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Generalized boundary and transition conditions and the uniqueness of solution

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1 Introduction

To better simulate the material properties of a surface or layer, a method that has attracted attention is to include higher derivatives in the boundary specification of the field. The result is a generalized boundary or transition condition whose order is specified by the highest (Mth) derivative present. The conditions are logical extensions of the first order ones corresponding to the usual surface impedance or sheet conditions, but when applied to a surface with a discontinuity either in the structure itself or the condition, they do not produce a unique solution to the problem if $M > 1$. The inclusion of the standard edge conditions is no longer sufficient, and additional constraints are necessary to have a well-posed problem.

For simplicity we consider the case of a planar surface $y = 0$ illuminated by an H-polarized plane wave. In Section 2 the connection between generalized sheet transition conditions (GSTCs) and generalized impedance boundary conditions (GIBCs) is discussed, and this allows us to confine attention to the latter. Some of the ways to generate the conditions are presented in Section 3. All are based on the expansion of the fields inside the simulated layer or coating in powers of a small parameter δ, and because of this, some restrictions can be placed on the form of the GIBC. The uniqueness of the solution of the resulting boundary value problem is examined in Section 4, first for a GIBC whose coefficients are continuous functions of position, and then for a line discontinuity in the properties. In the latter case, the role of the standard edge conditions is indicated, and the additional constraints necessary when the GIBC is of order $M > 1$ are developed. The physical interpretation of the constraints is discussed in Section 5.

2 Generalized Conditions

For a planar surface there is a simple generic form of a GSTC that includes a GIBC as a special case. To show this, consider an electromagnetic field whose only non-zero components are E_x, E_y and H_z (H polarization) incident on a surface $y = 0$ where x, y, z are Cartesian coordinates. At the surface the following transition (or jump) conditions are imposed:

$$\prod_{m=1}^{M}\left(ik\gamma_m + \frac{\partial}{\partial y}\right)E_y^+ + \prod_{m=1}^{M}\left(ik\gamma_m - \frac{\partial}{\partial y}\right)E_y^- = 0 \qquad (1)$$

$$\prod_{m=1}^{M}\left(ik\gamma_m' + \frac{\partial}{\partial y}\right)E_y^+ - \prod_{m=1}^{M}\left(ik\gamma_m' - \frac{\partial}{\partial y}\right)E_y^- = 0 \qquad (2)$$

for some γ_m and γ_m', where the superscripts refer to the upper (+) and lower (−) sides of the surface, and a time factor $e^{-i\omega t}$ has been suppressed. For M odd the conditions are identical to those considered by Senior [1992], but for M even the conditions are interchanged. For all M, even as well as odd, (1) and (2) specify magnetic and electric current sheets respectively. The magnetic current is a function of the γ_m alone, the electric current depends on the γ_m' alone, and there is no coupling between them. In general, the surface is partially transparent, but if $\gamma_m = \gamma_m'$ $(m = 1, 2, \ldots, M)$ the surface becomes opaque. By adding (1) and (2) we then obtain

$$\prod_{m=1}^{M}\left(\frac{\partial}{\partial y} \pm ik\gamma_m\right)E_y = 0 \qquad (3)$$

at $y = \pm 0$, and (3) represents Mth order GIBCs [Senior and Volakis, 1989].

More generally, since any planar magnetic current radiates a field E_y (or H_z) which is symmetric about the plane, i.e.

$$E_y^+ = E_y^-, \qquad \frac{\partial E_y^+}{\partial y} = -\frac{\partial E_y^-}{\partial y},$$

(1) is equivalent to the GIBC

$$\prod_{m=1}^{M}\left(\frac{\partial}{\partial y}+ik\gamma_m\right)E_y=0 \qquad (y=+0) \tag{4}$$

for the even component of the field. Similarly, the electric current radiates an antisymmetric field for which

$$E_y^+=-E_y^-, \qquad \frac{\partial E_y^+}{\partial y}=\frac{\partial E_y^-}{\partial y},$$

and (2) is then equivalent to the GIBC

$$\prod_{m=1}^{M}\left(\frac{\partial}{\partial y}+ik\gamma_m'\right)E_y=0 \qquad (y=+0) \tag{5}$$

for the odd component of the field. It is therefore sufficient to confine attention to GIBCs of the form (4) for specified γ_m $(m=1,2,\ldots,M)$.

