THE CONSTRUCTION OF A VECTOR POTENTIAL

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For the solution of certain low frequency scattering problems it is necessary to determine a vector potential $\bar{A}$ such that $\nabla \times \bar{A} = \nabla\phi$ where ϕ is a known exterior potential arising from some distribution (charge or dipole) over a surface S. This is a rather classical problem in potential theory and, as shown by Stevenson (Q. Appl. Math. 12, 194-197, 1954), a solution exists if $\nabla^2\phi$ outside and on S and $\int_S \partial\phi/\partial n \, dS = 0$ where $\hat{n}$ is a unit vector normal to the surface.

When S is a closed surface, a method for the construction of $\bar{A}$ has been given by Stevenson. If ϕ is a magnetostatic potential produced by a field incident on the perfectly conducting surface S, ϕ can be expressed as a double layer distribution and $\bar{A}$ is given immediately in terms of the boundary values of $\phi + \phi^{inc}$, where ϕ^{inc} is the incident field potential. The result is also applicable to a curved shell or plate of infinitesimal thickness. On the other hand, if ϕ is an electrostatic potential associated with an isolated conductor, ϕ is naturally expressed as a single layer distribution, and the determination of $\bar{A}$ then requires the solution of a subsidiary (interior Neumann) potential problem, equivalent to the representation of ϕ as a double layer distribution. The construction now fails if S has zero interior volume.

Methods for the construction of $\bar{A}$ in the case of a curved or flat shell (or plate) are discussed. Not surprisingly, each involves the solution of a subsidiary problem, and no way to avoid this has been found. To illustrate the methods, some results for a circular disk are presented.

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The Construction of a Vector Potential

A rather classical problem in vector analysis is the determination of a vector function of position $\bar{F}(\bar{r})$ such that

$$\nabla \wedge \bar{F} = \bar{f} \tag{1}$$

where f is defined outside and on some surface B. The problem is discussed in many texts, but probably the most complete treatment is that of Stevenson (1954) who shows that the necessary and sufficient conditions for the existence of a solution are

$$\nabla \cdot \bar{f} = 0 \quad , \quad (\bar{r} \in \text{ext } B) \tag{2}$$

and

$$\int_B \hat{n} \cdot \bar{f} \, dS = 0 \quad . \tag{3}$$

To construct a solution, Stevenson first extends the definition of $\bar{f}$ into the interior of B by choosing an interior potential Φ^{in} such that

$$\bar{f} = \nabla \Phi^{in} \tag{4}$$

with

$$\nabla^2 \Phi^{in} = 0 \quad , \quad (\bar{r} \in \text{int } B) \tag{5}$$

and

$$\frac{\partial \Phi^{in}}{\partial n} = \hat{n} \cdot \bar{f} \quad , \quad (\bar{r} \in B) \quad . \tag{6}$$

This is a standard Newmann problem for Φ^{in} and has a solution by virtue of (2). A solution of (1) is then

$$\bar{F} = \frac{1}{4\pi} \nabla \wedge \int_{\substack{\text{all} \\ \text{space}}} \frac{\bar{f}(\bar{r}')}{R} \, dV' \tag{7}$$

where $R = |\bar{r} - \bar{r}'|$, and the most general solution is obtained by adding to this the gradient of any scalar.

In the solution of certain low frequency scattering problems it is necessary to determine the function $\bar{F}$. $\bar{f}$ is then a scattered electro- or magnetostatic field resulting from the incidence of an electro- or magnetostatic field on, for example, a perfectly conducting surface B. Since $\bar{f}$ can be expressed as

$$\bar{f} = \nabla \Phi^{S} \tag{8}$$

where Φ^{S} is an exterior (scattered) Laplacian potential known outside and on B, the conditions for the existence of $\bar{F}$ are satisfied if the body is electrically neutral and, indeed, the above solution can be written as

$$\bar{F} = -\frac{1}{4\pi} \nabla \wedge \int_{B} \hat{n}' (\Phi^{S} - \Phi^{in}) \frac{1}{R} \, dS' \quad . \tag{9}$$

It might seem that this is now the end of the matter, but as we shall see, the above solution is not always convenient or even valid.

The simplest case to consider first is that in which Φ^{S} is a magnetostatic potential.

Magnetostatics

If Φ^s is the scattered magnetostatic potential produced by the potential Φ^i incident on the perfectly conducting surface B, the boundary condition is

$$\frac{\partial \Phi^s}{\partial n} = -\frac{\partial \Phi^i}{\partial n} , \quad (\bar{r} \in B) \tag{10}$$

and since Φ^i is defined everywhere including the interior of B, Φ^i is itself an interior potential. Comparison with (6) now shows that

$$\Phi^{in} = -\Phi^i \tag{11}$$

and hence

$$\bar{F} = \frac{1}{4\pi} \nabla \wedge \int_B \hat{n}' \Phi^t \frac{1}{R} dS' \tag{12}$$

where

$$\Phi^t = \Phi^i + \Phi^s \tag{13}$$

is the total (incident plus scattered) magnetostatic potential.

