UNIFORM ASYMPTOTIC EXPANSIONS

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1 Introduction

In studying the physical optics solution for the bistatic scattering of a plane wave by an infinite, two-dimensional, perfectly conducting, S-shaped surface, it is necessary to develop a uniform asymptotic expansion of the integral

$$\Gamma = -\sqrt{\frac{k}{2\pi}} \int_{-\infty}^{\infty} [f'(x)\cos\phi - \sin\phi] \; e^{jk[Cx+Sf(x)]} dx \qquad (1)$$

where

$$C = \cos\phi + \cos\phi_o, \quad S = \sin\phi + \sin\phi_o \qquad (2)$$

with $\quad 0 \leq \pi - \phi < \frac{\pi}{2}$.

The surface is $y = f(x)$, and as a typical example consider

$$f(x) = A \int_o^{cx} e^{-t^2} dt \qquad (3)$$

where A and $1/c$ have dimensions (length)^1. Then

$$f'(x) = Ac\, e^{-(cx)^2}$$

which is positive for all real x;

$$f''(x) = -2Ac^3\, xe^{-(cx)^2}$$

which is negative (positive) for $x > (<)\ 0$, and

$$f'''(x) = -2Ac^3\left(1 - 2c^2x^2\right)\, e^{-(cx)^2}$$

which is negative for $|x| < \frac{1}{c\sqrt{2}}$.

In this particular example the surface is an odd function of x, but it is convenient to carry out the analysis for a more general surface having the following properties:

$$f(x) \text{ is a monotonic function of } x, \qquad -\infty < x < \infty$$

$$f(x) \; > (<) 0 \text{ for } x > (<)\, 0, \quad \text{implying } f(0) = 0$$
$$f'(x) \; \geq 0 \quad \text{with max } f'(x) = f'(0)$$
$$f''(x) \; < (>)\, 0 \text{ for } x > (<) 0, \quad \text{implying } f''(0) = 0$$

and

$f'''(x) < 0$ over the range spanned by the stationary phase (SP) points for the angles ϕ, ϕ_o of interest to us. The SP points are such that

$$C + Sf'(x) = 0 \tag{4}$$

implying

$$f'(x) = -\frac{C}{S} = -\cot\frac{1}{2}(\phi + \phi_o), \tag{5}$$

and if

$$\pi - \phi > 2\tan^{-1}\left[\frac{1}{f'(0)}\right] - (\pi - \phi_o), \tag{6}$$

(5) defines the two distinct SP points x_1, x_2 with $x_1 > 0$ and $x_2 < 0$. However, when

$$\pi - \phi = 2\tan^{-1}\left[\frac{1}{f'(0)}\right] - (\pi - \phi_o) \tag{7}$$

the two points merge at the origin, and for smaller values of $\pi - \phi$ the SP points correspond to pure imaginary values of x. For given $\pi - \phi_o$ the angular region (6) is that where specular contributions to the field exist, and the boundary is defined by (7). Beyond this there are no specular contributions, and only a small field is expected. We seek asymptotic evaluations of the integral expression (1) that are uniform in angle throughout the specular and non-specular regions.

2 Specular Region

At angles well within the specular region there are two distinct real SP points x_1, x_2 at which $S|f''(x)|/k >> 1$, and the dominant contributions to the integral come from small ranges of integration about these points. Since

$$f'(x)\cos\phi - \sin\phi = -\frac{\cos\frac{1}{2}(\phi-\phi_o)}{\sin\frac{1}{2}(\phi+\phi_o)} \tag{8}$$

at the SP points, we have

$$\Gamma \simeq \sqrt{\frac{k}{2\pi}}\,\frac{\cos\frac{1}{2}(\phi-\phi_o)}{\sin\frac{1}{2}(\phi+\phi_o)}\,I \tag{9}$$

where

$$I = I_1 + I_2 \tag{10}$$

with

$$I_1 = \int_{x_1-\delta_1}^{x_1+\delta_1} e^{jk[Cx+Sf(x)]}dx \tag{11}$$

$$I_2 = \int_{x_2-\delta_2}^{x_2+\delta_2} e^{jk[Cx+Sf(x)]}dx. \tag{12}$$

Consider first I_1. Expanding $f(x)$ in a Taylor series about $x = x_1(>0)$ and using (4),