Equation (4) is a scalar condition for the component E_y, and is a natural generalization of the first order (or standard) impedance boundary condition expressed in the form

$$\left(\frac{\partial}{\partial y}+ik\gamma\right)E_y=0.$$

At an edge or other line discontinuity in the properties of the surface, $\frac{\partial E_y}{\partial y}$ may be as singular as $x^{-3/2}$, and its Fourier transform does not then exist in the classical sense. We can avoid this problem by integrating with respect to x [Senior, 1987], and since

$$E_y=-\frac{iZ}{k}\frac{\partial H_z}{\partial x},$$

the condition (4) can be written as

$$\prod_{m=1}^{M}\left(\frac{\partial}{\partial y}+ik\gamma_m\right)H_z=0 \tag{6}$$

applied at the surface $y=+0$, where the constant of integration has been set to zero. This is the GIBC which we will use. We note in passing that when $M=1$, (6) is simply

$$E_x=Z\gamma_1 H_z \tag{7}$$

which is the Leontovich boundary condition for an opaque surface having normalized surface impedance γ_1.

The form (6) is quite convenient for many analytical purposes, e.g. a Wiener-Hopf analysis, but there is an equivalent form that we will also use. As noted by Senior and Volakis [1989], (6) can be written as

$$\sum_{m=0}^{M} \frac{b_m}{(ik)^m} \frac{\partial^m}{\partial y^m} H_z = 0, \tag{8}$$

and boundary conditions of increasing order M are then as follows:

$M = 1$

$$\frac{\partial H_z}{\partial y} = -ik\frac{b_0}{b_1} H_z \tag{9}$$

$M = 2$

$$\frac{\partial H_z}{\partial y} = -ik\frac{b_0 + b_2}{b_1}\left\{1 + \frac{b_2}{b_0 + b_2}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\} H_z \tag{10}$$

$M = 3$

$$\left\{1 + \frac{b_3}{b_1 + b_3}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\}\frac{\partial H_z}{\partial y} = -ik\frac{b_0 + b_2}{b_1 + b_3}\left\{1 + \frac{b_2}{b_0 + b_2}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\} H_z \tag{11}$$

$M = 4$

$$\begin{aligned}\left\{1 + \frac{b_3}{b_1 + b_3}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\}\frac{\partial H_z}{\partial y} &= -ik\frac{b_0 + b_2 + b_4}{b_1 + b_3}\left\{1 + \frac{b_2 + 2b_4}{b_0 + b_2 + b_4}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right. \\ &\quad \left. + \frac{b_4}{b_0 + b_2 + b_4}\frac{1}{k^4}\frac{\partial^4}{\partial x^4}\right\} H_z,\end{aligned} \tag{12}$$

and so on. As M increases, higher order even derivatives with respect to x appear on first the right hand side and then the left, and only if $b_m = 0$ for all odd $m > 1$ are there no (x) derivatives on the left.

3 Derivation of the Coefficients

The GSTCs (1) and (2) were originally introduced to better simulate the scattering properties of a thin layer of material of thickness $\tau \ll \lambda$, and

(6) then represents a dielectric layer of thickness $\tau/2$ backed by either a perfect magnetic conductor (corresponding to the antisymmetric field due to an electric current sheet) or a perfect electric conductor (corresponding to the symmetric field due to a magnetic current sheet), as shown in Fig. 1.

In the case of a homogeneous material having relative permittivity ϵ_1 and relative permeability μ_1, the first order ($M = 1$) conditions are those for a resistive sheet of resistivity R and a conductive sheet of conductivity R^*, viz.

$$\frac{\partial H_z}{\partial y} = -ikY2RH_z \qquad (13)$$

with

$$R = \frac{iZ}{k\tau(\epsilon_1 - 1)}$$

and

$$\frac{\partial H_z}{\partial y} = -\frac{ikY}{2R^*}H_z \qquad (14)$$

with

$$R^* = \frac{iY}{k\tau(\mu_1 - 1)}.$$

Unless the layer is very thin and/or lossy, the simulation is not very good, but it can be improved by replacing the conductive sheet by a modified one

[Senior and Volakis, 1987] satisfying the boundary condition

$$\frac{\partial H_z}{\partial y} = -\frac{ikY}{2R^*}\left\{1 + \frac{R^*}{R_e^*}\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial y^2}\right)\right\} H_z \tag{15}$$

where

$$R_e^* = \frac{iY\epsilon_1}{k\tau(\epsilon_1 - 1)},$$

and (15) is a second order GIBC. The additional terms in (15) model the effect of the normal component of the electric polarization current within the dielectric.

