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## **Diffraction by a Material Junction**

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## Abstract

For a material discontinuity represented by the junction of any two resistive, conductive or impedance half planes, a simple formula is derived expressing the diffracted field in terms of the fields diffracted by the half planes in isolation.

# 1 Introduction

An important source of scattering is the field diffracted at a discontinuity in the material properties of a surface. The simplest model for such a discontinuity is the common edge of two half planes of different material properties, and if the materials are simulated using an impedance boundary condition, the edge diffracted field can be determined using Maliuzhinets' method [1959]. Alternatively, for thin layers of non-magnetic materials, a representation in terms of resistive sheets may be appropriate, and the corresponding diffraction coefficient has recently been obtained by Uzgören et al [1989]; and yet a third possibility is the junction of two conductive sheets which are the electromagnetic dual of resistive sheets.

The purpose of this paper is to show that there is a simple formula expressing the field diffracted at a junction in terms of the edge diffracted field of each half plane in isolation. The formula is applicable for any combination of resistive, conductive and impedance sheets and constitutes a useful design tool. In Section 2 we cite the known edge diffraction coefficients for the three sheets and show how these are related to each other. In Section 3 we then provide a simple derivation of the diffraction coefficient for the junction of two resistive half planes using the angular spectrum method [Clemmow, 1953], and show how this can be applied to any junction using the fact that co-planar electric and magnetic currents do not interact.

## 2 Half Planes in Isolation

Consider first a uniform resistive sheet of resistivity  $R$  ohms per square occupying the portion  $x \geq 0$  of the plane  $y = 0$  and illuminated by an E polarized plane wave having

$$\mathbf{E}^i = \hat{z} e^{-ik(x \cos \phi_o + y \sin \phi_o)} \quad (1)$$

(see Fig. 1) where a time factor  $e^{-i\omega t}$  has been assumed and suppressed. Without loss of generality it can be assumed that  $0 \leq \phi_o \leq \pi$  and  $0 \leq \phi \leq 2\pi$  where  $x = \rho \cos \phi$ ,  $y = \rho \sin \phi$ .

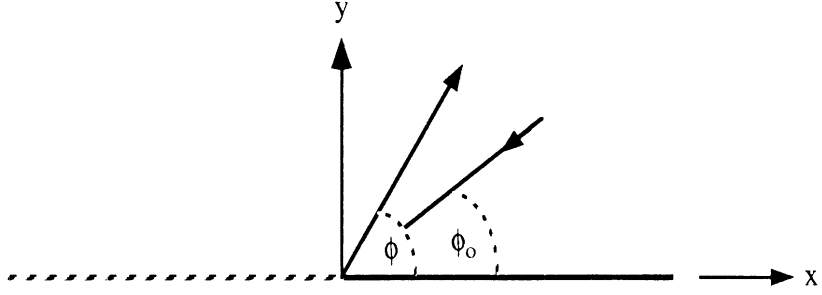


Figure 1: Fig. 1: Half plane geometry.

A resistive sheet supports only an electric current and the boundary conditions there are

$$E_z(x, +0) = E_z(x, -0) = -R \{H_x(x, +0) - H_x(x, -0)\}. \quad (2)$$

At large distances from the edge, the edge diffracted field can be written as

$$E_z^d = \sqrt{\frac{2}{\pi k \rho}} e^{i(k\rho - \pi/4)} D_E^{(1)}(\eta, \phi, \phi_0) \quad (3)$$

with  $\eta = 2R/Z$  where  $Z$  is the intrinsic impedance of free space, and as shown by Senior [1975a]

$$D_E^{(1)}(\eta, \phi, \phi_0) = \frac{i}{2} \frac{K(\eta, -k \cos \phi) K(\eta, -k \cos \phi_0)}{\cos \phi + \cos \phi_0}. \quad (4)$$

$K(\eta, \xi)$  is an upper half plane function associated with a Wiener-Hopf split and such that

$$K(\eta, \xi) K(\eta, -\xi) = \left\{ \eta + \frac{k}{\sqrt{k^2 - \xi^2}} \right\}^{-1}. \quad (5)$$

