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**A UNIFIED APPROACH TO INTEGRAL FORMULATIONS  
FOR ELECTROMAGNETIC SCATTERING**

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# A UNIFIED APPROACH TO INTEGRAL FORMULATIONS FOR ELECTROMAGNETIC SCATTERING

*(Tutorial Paper)*

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## ABSTRACT

A unified approach is presented to formulate various boundary and domain-boundary integral methods for electromagnetic scattering. Beginning with the weighted residual form of the wave equation, a variety of integral methods, including some well-known ones, are obtained by choosing different weighting and trial functions. Formulating different methods under such a common framework is important for instructional purposes and enables us to better understand their relationships. In fact, all known integral methods can be derived from the proposed general representation by choosing appropriate weighting and trial functions. To illustrate the approach, while maintaining simplicity, this paper deals only with the two-dimensional case.

## I. INTRODUCTION

Many numerical methods have been developed in the past for scattering computations. Interestingly, although the intent with all methods is to solve Maxwell's equations subject to given boundary conditions, their development has followed many different approaches. For example, the volume and/or surface integral equation methods are developed by invoking the equivalence principle [1], [2]. In contrast, the finite element methods have traditionally been based on variational principles which are closely related to the concept of conservation of energy [3], [4]. Clearly, although one should expect that all methods be inter-related, this is not apparent.

In this paper, we employ a unified approach to formulate various integral and finite element methods. The approach begins with the weighted residual equations for the fields inside and outside the scatterer(s). By choosing appropriate weighting and trial field expansion functions we then derive a variety of numerical formulations including some of the most well-known. This type of derivation is important for instructional purposes because it places all popular numerical formulations under a common framework. As a result, one can compare the similarities and differences of the methods making possible an evaluation of their effectiveness for a given application. In the following, we illustrate the unified approach by restricting our attention to the two-dimensional case for the only purpose of retaining the simplicity of the presentation. Extensions to the three-dimensional case are possible but are not considered.

In the next section we consider the scattering by a homogeneous cylinder whose formulation involves only boundary integrals. This is followed by the formulation for inhomogeneous cylinders which, as expected, involves both domain and boundary inte-

grals.

## II. BOUNDARY INTEGRAL FORMULATIONS

Consider the scattering problem illustrated in Figure 1, where a harmonic electromagnetic wave, having an angular frequency  $\omega$ , impinges upon a homogeneous, infinitely long cylinder immersed in an unbounded homogeneous medium. We shall assume that the cylinder has constitutive constants  $\epsilon_2$  and  $\mu_2$  and its axis is parallel to the  $z$ -axis. The constitutive constants of the surrounding medium will be denoted as  $\epsilon_1$  and  $\mu_1$ .

The electromagnetic field outside the cylinder, to be denoted as  $\phi_1$ , is comprised of the incident and scattered fields generated by the induced currents and charges in the volume and surface of the cylinder. It can be expressed as

$$\phi_1 = \phi^{inc} + \phi^s \quad \text{in } \Omega_\infty \quad (1)$$

in which  $\phi^{inc}$  and  $\phi^s$  denote the incident and scattered fields, respectively, and  $\Omega_\infty$  denotes the infinite region exterior to the cylinder. For  $E$ -polarization  $\phi = E_z$  and for  $H$ -polarization  $\phi = H_z$ . Since  $\phi^{inc}$  satisfies the wave equation, the scattered field satisfies the same equation

$$\nabla^2 \phi^s + k_1^2 \phi^s = 0 \quad \text{in } \Omega_\infty \quad (2)$$

subject to the Sommerfeld radiation condition

$$\lim_{\rho \rightarrow \infty} \sqrt{\rho} \left( \frac{\partial \phi^s}{\partial \rho} + j k_1 \phi^s \right) = 0 \quad (3)$$

where  $k_1 = \omega \sqrt{\mu_1 \epsilon_1}$  and  $\rho = \sqrt{x^2 + y^2}$ . Similarly, the field inside the cylinder,  $\phi_2$ , satisfies the Helmholtz equation

$$\nabla^2 \phi_2 + k_2^2 \phi_2 = 0 \quad \text{in } \Omega \quad (4)$$

where  $k_2 = \omega\sqrt{\mu_2\epsilon_2}$  and  $\Omega$  denotes the region occupied by the cylinder. The fields inside and outside the cylinder are then coupled by the boundary conditions

$$\phi_1 = \phi_2 = \phi, \quad u_1 \frac{\partial \phi_1}{\partial n} = u_2 \frac{\partial \phi_2}{\partial n} = q \quad \text{on } \Gamma \quad (5)$$

where  $u = 1/\mu_r$  for  $E$ -polarization and  $u = 1/\epsilon_r$  for  $H$ -polarization with  $\epsilon_r$  and  $\mu_r$  being the relative permittivity and permeability, respectively. Also,  $\Gamma$  denotes the boundary of the cylinder, separating  $\Omega$  and  $\Omega_\infty$ , and  $\hat{n}$  is the associated unit normal vector pointing from  $\Omega$  to  $\Omega_\infty$ . Equations (1)-(5) now completely define the boundary-value problem and we are interested in a solution of  $\phi^s$  and  $\phi_2$ .

Let us consider the scattered field first. By denoting the weighting function as  $W_1$ , the weighted residual form for (2) can be written as

$$\iint_{\Omega_\infty} W_1(\nabla^2 \phi^s + k_1^2 \phi^s) ds = 0 \quad \text{in } \Omega_\infty \quad (6)$$

where  $\phi^s$  now becomes the trial function for the scattered field. Invoking Green's second identity

$$\iint (W \nabla^2 \phi - \phi \nabla^2 W) ds = \oint \left( W \frac{\partial \phi}{\partial n} - \phi \frac{\partial W}{\partial n} \right) dl \quad (7)$$

(6) can then be written as

$$\begin{aligned} & \iint_{\Omega_\infty} \phi^s (\nabla^2 W_1 + k_1^2 W_1) ds - \oint_{\Gamma} \left( W_1 \frac{\partial \phi^s}{\partial n} - \phi^s \frac{\partial W_1}{\partial n} \right) dl \\ & + \oint_{\Gamma_\infty} \left( W_1 \frac{\partial \phi^s}{\partial \rho} - \phi^s \frac{\partial W_1}{\partial \rho} \right) dl = 0 \quad \text{in } \Omega_\infty \end{aligned} \quad (8)$$

where  $\Gamma_\infty$  denotes a circle of radius approaching infinity. Since  $\phi^s$  satisfies the Sommerfeld radiation condition (3), the integral over  $\Gamma_\infty$  vanishes provided we also choose  $W_1$

to satisfy the same condition. Thus, we have

$$\iint_{\Omega_\infty} \phi^s (\nabla^2 W_1 + k_1^2 W_1) ds - \oint_{\Gamma} \left( W_1 \frac{\partial \phi^s}{\partial n} - \phi^s \frac{\partial W_1}{\partial n} \right) dl = 0 \quad \text{in } \Omega_\infty \quad (9)$$

In a similar manner, we obtain the weighted residual form for (4) as

$$\iint_{\Omega} \phi_2 (\nabla^2 W_2 + k_2^2 W_2) ds + \oint_{\Gamma} \left( W_2 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial W_2}{\partial n} \right) dl = 0 \quad \text{in } \Omega \quad (10)$$

where  $W_2$  is another weighting function and  $\phi_2$  now becomes the trial function for the interior field.

