\alpha\ \psi(\alpha,\underline{\rho}_d)\cosh \omega(1-\alpha) \tag{A.9}$$

$$I_8 = \int_z^1 d\alpha\ \psi(\alpha,\underline{\rho}_d)\sinh \omega(1-\alpha)\ . \tag{A.10}$$

A plus or minus superscript notation will be used to indicate whether the unknown dependent variable is ψ^{++} or ψ^{--}, respectively. Now I_1 is approximated as (A.2) and I_2 through I_8 are given by

$$I_2 \approx -\frac{1}{2kL} \sum_{j=1}^{m-1} \psi(j\Delta)\left(\cos 2kL\, j + \frac{1}{2}\Delta - \cos 2kL\, j - \frac{1}{2}\Delta\right)$$

$$-\frac{1}{2kL}\psi(0)(\cos kL - 1) - \frac{1}{2kL}\psi(m\Delta)\left(\cos 2kLm\Delta - \cos 2kL\, m - \frac{1}{2}\Delta\right) \quad (A.11)$$

$$I_3 \approx \frac{1}{\omega} \sum_{j=1}^{m-1} \psi(j\Delta)\left(\sinh \omega\, j + \frac{1}{2}\Delta - \sinh \omega\, j - \frac{1}{2}\Delta\right)$$

$$+\frac{1}{\omega}\psi(0)\sinh\frac{\omega\Delta}{2} + \frac{1}{\omega}\psi(m\Delta)\left(\sinh \omega m\Delta - \sinh \omega\, m - \frac{1}{2}\Delta\right) \quad (A.12)$$

$$I_4 \approx \frac{1}{2kL} \sum_{j=1}^{m-1} \psi(j\Delta)\left(\sin 2kL\, j + \frac{1}{2}\Delta - \sin 2kL\, j - \frac{1}{2}\Delta\right)$$

$$+\frac{1}{2kL}\psi(0)\sin kL\Delta + \frac{1}{2kL}\psi(m\Delta)\left(\sin 2kLm\Delta - \sin 2kL\, m - \frac{1}{2}\Delta\right)$$

(A.13)

$$I_5 \approx -\frac{1}{2kL} \sum_{j=m+1}^{N1-1} \psi(j\Delta) \left(\cos 2kL \; j + \frac{1}{2} \; \Delta - \cos 2kL \; j - \frac{1}{2} \; \Delta\right)$$

$$- \frac{1}{2kL} \; \psi(m\Delta) \left(\cos 2kL \; m + \frac{1}{2} \; \Delta - \cos 2kLm\Delta\right) - \frac{1}{2kL} \; \psi(N\Delta)$$

$$\cdot \left(\cos 2kLN1\Delta - \cos 2kL \; N1 - \frac{1}{2} \; \Delta\right) \qquad (A.14)$$

$$I_6 \approx \frac{1}{2kL} \sum_{j=m+1}^{N1-1} \psi(j\Delta) \left(\sin 2kL \; j + \frac{1}{2} \; \Delta - \sin 2kL \; j - \frac{1}{2} \; \Delta\right)$$

$$= \frac{1}{2kL} \psi(m\Delta) \left(\sin 2kL \; m + \frac{1}{2} \; \Delta - \sin 2kLm\Delta\right) + \frac{1}{2kL} \psi(N1\Delta)$$

$$\cdot \left(\sin 2kLN1\Delta - \sin 2kL \; N1 - \frac{1}{2} \; \Delta\right) \qquad (A.15)$$

$$I_7 \approx -\frac{1}{\omega} \sum_{j=m+1}^{N1-1} \psi(j\Delta) \left(\sinh \omega \; N1 - j - \frac{1}{2} \; \Delta - \sinh \omega \; N1 - j + \frac{1}{2} \; \Delta\right)$$

$$- \frac{1}{\omega} \; \psi(m\Delta) \left(\sinh \omega \; N1 - m - \frac{1}{2} \; \Delta - \sinh \; \omega(nL - m)\Delta\right)$$

$$+ \frac{1}{\omega} \; \psi(N1\Delta) \quad \sinh \frac{\omega\Delta}{2} \qquad (A.16)$$

$$I_8 \approx -\frac{1}{\omega} \sum_{j=m+1}^{N1-1} \psi(j\Delta) \quad \cosh \omega \quad N1 - j - \frac{1}{2} \quad \Delta \quad - \cosh \omega \quad N1 - j + \frac{1}{2} \quad \Delta$$

$$- \frac{1}{\omega} \quad \psi(m\Delta) \quad \cosh \omega \quad N1 - m - \frac{1}{2} \quad \Delta \quad - \cosh \omega \; (nL - m)\Delta$$

$$- \frac{1}{\omega} \psi(N1\Delta) \quad 1 - \cos \frac{\omega\Delta}{2} \quad . \qquad \text{(A.17)}$$