To develop these (and higher order) conditions, one method is to expand the field component H_z inside the layer in a Taylor series in y and then match to the exterior field at the upper surface $y = \tau/2$. For the symmetric field we have

$$\begin{aligned} \frac{\partial}{\partial y} H_z^{\text{in}}(0) &= \frac{\partial}{\partial y} H_z^{\text{in}}\left(\frac{\tau}{2}\right) - \frac{\tau}{2}\frac{\partial^2}{\partial y^2} H_z^{\text{in}}\left(\frac{\tau}{2}\right) + \frac{\tau^2}{8}\frac{\partial^3}{\partial y^3} H_z^{\text{in}}\left(\frac{\tau}{2}\right) \\ &\quad - \frac{\tau^3}{48}\frac{\partial^4}{\partial y^4} H_z^{\text{in}}\left(\frac{\tau}{2}\right) + O(\tau^4) \end{aligned}$$

where the superscript 'in' denotes the interior field, and we have shown only the y dependence. But

$$\frac{\partial}{\partial y} H_z^{\text{in}}(0) = 0$$

from the boundary condition at a perfect electric conductor, and

$$\begin{aligned} \frac{\partial}{\partial y} H_z^{\text{in}}\left(\frac{\tau}{2}\right) &= \epsilon_1 \frac{\partial}{\partial y} H_z\left(\frac{\tau}{2}\right) \\ \frac{\partial^2}{\partial y^2} H_z^{\text{in}}\left(\frac{\tau}{2}\right) &= -\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right) H_z^{\text{in}}\left(\frac{\tau}{2}\right) = -\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right) H_z\left(\frac{\tau}{2}\right) \end{aligned}$$

etc., where $N = \sqrt{\epsilon_1\mu_1}$ is the complex refractive index of the layer material. Hence

$$\begin{aligned} &\epsilon_1\frac{\partial H_z}{\partial y} + \frac{\tau}{2}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right) H_z \\ &- \epsilon_1\frac{\tau^2}{8}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\frac{\partial H_z}{\partial y} - \frac{\tau^3}{48}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)^2 H_z + O(\tau^4) = 0, \end{aligned}$$

and accurate to the third order in τ, a boundary condition on the exterior field at $y = \tau/2$ is

$$\left\{1 - \frac{\tau^2}{8}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\right\}\frac{\partial H_z}{\partial y} = -\frac{\tau}{2\epsilon_1}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\left\{1 - \frac{\tau^2}{24}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\right\}H_z. \tag{16}$$

This is a fourth order GIBC expressed in the form (12), but to the same order in τ it can be written as

$$\frac{\partial H_z}{\partial y} = -\frac{\tau}{2\epsilon_1}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\left\{1 + \frac{\tau^2}{12}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\right\}H_z. \tag{17}$$

This is still a fourth order condition, but now $b_3 = 0$.

Similarly, for the antisymmetric field

$$\begin{aligned} H_z^{\text{in}}(0) &= H_z^{\text{in}}\left(\frac{\tau}{2}\right) - \frac{\tau}{2}\frac{\partial}{\partial y}H_z^{\text{in}}\left(\frac{\tau}{2}\right) + \frac{\tau^2}{8}\frac{\partial^2}{\partial y^2}H_z^{\text{in}}\left(\frac{\tau}{2}\right) \\ &\quad - \frac{\tau^3}{48}\frac{\partial^3}{\partial y^3}H_z^{\text{in}}\left(\frac{\tau}{2}\right) + \frac{\tau^4}{384}\frac{\partial^4}{\partial y^4}H_z^{\text{in}}\left(\frac{\tau}{2}\right) + O(\tau^5) \end{aligned}$$

and since $H_z^{\text{in}}(0) = 0$, we have

$$\begin{aligned} H_z - \epsilon_1\frac{\tau}{2}\frac{\partial H_z}{\partial y} - \frac{\tau^2}{8}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)H_z + \epsilon_1\frac{\tau^3}{48}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\frac{\partial H_z}{\partial y} & \\ + \frac{\tau^4}{384}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)^2 H_z + O(\tau^5) &= 0 \end{aligned}$$

i.e.