$$\begin{aligned} I_1 &\simeq e^{jk[Cx_1+Sf(x_1)]}\int_{-\delta_1}^{\delta_1} e^{j\frac{kS}{2}x^2 f''(x_1)}dx \\ &= e^{jk[Cx_1+Sf(x_1)]}\sqrt{\frac{2}{kS\{-f''(x_1)\}}}\int_{-\Delta_1}^{\Delta_1} e^{-jt^2}dt \end{aligned} \tag{13}$$

where

$$\Delta_1 = \sqrt{\frac{1}{2}kS\{-f''(x_1)\}}\;\delta_1. \tag{14}$$

Recalling that $f''(x_1) < 0$, the leading term in the asymptotic expansion of I_1 is obtained by allowing $\Delta_1 \to \infty$, giving

$$I_1 \simeq e^{jk[Cx_1+Sf(x_1)]}\sqrt{\frac{2}{kS\{-f''(x_1)\}}}\int_{-\infty}^{\infty} e^{-jt^2}dt$$

and therefore

$$I_1 \simeq e^{jk[Cx_1+Sf(x_1)]}\sqrt{\frac{2\pi}{kS\{-f''(x_1)\}}}\ e^{-j\frac{\pi}{4}}. \tag{15}$$

Similarly

$$I_2 \simeq e^{jk[Cx_2+Sf(x_2)]}\int_{-\delta_2}^{\delta_2} e^{j\frac{kS}{2}x^2 f(x_2)}dx, \tag{16}$$

and since $f''(x_2) > 0$,

$$I_2 \simeq e^{jk[Cx_2+Sf(x_2)]}\sqrt{\frac{2\pi}{kSf''(x_2)}}\ e^{j\frac{\pi}{4}}. \tag{17}$$

Hence, to the leading order,

$$I \simeq e^{jk[Cx_1+Sf(x_1)]}\sqrt{\frac{2\pi}{kS\{-f''(x_1)\}}}\ e^{-j\frac{\pi}{4}}$$

$$+e^{jk[Cx_2+Sf(x_2)]}\sqrt{\frac{2\pi}{kSf''(x_2)}}\ e^{j\frac{\pi}{4}} \tag{18}$$

which can be written more compactly as

$$I \simeq \sum_{i=1,2} e^{jk[Cx_i+Sf(x_i)]}\sqrt{\frac{2\pi}{kS|f''(x_i)|}}\ e^{-\frac{1}{2}[arg\ f''(x_i)-\frac{\pi}{2}]} \tag{19}$$

where $-\pi < arg\ f''(x_i) \leq \pi$.

3 Non-specular Region

In this region the values of x specified by (4) are pure imaginary, and if x_1 is such that $Im.x_1 > 0$, then $Im.x_2 < 0$ and $Im.f''(x_1) < 0$, $Im.f''(x_2) > 0$. The path of integration runs from $x = -\infty$ to $x = \infty$ but can be deformed to pass through either x_1 or x_2, and when this is done, the dominant contribution to the integral comes from the immediate vicinity of the point. For I_1 expansion of $f(x)$ about x_1 produces an integrand (see (13))

$$e^{\frac{1}{2}kS|f''(x_1)|x^2},$$

whereas I_2 gives (see (16))

$$e^{-\frac{1}{2}kS|f''(x_2)|x^2}$$

The points x_1, x_2 are now saddle points, and clearly x_2 is at a lower level than x_1. Accordingly, the saddle point appropriate for a steepest descent evaluation is x_2 and

$$\begin{aligned} I &\simeq e^{jk[Cx_2+Sf(x_2)]} \int_{-\delta_2}^{\delta_2} e^{-\frac{kS}{2}\{-jf''(x_2)\}x^2}\,dx \\ &= e^{jk[Cx_2+Sf(x_2)]} \sqrt{\frac{2}{kS\{-jf''(x_2)\}}} \int_{-\Delta_2}^{\Delta_2} e^{-t^2}\,dt \end{aligned} \qquad (20)$$

where

$$\Delta_2 = \sqrt{\frac{1}{2}kS\{-jf''(x_2)\}}\ \delta_2. \qquad (21)$$

The leading term in the asymptotic expansion is obtained by allowing $\Delta_2 \to \infty$ and is