Using the I_1 through I_8 notation the integral equations for ψ^{++} and ψ^{--} are written as

$$\psi^{++}(m\Delta, \underline{\rho}_d) = \frac{1}{2} \; [\chi(m\Delta, \underline{\rho}_d)(\omega I_1^+ + 2kLI_2^+ + \beta I_3^+ - \beta I_4^+) + (\beta \sin 2kL\Delta m$$

$$+ 2kL(\chi(m\Delta, \underline{\rho}_d) - \cos 2kL(m\Delta) \quad I_5^+ \; - (\beta(\chi(m\Delta, \underline{\rho}_d) - \cos 2kL)$$

$$- 2kL \sin 2kLm\Delta)I_6^+ - \frac{\phi_A}{2} \; \frac{\sinh \omega m\Delta}{\eta'} \; (\beta I_1^+ + \omega I_8^+)$$

$$- \chi(m\Delta, \underline{\rho}_d)(\omega I_1^- + 2kLI_2^- + \beta I_3^- - \beta I_4) - (\beta \sin^2 kLm\Delta + 2kL(\chi(m\Delta, \underline{\rho}_d)$$

$$- \cos 2kLm\Delta))I_5^- + (\beta(\chi(m\Delta, \underline{\rho}_d) - \cos 2kLm\Delta) - 2kL \sin^2 kLm\Delta)I_6^-$$

$$+ \frac{\phi_A}{2} \frac{\sinh \omega m\Delta}{\eta'} \; (\beta I_7^- + \beta I_8^-)] + \chi(m\Delta, \underline{\rho}_d)\psi^{++}(0, \underline{\rho}_d) \qquad \text{(A.18)}$$

and

$$\psi^{--}(m\Delta,\underline{\rho}_d) = \frac{1}{2}\kappa\left[\frac{\phi_A}{2}\frac{\sinh\omega(N1-m)\Delta}{\eta'}(\omega I_1^+ + 2kLI_2^+ + \beta I_3^+ - \beta I_4^+\right.$$

$$+ \left(2kL\frac{\phi_A}{2}\frac{\sinh\omega(N1-m)\Delta}{\eta'} + \cos 2kLM\Delta + \beta\sin 2kLm\Delta\right) I_5^+$$

$$- \left(2kL\sin 2kLm\Delta + \beta\frac{\phi_A}{2}\frac{\sinh\omega(N1-m)\Delta}{\eta'} - \cos 2kLM\Delta\right)$$

$$- \hat{\chi}(m\Delta,\underline{\rho}_d)(\beta I_7^+ + \beta I_8^+) - \frac{\phi_A}{2}\frac{\sinh\omega(n1-m)\Delta}{\eta'}(\omega I_1^- + 2kLI_2^- + \beta I_3^- - \beta I_4^-)$$

$$- \left(2kL\frac{\phi_A}{2}\frac{\sinh\omega(N1-m)\Delta}{\eta'} + \cos 2kLm\Delta) + \beta\sin 2kLm\Delta\right) I_5^-$$

$$+ \left(2kL\sin 2kLm\Delta + \beta\frac{\phi_A}{2}\frac{\sinh\omega(N1-m)\Delta}{\eta'} - \cos 2kLm\Delta\right) I_6^-$$

$$\left. + \hat{\chi}(m\Delta,\underline{\rho}_d)(\beta I_7^- + \beta I_8^-)\right] + \frac{\phi_A}{2}\frac{\sinh\omega(N1-m)\Delta}{\eta'}\psi^{++}(0,\underline{\rho}_d) \quad . \qquad (A.19)$$

