A Method of Resolving Data into Two Maximally Smooth Components

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ABSTRACT

An algorithm is described which resolves data into maximally smooth components given some knowledge of the relative behavior of these components. The algorithm is used to analyze the specific problem of resolving frequency domain scattering data from a complex target into two components associated with different areas of the target. The resulting matrix problem is not only well conditioned but also amenable to extremely rapid solution.

1. INTRODUCTION

Frequency domain data are often strongly periodic. This is true, for example, in the case of the backscattered field of a target where the periodicity is attributable to the interference between the signals produced by localized scattering centers on the target, and the determination of the components due to each is then important for diagnostic and other purposes. The simplest situation is that of a single dominant period indicative of two such centers whose physical separation can be deduced from the period or from an examination of the geometry of the target. The task of extracting these components becomes well posed when they are assumed to be in some sense maximally smooth and this is the problem treated.

2. FORMULATION

Assume $p(\omega)$ to be strongly periodic, frequency domain data. The periodicity is assumed to arise from interference resulting from a difference, d, in path lengths associated with the two primary scatterers. To estimate what portion of the data $p(\omega)$ is associated with each of the two scatterers, $p_1(\omega)$ and $p_2(\omega)$, the periodic properties of $p(\omega)$ must be exploited. Thus, writing the phase relationship of $p(\omega)$, $p_1(\omega)$, and $p_2(\omega)$ explicitly,

$$p(\omega) = p_1(\omega) + exp(i2\omega d/c)p_2(\omega) \tag{1}$$

where $p_1(\omega)$ and $p(\omega)$ are phase referenced to the forward scatterer and $p_2(\omega)$ is phase referenced to the rear scatterer. Requiring $p_1(\omega)$ and $p_2(\omega)$ to be smooth in addition to satisfying (1) permits a unique decomposition which is easy to implement. For the purpose of defining the decomposition, smoothness is measured using a weighted Euclidean norm of the second derivatives of $p_1(\omega)$ and $p_2(\omega)$ and the maximally smooth solution will have the smallest norm. The weighting of the norm may be employed to specify the smoothness of different portions of $p_1(\omega)$

and $p_2(\omega)$.

To affect a rapid solution of (1) subject to the constraint of maximally smooth components, it is necessary to linearize the problem. Linear problems, in turn, are most easily discussed using matrix terminology. To this end, let $\mathbf{p}$ represent an N element vector containing $p(\omega)$ sampled over N frequencies where the ith element of $\mathbf{p}$ is $p(\omega_i)$. Similarly, p_1 and p_2 are replaced by $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. Thus,

$$\mathbf{p}_1 = [p_1(\omega_1), p_1(\omega_2), p_1(\omega_3), \ldots, p_1(\omega_N)]^T \tag{2}$$

and

$$\ddot{\mathbf{p}}_1 = \left[\frac{p_1(\omega_3)-2p_1(\omega_2)+p_1(\omega_1)}{(\Delta\omega)^2}, \frac{p_1(\omega_4)-2p_1(\omega_3)+p_1(\omega_2)}{(\Delta\omega)^2}, \cdots \right]^T \tag{3}$$

where $\mathbf{p}_1$ is an N element vector; $\ddot{\mathbf{p}}_1$ is an N-2 element vector and represents the second derivative of $\mathbf{p}_1$; and $\mathbf{p}_2$ and $\ddot{\mathbf{p}}_2$ are defined similarly. The precise sampling rate is unimportant subject to two constraints.

$$N \geq 4 \tag{4}$$

since using less than 4 data points makes the problem singular. Secondly,

$$\Delta\omega \neq m(\pi c / d) \quad , \quad m = 1, 2, 3, \cdots \tag{5}$$

since sampling at a multiple of $\pi c / d$ would cause the exponential in (1) to become unity and eliminate the only observable difference between $\mathbf{p}_1$ and $\mathbf{p}_2$. The data are assumed to have been sampled uniformly and thus the $(\Delta\omega)^2$ is factored out and ignored. Proceeding, then, to define the matrix problem, (1) becomes