Equations (9) and (10), together with the boundary conditions (5), form a complete set of integral equations for  $\phi^s$  and  $\phi_2$ . It remains to choose the weighting functions  $W_1$  and  $W_2$  and expansions for the trial functions  $\phi^s$  and  $\phi_2$  to obtain a system of algebraic equations for a numerical solution. It will be seen below that different choices of the weighting and trial functions lead to different formulations, some of which are well-known and commonly used. Note that since  $\Omega_\infty$  and  $\Omega$  are homogeneous it would be advantageous to choose the weighting functions  $W_1$  and  $W_2$  so that the area integrals in (9) and (10) are eliminated and this is the principle followed in the formulations described below.

### Method 1: Green's Functions for Weighting and Polynomials for Expansion

In this approach, we choose  $W_1 = G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')$  and  $W_2 = G_2(\boldsymbol{\rho}, \boldsymbol{\rho}')$  where  $G_1$  and  $G_2$  are the two-dimensional unbounded space Green's functions satisfying the partial differential equation

$$\nabla^2 G_{1,2} + k_{1,2}^2 G_{1,2} = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \quad (11)$$

where  $\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')$  is the usual Dirac delta function. Substituting (11) into (9) and (10) yields

$$\oint_{\Gamma} \left[ G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{\partial \phi^s(\boldsymbol{\rho}')}{\partial n'} - \phi^s(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right] dl' = - \begin{cases} 0 & \text{for } \boldsymbol{\rho} \in \Omega \\ (1-c)\phi^s(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \text{ on } \Gamma \\ \phi^s(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \in \Omega_{\infty} \end{cases} \quad (12)$$

and

$$\oint_{\Gamma} \left[ G_2(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{\partial \phi_2(\boldsymbol{\rho}')}{\partial n'} - \phi_2(\boldsymbol{\rho}') \frac{\partial G_2(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right] dl' = \begin{cases} \phi_2(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \in \Omega \\ c\phi_2(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \text{ on } \Gamma \\ 0 & \text{for } \boldsymbol{\rho} \in \Omega_{\infty} \end{cases} \quad (13)$$

where  $\oint$  denotes the Cauchy principle value integral with the singularities removed, and  $c = 1/2$  if  $\boldsymbol{\rho}$  is on the smooth portion of  $\Gamma$  or  $c = \alpha/2\pi$  if  $\boldsymbol{\rho}$  is at a sharp corner having an internal angle of  $\alpha$  (see Figure 2). From (12) it can be further shown that

$$\begin{aligned} & \phi^{inc}(\boldsymbol{\rho}) - \oint_{\Gamma} \left[ G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{\partial \phi_1(\boldsymbol{\rho}')}{\partial n'} - \phi_1(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right] dl' \\ &= \begin{cases} 0 & \text{for } \boldsymbol{\rho} \in \Omega \\ (1-c)\phi_1(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \text{ on } \Gamma \\ \phi_1(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \in \Omega_{\infty} \end{cases} \end{aligned} \quad (14)$$

upon making use of (1) and

$$\oint_{\Gamma} \left[ G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{\partial \phi^{inc}(\boldsymbol{\rho}')}{\partial n'} - \phi^{inc}(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right] dl' = \begin{cases} \phi^{inc}(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \in \Omega \\ c\phi^{inc}(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \text{ on } \Gamma \\ 0 & \text{for } \boldsymbol{\rho} \in \Omega_{\infty} \end{cases} \quad (15)$$

Equations (13) and (14) are recognized as the well-known boundary integral equations. To solve for  $\phi^s$  and  $\phi_2$  we may invoke the boundary conditions (5) in (13) and

(14) to obtain the coupled set of integral equations

$$\left\{ \begin{array}{l} (1-c)\phi(\boldsymbol{\rho}) + \oint_{\Gamma} \left[ \frac{1}{u_1} G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') q(\boldsymbol{\rho}') - \phi(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right] dl' = \phi^{inc}(\boldsymbol{\rho}) \\ c\phi(\boldsymbol{\rho}) - \oint_{\Gamma} \left[ \frac{1}{u_2} G_2(\boldsymbol{\rho}, \boldsymbol{\rho}') q(\boldsymbol{\rho}') - \phi(\boldsymbol{\rho}') \frac{\partial G_2(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right] dl' = 0 \end{array} \right. \quad \begin{array}{l} (16a) \\ (16b) \end{array}$$

valid for  $\boldsymbol{\rho}$  on  $\Gamma$ .

Equation (16) has been widely used in moment method implementations [5], [6] and to illustrate this procedure, we expand  $\phi$  and  $q$  as

$$\phi = \sum_{j=1}^N L_j \phi_j, \quad q = \sum_{j=1}^N L'_j q_j \quad (17)$$

with  $\phi_j$  and  $q_j$  being the unknown expansion coefficients whereas  $L_j$  and  $L'_j$  are either entire domain or subdomain polynomial expansion functions defined on  $\Gamma$ . Substituting (17) into (16) and applying Galerkin's technique yields the matrix system

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} \phi \\ q \end{Bmatrix} = \begin{Bmatrix} b \\ 0 \end{Bmatrix} \quad (18)$$

where the elements of the submatrices  $[A]$ ,  $[B]$ ,  $[C]$  and  $[D]$  are given by

$$A_{ij} = (1-c) \oint_{\Gamma} L_i(\boldsymbol{\rho}) L_j(\boldsymbol{\rho}) dl - \oint_{\Gamma} \left[ L_i(\boldsymbol{\rho}) \oint_{\Gamma} L_j(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} dl' \right] dl \quad (19)$$

$$B_{ij} = \frac{1}{u_1} \oint_{\Gamma} \left[ L_i(\boldsymbol{\rho}) \oint_{\Gamma} L'_j(\boldsymbol{\rho}') G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') dl' \right] dl \quad (20)$$

$$C_{ij} = c \oint_{\Gamma} L_i(\boldsymbol{\rho}) L_j(\boldsymbol{\rho}) dl + \oint_{\Gamma} \left[ L_i(\boldsymbol{\rho}) \oint_{\Gamma} L_j(\boldsymbol{\rho}') \frac{\partial G_2(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} dl' \right] dl \quad (21)$$

$$D_{ij} = -\frac{1}{u_2} \oint_{\Gamma} \left[ L_i(\boldsymbol{\rho}) \oint_{\Gamma} L'_j(\boldsymbol{\rho}') G_2(\boldsymbol{\rho}, \boldsymbol{\rho}') dl' \right] dl \quad (22)$$

and the elements of  $\{b\}$  are given by

$$b_i = \oint_{\Gamma} L_i(\boldsymbol{\rho}) \phi^{inc}(\boldsymbol{\rho}) dl \quad (23)$$



If we replace the weighting functions  $L_i(\boldsymbol{\rho})$  in (19)-(23) by  $\delta(\boldsymbol{\rho} - \boldsymbol{\rho}_i)$ , (18) becomes the well-known point-matching system [2].