The integer m in (A.18) and (A.19) ranges from 1 to N1 - 1, and equals zero or N1 must be considered separately and are given by

$$\psi^{++}(N1\Delta,\underline{\rho}_d) = \frac{1}{2}\kappa\,(N1\Delta,\underline{\rho}_d)\left[\omega I^+ + 2kLI^+ + \beta I_3^+ - \beta I_4^+\right.$$

$$\left. - \omega I_1^- - 2kLI_2^- - \beta I_3^- + \beta I_4^-\right] + \chi(N1\Delta,\underline{\rho}_d)\psi^{++}(0,\underline{\rho}_d) \qquad (A.20)$$

and

$$\psi^{--}(0,\underline{\rho}_d) = \frac{1}{2}\kappa\Bigg[2kL\left(\frac{\phi_A}{2}\,\frac{\sinh\omega}{\eta'}+1\right)I_5^+ - \beta\left(\frac{\phi_A}{2}\,\frac{\sinh\omega}{\eta'}-1\right)I_6^+$$

$$-\hat{\chi}(0,\underline{\rho}_d)(\beta I_7^+ + \omega I_8^+) - 2kL\left(\frac{\phi_A}{2}\,\frac{\sinh\omega}{\eta'}+1\right)I_5^- + \beta\left(\frac{\phi_A}{2}\,\frac{\sinh\omega}{\eta'}-1\right)I_6^-$$

$$+\beta\left(\frac{\phi_A}{2}\,\frac{\sinh\omega}{\eta'}-1\right)I_6^- + \hat{\chi}(0,\underline{\rho}_d)(\beta I_7^- + \omega I_8^-) + \frac{\phi_A}{2}\,\frac{\sinh\omega}{\eta'}\,\psi^{++}(0,\underline{\rho}_d)\Bigg].$$

(A.21)

In the above representation if the lower limit of a summation is greater than the upper limit that term is understood to be zero.

Equations (A.18) through (A.21) form a system of 2N1 simultaneous equations for $\psi^{++}(\Delta,\underline{\rho}_d),\ldots,\psi^{++}(N1\Delta,\underline{\rho}_d)$, $\psi^{--}(0,\underline{\rho}_d),\ldots,\psi^{--}(N1-1)\Delta,\underline{\rho}_d)$ with driving force $\psi^{++}(0,\underline{\rho}_d)$. A program to solve (A.18) through (A.21) for the desired unknowns consists of two elements. The first element loads the coefficients of $\psi^{++}(\Delta,\underline{\rho}_d)$, $\ldots,\psi^{++}(N1\Delta,\underline{\rho}_d)$, $\psi^{--}(0,\underline{\rho}_d),\ldots,\psi^{--}(N1-\Delta,\underline{\rho}_d)$ as determined by (A.18) through (A.21) into the first 2N1 rows and columns and the coefficients of $\psi^{++}(0,\underline{\rho}_d)$ into the 2N1 + 1'th column of a 2N1 by 2N1 + 1 matrix, A1. There is a great deal of symmetry in the coefficient kernels (terms such as cos 2kL $(j + 1/2)\Delta$ - cos 2kL $\cdot(j - 1/2)\Delta$ and sinh $\omega(j + 1/2)\Delta$ - sinh $\omega(j - 1/2)\Delta$) of I_1^+ through I_8^+ and I_1^- through I_8^- and the other coefficient components of (A.18) through (A.21) which can be exploited to ease the programming chore. The second element of the program solves the 2N1 equations represented by A1. Rather than compute the inverse of the first 2N1 rows and columns of A1 and multiply the 2N1 + 1'th column of A1 by the terms in the inverse to arrive at the values of ψ^{++} and ψ^{--}, methods of solving systems of linear equations which realize computational efficiency at the expense of mathematical elegance are employed. The method used is a variation of the Gaussian elimination technique. It consists of an LU decomposition with backward substitution using iterative refinement. Briefly, A, the coefficient matrix consisting of the first 2N1 rows and columns of A1, is factored into a lower triangular matrix, L, and an upper triangular matrix, U, such that LU = A. Then Ax = b can be solved by first solving Ly = b and then Ux = y. Since L and U are triangular the solutions for y and then x can be realized by a simple procedure called a backward substitution. Now the solution vector Ux, will contain roundoff errors. The iterative refinement procedure is used which multiplies the solution vector by A to arrive at an estimate of b which is then used to update the solution vector. The details of these numerical techniques are available in Jones.