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{U}\,\mathbf{p}_2 \tag{6}$$

where

$$\mathbf{U} = \mathrm{diag}\left[exp(i2\omega_1 d / c), exp(i2\omega_2 d / c), \ldots, exp(i2\omega_N d / c)\right] \tag{7}$$

and

$$\ddot{\mathbf{p}}_1 = \mathbf{A}\,\mathbf{p}_1 \tag{8}$$

$$\ddot{\mathbf{p}}_2 = \mathbf{A}\,\mathbf{p}_2 \tag{9}$$

with

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & -1 & 2 & -1 \end{bmatrix} \tag{10}$$

From (6) and (9)

$$\ddot{\mathbf{p}}_2 = \mathbf{A}\,\bar{\mathbf{U}}\,\mathbf{p} - \mathbf{A}\,\bar{\mathbf{U}}\,\mathbf{p}_1 \tag{11}$$

To minimize $\ddot{\mathbf{p}}_1$ and $\ddot{\mathbf{p}}_2$ and thence obtain the optimal $\mathbf{p}_1$ and $\mathbf{p}_2$, define the error vector

$$\mathbf{e} = \begin{bmatrix} \ddot{\mathbf{p}}_1 \\ \cdots \\ \ddot{\mathbf{p}}_2 \end{bmatrix} \tag{12}$$

(i.e., the concatenation of $\ddot{\mathbf{p}}_1$ with $\ddot{\mathbf{p}}_2$) and minimize $\mathbf{e}$. As mentioned above, this minimization is with respect to a weighted, Euclidean norm, which is now defined as

$$||\mathbf{e}|| = \bar{\mathbf{e}}^T\,\mathbf{M}\,\mathbf{e} \tag{13}$$

where M is a diagonal matrix of order 2(N-2) whose elements are positive real numbers which specify the weighting at each point of $\ddot{\mathbf{p}}_1$ and $\ddot{\mathbf{p}}_2$. Since the weighting of each point may be specified independently, there is considerable freedom in defining how the components should be smoothed. From (8), (11) and (12)

$$\mathbf{e} = \begin{bmatrix} \mathbf{A}\mathbf{p}_1 \\ \cdots\cdots\cdots \\ \mathbf{A}\,\bar{\mathbf{U}}\,\mathbf{p} - \mathbf{A}\,\bar{\mathbf{U}}\,\mathbf{p}_1 \end{bmatrix} = \mathbf{K}\,\mathbf{p}_1 - \mathbf{g} \tag{14}$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} \\ \cdots\cdots \\ -\mathbf{A}\,\bar{\mathbf{U}} \end{bmatrix} \tag{15}$$

and

$$\mathbf{g} = \begin{bmatrix} 0 \\ \cdots\cdots \\ -\mathbf{A}\,\bar{\mathbf{U}}\,\mathbf{p} \end{bmatrix} \tag{16}$$

By the projection theorem [1], the minimum $\mathbf{e}$, call it $\mathbf{e}_o$, will be perpendicular to the range of $\mathbf{K}$, where perpendicular is in the sense of the inner product space associated with the norm defined in (13)

$$\bar{\mathbf{e}}_o^T\,\mathbf{M}\,\mathbf{K}\,\mathbf{p}_1 = 0, \quad \text{for all } \mathbf{p}_1 \tag{17}$$

$$\bar{\mathbf{K}}^T\,\mathbf{M}\,(\mathbf{K}\,\mathbf{p}_{1o} - \mathbf{g}) = 0 \tag{18}$$

where $\mathbf{p}_{1o}$ is the $\mathbf{p}_1$ associated with the maximally smooth solution. All that remains is to solve

$$\mathbf{H}\,\mathbf{p}_{1o} = \bar{\mathbf{K}}^T\,\mathbf{M}\,\mathbf{g} \tag{19}$$

for $\mathbf{p}_{1o}$, where

$$\mathbf{H} = \bar{\mathbf{K}}^T\,\mathbf{M}\,\mathbf{K} \tag{20}$$

and then obtain $\mathbf{p}_2$ from (6). As may be seen from (7), (10), and (15), $\mathbf{H}$ is an Hermitian banded matrix of half-bandwidth 2. It may be shown that $\mathbf{H}$ is also non-singular and positive-definite if $\mathbf{M}^{1/2}\,\bar{\mathbf{K}}$ has independent columns. This is guaranteed when sampling is constrained by (4) and (5). Since $\mathbf{H}$ is Hermitian and positive-definite with half-bandwidth of 2, highly efficient algorithms such as the Cholesky decomposition [2] may be employed. Further, the number of matrix elements which must be computed to solve for $\mathbf{p}_1$ and $\mathbf{p}_2$ grows linearly with N, instead of N^2. Thus to decompose 50 data points, only 3 × 50 elements need be computed and stored as opposed to the usual 50 × 50 which result from an unbanded matrix problem.