$$I \simeq e^{jk[Cx_2+Sf(x_2)]} \sqrt{\frac{2\pi}{kS\{-jf''(x_2)\}}} \qquad (22)$$

where x_2 is such that $Im.x_2 < 0$. With the previous definition of arg $f''(x_i)$, (22) can be written alternatively as

$$I \simeq e^{jk[Cx_2+Sf(x_2)]} \sqrt{\frac{2\pi}{kS|f''(x_2)|}}\ e^{-\frac{j}{2}\left[\arg f''(x_2)-\frac{\pi}{2}\right]} \qquad (23)$$

and the result is equivalent to retaining only that term in (19) corresponding to the SP point whose imaginary part is negative.

4 Boundary

The boundary separating the specular and non-specular regions is defined by (7), and corresponds to the merger of x_1 and x_2 at the origin where $f''(x) = 0$. Accordingly, (19) and (23) fail when the observation point is on the boundary, but it is a trivial matter to develop an asymptotic approximation valid in this case.

Since there is now only a single SP point at $x = 0$,

$$I \simeq \int_{-\delta}^{\delta} e^{jk[Cx+Sf(x)]}dx \tag{24}$$

and by expanding $f(x)$ in a Taylor series about $x = 0$, we obtain

$$I \simeq e^{jkSf(0)} \int_{-\delta}^{\delta} e^{j\frac{kS}{6}x^3 f'''(0)}dx \tag{25}$$

where we have used the fact that $f'(0) = f''(0) = 0$. By assumption $f'''(0) < 0$ and hence

$$I \simeq e^{jkSf(0)} \sqrt{\frac{2}{kS\{-f'''(0)\}}} \int_{-\Delta}^{\Delta} e^{-j\frac{t^3}{3}}dt$$

where

$$\Delta = \sqrt{\frac{1}{2}kS\{-f'''(0)\}}\ \delta. \tag{26}$$

On allowing $\Delta \to \infty$ the leading term in the asymptotic expansion is found to be

$$I \simeq e^{jkSf(0)} \sqrt{\frac{2}{kS\{-f'''(0)\}}} \int_{-\infty}^{\infty} e^{-j\frac{t^3}{3}}dt,$$

that is

$$I \simeq e^{jkSf(0)} \sqrt{\frac{2}{kS\{-f'''(0)\}}}\ 2\pi Ai(0) \tag{27}$$

where

$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j(\frac{t^3}{3}+zt)}dt \tag{28}$$

is the Airy integral of the first kind. We note that [Abramowitz and Stegun, 1964]

$$Ai(0) = 0.355028....$$

5 Uniform Asymptotic Expansion, Specular Region

The asymptotic expressions (19) and (23) in the specular and non-specular regions respectively are valid only in those parts of the regions which are bounded away from the boundary at which (27) applies, and we now seek expressions which are uniform in angle and which match (27) into (19) and (23). We consider the specular region first.

Since $f''(x)$ vanishes at the SP point(s) corresponding to the boundary, it is necessary to retain an additional term in the Taylor series expansion of $f(x)$ about $x = x_1$ and x_2. In the case of I_1, (13) is replaced by

$$\begin{aligned} I_1 &\simeq e^{jk[Cx_1+Sf(x_1)]} \int_{-\delta_1}^{\delta_1} e^{j\frac{kS}{2}x^2\left[f''(x_1)+\frac{x}{3}f'''(x_1)\right]}dx \qquad (29) \\ &= e^{jk[Cx_1+Sf(x_1)]} \int_{-\delta+\epsilon_1}^{\delta_1+\epsilon_1} e^{j\frac{kS}{2}(y-\epsilon_1)^2\left[f''(x_1)+\frac{1}{3}(y-\epsilon_1)f'''(x_1)\right]}dy. \end{aligned}$$