The Fortran program used is as follows:

```
C-----------------------------------------------------------------
C PROGRAM  TO SOLVE A SET OF SIMUTANEOUS
C INTEGRAL EQUATIONS FOR THE
C POSITIVE AND NEGATIVE PROPAGATING COHERENCE
C FUNCTIONS  IN A RANDOM SLAB INFINITE IN THE
C X-Y PLANE AND FINITE IN THE Z DIRECTION. A
C PLANE WAVE IS NORMALLY INCIDENT ON THE  SLAB.
C THE SOLUTION USES THE METHOD OF MOMENTS WHERE
C THE FORMULATION HOLDS FOR A FIXED TRANSVERSE
C COORDINATE, RHO. N1 IS THE NUMBER OF SAMPLING
C POINTS IN THE SLAB ALONG THE Z DIRECTION.
C  INPUTS REQUIRED ARE:
C       N1= NUMBER OF SAMPLE POINTS. ALL
C ARGUMENTS OF REAL*8 AND INTEGER
C MUST BE CHANGED. ALL ARGUMENTS
C OF REAL*8 ARE 2N1 EXCEPT THE SECOND ARGUMENT
C OF A1 WHICH IS 2N1+1.THE ARGUMENT OF
C IV IS 2N1.
C       GAMMA= SLAB THICKNESS/ WAVELENGTH
C       RHO= MAGNITUDE OF TRANSVERSE SEPARATION
C       VAR=VARIANCE OF PERMITTIVITY FLUCTUATIONS/
C                                 WAVELENGTH
C       PHSUBA=CORRELATION FUNCTION OF
C                         PERMITTIVITY FLUCTUATIONS
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 A1(192,193),A(192,192),T(192,192),
     1B(192),G(192),R(192)
      INTEGER ADIM,TDIM,IV(192)
      REAL*8 NEW
      N1=96
      N=2*N1
      ADIM=2*N1
      TDIM=2*N1
      M1=2*N1
      M1M=M1+1
C---------------------------------------------------------------
C LOOP 10 INITIALIZES A COEFFICIENT MATRIX,
C A1,  WHICH IS A 2N1 BY 2N1+1 MATRIX TO
C A MATRIX WHICH IS ALL ZEROES EXCEPT FOR THE
C A1(L,L) ENTRIES WHICH ARE ONES.
C---------------------------------------------------------------
      DO 10 L=1,M1
      DO 10 J=1,M1M
      IF (L .NE.J) GO TO 5
      A1(L,J)=1.0
      GO TO 10
    5 A1(L,J)=0.0
   10 CONTINUE
```

```
C-----------------------------------------------------------------
C STORES CONSTANTS USED IN CALCULATION OF
C COEFFICIENT MATRIX.
C-----------------------------------------------------------------
      CL=.002
      PI=3.141592653589793
      VAR=100000.
      GAMMA=.00001
      BETA=2.0*PI**2*GAMMA*VAR
      RHO=.0036
      PHSUBA=DEXP((-1.0)*(RHO**2/CL**2))
      OMEGA=BETA*DSQRT(1.0-PHSUBA)
      X=OMEGA**2+16.0*PI**2*GAMMA**2
      DELTA=1.0/(2*N1)
      OMEGA1=OMEGA*DELTA
      GAMMA1=4.0*PI*GAMMA*DELTA
      NEW=DSQRT(1.0-PHSUBA)*DCOSH(OMEGA)+
     1(1.0-.5*PHSUBA)*DSINH(OMEGA)
      CD=BETA/(4.0*PI*GAMMA)
      C1=.5*OMEGA**2/X
      C2=C1*CD
      C3=.5*OMEGA/X*BETA
C-----------------------------------------------------------------
C LOOP 100 LOADS THE A1 MATRIX. ALL THE ENTRIES
C EXCEPT THE LAST COLUMN ARE COEFFICIENTS
C OF THE DESIRED UNKNOWNS. THE LAST COLUMN IS
C THE DRIVING FORCE.
C-----------------------------------------------------------------
      MN=N1+1
      DO 100 L=1,MN
      ZETA=(DSQRT(1.0-PHSUBA)*DCOSH(OMEGA1*
     1(N1+1.0-L))+(1.0-.5*PHSUBA)
     2*DSINH(OMEGA1*(N1+1.0-L)))/NEW
      PHAS=PHSUBA/(2.0*NEW)*DSINH(OMEGA1*(N1+1.0-L)
      PHASN=PHSUBA/(2.0*NEW)*DSINH(OMEGA1*(L-1.0))
      ZETAN=(DSQRT(1.0-PHSUBA)*DCOSH(OMEGA1*(L-1.0)
     1+(1.0-.5*PHSUBA)*
     2DSINH(OMEGA1*(L-1.0)))/NEW
      ZYN=PHAS+DCOS(GAMMA1*(L-1.0))+
     1CD*DSIN(GAMMA1*(L-1.0))
      ZYNN=CD*(PHAS-DCOS(GAMMA1*(L-1.0)))+
     1DSIN(GAMMA1*(L-1.0))
      PARS=CD*DSIN(GAMMA1*(L-1.0))+ZETA-