$$\begin{aligned} &\left\{1 - \frac{\tau^2}{24}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)\right\}\frac{\partial H_z}{\partial y} \\ &= \frac{2}{\epsilon_1\tau}\left\{1 - \frac{\tau^2}{8}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right) + \frac{\tau^4}{384}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)^2\right\}H_z \end{aligned} \tag{18}$$

This is also a fourth order GIBC. It is accurate to the fourth order in τ, and to this same accuracy it might seem that it could be written as

$$\frac{\partial H_z}{\partial y} = \frac{2}{\epsilon_1\tau}\left\{1 - \frac{\tau^2}{12}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right) - \frac{\tau^4}{1152}\left(N^2k^2 + \frac{\partial^2}{\partial x^2}\right)^2\right\}H_z. \tag{19}$$

A slightly different method has been employed by Ljalinov [1992] to treat the problem of a curved homogeneous layer of constant thickness using two-dimensional curvilinear coordinates. It also produces an expansion in powers of τ, but involves the explicit solution of the wave equation in the dielectric. To illustrate, consider the case of the antisymmetric field for a planar layer, and introduce the normalized coordinates

$$\xi = kx, \qquad \eta = 2y/\tau \qquad \text{with } \delta = k\tau/2.$$

If $U = H_z^{\text{in}}$, the equation satisfied by $U(\xi, \eta)$ is

$$\left\{ \frac{\partial^2}{\partial \eta^2} + \delta^2 \left(N^2 + \frac{\partial^2}{\partial \xi^2} \right) \right\} U(\xi, \eta) = 0$$

with $U(\xi, 0) = 0$. Expand U as

$$U = U_0 + \delta U_1 + \delta^2 U_2 + \delta^3 U_3 + \delta^4 U_4 + O(\delta^5).$$

Then

$$\frac{\partial^2 U_0}{\partial \eta^2} = 0$$

implying

$$U_0(\xi, \eta) = c_0 + c_1 \eta$$

and since $U(\xi, 0) = 0$,

$$U_0(\xi, \eta) = c_1 \eta = \eta\, U_0(\xi, 1).$$

Similarly

$$U_1(\xi, \eta) = \eta\, U_1(\xi, 1).$$

For $U_2(\xi, \eta)$ we have

$$\frac{\partial^2 U_2}{\partial \eta^2} = -\left(N^2 + \frac{\partial^2}{\partial \xi^2} \right) U_0(\xi, \eta) = -\eta \left(N^2 + \frac{\partial^2}{\partial \xi^2} \right) U_0(\xi, 1)$$

implying

$$U_2(\xi, \eta) = c_0 + c_1 \eta - \frac{1}{6} \eta^3 \left(N^2 + \frac{\partial^2}{\partial \xi^2} \right) U_0(\xi, 1),$$

and since $U_2(\xi,0)=0$,

$$U_2(\xi,\eta)=c_1\eta-\frac{1}{6}\eta^3\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)U_0(\xi,1).$$

But

$$U_2(\xi,1)=c_1-\frac{1}{6}\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)U_0(\xi,1)$$

and therefore

$$U_2(\xi,\eta)=\eta\,U_2(\xi,1)-\frac{\eta}{6}\left(\eta^2-1\right)\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)U_0(\xi,1).$$

Likewise

$$U_3(\xi,\eta)=\eta\,U_3(\xi,1)-\frac{\eta}{6}\left(\eta^2-1\right)\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)U_1(\xi,1)$$

and finally

$$\begin{aligned}U_4(\xi,\eta) &= \eta\,U_4(\xi,1)-\frac{\eta}{6}\left(\eta^2-1\right)\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)U_2(\xi,1)\\ &\quad+\frac{\eta}{360}\left(3\eta^2-1\right)\left(\eta^2-1\right)\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)^2U_0(\xi,1).\end{aligned}$$