Choosing

$$\epsilon_1 = \frac{f''(x_1)}{f'''(x_1)}$$

the exponent in the integrand becomes

$$j\frac{kS}{2}\left[\frac{1}{3}y^3 f'''(x_1) - y\frac{\{f''(x_1)\}^2}{f'''(x_1)} + \frac{2}{3}\frac{\{f''(x_1)\}^3}{\{f'''(x_1)\}^2}\right],$$

and therefore

$$\begin{aligned} I_1 \simeq\ & e^{jk[Cx_1+Sf(x_1)]+j\frac{kS}{3}\{f''(x_1)\}^3/\{f'''(x_1)\}^2} \\ & \cdot \int_{-\delta_1+f''(x_1)/f'''(x_1)}^{\delta_1+f''(x_1)/f'''(x_1)} e^{j\frac{kS}{2}\left[\frac{1}{3}y^3 f'''(x_1)-y\{f''(x_1)\}^2/f'''(x_1)\right]}dy. \qquad (30) \end{aligned}$$

Recognizing that $f''(x_1) \leq 0$ and $f'''(x_1) < 0$, this can be written as

$$\begin{aligned} I_1 \simeq\ & e^{jk[Cx_1+Sf(x_1)]-j\frac{2}{3}\gamma_1^{\frac{3}{2}}} \left[\frac{2}{kS\{-f'''(x_1)\}}\right]^{\frac{1}{3}} \\ & \cdot \int_{\Delta_1\left[-1+\frac{1}{\delta_1}f''(x_1)/f'''(x_1)\right]}^{\Delta_1\left[1+\frac{1}{\delta_1}f''(x_1)/f'''(x_1)\right]} e^{-j\left(\frac{t^3}{3}-\gamma_1 t\right)}dt \end{aligned}$$

where

$$
\begin{aligned}
\Delta_1 &= \left[\frac{1}{2}kS\left\{-f'''(x_1)\right\}\right]^{\frac{1}{3}}\delta_1 \\
\gamma_1 &= \left[\frac{kS}{2\left\{-f'''(x_1)\right\}^2}\right]^{\frac{2}{3}}\left\{-f''(x_1)\right\}^2. \qquad (31)
\end{aligned}
$$

To ensure that the range of integration does not include negative values of t we now choose

$$
\delta_1 = \frac{f''(x_1)}{f'''(x_1)} \qquad (= \epsilon_1)
$$

Then, for $S|f'''(x_1)|/k >> 1$ the leading term in the asymptotic expansion is

$$
I_1 \simeq e^{jk[Cx_1+Sf(x_1)]}\left[\frac{2}{kS\left\{-f'''(x_1)\right\}}\right]^{\frac{1}{3}} e^{-j\frac{2}{3}\gamma_1^{\frac{3}{2}}}\int_o^\infty e^{-j(\frac{t^3}{3}-\gamma_1 t)}dt, \qquad (32)
$$

which can be expressed in terms of Airy integrals as

$$
\begin{aligned}
I_1 \simeq\ & e^{jk[Cx_1+Sf(x_1)]}\left[\frac{2}{kS\left\{-f'''(x_1)\right\}}\right]^{\frac{1}{3}} e^{-j\frac{2}{3}\gamma_1^{\frac{3}{2}}} \\
& \cdot \pi\left\{Ai(-\gamma_1) - jBi(-\gamma_1) + jHi(-\gamma_1)\right\} \qquad (33)
\end{aligned}
$$

where

$$
Bi(z) = \frac{1}{\pi}\int_o^\infty \sin\left(\frac{t^3}{3}+zt\right)dt + \frac{1}{\pi}\int_o^\infty e^{-\frac{t^3}{3}+zt}dt \qquad (34)
$$

is the Airy integral of the second kind and [Abramowitz and Stegun, 1964]

$$
Hi(z) = \frac{1}{\pi}\int_o^\infty e^{-\frac{t^3}{3}+zt}dt. \qquad (35)
$$

The treatment of the integral I_2 is similar. In place of (16) we now have

$$
I_2 \simeq e^{jk[Cx_2+Sf(x_2)]}\int_{-\delta_2}^{\delta_2} e^{j\frac{kS}{2}x^2[f''(x_2)+\frac{x}{3}f'''(x_2)]}dx \qquad (36)
$$

which can be written as

$$I_2 \simeq e^{jk[Cx_2+Sf(x_2)]+j\frac{kS}{3}\{f''(x_2)\}^3/\{f'''(x_2)\}^2} \cdot \int_{-\delta_2+f''(x_2)/f'''(x_2)}^{\delta_2+f''(x_2)/f'''(x_2)} e^{j\frac{kS}{2}\left[\frac{1}{3}y^3 f'''(x_2)-y\{f''(x_2)\}^2/f'''(x_2)\right]}dy. \quad (37)$$