     1DCOS(GAMMA1*(L-1.0))
      PARSN=(-1.0)*DSIN(GAMMA1*(L-1.0))+
     1CD*(ZETA-DCOS(GAMMA1*(L-1.0)))
   65 IF (L .EQ. 1) GO TO 74
```

```
C--------------------------------------------------------------
C LOOP 73 COMPRISES COEFFICIENTS FORMED BY
C INTEGRALS I1 THROUGH I4.
C--------------------------------------------------------------
      DO 73 J=1, L
C--------------------------------------------------------------
C COMPUTATION OF COEFFICIENT KERNEL.
C--------------------------------------------------------------
      IF (J .NE. 1) GO TO 66
      ALPHA=.5
      CGAMMA=.5
      GO TO 68
   66 IF (J .NE. L) GO TO 67
      ALPHA=0.0
      CGAMMA=.5
      GO TO 68
   67 ALPHA=.5
      CGAMMA=1.0
   68 ARG1D=OMEGA1*(J-1.0+ALPHA)
      ARG2D=OMEGA1*(J-1.0+ALPHA-CGAMMA)
      ARG3D=GAMMA1*(J-1.0+ALPHA)
      ARG4D=GAMMA1*(J-1.0+ALPHA-CGAMMA)
      D3=DSINH(ARG1D)-DSINH(ARG2D)
      D1=DCOSH(ARG1D)-DCOSH(ARG2D)
      D4=DSIN(ARG3D)-DSIN(ARG4D)
      D2=DCOS(ARG3D)-DCOS(ARG4D)
C--------------------------------------------------------------
C SUBTRACTION OF PARTIAL COEFFICIENT INTO
C COEFFICIENT MATRIX. DOES NOT LOAD THE LAST
C COLUMN OF A1 WHICH IS THE DRIVING FORCE.
C--------------------------------------------------------------
      IF (J .NE. 1) GO TO 70
      A1(L-1,2*N1+1)=A1(L-1,2*N1+1)+
     1ZETA*(C1*(D1-D2-CD*D4)+C3*D3)
      A1(L-1,N1+1)=A1(L-1,N1+1)+
     1ZETA*(C1*(D1-D2-CD*D4)+C3*D3)
      IF (L .EQ. MN) GO TO 69
      A1(L+N1,2*N1+1)=A1(L+N1,2*N1+1)+
     1PHAS*(C1*(D1-D2-CD*D4)+C3*D3)
      A1(L+N1,N1+1)=A1(L+N1,N1+1)+
     1PHAS*(C1*(D1-D2-CD*D4)+C3*D3)
   69 GO TO 73
   70 A1(L-1,J-1)=A1(L-1,J-1)-
     1ZETA*(C1*(D1-D2-CD*D4)+C3*D3)
      IF (L .NE. MN) GO TO 71
      IF (J .NE. MN)  GO TO 72
      GO TO 73
   71 A1(L+N1,J-1)=A1(L+N1,J-1) -
     1PHAS*(C1*(D1-D2-CD*D4)+C3*D3)
      A1(L+N1,J+N1)=A1(L+N1,J+N1) +
     1PHAS*(C1*(D1-D2-CD*D4)+C3*D3)
```

```
      72 A1(L-1,J+N1)=A1(L-1,J+N1)+
        1ZETA*(C1*(D1-D2-CD*D4)+C3*D3)
      73 CONTINUE
      74 IF (L .EQ. MN) GO TO 86
C-------------------------------------------------------------------
C LOOP 85 COMPUTES COEFFICIENTS FORMED BY
C INTEGRALS I5 THROUGH I8.
C-------------------------------------------------------------------
         DO 85 J=L, MN
C-----------------------------------------------------------------
C COMPUTATION OF COEFFICIENT KERNEL.
C -----------------------------------------------------------------
         IF (J .NE. L) GO TO 75
         ALPHA=.5
         CGAMMA=.5
         GO TO 77
      75 IF (J .NE. MN) GO TO 76
         ALPHA=0.0
         CGAMMA=.5
         GO TO 77
      76 ALPHA=.5
         CGAMMA=1.0
      77 D5=DCOS(GAMMA1*(J-1.0+ALPHA))-
        1DCOS(GAMMA1*(J-1.0+ALPHA-CGAMMA))
         D6=DSIN(GAMMA1*(J-1.0+ALPHA))-
        1DSIN(GAMMA1*(J-1.0+ALPHA-CGAMMA))
         D8=DCOSH(OMEGA1*(N1+1.0-J-ALPHA))-
        1DCOSH(OMEGA1*(N1+1.0-J-ALPHA+CGAMMA))
         D7=DSINH(OMEGA1*(N1+1.0-J-ALPHA)) -
        1DSINH(OMEGA1*(N1+1.0-J-ALPHA+CGAMMA))
C---------------------------------------------------------------------
C SUBTRACTION OF PARTIAL COEFFICIENT INTO
C COEFFICIENT MATRIX. DOES NOT LOAD THE LAST
C COLUMN OF A1 WHICH IS THE DRIVING FORCE.
C---------------------------------------------------------------------