## Method 2: Eigenfunctions for Weighting and Polynomials for Expansion

In this approach, we choose  $W_1 = \Psi_{1i}$  and  $W_2 = \Psi_{2i}$  where  $\Psi_{1i}$  and  $\Psi_{2i}$  satisfy the Helmholtz equations

$$\nabla^2 \Psi_{1i} + k_1^2 \Psi_{1i} = 0 \quad \text{for } \boldsymbol{\rho} \text{ in } \Omega_\infty \quad (24a)$$

and

$$\nabla^2 \Psi_{2i} + k_2^2 \Psi_{2i} = 0 \quad \text{for } \boldsymbol{\rho} \text{ in } \Omega \quad (24b)$$

respectively. Provided we choose  $\Psi_{1i}$  to satisfy the Sommerfeld radiation condition (3), from (9) and (10) we then obtain

$$\oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi^s}{\partial n} - \phi^s \frac{\partial \Psi_{1i}}{\partial n} \right) dl = 0 \quad \text{in } \Omega_\infty \quad (25)$$

and

$$\oint_{\Gamma} \left( \Psi_{2i} \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \Psi_{2i}}{\partial n} \right) dl = 0 \quad \text{in } \Omega \quad (26)$$

Further, by invoking the boundary conditions (5), (25) and (26) become

$$\left\{ \begin{array}{l} \oint_{\Gamma} \left( \frac{1}{u_1} \Psi_{1i} q - \phi \frac{\partial \Psi_{1i}}{\partial n} \right) dl = \oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial \Psi_{1i}}{\partial n} \right) dl \\ \oint_{\Gamma} \left( \frac{1}{u_2} \Psi_{2i} q - \phi \frac{\partial \Psi_{2i}}{\partial n} \right) dl = 0 \end{array} \right. \quad (27a)$$

$$\left. \right\} \quad (27b)$$

for  $\boldsymbol{\rho}$  on  $\Gamma$ , representing two coupled integral equations which can also be solved via the numerical procedure illustrated in Method 1. In particular, upon substitution of the expansion of  $\phi$  and  $q$  given in (17), we obtain a system of algebraic equations of the same

form as in (18). However, the elements of the submatrices are different and in this case we find

$$A_{ij} = - \oint_{\Gamma} \frac{\partial \Psi_{1i}}{\partial n} L_j dl \quad (28)$$

$$B_{ij} = \frac{1}{u_1} \oint_{\Gamma} \Psi_{1i} L'_j dl \quad (29)$$

$$C_{ij} = - \oint_{\Gamma} \frac{\partial \Psi_{2i}}{\partial n} L_j dl \quad (30)$$

$$D_{ij} = \frac{1}{u_2} \oint_{\Gamma} \Psi_{2i} L'_j dl \quad (31)$$

and

$$b_i = \oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial \Psi_{1i}}{\partial n} \right) dl \quad (32)$$

It remains to specify  $\Psi_{1i}$  and  $\Psi_{2i}$  and this may depend on the geometry of the scatterer; however, satisfactory results can be obtained for smooth convex scatterers having nearly circular cross sections by setting  $\Psi_{1i}$  and  $\Psi_{2i}$  equal to the cylindrical eigenfunctions  $\Psi_{1i}^{(1)}$  and  $\Psi_{2i}^{(4)}$ , respectively, defined later in (37) and (39). The approach with this choice of weighting functions is designated as Method 2. We remark that in comparison with (19)-(22), the integrals in (28)-(31) do not have singular integrands and can, thus, be evaluated numerically without difficulty. In addition, this method has the added advantage of not suffering from the internal resonances which are associated with (16). It has recently been employed for a solution of the eigenvalue problem pertaining to optical waveguides [7]. However, the method is not popular and possible reasons for this are (i) it requires the computation of high order Bessel and Hankel functions, (ii) the procedure for treating the integrand singularities in the evaluation of (19)-(22) has been well-established and (iii) Method 1 has been successfully applied to a variety of

geometries.

In passing, we note that the eigenfunctions are not the only choice for  $\Psi_{1i}$  and  $\Psi_{2i}$  and particularly, for cylinders whose cross sections deviate substantially from a circle, they may result in ill-conditioned systems. In that case a different choice may be more appropriate. Observing that (25) and (26) are valid for any  $\Psi_{1i}$  and  $\Psi_{2i}$  provided they satisfy (24), we may choose  $\Psi_{1i}$  to be  $H_0^{(2)}(k_1|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  and  $\Psi_{2i}$  to be  $J_0(k_2|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  where  $H_0^{(2)}(\bullet)$  denotes the zeroth order Hankel function of the second kind,  $J_0(\bullet)$  denotes the zeroth order Bessel function, and  $\boldsymbol{\rho}_i$  denotes the location of the  $i$ th testing point residing on or inside  $\Gamma$ . The only concern with such a choice is that  $\boldsymbol{\rho}_i$  ( $i = 1, 2, 3, \dots$ ) must be judiciously chosen so that  $H_0^{(2)}(k_1|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  form a complete set for representing  $\phi^s$  and so do  $J_0(k_2|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  for  $\phi_2$ . We may also choose  $\Psi_{2i}$  to be  $H_0^{(2)}(k_2|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  instead of  $J_0(k_2|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  but  $\boldsymbol{\rho}_i$  must now reside on or outside  $\Gamma$  so that  $\Psi_{2i}$  is non-singular in  $\Omega$ . Finally, we remark that if we choose  $\Psi_{1i}$  to be  $H_0^{(2)}(k_1|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  and  $\Psi_{2i}$  to be  $H_0^{(2)}(k_2|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  with  $\boldsymbol{\rho}_i$  residing on  $\Gamma$ , then (27) becomes (16) and the resulting system given by (28)-(32) becomes identical to that given by (19)-(23) with point-matching.