Hence

$$\begin{aligned}U(\xi,\eta) &= \eta\,U(\xi,1)-\delta^2\frac{\eta}{6}\left(\eta^2-1\right)\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)U(\xi,1)\\ &\quad+\delta^4\frac{\eta}{360}\left(3\eta^2-1\right)\left(\eta^2-1\right)\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)^2U(\xi,1)+O(\delta^6)\end{aligned}$$

giving

$$\frac{\partial}{\partial\eta}U(\xi,\eta)\bigg|_{\eta=1}=\left\{1-\frac{\delta^2}{3}\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)-\frac{\delta^4}{45}\left(N^2+\frac{\partial^2}{\partial\xi^2}\right)^2\right\}U(\xi,1).$$

On matching to the exterior field at $\eta=1$ and then reverting to the original coordinates, we obtain

$$\frac{\partial H_z}{\partial y}=\frac{2}{\epsilon_1\tau}\left\{1-\frac{\tau^2}{12}\left(N^2k^2+\frac{\partial^2}{\partial x^2}\right)-\frac{\tau^4}{720}\left(N^2k^2+\frac{\partial^2}{\partial x^2}\right)^2\right\}H_z \qquad (20)$$

accurate to the fourth order in τ. This differs from (19) in the coefficient of τ^4, but when Ljalinov's method is applied to the symmetric field, the result is identical to (17).

A knowledge of the reflection coefficient can also be used to generate a GIBC. For the homogeneous layer in Fig. 1, the reflection coefficient at the upper surface $y = \tau/2$ is

$$R_s = -\frac{\sqrt{N^2 - \cos^2\phi_0}\,\tan\left(\frac{k\tau}{2}\sqrt{N^2 - \cos^2\phi_0}\right) - i\epsilon_1 \sin\phi_0}{\sqrt{N^2 - \cos^2\phi_0}\,\tan\left(\frac{k\tau}{2}\sqrt{N^2 - \cos^2\phi_0}\right) + i\epsilon_1 \sin\phi_0} \tag{21}$$

for the symmetric field, and

$$R_a = -\frac{\sqrt{N^2 - \cos^2\phi_0}\,\cot\left(\frac{k\tau}{2}\sqrt{N^2 - \cos^2\phi_0}\right) + i\epsilon_1 \sin\phi_0}{\sqrt{N^2 - \cos^2\phi_0}\,\cot\left(\frac{k\tau}{2}\sqrt{N^2 - \cos^2\phi_0}\right) - i\epsilon_1 \sin\phi_0} \tag{22}$$

for the antisymmetric field. If x is small

$$\begin{aligned}\tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \\ \cot x &= \frac{1}{x}\left(1 - \frac{x^2}{3} - \frac{x^4}{45} - \cdots\right)\end{aligned}$$

and hence, for small $k\tau$,

$$R_s = -\frac{\frac{k\tau}{2}N^2 - \cos^2\phi_0\left\{1 + \frac{(k\tau)^2}{12}\left(N^2 - \cos^2\phi_0\right) + \cdots\right\} - i\epsilon_1\sin\phi_0}{\frac{k\tau}{2}N^2 - \cos^2\phi_0\left\{1 + \frac{(k\tau)^2}{12}\left(N^2 - \cos^2\phi_0\right) + \cdots\right\} + i\epsilon_1\sin\phi_0} \tag{23}$$

$$R_a = -\frac{\frac{2}{k\tau}\left\{1 - \frac{(k\tau)^2}{12}\left(N^2 - \cos^2\phi_0\right) - \cdots\right\} + i\epsilon_1\sin\phi_0}{\frac{2}{k\tau}\left\{1 - \frac{(k\tau)^2}{12}\left(N^2 - \cos^2\phi_0\right) - \cdots\right\} - i\epsilon_1\sin\phi_0}. \tag{24}$$

In either case, the only odd power of $\sin\phi_0$ present is the first, and the corresponding boundary conditions then have $b_3 = b_5 = \cdots = 0$. If this is so, (8) can be written as

$$\left\{b_0 + \frac{b_1}{ik}\frac{\partial}{\partial y} + \frac{b_2}{k^2}\left(k^2 + \frac{\partial^2}{\partial x^2}\right) + \frac{b_4}{k^4}\left(k^2 + \frac{\partial^2}{\partial x^2}\right)^2 + \cdots\right\} H_z = 0 \tag{25}$$

and the corresponding reflection coefficient is

$$R = -\frac{b_0 + b_2(1-\cos^2\phi_0) + b_4(1-\cos^2\phi_0)^2 + \cdots - b_1\sin\phi_0}{b_0 + b_2(1-\cos^2\phi_0) + b_4(1-\cos^2\phi_0)^2 + \cdots + b_1\sin\phi_0} \tag{26}$$