         IF (J .NE. MN) GO TO 82
         IF ( L .EQ. 1) GO TO 79
         A1(L-1,N1)=A1(L-1,N1)+C1*PARS*D5+
        1C1*PARSN*D6-C3*PHASN*D7-C1*PHASN*D8
      79 A1(L+N1,N1)=A1(L+N1,N1)+C1*ZYN*D5+
        1C1*ZYNN*D6-ZETAN*(C3*D7+C1*D8)
         GO TO 85
      82 IF (L .EQ. 1) GO TO 83
         A1(L-1,J-1)=A1(L-1,J-1)+
        1C1*(PARS*D5+PARSN*D6-PHASN*D8)-C3
        2*PHASN*D7
         A1(L-1,J+N1)=A1(L-1,J+N1)-
        1C1*(PARS*D5+PARSN*D6-PHASN*D8)+
        2C3*PHASN*D7
      83 IF (J .EQ. 1) GO TO 84
```

```
      A1(L+N1,J-1)=A1(L+N1,J-1)+
     1C1*(ZYN*D5+ZYNN*D6-ZETAN*D8)
     2C3*ZETAN*D7
      A1(L+N1,J+N1)=A1(L+N1,J+N1)-
     1C1*(ZYN*D5+ZYNN*D6-ZETAN*D8)+
     2C3*ZETAN*D7
      GO TO 85
   84 A1(N1+1,2*N1+1)=A1(N1+1,2*N1+1)-
     1C1*(ZYN*D5+ZYNN*D6+ZETAN*D8)+
     2C3*ZETAN*D7
      A1(N1+1,N1+1)=A1(N1+1,N1+1)-
     1C1*(ZYN*D5+ZYNN*D6-ZETAN*D8)+
     2C3*ZETAN*D7
   85 CONTINUE
C-----------------------------------------------
C LOADING OF DRIVING FORCE TERMS INTO LAST
C COLUMN OF COEFFICIENT MATRIX, A1.
C-----------------------------------------------
   86 IF (L .NE. 1) GO TO 87
      A1(N1+1,2*N1+1)=A1(N1+1,2*N1+1)+PHAS
      GO TO 100
   87 IF (L .NE. MN) GO TO 88
      A1(N1,2*N1+1)=A1(N1,2*N1+1)+ ZETA
      GO TO 100
   88 A1(L-1,2*N1+1)=A1(L-1,2*N1+1)+ZETA
      A1(L+N1,2*N1+1)=A1(L+N1,2*N1+1)+PHAS
  100 CONTINUE
C-----------------------------------------------
C TRANSVERSE COORDINATE PRINTOUT.
C-----------------------------------------------
      WRITE (6,115) RHO
C-----------------------------------------------
C FORMATTING MATRIX DATA PREPATORY TO INVERSION.
C A1, EXCEPT FOR THE LAST COLUMN, IS LOADED
C INTO A, A 2N1 BY 2N1 MATRIX. THE LAST
C COLUMN OF A1 IS LOADED INTO B.
C-----------------------------------------------
      DO 102 I=1,N
      DO 102 J=1,N
      A(I,J)=A1(I,J)
  102 CONTINUE
      DO 103 I=1,N
      B(I)=A1(I,M1M)
  103 CONTINUE

C-----------------------------------------------
C DOUBLE PRECISION LU DECOMPOSITION OF A.
C-----------------------------------------------
      CALL DLUD(N,ADIM,A,IDIM,T,IV)
      IF (IV(N) .EQ. 0) GO TO 104
```

```
C-----------------------------------------------------------------
C DOUBLE PRECISION BACKWARD SUBSTITUTION WITH
C ITERATIVE REFINEMENTS USING B AS A DRIVING
C FORCE.
C-----------------------------------------------------------------
      CALL DIR(N,ADIM,A,TDIM,T,IV,Q,B,R,IER)
C-----------------------------------------------------------------
C UNKNOWN OUTPUT Q(I)
C-----------------------------------------------------------------
      WRITE (6,110) (Q(I),I=1,M1)
C-----------------------------------------------------------------
C IER IS ITERATIVE REFINEMENT FLAG INDICATING
C HOW MANY ITERATIONS IT TOOK FOR THE
C SOLUTION TO NUMERICALLY CONVERGE.
C-----------------------------------------------------------------
      PRINT 120,IER
      GO TO 106
  104 CONTINUE
      WRITE (6,105)
C-----------------------------------------------------------------
C WARNING FLAG TO TELL OF COMPUTATIONALLY
C  SINGULAR MATRIX.
C-----------------------------------------------------------------
  105 FORMAT ('SINGULAR DETERMINATE')
  106 CONTINUE
  200 CONTINUE
  110 FORMAT (1X,8F15.7)
  115 FORMAT (1X,E15.7)
  120 FORMAT (1X,I5)
      END
```