3. RESULTS

The algorithm was programmed and tested on data resulting from back-scattering from a resistive strip under edge-on illumination. The strip was of width d, and 41 data points were obtained with $\omega = n\pi c/(8d)$, $n = 8, 9, 10, \ldots, 48$. It was desired to determine what portion of the back-scattering was due to the leading edge and what portion was due to the trailing edge. The periodic nature of the data (see Figure 1), indicated that the data might be amenable to the decomposition described in the previous section, and, in fact, was the motivation for the development of the current method. The data was assumed to have the form

$$p(n) = p_1(n) + exp(in\pi/4)\, p_2(n) \qquad (21)$$

and the program was run on 41 data points associated with n varying from 8 to 48. Although the approximation of two independent scatterers deteriorates for wavelengths less than the strip width, d, (n less than 16), it was desired to test the robustness of the algorithm. As can be seen from Figure 2, the periodicity which prompted the decomposition does not appear in the components. Consideration of the radiative interaction between the two scatterers indicated that p_1 should be smoother than p_2. Thus the problem was solved with the smoothness of p_1 weighted more heavily than that of p_2. Several weights were tried, and the results of the extremal weights of 1 and 100 are presented in Figure 2. An interesting result is that although the 100 weighting case did produce a very smooth p_1, p_2 did not deteriorate significantly. For the example presented, the matrix problem was formulated and solved using only 0.01 seconds of cpu time on an Amdahl 5860. Finally, it should be noted that the matrix is well conditioned, with estimated condition numbers of 10^3 and 10^4 for the matrices associated with the uniform weighting and 100 to 1 weighting, respectively. These condition numbers, for a matrix of order 40, are quite good and indicate a highly stable matrix which is relatively insensitive to numerical round-off.

4. DISCUSSION

The example treated in this communication was solved by using the periodic nature of $p(\omega)$ and knowledge of the physical distance separating the two scatterers. If phase information is not available, then the present algorithm is inadequate due to the extra degree of freedom in the relationship between p, p_1, and p_2. Conversely, additional information may be incorporated into the maximally smooth decomposition so as to produce a decomposition of the form

$$p(\omega) = a(\omega)\, p_1(\omega) + b(\omega)\, p_2(\omega) \qquad (22)$$

where $a(\omega)$ and $b(\omega)$ are completely arbitrary and may be chosen to explicitly factor out of $p_1(\omega)$ and $p_2(\omega)$, respectively, any behavior which is known a priori. The decomposition into $p_1(\omega)$ and $p_2(\omega)$ may be obtained by following the analysis of section 2. The derivation of the maximally smooth decomposition resulting from (22) proceeds much as before with U still a diagonal matrix whose ith element is now $b(\omega_i)/a(\omega_i)$. Since the algorithm presented in this communication is not only efficient but also well conditioned, a generalization to more than two components should be feasible. Although such additions would complicate the **K** matrix and increase the bandwidth of **H**, the matrix **H** should still be sufficiently ordered to permit an efficient solution.

Finally, an additional use of the algorithm is to interpolate the original data. Since the original data is rather sparse and oscillatory it is difficult to gauge the locations of minima and maxima. Conversely, working with the components, each of which is very smooth, interpolation is easy; the interpolated components may then be recombined to yield a highly accurate interpolation of the original data.