Recognizing that $f''(x_2) \geq 0$ but $f'''(x_2) < 0$ we choose

$$\delta_2 = -\frac{f''(x_2)}{f'''(x_2)} \qquad (\geq 0)$$

to ensure that the integration is confined to negative values of y, and then

$$I_2 \simeq e^{jk[Cx_2+Sf(x_2)]}\left[\frac{2}{kS\{-f'''(x_1)\}}\right]^{\frac{1}{3}} e^{j\frac{2}{3}\gamma_2^{\frac{3}{2}}} \int_{-2\Delta_2}^{o} e^{-j(\frac{t^3}{3}-\gamma_2 t)}dt$$

where

$$\Delta_2 = \left[\frac{1}{2}kS\{-f'''(x_2)\}\right]^{\frac{1}{3}}\delta_2 \quad (38)$$

$$\gamma_2 = \left[\frac{kS}{2\{-f'''(x_1)\}^2}\right]^{\frac{2}{3}}\{f''(x_2)\}^2 \quad (39)$$

The leading term in the asymptotic expansion is obtained by allowing $\Delta_2 \rightarrow \infty$, giving

$$I_2 \simeq e^{jk[Cx_2+Sf(x_2)]}\left[\frac{2}{kS\{-f'''(x_2)\}}\right]^{\frac{1}{3}} e^{j\frac{2}{3}\gamma_2^{\frac{3}{2}}} \int_{-\infty}^{o} e^{-j(\frac{t^3}{3}-\gamma_2 t)}dt \quad (40)$$

which can be expressed in terms of Airy integrals as

$$I_2 \simeq e^{jk[Cx_2+Sf(x_2)]}\left[\frac{2}{kS\{-f''(x_2)\}}\right]^{\frac{1}{3}} e^{j\frac{2}{3}\gamma_2^{\frac{3}{2}}} \cdot\pi\{Ai(-\gamma_2)+jBi(-\gamma_2)-jHi(-\gamma_2)\}. \quad (41)$$

The desired uniform asymptotic expression for I is therefore

$$I \simeq e^{jk[Cx_1+Sf(x_1)]}\left[\frac{2}{kS\{-f'''(x_1)\}}\right]^{\frac{1}{3}} e^{-j\frac{2}{3}\gamma_1^{\frac{3}{2}}}\pi\{Ai(-\gamma_1)-jBi(-\gamma_1) +jHi(-\gamma_1)\} + e^{jk[Cx_2+Sf(x_2)]}\left[\frac{2}{kS\{-f'''(x_2)\}}\right]^{\frac{1}{3}} \cdot e^{j\frac{2}{3}\gamma_2^{\frac{3}{2}}}\pi\{Ai(-\gamma_2)+jBi(-\gamma_2)-jHi(-\gamma_2)\} \quad (42)$$

where γ_1 and γ_2 are given in (31) and (39) respectively. On the boundary of the specular region, $x_1 = x_2 = 0$, and since $f''(0) = 0$ we have $\gamma_1 = \gamma_2 = 0$ there. The expression (42) for I then reduces to that in (27). Well within the specular region, $\gamma_1, \gamma_2 >> 1$, and for $\gamma >> 1$

$$\pi \{Ai(-\gamma) \mp jBi(-\gamma) \pm jHi(-\gamma)\} \sim \frac{\sqrt{\pi}}{\gamma^{\frac{1}{4}}} e^{\pm j\left(\frac{2}{3}\gamma^{\frac{3}{2}} - \frac{\pi}{4}\right)} \tag{43}$$

[Abramowitz and Stegun, 1964]. When this is substituted into (42), we recover (18) precisely.

6 Uniform Asymptotic Expansion, Non-specular Region

The final task is to develop a uniform expression which matches (27) into the formula (23) in the non-specular region, and the procedure is similar to that given above.