The program is written in double precision using real arithmetic. It computes $\psi^{++}(\Delta,\underline{\rho}_d),\ldots, \psi^{++}(N1\Delta,\underline{\rho}_d),\psi^{--}(0,\underline{\rho}_d),\ldots, \psi^{--}((N1 - 1)\Delta,\underline{\rho}_d)$ for a fixed ρ_d. Required inputs are: N1 the number of sampling points along the direction of propagation of a normally incident plane wave in a slab of random material, GAMMA the slab thickness per unit wavelength, VAR the variance of the permittivity fluctuations per unit wavelength, RHO the magnitude of the separation distance between the two points at which the field is being examined to determine the coherence functions and PHSUBA the correlation function of the permitivity fluctuations. Note that since N1 sampling points involve 2N1 unknows matrices A and T must be dimensioned 2N1 by 2N1 while vectors, B, G, R and IV are dimensioned 2N1. The matrix A1 which is used to store the coefficients of the integral equations must be dimensioned 2N1 by 2N1 + 1 to accommodate the driving force term which appears in each of the 2N1 equations. The LU decomposition and backward substitution using iterative refinement is accomplished by using DLUD and DIR, two subroutines in the system software package entitled Numerical Analysis and Application Softward package of the Michigan Terminal System. The first of these, DLUD, is a double precision subroutine which computes the LU decomposition of a matrix stored in A and returns the decomposition in matrix T and vector IV. The subroutine DIR computes the solution of a system of equations given by Ax = B. The details of these subroutines can be found in reference [18].