### Method 3: Eigenfunctions for both Weighting and Expansion

In this method, we again choose the eigenfunction  $\Psi_{1i}$  satisfying (24a) as the weighting function  $W_1$ . With this choice, (9) becomes (27a) and can be written as

$$\oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \Psi_{1i}}{\partial n} \right) dl = \oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial \Psi_{1i}}{\partial n} \right) dl \quad (33)$$

assuming again that  $\Psi_{1i}$  satisfies the radiation condition (3). We observe that all quantities associated with the right hand side integral of (33) are known and to facilitate its evaluation we may introduce a fictitious circular boundary  $\Gamma_1$ , sufficiently large to

enclose the entire scatterer (see Figure 3). By employing Green's second identity it can be shown that

$$\oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial \Psi_{1i}}{\partial n} \right) dl = \oint_{\Gamma_1} \left( \Psi_{1i} \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial \Psi_{1i}}{\partial n} \right) dl \quad (34)$$

provided no sources of the incident field exist in the region enclosed by  $\Gamma$  and  $\Gamma_1$  and thus, (33) can be rewritten as

$$\oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \Psi_{1i}}{\partial n} \right) dl = \oint_{\Gamma_1} \left( \Psi_{1i} \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial \Psi_{1i}}{\partial n} \right) dl. \quad (35)$$

The evaluation of the right hand side integral can now be substantially simplified if  $\phi^{inc}$  is expanded in terms of basis which are orthogonal to  $\Psi_{1i}$ . We may choose to expand  $\phi^{inc}$  as

$$\phi^{inc} = \sum_{i=1}^{\infty} a_i \Psi_{1i}^{(4)} = \sum_{i=1}^{\infty} a_i J_m(k_1 \rho)_{\cos}^{\sin} m \phi; \quad m = \text{int}(i/2) \quad (36)$$

where  $\sin m\phi$  is used for even  $i$  and  $\cos m\phi$  is used for odd  $i$ . Also,  $J_m(\bullet)$  denote the  $m$ th order Bessel functions and the expansion constants  $a_i$  can be found by invoking the orthogonality of the expansion functions. Substituting (36) into (35) and choosing

$$\Psi_{1i} = \Psi_{1i}^{(1)} = H_m^{(2)}(k_1 \rho)_{\cos}^{\sin} m \phi \quad (37)$$

with  $H_m^{(2)}(\bullet)$  denoting the  $m$ th order Hankel functions of the second kind, we obtain

$$-\frac{j}{2} \oint_{\Gamma} \left( \Psi_{1i}^{(1)} \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \Psi_{1i}^{(1)}}{\partial n} \right) dl = a_i \quad (38)$$

upon invoking the orthogonality of the sine and cosine functions and making use of the Wronskian for the Bessel functions. This provides a global relation between the field and its normal derivative on the boundary.

Equation (38) is sufficient for the solution of  $\phi_1$  or  $\partial\phi_1/\partial n$  provided either of these vanishes on  $\Gamma$  as is the case for a perfectly conducting boundary. For the general case, however, a solution for both  $\phi_1$  and  $\partial\phi_1/\partial n$  requires that (38) be supplemented by an additional equation which can be obtained from the interior field formulation. An appropriate eigenfunction representation for the interior field  $\phi_2$  is

$$\phi_2 = \sum_{i=1}^{\infty} g_i \Psi_{2i}^{(4)} = \sum_{i=1}^{\infty} g_i J_m(k_2 \rho)_{\cos}^{\sin} m \phi \quad (39)$$

where again  $m = \text{int}(i/2)$  and  $g_i$  are unknown expansion constants. Strictly speaking, this expansion is only valid for  $\rho$  residing on or inside a circle which is completely enclosed by  $\Gamma$ . However, Waterman [11] showed that the expansion functions  $\Psi_{2i}^{(4)}$  also form a closed complete set capable of representing the unknown surface field on  $\Gamma$ . He further showed that the expansion (39) converges in the mean on  $\Gamma$  and is differentiable there. This was proven through the identity

$$\oint_{\Gamma} \left( W_2 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial W_2}{\partial n} \right) dl = \oint_{\Gamma_2} \left( W_2 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial W_2}{\partial n} \right) dl \quad (40)$$

valid for  $W_2$  being non-singular within the region enclosed by  $\Gamma$  and  $\Gamma_2$ , where  $\Gamma_2$  is a circular contour completely enclosed by  $\Gamma$  as shown in Figure 3, and by choosing  $W_2$  to be  $\Psi_{2i}^{(1)}$  and  $\Psi_{2i}^{(4)}$ .

Thus, (39) can be used to approximately represent the boundary field and its normal derivative on  $\Gamma$  and by invoking the boundary conditions (5) and substituting (39) into (38) we obtain

$$[Q]\{g\} = \{a\} \quad (41)$$

where

$$Q_{ij} = -\frac{j}{2} \oint_{\Gamma} \left( \frac{u_2}{u_1} \Psi_{1i}^{(1)} \frac{\partial \Psi_{2j}^{(4)}}{\partial n} - \Psi_{2j}^{(4)} \frac{\partial \Psi_{1i}^{(1)}}{\partial n} \right) dl \quad (42)$$

This matrix system can be solved for the expansion coefficients  $g_i$  which can then be used in (39) to obtain the boundary field and its normal derivative. Elsewhere, the scattered field can be found by evaluating the pertinent boundary integral (see for example (14)).

An alternative way to evaluate the scattered field is to expand it as

$$\phi^s = \sum_{i=1}^{\infty} f_i \Psi_{1i}^{(1)} = \sum_{i=1}^{\infty} f_i H_m^{(2)}(k_1 \rho)_{\cos}^{\sin} m \phi \quad \rho \geq \rho_1 \quad (43)$$

where  $f_i$  are the unknown constant coefficients to be determined and  $\rho_1$  is the radius of the circle  $\Gamma_1$ . Substituting (43) into the integral identity

$$\begin{aligned} & \oint_{\Gamma} \left( W_1 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial W_1}{\partial n} \right) dl \\ &= \oint_{\Gamma_1} \left( W_1 \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial W_1}{\partial n} \right) dl + \oint_{\Gamma_1} \left( W_1 \frac{\partial \phi^s}{\partial n} - \phi^s \frac{\partial W_1}{\partial n} \right) dl \end{aligned} \quad (44)$$

(which can be proved using Green's second identity) and setting  $W_1 = \Psi_{1i}^{(4)}$  we obtain

$$f_i = \frac{j}{2} \oint_{\Gamma} \left( \Psi_{1i}^{(4)} \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \Psi_{1i}^{(4)}}{\partial n} \right) dl \quad (45)$$

since the first right hand side integral in (44) vanishes when the expansion (36) is introduced. Further, by invoking the boundary conditions (5) and making use of the expansion (39) in (45), we find

$$\{f\} = [P]\{g\} \quad (46)$$

where the elements of  $[P]$  are given by

$$P_{ij} = \frac{j}{2} \oint_{\Gamma} \left( \frac{u_2}{u_1} \Psi_{1i}^{(4)} \frac{\partial \Psi_{2j}^{(4)}}{\partial n} - \Psi_{2j}^{(4)} \frac{\partial \Psi_{1i}^{(4)}}{\partial n} \right) dl \quad (47)$$

When (46) is combined with (41) we then obtain the system

$$\{f\} = [T]\{a\}, \quad [T] = [P][Q]^{-1} \quad (48)$$

which can be solved by properly truncating the infinite series in (36), (39) and (43).