This result is convenient, since Φ^t is known, and it is also a standard result. It can be obtained, for example, from the Stratton-Chu representation of an electromagnetic field by expanding in powers of ik and recognizing that $\bar{F}$ is simply a first order scattered electric field. Alternatively, if we seek a representation of $\bar{F}$ in the form

$$\bar{F} = -\frac{1}{4\pi} \nabla \wedge \int_B \bar{g}(\bar{r}') \frac{1}{R} dS' \tag{14}$$

then $\nabla \wedge \bar{F} = \nabla \Phi^S$ if

$$\Phi^S = \frac{1}{4\pi} \int_B \bar{g}(\bar{r}') \cdot \nabla' \frac{1}{R} dS' \quad , \tag{15}$$

and since Φ^S is naturally expressible as a double layer distribution, viz

$$\Phi^S = \frac{1}{4\pi} \int_B \Phi^t(\bar{r}') \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) dS' \quad , \tag{16}$$

it is evident that a solution is obtained by choosing

$$\bar{g}(\bar{r}') = \hat{n}' \Phi^t(\bar{r}') \quad . \tag{17}$$

There are, of course, other ways to obtain this same result, but a key point is that if the interior of B shrinks to zero, so that B becomes a shell or plate of infinitesimal thickness, the solution remains valid. Thus, for a shell,

$$\bar{F} = -\frac{1}{4\pi} \nabla \wedge \int_B \hat{n}' \Phi^t \Big|_-^+ \frac{1}{R} dS' \quad , \tag{18}$$

where the integration is over the upper (positive) side only, and the vertical line denotes the discontinuity across B. In particular, for a (planar) shell or plate in the plane $z = 0$, $\hat{n}' = z$ and $F_z = 0$.

Unfortunately, the electrostatic case is not quite so simple.

Electrostatics

Φ^s is now assumed to be a scattered electrostatic potential produced by a potential Φ^i incident on the solid perfectly conducting body B. Φ^s is therefore attributable to a charge distribution set up on the surface, and if the body is electrically neutral, the conditions for the existence of a solution $\bar{F}$ are satisfied. As before, a solution is

$$\bar{F} = -\frac{1}{4\pi} \nabla \wedge \int_B \hat{n}' (\Phi^s - \Phi^{in}) \frac{1}{R} dS' \qquad (19)$$

where Φ^{in} is the corresponding interior potential whose determination now requires the explicit solution of an interior Neumann problem. This is a task which is inherent in the form of (19) in the electrostatic case. Recognizing that $\bar{F}$ is a first order (in ik) scattered magnetic field associated with the electrostatic field $\nabla\Phi^s$, (19) is a representation of $\bar{F}$ in terms of a surface distribution of electric dipoles with moments normal to the surface. This is only possible if the surface is electrcally neutral locally as well as globally, and the interior potential achieves this end by placing on the inside surface of B a charge distribution equal and opposite in sign to that on the outside surface.

Alternatively expressed, since Φ^s is naturally represented as a single layer distribution, and the general form of $\bar{F}$ implies a double layer distribution, it is necessary to express Φ^s as a double layer distribution. This involves the solution of an interior potential problem of Neumann type. Although it is a task which is always

possible for a solid body, it is certainly inconvenient to have to solve a new potential problem. Worse still, for a shell of infinitesimal thickness the task is meaningless, and our solution is no longer valid.

Realizing that $\bar{F}$ is not unique, it is certainly reasonable to seek other forms of solution which, for a solid body, do not entail the solution of an additional potential problem and which would hopefully carry over in the limiting case of a shell. For a few particular solid bodies it is relatively trivial to construct a function $\bar{F}$. Thus, for a sphere of radius r_0 with $\Phi^i = \hat{a} \cdot \bar{r}$

$$\Phi^s = \hat{a} \cdot \nabla \left(\frac{r_0^3}{r} \right)$$

and a possible solution is

$$\bar{F} = \nabla \wedge \left(-a \frac{r_0^3}{r} \right) . \tag{20}$$

Similarly, for a body of revolution about the z axis, a knowledge of the dependence of the incident potential on the azimuthal angle ϕ specifies the dependence of Φ^s and enables a simple expression for $\bar{F}$ to be found. Thus, if $\Phi^i = x$, then

$$\Phi^s = \cos \phi \; f(\rho, z)$$

where ρ, ϕ, z are cylindrical polar coordinates, and it is easy to show that a possible solution is

$$\bar{F} = y\hat{\phi} \wedge \nabla f \quad . \tag{21}$$

In contrast to our original solution, $\nabla \cdot \bar{F} \neq 0$, so that $\bar{F}$ does not represent a magnetic field. Though we can construct a magnetic field by adding to $\bar{F}$ the gradient of potential, the determination of this is a separate potential problem that must be solved.