In terms of Maliuzhinets' half plane function  $\psi_\pi(\alpha)$

$$K(\eta, -k \cos \phi) = \frac{1}{\sqrt{\eta}} \frac{\sin \frac{\phi}{2}}{\cos \frac{\chi}{2} - \cos \frac{1}{2} \left( \frac{\pi}{2} + \phi \right)} \frac{\psi_\pi(\pi - \phi + \chi) \psi_\pi(\pi - \phi - \chi)}{\psi_\pi(\phi - \chi) \psi_\pi(\phi + \chi)} \quad (6)$$

[Senior, 1975a] with  $\cos \chi = 1/\eta$ . When  $R = 0$  implying  $\eta = 0$  the resistive sheet is perfectly conducting and

$$K(0, \xi) = \sqrt{\frac{k + \xi}{k}} \quad (7)$$

giving

$$D_E^{(1)}(0, \phi, \phi_o) = i \frac{\sin \frac{\phi}{2} \sin \frac{\phi_o}{2}}{\cos \phi + \cos \phi_o} \quad (8)$$

in agreement with the known result [Bowman et al, 1987; p. 315]. On the other hand, when  $R = \infty$  implying  $\eta = \infty$ , the resistive sheet ceases to exist, and since

$$K(\eta, \xi) \sim \frac{1}{\sqrt{\eta}} \quad (9)$$

as  $|\eta| \rightarrow \infty$ ,  $D_E^{(1)}(\infty, \phi, \phi_o) = 0$  as required.

The electromagnetic dual of a resistive sheet is a conductive one. This supports only a magnetic current and for a sheet occupying the portion  $x \geq 0$  of the plane  $y = 0$  the boundary conditions are

$$H_x(x, +0) = H_x(x, -0) = R^* \{E_z(x, +0) - E_z(x, -0)\} \quad (10)$$

where  $R^*$  siemens per square is the conductivity. The edge diffraction coefficient is [Senior, 1975a]

$$D_E^{(2)}(\eta, \phi, \phi_o) = -Z\eta \cos \frac{\phi}{2} \cos \frac{\phi_o}{2} D_E^{(1)}(\eta, \phi, \phi_o) \quad (11)$$

with  $\eta = 1/(2R^*Z)$ . When  $R^* = 0$  implying  $\eta = \infty$  the sheet is a perfect magnetic conductor, and from (4), (9) and (11)

$$D_E^{(2)}(\infty, \phi, \phi_o) = -i \frac{\cos \frac{\phi}{2} \cos \frac{\phi_o}{2}}{\cos \phi + \cos \phi_o}, \quad (12)$$

whereas if  $R^* = \infty$  implying  $\eta = 0$ , the sheet ceases to exist, and  $D_E^{(2)}(0, \phi, \phi_o) = 0$  as required.

The last case to be considered is that of an opaque sheet having an impedance boundary condition imposed on both sides. If the surface impedance is  $\eta Z$ , the boundary conditions are

$$E_z(x, \pm 0) = \mp \eta Z H_x(x, \pm 0), \quad (13)$$

and as noted by Senior [1975a], the sheet is equivalent to a combination of resistive and conductive sheets whose properties are such that

$$\eta = \frac{2R}{Z} = \frac{1}{2R^*Z}. \quad (14)$$

Since the two sheets do not interact, the edge diffraction coefficient is

$$D_E^{(3)}(\eta, \phi, \phi_o) = D_E^{(1)}(\eta, \phi, \phi_o) + D_E^{(2)}(\eta, \phi, \phi_o)$$

i.e.

$$D_E^{(3)}(\eta, \phi, \phi_o) = \left(1 - 2\eta \cos \frac{\phi}{2} \cos \frac{\phi_o}{2}\right) D_E^{(1)}(\eta, \phi, \phi_o). \quad (15)$$