The formulation described herein is the two-dimensional version of the so-called extended boundary condition method which is also known as the  $T$ -matrix method or the null field method. The method was originally developed by expanding the Green's dyadic and employing the concept of analytic continuation [8]. It can be also developed by invoking Huygens' principle [9] or, alternatively, by using Schelkunoff's equivalence principle [10]. The formulation presented here is, however, more parallel to the one described in [11] which employs the concept of flux conservation. Similarly to Method 2, this formulation does not exhibit difficulties associated with internal resonances. Furthermore, for scatterers having nearly circular cross sections the number of unknowns required for a given accuracy can be smaller than that for the previous two methods. However, for elongated cylinders, the field representations (39) and (43) may be inadequate or slowly converging (consequently the system may be ill-conditioned) making the method much less attractive. It was seen, though, that (43) can be avoided if (41) is solved independent of (46) and (14) is then used to compute the scattered field. Nevertheless, the approximate representation of (39) for the boundary field and its normal derivative is essential to the method and its use is the major source of error when simulating scatterers with elongated cross sections.

The aforementioned three different methods for the solution of the boundary-value problem defined by (1)-(5) are summarized in Table 1. These methods share a common

feature that the same type of weighting or trial field expansion functions are employed for both fields inside and outside the scatterer. This is, of course, not required and in fact we may choose to formulate the interior field using Method 1 and the exterior field in accordance with Method 2 or 3. Such combinations lead to a variety of formulations (not necessarily more advantageous than those just presented). In passing, we should also note that the methods described herein are equally applicable to multilayered, conducting and impedance scatterers provided the boundary conditions are accordingly modified. In particular for perfectly conducting and impedance cylinders the internal field vanishes leading to substantial simplification in formulations and a reduction in unknowns.

Table 1: Different Choices for Weighting and Expansion Functions

Method	Weighting Functions	Expansion Functions	Eqn. #
Method 1	Green's Functions	Polynomials	(18), (19)-(23)
Method 2	Eigenfunctions	Polynomials	(18), (28)-(32)
Method 3	Eigenfunctions	Eigenfunctions	(42), (47), (48)

### III. DOMAIN-BOUNDARY INTEGRAL FORMULATIONS

The solution methods presented in the previous section are restricted to homogeneous scatterers, and whereas the expressions involving the exterior field  $\phi_1$  are still valid for inhomogeneous scatterers, those associated with the interior field  $\phi_2$  must be modified to account for the inhomogeneity. When the scatterer is inhomogeneous, the appropriate wave equation for the interior field is

$$\nabla \cdot (u_2 \nabla \phi_2) + k_0^2 v_2 \phi_2 = 0 \quad \text{in } \Omega \quad (49)$$



where  $\phi$  and  $u$  are the same as those defined earlier,  $k_0$  is the free-space wavenumber, and  $v = \epsilon_r$  for  $E$ -polarization and  $v = \mu_r$  for  $H$ -polarization. The boundary-value problem is then completely defined by (1)-(3), (5) and (49).

To derive integral equations for  $\phi_2$ , we can follow a similar procedure to that outlined for the homogeneous scatterers. The integral of the weighted residual of (49) can be written as

$$\iint_{\Omega} W_2 [\nabla \cdot (u_2 \nabla \phi_2) + k_0^2 v_2 \phi_2] ds = 0 \quad \text{in } \Omega \quad (50)$$

and by invoking the identity

$$\nabla \cdot (W_2 u_2 \nabla \phi_2) = W_2 [\nabla \cdot (u_2 \nabla \phi_2)] + u_2 \nabla \phi_2 \cdot \nabla W_2 \quad (51)$$

and the divergence theorem we obtain

$$\iint_{\Omega} (u_2 \nabla W_2 \cdot \nabla \phi_2 - k_0^2 v_2 W_2 \phi_2) ds - \oint_{\Gamma} u_2 W_2 \frac{\partial \phi_2}{\partial n} dl = 0 \quad \text{in } \Omega \quad (52)$$

As before, different choices for  $W_2$  lead to different integral formulations and some of these are discussed below.

### Method 1: Green's Function for Weighting

The Green's function  $G_2$  associated with  $(u_2, v_2)$  is usually not available. However, we may choose  $G_1$ , defined in (11), for  $W_2$  to rewrite (52) as

$$\begin{aligned} & \iint_{\Omega} \left[ u_2(\boldsymbol{\rho}) \nabla G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \cdot \nabla \phi_2(\boldsymbol{\rho}) - k_0^2 v_2(\boldsymbol{\rho}) G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \phi_2(\boldsymbol{\rho}) \right] ds \\ & - \oint_{\Gamma} u_2(\boldsymbol{\rho}) G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{\partial \phi_2(\boldsymbol{\rho})}{\partial n} dl = 0 \end{aligned} \quad (53)$$

and through the repeated use of the identity (51) and the divergence theorem we find

$$\iint_{\Omega} \left[ k_0^2 v_2(\boldsymbol{\rho}) G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') + u_2(\boldsymbol{\rho}) \nabla^2 G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') + \nabla u_2(\boldsymbol{\rho}) \cdot \nabla G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \right] \phi_2(\boldsymbol{\rho}) ds$$

$$+ \oint_{\Gamma} u_2(\boldsymbol{\rho}) \left[ G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{\partial \phi_2(\boldsymbol{\rho})}{\partial n} - \phi_2(\boldsymbol{\rho}) \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n} \right] dl = 0 \quad (54)$$