For a solid body without rotational symmetry, the only method we have found that does not involve the determination of an interior potential is to introduce a potential (harmonic) function of zero degree as defined, for example, by Hobson (1931). Let

$$K(\bar{r},\bar{r}') = \ln \frac{R - \hat{r}' \cdot (\bar{r} - \bar{r}')}{r - \hat{r}' \cdot \bar{r}} \tag{22}$$

where $R = |\bar{r} - \bar{r}'|$ as before. Then

$$\hat{r}' \cdot \nabla K = \frac{1}{r} - \frac{1}{R} \tag{23}$$

$$\nabla^2 K = 0 \quad , \qquad (\bar{r} \neq 0, \bar{r}') \tag{24}$$

and

$$K \sim -\frac{r'}{r} \quad \text{as } r \to \infty \quad . \tag{25}$$

Consider

$$\bar{F} = \frac{1}{4\pi} \nabla \wedge \int_B \frac{\partial \Phi^t}{\partial n'} \hat{r}' K(\bar{r},\bar{r}') \, dS' \tag{26}$$

where the origin of coordinates is inside B. For $\bar{r} \in$ ext B

$$\nabla \wedge \bar{F} = \frac{1}{4\pi} \int_B \frac{\partial \Phi^t}{\partial n'} \hat{r}' \cdot \nabla K \, dS'$$

$$= \nabla \frac{1}{4\pi} \int_B \frac{\partial \Phi^t}{\partial n'} \left(\frac{1}{r} - \frac{1}{R} \right) dS'$$

$$= \nabla \Phi^s \qquad (27)$$

as required, provided the zero induced charge condition is satisfied. Thus (26) represents a valid solution which is also a possible magnetic field, and the nice thing is that it can be computed directly from a knowledge of the surface charge distribution. In the particular case of a sphere with $\phi^i = -\hat{a} \cdot \bar{r}$,

$$\bar{F} = \nabla \wedge \left(-\hat{a} \frac{r_o}{r} \right)$$

as before.

When I prepared the abstract for this talk, I had hoped to prove that (26) is also a valid solution of (1) when the interior of B shrinks to zero, so that for a shell or plate

$$\bar{F} = \frac{1}{4\pi} \nabla \wedge \int_B \left. \frac{\partial \Phi^t}{\partial n'} \right|_-^+ \hat{r}' K(\bar{r}, \bar{r}') \, dS' \quad ; \qquad (28)$$

and to demonstrate its validity by computing $\bar{F}$ for, say, circular and rectangular plates. Unfortunately, the task is not trivial, and because of some convergence difficulties that arise when the origin

of coordinates is on the surface, the problem of finding a function $\bar{F}$ even for a flat plate is not yet fully resolved.

The only plate for which a simple analytical expression for $\bar{F}$ is available is a circular disk of radius r_o, lying in the plane $z = 0$. If the incident potential is

$$\Phi^i = -\hat{a} \cdot \bar{r}$$

then

$$\left.\frac{\partial \Phi^t}{\partial z}\right|_-^+ = -\frac{8}{\pi} \hat{a} \cdot \bar{\rho} \frac{1}{\sqrt{r_o^2 - \rho^2}} \tag{29}$$

$$= \hat{z} \cdot \nabla \wedge \bar{h}$$

where

$$\bar{h} = -2\hat{z} \wedge \hat{a}\psi\Big|_-^+ \tag{30}$$

with

$$\psi\Big|_-^+ = -\frac{4}{\pi} \sqrt{r_o^2 - \rho^2} \ . \tag{31}$$

We remark that ψ is simply the magnetostatic potential for the same geometry, and note that $\bar{h}$ (which is the discontinuity in the tangential magnetic field across the plate) vanishes at the edges in accordance with the edge condition there. An expression for $\bar{F}$ is then

$$\bar{F} = \frac{1}{4\pi} \nabla \wedge \int_B \hat{z} \wedge \bar{h} \frac{1}{R} dS'$$

$$= -\frac{1}{2\pi} \nabla \wedge \int_B \hat{z} \wedge (\hat{z} \wedge \hat{a}) \frac{1}{R} \psi\Big|_-^+ dS' \tag{32}$$

and thus

$$\bar{F} = 2\hat{a}\wedge\hat{z}\psi + \{\hat{z}\wedge(\hat{z}\wedge\hat{a})\}\wedge\nabla\left\{\frac{1}{2\pi}\int_B \psi\Big|_-^+ \frac{1}{R}\,dS'\right\}\hat{z} \quad . \tag{33}$$

In closing, I want to emphasize that the main objective of this work was to seek a solution $\bar{F}$ of the equation $\nabla\wedge\bar{F} = \nabla\Phi$ valid for a shell or plate when Φ is an electrostatic potential, and to do so within the general context of classical potential theory. For a flat perfectly conducting plate it is obvious that we could set up the coupled integral equations for the components of the total current induced in the plate and simply solve these numerically. We can do this even in the dynamic case, and by expanding all quantities in power of ik, derive the coupled integral equations for the first order currents, leading then to numerical values for our function $\bar{F}$. But the whole purpose of a low frequency expansion is to simplify the numerical tasks involved. If the determination of $\bar{F}$ alone requires us to solve integral equations which are just as complicated as for the entire dynamic case, there is no longer any rationale for considering a low frequency expansion.