For an isolated sheet occupying the half plane  $x \leq 0$  the diffraction coefficients differ from the above only in having  $\phi_o$  replaced by  $\pi - \phi_o$  and  $\phi$  replaced by  $\pi \mp \phi$  for  $y < > 0$  respectively. We can also obtain the results for H polarization using duality. For H polarization we write

$$H_z^d = \sqrt{\frac{2}{\pi k \rho}} e^{i(k\rho - \pi/4)} D_H(\eta, \phi, \phi_o), \quad (16)$$

and recognizing that resistive and conductive sheets are the duals of each other,

$$D_H^{(1)}(\eta, \phi, \phi_o) = D_E^{(2)}\left(\frac{1}{\eta}, \phi, \phi_o\right), \quad (17)$$

$$D_H^{(2)}(\eta, \phi, \phi_o) = D_E^{(1)}\left(\frac{1}{\eta}, \phi, \phi_o\right). \quad (18)$$

In particular, for a perfectly conducting half plane (a resistive sheet having  $R = 0$  implying  $\eta = 0$ ) in  $x \geq 0$ , (12) shows that

$$D_H^{(1)}(0, \phi, \phi_o) = -i \frac{\cos \frac{\phi}{2} \cos \frac{\phi_o}{2}}{\cos \phi + \cos \phi_o} \quad (19)$$

is agreement with the known result [Bowman et al, 1987; p. 322].

### 3 Resistive Sheet Junction

Consider two resistive sheets of resistivities  $R_1(= \eta_1 Z/2)$  and  $R_2(= \eta_2 Z/2)$  occupying the portions  $x < 0$  and  $x > 0$ , respectively, of the plane  $y = 0$ . We seek the diffraction coefficient attributable to the junction at  $x = 0$  when the incident field is the E polarized plane wave (1).

If the sheet having resistivity  $R_1$  occupies the entire plane  $y = 0$ , a simple analysis shows

$$E_z = \begin{cases} e^{-ik(x \cos \phi_o + y \sin \phi_o)} - (1 + \eta_1 \sin \phi_o)^{-1} e^{-ik(x \cos \phi_o - y \sin \phi_o)} \\ \eta_1 \sin \phi_o (1 + \eta_1 \sin \phi_o)^{-1} e^{-ik(x \cos \phi_o + y \sin \phi_o)} \end{cases}$$

implying

$$H_x = \begin{cases} -Y \sin \phi_o \left[ e^{-ik(x \cos \phi_o + y \sin \phi_o)} + (1 + \eta_1 \sin \phi_o)^{-1} e^{-ik(x \cos \phi_o - y \sin \phi_o)} \right] \\ -Y \sin \phi_o \eta_1 \sin \phi_o (1 + \eta_1 \sin \phi_o)^{-1} e^{-ik(x \cos \phi_o + y \sin \phi_o)} \end{cases}$$

for  $y > 0$  and  $y < 0$  respectively, and we will denote this field with a superscript 'o'. When the resistivity in  $x > 0$  is  $R_2$ , the total field is written as

$$\mathbf{E} = \mathbf{E}^o + \mathbf{E}^s, \quad \mathbf{H} = \mathbf{H}^o + \mathbf{H}^s,$$

and since the 'scattered' field is attributable to an electric current in the plane  $y = 0$ , we have [Clemmow, 1953]

$$E_z^s = \int_{-\infty}^{\infty} P(\xi) e^{i(\xi x + |y| \sqrt{k^2 - \xi^2})} \frac{d\xi}{\sqrt{k^2 - \xi^2}} \quad (20)$$

$$H_x^s = \frac{Y}{k} \frac{y}{|y|} \int_{-\infty}^{\infty} P(\xi) e^{i(\xi x + |y| \sqrt{k^2 - \xi^2})} d\xi.$$

By virtue of this choice, the continuity of  $E_z$  across the plane  $y = 0$  is assured, and from the remaining boundary conditions in  $x < 0$  and  $x > 0$  we obtain

$$\int_{-\infty}^{\infty} \left( \eta_1 + \frac{k}{\sqrt{k^2 - \xi^2}} \right) P(\xi) e^{i\xi x} d\xi = 0 \quad (x < 0) \quad (21)$$

$$\int_{-\infty}^{\infty} \left( \eta_2 + \frac{k}{\sqrt{k^2 - \xi^2}} \right) P(\xi) e^{i\xi x} d\xi = \frac{k(\eta_2 - \eta_1)}{\eta_1 + \frac{k}{\sqrt{k^2 - \xi_o^2}}} e^{-i\xi_o x} \quad (x > 0) \quad (22)$$

where  $\xi_o = k \cos \phi_o$ .