The final topic to be discussed in this appendix is the number of sample points required to provide a desired level of solution accuracy. The level of accuracy demanded for this work is not stringent since the goal was to determine the behavior of coherence as the parameters of slab thickness and variance of permittivity fluctuations were varied not to determine accurate absolute levels of coherence. The tradeoff of cost versus number of sampling points and the information gained by increasing sampling was often a factor in determining the data run. Difficulties in accuracy occur because of the very rapid change in the values of cosh and sinh over small ranges of argument. Since the arguments of cosh and sinh contain

$$\omega = 2\pi^2 \frac{\sigma_\varepsilon^2}{\lambda} \frac{L}{\lambda} (1 - \phi_A)^{1/2}$$

where $\phi_A \to 0$ for increasing ρ_d, large values of $(\sigma_\varepsilon^2/\lambda)(L/\lambda)$ for moving ρ_d away from zero will require an increasing number of samples. The worst case condition is then when $\phi_A = 0$ or at $\rho_d = \infty$. This provides a method of checking the accuracy of the numerical results. Since the forward and backscatter fields contain an average component the coherence functions will decay to the energy contained in those average fields with increasing ρ_d. The energy in the average fields can be computed exactly by using the results in Chapter III. The coherence functions can be calculated numerically at $\rho_d = \infty$ by setting $\phi_A = 0$ in the program. If the two results compare well for a given number of samples then the accuracy at smaller values of ρ_d is good. The maximum number of samples used to run the data contained in this thesis was 96. This represents a system of 192 equations. The most severe restriction on the kind of data which could be run was the computationally maximum allowable argument for cosh and sinh. The L/λ and $\sigma_\varepsilon^2/\lambda$ parameters had to be adjusted such that the arguments of cosh and sinh fell below the allowable maximum of 175.

APPENDIX B. BORN APPROXIMATION

An expression for the Born or single scatter approximation is desirable which is compatible with the formulation presented within the main body of this thesis. Beginning with the equation for the complex amplitude of the backscatter field, $u^-(\underline{r})$, as written in Eq. (2.14), Section 2.1 and noticing that $u^-(\underline{r})$ can be considered as a perturbation of the free space backscatter field it is possible to write

$$u^-(\underline{r}) = \int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; Z^-(z,z',\underline{\rho},\underline{\rho}') \left(-\frac{ik}{2}\tilde{\varepsilon}(\underline{r}')u^-(\underline{r}') + \frac{1}{2ik}\nabla_T'^2 u^+(\underline{r}') - \frac{ik}{2}\tilde{\varepsilon}(\underline{r}')u^+(\underline{r}') \quad e^{2ikz} \right) , \quad (B.1)$$

where $Z^-(z,z',\underline{\rho},\underline{\rho}')$ is the fundamental solution of the backward propagating wave in free space as presented in Section 2.2. The single scattering approximation assumes that the extinction of free space fields within the body of the medium of interest is negligible. The implication is that the fields on the right-hand side of (B.1) can be replaced by their free space values. Since the free space value of $u^-(\underline{r})$ is zero and that of $u^+(\underline{r})$ is one for an incident plane wave,

$$u^-(\underline{r}) = -\frac{ik}{2}\int_z^L dz' \int_{-\infty}^{\infty} d\underline{\rho}' \; Z^-(z,z',\underline{\rho},\underline{\rho}')\tilde{\varepsilon}(\underline{r}')e^{2ikz'} \qquad (B.2)$$

results. The average energy in the backscatter field is given by

$$<|u^-(\underline{r})|^2> \frac{k^2}{4}\int_z^L dz_1 \int_z^L dz_2 \int_{-\infty}^{\infty} d\underline{\rho}_1 \int_{-\infty}^{\infty} d\underline{\rho}_2 \; Z^-(z,z_1,\underline{\rho},\underline{\rho}_1')$$

$$\cdot Z^{-*}(z_1 z_2,\underline{\rho}_1,\underline{\rho}_2)<\tilde{\varepsilon}(\underline{r}_1)\tilde{\varepsilon}(\underline{r}_2)> e^{2ik(z_1-z_2)} \quad . \qquad (B.3)$$

Making use of equation (3.4) of Section 3.1 to write the correlation of permittivity fluctuations results in

$$<|u^-(\underline{r})|^2> = \frac{k^2}{4}\int_z^L dz_1 \int_{-\infty}^{\infty} d\underline{\rho}_1 \int_{-\infty}^{\infty} d\underline{\rho}_2$$

$$\cdot Z^-(z,z_1,\underline{\rho},\underline{\rho}_1)Z^{-*}(z,z_1,\underline{\rho},\underline{\rho}_2)A(\underline{\rho}_1 - \underline{\rho}_2) \quad . \qquad (B.4)$$

Performing two stationary phase integrations as in Section 4.2 results in

$$<|u^-(\underline{r})|^2> = \frac{k^2}{4} A(0)\int_z^L dz' \qquad (B.5)$$

or

$$<|u^-(\underline{r})|^2> = \frac{\beta}{2}\,(1 - \zeta) \qquad (B.6)$$

where ζ is the z-coordinate normalized to the slab thickness. Equation (B.6) evaluated at $\zeta = 0$ represents the Born approximation to the backscattered energy.

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