The relevant SP point is now the saddle point x_2, and by including an additional term in the Taylor series expansion of $f(x)$ about $x = x_2$ we again obtain (37). However, $f''(x_2)$ is now pure imaginary with $Im.f(x_2) \geq 0$, and therefore

$$\begin{aligned} I \quad &\simeq \quad e^{jk[Cx_2 + Sf(x_2)]} \left[\frac{2}{kS\{-f'''(x_2)\}}\right]^{\frac{1}{3}} e^{\frac{2}{3}(-\gamma_2)^{\frac{3}{2}}} \\ &\cdot \int_{\Delta_2\left[-1-\frac{1}{\delta_2}\frac{f''(x_2)}{f'''(x_2)}\right]}^{\Delta_2\left[1-\frac{1}{\delta_2}\frac{f''(x_2)}{f'''(x_2)}\right]} e^{-j(\frac{t^3}{3} - \gamma_2 t)} dt \end{aligned}$$

where Δ_2 is given in (38) and

$$\gamma_2 = -\left[\frac{kS}{2\{-f'''(x_2)\}^2}\right]^{\frac{2}{3}} \{-jf''(x_2)\}^2 \quad \leq 0. \tag{44}$$

An asymptotic expression is obtained by allowing $\Delta_2 \to \infty$, and the result is

$$I \simeq e^{jk[Cx_2 + Sf(x_2)]} \left[\frac{2}{kS\{-f'''(x_2)\}}\right]^{\frac{1}{3}} e^{\frac{2}{3}(-\gamma_2)^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{-j\left(\frac{t^3}{3} - \gamma_2 t\right)} dt,$$

which can be expressed in terms of an Airy integral as

$$I \simeq e^{jk[Cx_2+Sf(x_2)]}\left[\frac{2}{kS\{-f'''(x_2)\}}\right]^{\frac{1}{3}} e^{\frac{2}{3}(-\gamma_2)^{\frac{3}{2}}} 2\pi Ai(-\gamma_2) \tag{45}$$

with γ_2 as shown in (44).

On the boundary between the non-specular and specular regions, $x_2 = 0$ with $f(x_2) = f''(x_2) = 0$. Thus, $\gamma_2 = 0$ and (45) is then identical to (27). On the other hand, well within the non-specular region where $\gamma_2 < 0$, the Airy integral can be replaced by the leading term in its asymptotic expansion, viz

$$Ai(-\gamma_2) \sim \frac{1}{2\sqrt{\pi}}(-\gamma_2)^{-\frac{1}{4}} e^{-\frac{2}{3}(-\gamma_2)^{\frac{3}{2}}}, \tag{46}$$

and when this is inserted into (45) we recover (22).

7 Summary

In the specular region a uniform asymptotic expression valid up to and including the boundary with the non-specular region is

$$\begin{aligned} I \;\simeq\; & \sum_{i=1,2} e^{jk[Cx_i+Sf(x_i)]}\left[\frac{2}{kS|f'''(x_i)|}\right]^{\frac{1}{3}} e^{\mp j\frac{2}{3}\gamma_i^{\frac{3}{2}}} \\ & \cdot\, \pi\left\{Ai(-\gamma_i) \mp jBi(-\gamma_i) \pm jHi(-\gamma_i)\right\} \end{aligned} \tag{47}$$

with

$$\gamma_i = \left[\frac{2}{kS|f'''(x_i)|^2}\right]^{\frac{1}{3}} \{f''(x_i)\}^2 \tag{48}$$

and the upper (lower) signs for $x_i > (<) 0$. The analogous result for the non-specular region is

$$I \simeq e^{jk[Cx_2+Sf(x_2)]}\left[\frac{2}{kS|f'''(x_2)|}\right]^{\frac{1}{3}} e^{\frac{2}{3}(-\gamma_2)^{\frac{3}{2}}} 2\pi Ai(-\gamma_2) \tag{49}$$

with

$$\gamma_2 = -\left[\frac{2}{kS|f'''(x_2)|^2}\right]^{\frac{1}{3}} \{-jf''(x_2)\}^2 \tag{50}$$

where x_2 is the pure imaginary saddle point having $Im.x_2 < 0$ and $Im.f''(x_2) \geq 0$.

Reference

Abramowitz, M., and I.A. Stegun (1964), Handbook of Mathematical Functions, National Bureau of Standard, pp. 446-7.