Equation (21) demands that the non-exponential portion of the integrand be a function analytical and free of zeros in a lower half plane (an ‘L’ function) and therefore

$$P(\xi) = K(\eta_1, \xi)L(\xi).$$

Similarly, apart from a pole at  $\xi = -\xi_o$ , the non-exponential portion of the integrand in (22) must be an upper half plane (or ‘U’) function so that

$$P(\xi) = \frac{K(\eta_2, -\xi)}{\xi + \xi_o} U(\xi).$$

Hence

$$P(\xi) = \frac{K(\eta_1, \xi)K(\eta_2, -\xi)}{\xi + \xi_o} A(\xi)$$

where  $A(\xi)$  is a function analytic everywhere. To reproduce the right hand side of (22) it is necessary that

$$A(\xi_o) = \frac{ik}{2\pi} (\eta_1 - \eta_2) K(\eta_1, \xi_o) K(\eta_2, -\xi_o),$$

and since order considerations show that  $A(\xi)$  is at most a constant, the final expression for  $P(\xi)$  is

$$P(\xi) = \frac{ik}{2\pi} (\eta_1 - \eta_2) \frac{K(\eta_1, \xi)K(\eta_1, \xi_o)K(\eta_2, -\xi)K(\eta_2, -\xi_o)}{\xi + \xi_o}. \quad (23)$$

When (23) is inserted into (20) the substitution  $\xi = k \cos \alpha$  gives

$$\begin{aligned} E_z^s &= \frac{i}{2\pi} (\eta_1 - \eta_2) \\ &\cdot \int_S \frac{K(\eta_1, k \cos \alpha)K(\eta_1, k \cos \phi_o)K(\eta_2, -k \cos \alpha)K(\eta_2, -k \cos \phi_o)}{\cos \alpha + \cos \phi_o} \\ &\cdot e^{ik(x \cos \alpha + |y| \sin \alpha)} d\alpha \end{aligned}$$



where  $S$  is a path on which  $\cos \alpha$  runs from  $\infty$  to  $-\infty$ , and from a stationary phase analysis the diffraction coefficient of the junction is found to be

$$D_E^{(1)}(\eta_1, \eta_2, \phi, \phi_o) = \frac{i}{2}(\eta_1 - \eta_2) \cdot \frac{K(\eta_1, k \cos \phi)K(\eta_1, k \cos \phi_o)K(\eta_2, -k \cos \phi)K(\eta_2, -k \cos \phi_o)}{\cos \phi + \cos \phi_o}. \quad (24)$$

If only the resistive sheet on the right is present,

$$D_E^{(1)}(\infty, \eta_2, \phi, \phi_o) = \frac{i}{2} \frac{K(\eta_2, -k \cos \phi)K(\eta_2, -k \cos \phi_o)}{\cos \phi + \cos \phi_o}$$

in agreement with (4), and when only the left hand sheet is there,

$$D_E^{(1)}(\eta_1, \infty, \phi, \phi_o) = -\frac{i}{2} \frac{K(\eta_1, k \cos \phi)K(\eta_1, k \cos \phi_o)}{\cos \phi + \cos \phi_o}$$

consistent with the substitution described in Section 2. As evident from these

$$D_E^{(1)}(\eta_1, \eta_2, \phi, \phi_o) = 2i(\eta_1 - \eta_2)(\cos \phi + \cos \phi_o) \cdot D_E^{(1)}(\eta_1, \infty, \phi, \phi_o)D_E^{(1)}(\infty, \eta_2, \phi, \phi_o) \quad (25)$$

and therefore

$$D_E^{(1)}(\eta_1, \eta_2, \phi, \phi_o) = 2i(\eta_1 - \eta_2)(\cos \phi + \cos \phi_o) \cdot D_E^{(1)}(\eta_1, \pi \mp \phi, \pi - \phi_o)D_E^{(1)}(\eta_2, \phi, \phi_o). \quad (26)$$

This is the required expression for the junction contribution in terms of the edge diffraction coefficients of the sheets in isolation.