Further, by making use of (11), (54) becomes

$$\begin{aligned} & \iint_{\Omega} \left\{ [k_2^2(\boldsymbol{\rho}') - k_1^2] u_2(\boldsymbol{\rho}') G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') + \nabla' u_2(\boldsymbol{\rho}') \cdot \nabla' G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \right\} \phi_2(\boldsymbol{\rho}') ds' \\ & + \oint_{\Gamma} u_2(\boldsymbol{\rho}') \left[ G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{\partial \phi_2(\boldsymbol{\rho}')}{\partial n'} - \phi_2(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} \right] dl' \\ & = \begin{cases} u_2(\boldsymbol{\rho}) \phi_2(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \in \Omega \\ cu_2(\boldsymbol{\rho}) \phi_2(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \text{ on } \Gamma \\ 0 & \text{for } \boldsymbol{\rho} \in \Omega_{\infty} \end{cases} \quad (55) \end{aligned}$$

where we have also interchanged the primed and unprimed coordinates. Combining this with (14), which is still valid for the exterior field, through the boundary conditions (5), we finally obtain

$$\begin{aligned} & \iint_{\Omega} \left\{ [k_2^2(\boldsymbol{\rho}') - k_1^2] u_2(\boldsymbol{\rho}') G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') + \nabla' u_2(\boldsymbol{\rho}') \cdot \nabla' G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') \right\} \phi_2(\boldsymbol{\rho}') ds' \\ & + \oint_{\Gamma} [u_1 - u_2(\boldsymbol{\rho}')] \phi(\boldsymbol{\rho}') \frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial n'} dl' + u_1 \phi^{inc}(\boldsymbol{\rho}) \\ & = \begin{cases} u_2(\boldsymbol{\rho}) \phi_2(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \in \Omega \\ [(1 - c)u_1 + cu_2(\boldsymbol{\rho})] \phi(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \text{ on } \Gamma \\ u_1 \phi_1(\boldsymbol{\rho}) & \text{for } \boldsymbol{\rho} \in \Omega_{\infty} \end{cases} \quad (56) \end{aligned}$$

which is the integral equation derived in [12]. We note that this integral equation involves only a single unknown field component within the inhomogeneous scatterer and along the contours defining an abrupt change in material constants. It has been shown, however, to be equivalent to the volume integral equations involving the vector polarization currents given in [13]-[15] which require three unknowns for a simulation of general inhomogeneous scatterers and are also associated with higher singular kernels. A solution of (56) for

general scatterers has been given in [16] using pulse basis and point matching and has also been implemented with isoparametric elements [17] to yield a more accurate simulation in the case of high contrast dielectrics.

## Method 2: Eigenfunctions for Weighting

The eigenfunctions  $\Psi_{2i}$  associated with  $(u_2, v_2)$  are usually not available. However, if we choose  $W_2$  to be the eigenfunctions  $\Psi_i$  satisfying the Helmholtz equation

$$\nabla^2 \Psi_i + k^2 \Psi_i = 0 \quad \text{for } \rho \text{ in } \Omega \quad (57)$$

(52) becomes

$$\iint_{\Omega} \left( u_2 \nabla \Psi_i \cdot \nabla \phi_2 - k_0^2 v_2 \Psi_i \phi_2 \right) ds - \oint_{\Gamma} u_2 \Psi_i \frac{\partial \phi_2}{\partial n} dl = 0 \quad \text{in } \Omega \quad (58)$$

In (57),  $k$  is an arbitrary constant and could be chosen as the average value of  $k_2$  for best result. Following the same procedure as described above under Method 1, (58) can be rewritten as

$$\begin{aligned} \iint_{\Omega} \left[ (k_2^2 - k^2) u_2 \Psi_i + \nabla u_2 \cdot \nabla \Psi_i \right] \phi_2 ds \\ + \oint_{\Gamma} u_2 \left( \Psi_i \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \Psi_i}{\partial n} \right) dl = 0 \quad \text{in } \Omega \end{aligned} \quad (59)$$

which parallels that in (55). This together with (25) for the external fields and the boundary conditions (5) yield the two coupled equations

$$\left\{ \oint_{\Gamma} \left( \frac{1}{u_1} \Psi_{1i} q - \phi \frac{\partial \Psi_{1i}}{\partial n} \right) dl = \oint_{\Gamma} \left( \Psi_{1i} \frac{\partial \phi^{inc}}{\partial n} - \phi^{inc} \frac{\partial \Psi_{1i}}{\partial n} \right) dl \right. \quad (60a)$$

$$\left. \iint_{\Omega} \left[ (k_2^2 - k^2) u_2 \Psi_i + \nabla u_2 \cdot \nabla \Psi_i \right] \phi_2 ds + \oint_{\Gamma} \left( \Psi_i q - u_2 \phi \frac{\partial \Psi_i}{\partial n} \right) dl = 0 \right. \quad (60b)$$

These integral equations can now be discretized by expanding the interior field, the boundary field and its normal derivative. Clearly, the integrands of (60) have no singularity and their evaluation can therefore be carried out numerically without difficulty. It should be noted however that in most cases (56) yields a more efficient solution because it does not involve the normal derivative of the boundary field.

Similarly to that remarked earlier in Method 2 of Section II, we may also choose other solutions to (24a) and (57) for  $\Psi_{1i}$  and  $\Psi_i$  (other than the eigenfunctions). Particularly, if we choose both  $\Psi_{1i}$  and  $\Psi_i$  to be  $H_0^{(2)}(k_1|\boldsymbol{\rho} - \boldsymbol{\rho}_i|)$  with  $\boldsymbol{\rho}_i$  residing on  $\Gamma$ , then (60a) and (60b) can be combined to yield an integral equation identical to (56).

### Method 3: Subdomain Basis for Weighting and Expansion

If region  $\Omega$  is subdivided into a number of small elements and within each element a set of polynomials is used for the weighting and field expansion functions, we then obtain the well-known finite element method. Specifically, the field in  $\Omega$  and its normal derivative on  $\Gamma$  are expressed as

$$\phi_2 = \sum_{j=1}^{N_T} N_j^g \phi_j, \quad q = u_2 \frac{\partial \phi_2}{\partial n} = \sum_{j=1}^{N_B} L_j^g q_j \quad (61)$$

where  $N_T$  denotes the total number of nodes in  $\Omega$  (including boundary nodes) and  $N_B$  denotes the number of boundary nodes. Also,  $N_j^g$  and  $L_j^g$  are known expansion or shape functions chosen so that  $\phi_j$  and  $q_j$  represent, respectively, the unknown  $\phi_2$  and  $q$  at the  $j$ th node. Substituting (61) into (52) and choosing  $W_2$  to be  $N_i^g$ , we obtain the finite element matrix equation

$$\begin{bmatrix} K_{\Gamma\Gamma} & K_{\Gamma I} \\ K_{I\Gamma} & K_{II} \end{bmatrix} \begin{Bmatrix} \phi_{\Gamma} \\ \phi_I \end{Bmatrix} = \begin{bmatrix} H_{\Gamma\Gamma} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} q \\ 0 \end{Bmatrix} \quad (62)$$

where the subscripts  $\Gamma$  and  $I$  are used to denote the fields at the nodes on  $\Gamma$  and those interior of  $\Gamma$ , respectively. The elements of the matrices  $[K]$  and  $[H]$  are given by

$$K_{ij} = \iint_{\Omega} \left( u_2 \nabla N_i^g \cdot \nabla N_j^g - k_0^2 v_2 N_i^g N_j^g \right) ds \quad (63)$$

$$H_{ij} = \oint_{\Gamma} N_i^g L_j^g dl \quad (64)$$

Clearly, the system (62) provides a relation between the boundary field and its normal derivative in a discrete form and to solve this we must introduce an additional relation between the two quantities at the boundary. This must be derived from the exterior field formulation and can take the form of an integral relation based on one of the methods described earlier. Below we discuss various approaches.