It now follows that for the symmetric field the fourth order boundary condition is

$$\left\{\frac{\tau}{2}\left[N^2K^2\left(1+\frac{\tau^2}{12}N^2k^2\right)+\left(1+\frac{\tau^2}{6}N^2k^2\right)\frac{\partial^2}{\partial x^2} + \frac{\tau^2}{12}\frac{\partial^4}{\partial x^4}+O(\tau^4)\right]+\epsilon_1\frac{\partial}{\partial y}\right\}H_z = 0 \tag{27}$$

which is identical to (17), and for the antisymmetric field

$$\left\{\frac{2}{\tau}\left[1-\frac{\tau^2}{12}N^2k^2-\frac{\tau^4}{720}N^2k^2-\frac{\tau^2}{12}\left(1+\frac{\tau^2}{30}N^2k^2\right)\frac{\partial^2}{\partial x^2} - \frac{\tau^4}{720}\frac{\partial^4}{\partial x^4}+O(\tau^6)\right]-\epsilon_1\frac{\partial}{\partial y}\right\}H_z = 0 \tag{28}$$

in agreement with (20) but not with (19). It would therefore appear that the inversion of the differential operator in going from (18) to (19) is only valid to the leading order in τ^2.

The preceding analyses have all involved Taylor series in τ, and the resulting boundary conditions are most appropriate for a layer of low contrast material whose complex refractive index N is not large in magnitude. Using the reflection coefficients (21) and (22) we can also develop boundary conditions applicable to a high contrast material for which $|N| \gg 1$, and fourth order GIBCs of this type were derived by Senior and Volakis [1989]. These have proved to be remarkably accurate, and there is almost no reduction in accuracy if the coefficient b_4 is put equal to zero. The general form of the resulting third order condition is then

$$\left\{b_0 + \frac{b_1}{ik}\frac{\partial}{\partial y} + \frac{b_2}{k^2}\left(k^2+\frac{\partial^2}{\partial x^2}\right) + \frac{b_3}{ik^3}\left(k^2+\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y}\right\}H_z = 0 \tag{29}$$

for specified b_m $(m = 0,1,2,3)$, and the fact that $b_3 \neq 0$ is due to writing $\tan x$ as $\frac{\sin x}{\cos x}$ and expanding numerator and denominator separately. Equation (29) is simply

$$\left\{b_0\left[1+\frac{b_2}{b_0}\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right)\right]+\frac{b_1}{ik}\left[1+\frac{b_3}{b_1}\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y}\right]\right\}H_z=0,$$

and to the same order in $|N|$ it can be rewritten as

$$\left\{b_0\left[1+\frac{b_2'}{b_0}\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right)\right]+\frac{b_1}{ik}\frac{\partial}{\partial y}\right\}H_z=0 \tag{30}$$

with

$$b_2'=\frac{b_2b_1-b_3b_0}{b_1}.$$

This is a second order condition, and it has been verified that it provides almost the same accuracy as the previous third order condition. Indeed, a more consistent approach would never have produced a GIBC of third order, and suggests that a second order condition whose coefficients are chosen appropriately may often suffice.

More generally, for the two-dimensional problem of a curved layer of high contrast material, asymptotic analyses [Senior, 1990; Buldyrev et al., 1990] based on Rytov's [1940] technique produce boundary conditions of the form

$$\frac{\partial H_z}{\partial n}=-\alpha H_z+\frac{\partial}{\partial s}\left(\beta\frac{\partial H_z}{\partial s}\right)-\frac{\partial^2}{\partial s^2}\left(\gamma\frac{\partial^2 H_z}{\partial s^2}\right) \tag{31}$$

where n and s are variables normal and tangential to the surface respectively, and α, β and γ are functions of the surface shape and material properties. In the particular case of a planar layer, (31) reduces to

$$\frac{\partial H_z}{\partial y}=-\left(\alpha-\beta\frac{\partial^2}{\partial x^2}+\gamma\frac{\partial^4}{\partial x^4}\right)H_z, \tag{32}$$

which is a fourth order GIBC in agreement with the boundary conditions above. When expressed in the form (8), the coefficient b_3 is zero.

