A NOTE ON THE EXTINCTION EFFICIENCY

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An important parameter in electromagnetic scattering is the extinction efficiency Q_{ext} defined as the ratio of the sum of the scattering and absorption cross sections and the geometrical cross section. It is accepted that in the limit as the electrical size of the particle becomes infinite, $Q_{ext} = 2$, and in a recent communication [1] an attempt was made to prove this result for a perfectly reflecting sphere by approximating the individual terms in the Mie series expression for Q_{ext}. There are, however, rigorous proofs based on the application of a Watson transformation already available in the microwave literature, and the purpose of this note is to summarize one of these.

If $x = 2\pi a/\lambda$ where a is the radius of the perfectly reflecting sphere and λ is the wavelength,

$$Q_{ext} = \frac{4}{x^2} \operatorname{Re} S \tag{1}$$

with

$$S = \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right)(a_n + b_n) \ . \tag{2}$$

The a_n and b_n are the Mie series coefficients defined as

$$a_n = \psi_n(x)/\xi_n^{(1)}(x) \quad , \quad b_n = \psi_n'(x)/\xi_n^{(1)'}(x) \quad ,$$

where $\psi_n(x)$ and $\xi_n^{(1)}(x)$ are proportional to the spherical Bessel and Hankel functions of the first kind:

$$\psi_n(x) = xj_n(x) \quad , \quad \xi_n^{(1)}(x) = xh_n^{(1)}(x) \quad ,$$

and the prime denotes differentiation.

By application of a Watson transformation to (2) we have

$$S = \int_C \left\{ \frac{\psi_{\nu-1/2}(x)}{\xi^{(1)}_{\nu-1/2}(x)} + \frac{\psi'_{\nu-1/2}(x)}{\xi^{(1)'}_{\nu-1/2}(x)} \right\} \frac{\nu\, d\nu}{1 + e^{2\pi i\nu}} - \frac{1}{2} \quad ,$$

where C is a path in the complex ν plane running from $\nu = \infty - i\varepsilon$ to $\nu = 0$ and thence to $\nu = \infty + i\varepsilon$ with $\varepsilon > 0$. The path encloses in a clockwise sense the zeros of $1 + e^{2\pi i\nu}$ for $\nu \geq 1/2$, and the term $-1/2$ serves to cancel the residue at $\nu = 1/2$. If the lower half of the path is now reflected in the origin, the resulting integrals can be manipulated to give

$$S = S^{o} + S^{c(1)} + S^{c(2)} \tag{3}$$

where

$$S^{o} = \int_0^x \nu d\nu - \frac{1}{2} + \int_{\infty - i\varepsilon}^{0} \frac{e^{-2\pi i\nu}}{1 + e^{-2\pi i\nu}}\, \nu d\nu - \int_0^{\infty + i\varepsilon} \frac{e^{2\pi i\nu}}{1 + e^{2\pi i\nu}}\, \nu d\nu \tag{4}$$

$$S^{c(1)} = -\frac{1}{2} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \left\{ \frac{\xi^{(2)}_{\nu-1/2}(x)}{\xi^{(1)}_{\nu-1/2}(x)} + \frac{\xi^{(2)'}_{\nu-1/2}(x)}{\xi^{(1)'}_{\nu-1/2}(x)} \right\} \frac{e^{2\pi i\nu}}{1 + e^{2\pi i\nu}}\, \nu d\nu \tag{5}$$

$$S^{c(1)} = \frac{1}{2} \int_{-\infty}^{x} \left\{ \frac{\xi^{(2)}_{\nu-1/2}(x)}{\xi^{(1)}_{\nu-1/2}(x)} + \frac{\xi^{(2)'}_{\nu-1/2}(x)}{\xi^{(1)'}_{\nu-1/2}(x)} \right\} \nu d\nu$$

$$+ \int_x^{\infty} \left\{ \frac{\psi_{\nu-1/2}(x)}{\xi^{(1)}_{\nu-1/2}(x)} + \frac{\psi'_{\nu-1/2}(x)}{\xi^{(1)'}_{\nu-1/2}(x)} \right\} \nu d\nu \quad . \tag{6}$$

As split up in this manner, S^o contains terms which are geometrical in character, whereas $S^{c(1)}$ and $S^{c(2)}$ represent the contributions from the creeping waves and the vestiges thereof.

The evaluation of the expression for S^o is straightforward. The last two integrals in (4) are equal and opposite in sign, and can be computed by expanding the denominators in powers of the exponential. The result is

$$S^o = \frac{1}{2}\left(x^2 - \frac{11}{12}\right) . \tag{7}$$

$S^{c(1)}$ can be evaluated as a sum of residues at the poles of the Hankel function ratios in the upper half plane, leading to the familiar expression in terms of creeping waves which have travelled at least once around the sphere, and $S^{c(1)}$ decreases exponentially with increasing x. For the integrals comprising $S^{c(2)}$, no closure of the paths is possible. $S^{c(2)}$ represents the contribution of rays which have merely grazed the sphere, and by introducing the Airy integral representations of the functions involved, the integrals can be reduced to known forms. The details of the calculations are given in [2], and the result is

$$S^{c(2)} = x^{4/3}(0.082972 + i0.144019) + O(x^{2/3}) . \tag{8}$$

From (1), (2), (3), (7) and (8) it now follows immediately that

$$\lim_{x \to \infty} Q_{ext} = 2 \ .$$

In principle at least a similar approach is possible (see [3]) for a homogeneous dielectric sphere whose Mie coefficients are

$$a_n = \frac{\alpha_n \psi_n(x) - \psi_n'(x)}{\alpha_n \xi_n^{(1)}(x) - \xi_n^{(1)'}(x)}$$

$$b_n = \frac{\beta_n \psi_n(x) - \psi_n'(x)}{\beta_n \xi_n^{(1)}(x) - \xi_n^{(1)'}(x)}$$

where

$$\alpha_n = m^2 \beta_n = m \frac{\psi_n'(mx)}{\psi_n(mx)}$$

and m is the refractive index, and the only real difficulty is the asymptotic estimate of the terms corresponding to $S^{c(1)}$ and $S^{c(2)}$ above.

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REEFERENCES

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