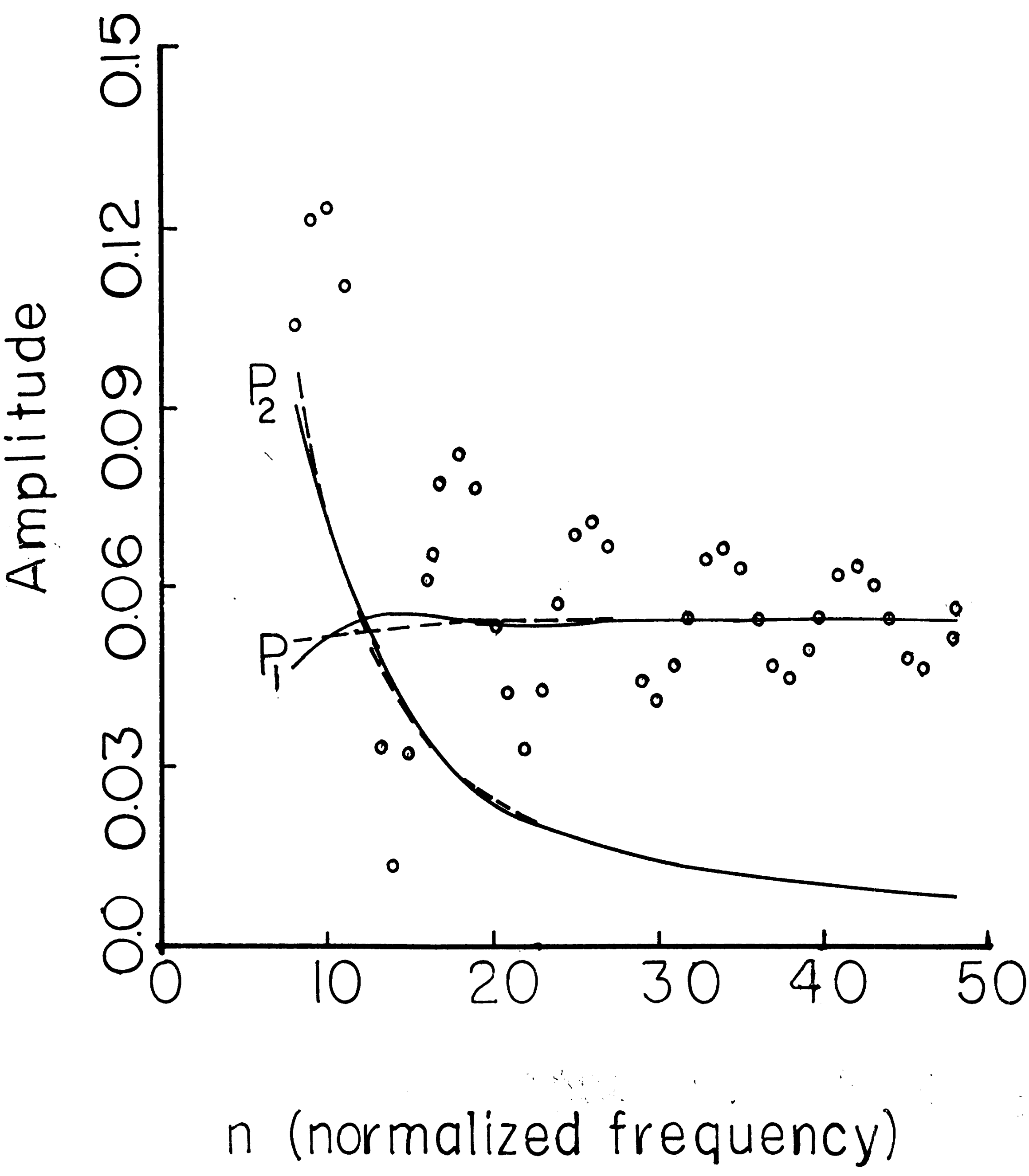
REFERENCES

[1] D.G. Luenberger, *Optimization by Vector Space Methods*. New York: Wiley, 1969.