#### A. Finite element/boundary element method

If the exterior field is formulated in accordance with Methods 1 and 2 described in Section II, the resulting relation between the exterior boundary field and its normal derivative is the first row of (18). When this is combined with (62), we obtain the system

$$\begin{bmatrix} A & 0 & B \\ K_{\Gamma\Gamma} & K_{\Gamma I} & -H_{\Gamma\Gamma} \\ K_{I\Gamma} & K_{II} & 0 \end{bmatrix} \begin{Bmatrix} \phi_{\Gamma} \\ \phi_I \\ q \end{Bmatrix} = \begin{Bmatrix} b \\ 0 \\ 0 \end{Bmatrix} \quad (65)$$

where the elements of  $[A]$ ,  $[B]$  and  $\{b\}$  are given by (19)-(23) or (28)-(32), depending on the method used to formulate the exterior field. We observe that the  $[K]$  submatrices are sparse and banded whereas  $[A]$  and  $[B]$  are fully populated matrices. However, substantial memory reduction can be achieved if we first solve (62) to obtain  $\{\phi_{\Gamma}\} = [E]\{q\}$  and then solve the equation  $([A][E] + [B])\{q\} = \{b\}$  for  $\{q\}$ . This finite element/boundary element method has been recently applied to a variety of two-dimensional scattering com-

putations [18]-[21] and has also been found to be particularly attractive for scattering by grooves and slots in thick conducting planes [22]-[24].

## B. Finite element/extended boundary condition method

This method was proposed by Morgan *et al.* [25] to formulate the scattering by a body of revolution. The exterior fields are formulated following Method 3 of Section II. In particular, from (38) we have

$$-\frac{j}{2} \oint_{\Gamma} \left( \frac{1}{u_1} \Psi_{1i}^{(1)} q - \phi_{\Gamma} \frac{\partial \Psi_{1i}^{(1)}}{\partial n} \right) dl = a_i \quad (66)$$

where  $a_i$  correspond to the expansion coefficients associated with the incident field. Equations (62) and (66) now provide a coupled pair of relations between the field and its normal derivative at the boundary. To solve this coupled set of equations we can first expand  $q$  and  $\phi_{\Gamma}$  in a form compatible with (61) and this results in the same finite element/boundary element method described above with the exterior fields formulated in accordance with Method 2 of Section II. An alternative is to generate from (62) a set of coupled expansion bases for  $q$  and  $\phi_{\Gamma}$  to be used in (66) for a solution of the expansion coefficients. Specifically, we first choose a set of known bases for  $\{q\}$ , denoted by  $\hat{q}_j$ , and for each of these we can use (62) to compute the corresponding coupled basis for  $\{\phi_{\Gamma}\}$ , denoted as  $\hat{\phi}_j$ . Since the system (62) is banded and sparse, the computational demands associated with this repetitive process are quite manageable. Thus, we can expand  $q$  and  $\phi_{\Gamma}$  using a single set of the expansion coefficients  $g_j$  as

$$q = \sum_{j=1}^N g_j \hat{q}_j, \quad \phi_{\Gamma} = \sum_{j=1}^N g_j \hat{\phi}_j \quad (67)$$

where  $(\hat{\phi}_j, \hat{q}_j)$  are the known numerical bases. When these expansions are substituted

into (66) we obtain the system

$$[Q]\{g\} = \{a\} \quad (68)$$

which is of the same form as in that (41) with the elements of  $[Q]$  given by

$$Q_{ij} = -\frac{j}{2} \oint_{\Gamma} \left( \frac{1}{u_1} \Psi_{1i}^{(1)} \hat{q}_j - \hat{\phi}_j \frac{\partial \Psi_{1i}^{(1)}}{\partial n} \right) dl \quad (69)$$

It remains to choose  $\hat{q}_j$  and this can be done rather arbitrarily as long as a set of  $\hat{q}_j$  are capable of representing the true solution of  $q$  on  $\Gamma$ . For example, we may choose the entire domain basis  $\hat{q}_j = \sin \frac{j}{2} \phi$  (for odd  $j$ ) and  $\hat{q}_j = \cos \frac{j-1}{2} \phi$  (for even  $j$ ) for certain geometries.

Once the expansion coefficients  $g_j$  are solved from (68), the scattered field can be computed by evaluating the integrals involving the boundary field and its normal derivative given in (67). Alternatively, we can follow the procedure illustrated in Method 3 of Section II by first expanding the scattered field as in (43) and then solving for the expansion coefficients  $f_i$  from (46), with  $P_{ij}$  now given by

$$P_{ij} = \frac{j}{2} \oint_{\Gamma} \left( \frac{1}{u_1} \Psi_{1i}^{(4)} \hat{q}_j - \hat{\phi}_j \frac{\partial \Psi_{1i}^{(4)}}{\partial n} \right) dl \quad (70)$$

### C. Unimoment method

The unimoment method was first proposed by Mei [26], [27] and is very similar to the finite element/extended boundary condition method. In fact, the latter is an extension of the former. Based on this formulation the boundary  $\Gamma$  is chosen to be a circle, say  $\Gamma_1$ , of radius  $\rho_1$ . As a result, the scattered field can be represented by (43) and through the same procedure described above, a set of coupled bases similar to those given in (67) can

be generated from (62) for the fields inside  $\Gamma_1$ . The boundary conditions (5) are then imposed through their weighted residual forms

$$\left\{ \begin{array}{l} \oint_{\Gamma_1} W (\phi^s + \phi^{inc}) dl = \oint_{\Gamma_1} W \phi_{\Gamma} dl \\ \oint_{\Gamma_1} W u_1 \left( \frac{\partial \phi^s}{\partial n} + \frac{\partial \phi^{inc}}{\partial n} \right) dl = \oint_{\Gamma_1} W q dl \end{array} \right. \quad \begin{array}{l} (71a) \\ (71b) \end{array}$$

where  $W$  denotes the weighting function. Substituting (43) and (67) into the above and choosing a set of weighting functions for  $W$  yields a system of equations for a solution of  $f_i$  and  $g_i$ . It is then seen that the unimoment method results in a  $2N \times 2N$  matrix whereas the finite element/extended boundary condition method leads to an  $N \times N$  matrix. Further, this approach introduces additional nodal fields to be computed in the region between the structure's boundary and the fictitious circular boundary  $\Gamma_1$ .