For two conductive sheets having conductivity  $R_1^* = 1/(2\eta_1 Z)$  on the left and  $R_2^* = 1/(2\eta_2 Z)$  on the right, a similar analysis shows

$$D_E^{(2)}(\eta_1, \eta_2, \phi, \phi_o) = \pm \frac{i}{2}(\eta_1 - \eta_2) \cdot \frac{K(\eta_1, k \cos \phi)K(\eta_1, k \cos \phi_o)K(\eta_2, -k \cos \phi)K(\eta_2, -k \cos \phi_o)}{\cos \phi + \cos \phi_o} \quad (27)$$

with the upper (lower) sign for  $y > (<)0$ , and this differs from (24) only in the sign alternative. The formula can be written in terms of the diffraction coefficients for conductive sheets in isolation, but it is simpler to express it in terms of the coefficients for *resistive* sheets, *viz.*

$$D_E^{(2)}(\eta_1, \eta_2, \phi, \phi_o) = \pm 2i(\eta_1 - \eta_2)(\cos \phi + \cos \phi_o) \cdot D_E^{(1)}(\eta_1, \pi \mp \phi, \pi - \phi_o) D_E^{(1)}(\eta_2, \phi, \phi_o) \quad (28)$$

which should be compared with (26). It now follows that for two impedance sheets

$$D_E^{(3)}(\eta_1, \eta_2, \phi, \phi_o) = 4i(\eta_1 - \eta_2)(\cos \phi + \cos \phi_o) \cdot D_E^{(1)}(\eta_1, \pi - \phi, \pi - \phi_o) D_E^{(1)}(\eta_2, \phi, \phi_o) \quad (29)$$

for  $0 \leq \phi \leq \pi$  and zero otherwise. This is logical since the sheets are opaque.

## 4 Concluding Remarks

The preceding junction diffraction coefficients encompass all possible cases of two abutting sheets subject to first order transition or boundary conditions. If, for example, the right hand sheet is an impedance one with surface impedance  $\eta_2 Z$  and the left hand sheet is resistive with resistivity  $R = \eta_1 Z/2$ , the fact that electric and magnetic current sheets do not interact implies

$$D_E(\eta_1, \eta_2, \phi, \phi_o) = D_E^{(1)}(\eta_1, \eta_2, \phi, \phi_o) + D_E^{(2)}(\eta_2, \phi, \phi_o) \quad (30)$$

where the coefficients on the right hand side are given in (11) and (26). It can be verified that the diffraction coefficient derived by Uzgören et al [1989] has this form. Similarly, if the impedance sheet is replaced by a conductive one with conductivity  $R^* = 1/(2\eta_2 Z)$ . then

$$D_E(\eta_1, \eta_2, \phi, \phi_o) = D_E^{(1)}(\eta_1, \pi \mp \phi, \pi - \phi_o) + D_E^{(2)}(\eta_2, \phi, \phi_o). \quad (31)$$

There are two limitations that should be noted. The first is that all of the diffraction coefficients are non-uniform ones which are infinite at the reflected and transmitted wave boundaries. Since the infinities are associated with the optics field, they could be eliminated using the same procedure employed to derive a uniform diffraction coefficient for an isolated half plane. We have also restricted attention to incidence and diffraction in the  $xy$  plane perpendicular to the common edge of the half planes. For skew incidence the diffraction coefficient for a single half plane is much more complicated [Senior, 1975b], and it is not evident that there are formulas for the junction effect analogous to those presented here.

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