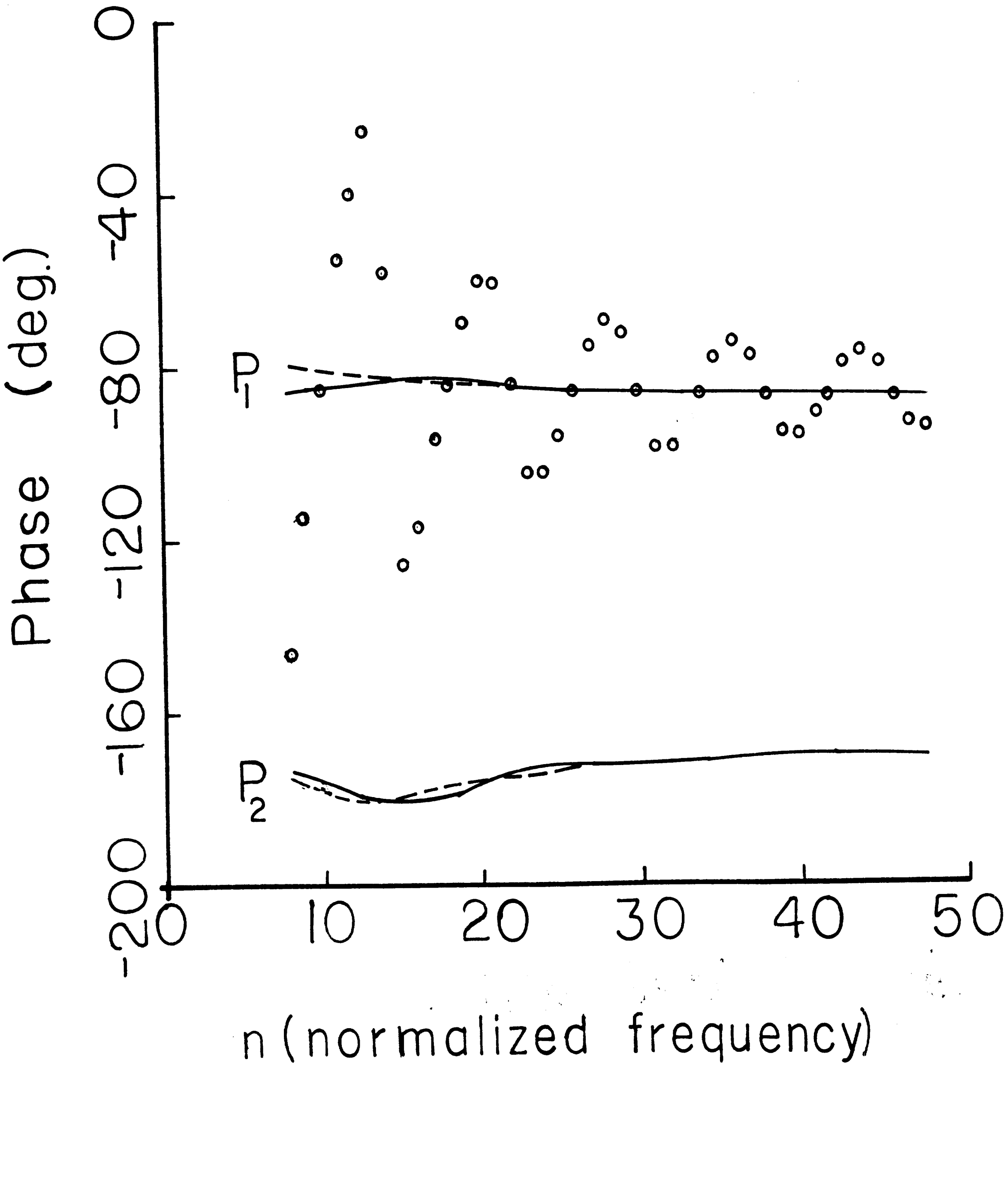
[2] G. Dahlquist and A. Bjorck, *Numerical Methods*. Englewood Cliffs, New Jersey: Prentice Hall, 1974.

FIGURE CAPTION

1 Original data(oo). Maximally smooth decomposition: Solid line - decomposition with uniform weighting, Dashed line - decomposition with weighting of $\ddot{p}_1$ multiplied by 100.



Amplitude
0.0
0.03
0.06
0.09
0.12
0.15
0
10
20
30
40
50
P_2
P_1
n (normalized frequency)



Phase (deg.)
0
-40
-80
-120
-160
-200
P_1
P_2
0
10
20
30
40
50
n (normalized frequency)