In passing, we note that the finite element/extended boundary condition method and the unimoment method usually result in a smaller size matrix than the finite element/boundary element method. This has often been attributed to the expansion of the scattered field in the form of (43), but is not true. In the formulation of the finite element/extended boundary condition method it is clearly seen that the expansion (43) is not an essential element since a solution can be obtained directly from (68). The smaller size matrix should actually be attributed to the use of coupled bases. A matrix of the same size can also be obtained when the boundary integral formulation (16a) or (27a) is used in conjunction with the coupled bases and this becomes obvious when the coupled basis expansion (67) is substituted into (16a) or (27a). The only concern with the use of the coupled bases is how to choose the known bases  $\hat{q}_j$  which should be capable of representing the true solution of  $q$  on the boundary to be used to compute the coupled



bases  $\hat{\phi}_j$ . The safest choice is to choose a set of orthogonal pulse basis, i.e.,  $q_j = 1$  and  $q_i = 0$  for  $i \neq j$ . However, this choice offers no advantage over the standard approach adopted in the finite element/boundary element method since the resulting full matrix will have the same size. The coupled bases will have an advantage only if the number of the coupled bases is smaller than the number of the boundary nodes.

Table 2: Different Choices for Weighting Function  $W_2$

Method	Weighting Function $W_2$	Resulting Formulation	Eqn. #
Method 1	Green's Function	Singular Integral Equation	(56)
Method 2	Eigenfunctions	Non-singular Integral Equation	(60)
Method 3	Subdomain Basis	Finite Element Method	(62)

The three methods presented in this section are summarized in Table 2. In Method 1, subsectional bases are usually chosen to expand the boundary and interior fields, and similar expansion functions could be used in Method 2. For Method 3, subsectional expansion and weighting functions must be chosen and three variations of this method were presented depending on how the exterior fields are formulated.

#### IV. CONCLUDING REMARKS

In this paper, we presented a unified approach to formulate various boundary and domain-boundary integral methods, including the finite element method, for electromagnetic scattering. First, we introduced weighting and trial functions to formulate two weighted residual equations for the fields inside and outside the scatterer. It was shown

that different choices for the weighting and trial functions result in different integral equations or methods for the solution of the exterior and interior fields. Among the various methods presented in this paper, those employing the unbounded space Green's functions in conjunction with subsectional field expansions (Method 1 of Sections II and III) have been most frequently used primarily because they are capable of treating arbitrarily shaped geometries. Method 2 of Sections II and III, where eigenfunctions are employed for weighting, was presented to merely show that there are other possible ways to formulate the problem. The extended boundary condition method (Method 3 of Section II) could be efficient for far field computations, but it is not likely to provide accurate result for the induced field particularly for those structures which deviate substantially from a circular cross-section. This is primarily due to the approximate representation of the field and its normal derivative at the boundary in terms of cylindrical eigenfunctions. However, by employing the finite element method to formulate the interior fields, the accuracy of the method can be restored. The resulting formulation would be the finite element/extended boundary condition method (Method 3B of Section III). Nevertheless, to date, among the formulations which incorporate the finite element method those discussed under Method 3A of Section III has been most widely used.

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## FIGURE CAPTIONS

**Fig. 1** Geometry of the scattering problem.

**Fig. 2** Illustration of the internal angle.

**Fig. 3** Illustration of  $\Gamma_1$  and  $\Gamma_2$ .

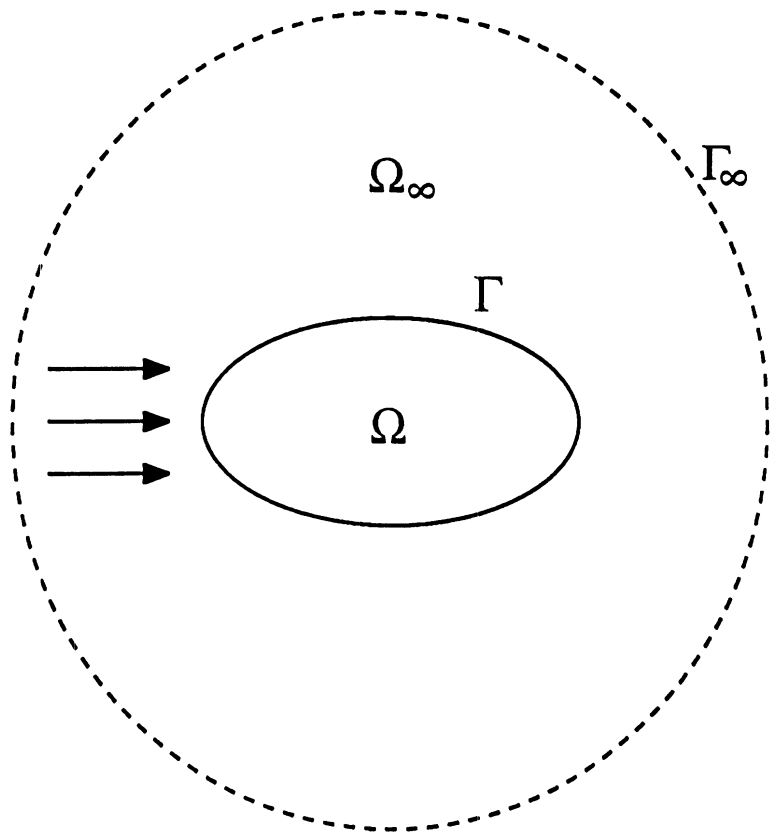


Fig. 1

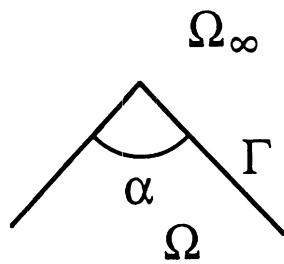


Fig. 2

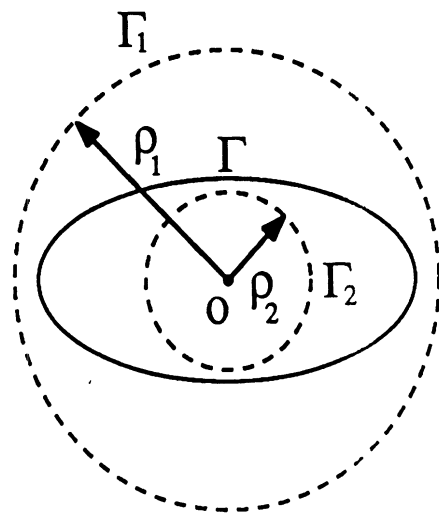


Fig. 3