There is one final matter to discuss. All of the boundary conditions considered so far have been applied at the upper ($y=\tau/2$) surface of the layer, but in practice it may be more desirable to impose the condition at the location of the backing, i.e. at the mid-surface $y=0$ of the original layer. If R' is the reflection coefficient at $y=0$, then

$$R'=e^{ik\tau\sin\phi_0}R$$

where, for a homogeneous layer, R is given in (21) or (22). By a Taylor expansion in powers of τ

$$
\begin{aligned}
e^{ik\tau \sin\phi_0} &= \frac{e^{i\frac{k\tau}{2}\sin\phi_0}}{e^{-i\frac{k\tau}{2}\sin\phi_0}} \\
&= \frac{1-\frac{(k\tau)^2}{8}\sin^2\phi_0 + i\frac{k\tau}{2}\sin\phi_0\left\{1-\frac{(k\tau)^2}{24}\sin^2\phi_0\right\}+O(\tau^4)}{1-\frac{(k\tau)^2}{8}\sin^2\phi_0 - i\frac{k\tau}{2}\sin\phi_0\left\{1-\frac{(k\tau)^2}{24}\sin^2\phi_0\right\}+O(\tau^4)},
\end{aligned}
$$

and by dividing numerator and denominator by $1-\frac{(k\tau)^2}{24}\sin^2\phi_0$, we obtain

$$
e^{ik\tau\sin\phi_0} = \frac{1-\frac{(k\tau)^2}{12}(1-\cos^2\phi_0)+i\frac{k\tau}{2}\sin\phi_0}{1-\frac{(k\tau)^2}{12}(1-\cos^2\phi_0)-i\frac{k\tau}{2}\sin\phi_0}+O(\tau^4). \tag{33}
$$

This has the form of (23) and (24), and shows that we can displace the location of the simulating surface and still keep b_3 (and all subsequent b_m for m odd) zero.

In view of these considerations we shall henceforth restrict attention to the GIBC (32) for specified α, β and γ, or (equivalently) to (8) with $M \leq 4$ and $b_3 = 0$. Comparison of (32) and (8) for $M = 4$ then shows

$$
\alpha = ik\frac{b_0+b_2+b_4}{b_1}, \qquad \beta = -\frac{i}{k}\frac{b_2+2b_4}{b_1}, \qquad \gamma = \frac{i}{k^3}\frac{b_4}{b_1} \tag{34}
$$

and it is sufficient to choose $b_4 = 1$.

4 Uniqueness

Consider the region Σ bounded internally by a cylindrical surface with boundary curve C and externally by a cylinder of infinitely large radius whose boundary curve is C_∞. Application of Green's theorem (or the divergence theorem with $\bar{u} = f\nabla g$) to this region gives

$$
\iint_\Sigma (\nabla f.\nabla g + f\nabla^2 g)\, d\Sigma = -\left(\int_C + \int_{C_\infty}\right) f\frac{\partial g}{\partial n}\, ds
$$

provided f and ∇g are continuous with continuous first derivatives inside and on the boundaries of Σ, and $\hat{n}$ is the unit vector normal directed into Σ.

Let U be a scalar function of position satisfying the two-dimensional scalar wave equation

$$(\nabla^2 + k^2)U = 0$$

and choose $g = U,\ f = U^*$ where the asterisk denotes the complex conjugate. Then

$$\nabla^2 g = -k^2 U$$

and

$$\nabla f . \nabla g = |\nabla U|^2$$

so that

$$k^2 \iint_\Sigma |U|^2 \, d\Sigma - \iint_\Sigma |\nabla U|^2 \, d\Sigma = \left(\int_C + \int_{C_\infty} \right) U^* \frac{\partial U}{\partial n} \, ds.$$

Assume U originates from sources of finite extent in Σ, and let k have a small positive imaginary part, corresponding to some loss in Σ. Since U must decrease as $\exp\{-\rho \mathrm{Im}.k\}$ at infinity, the integral over C_∞ vanishes by virtue of the radiation condition, and

$$\mathrm{Im}.k^2 \iint_\Sigma |U|^2 \, d\Sigma = \mathrm{Im}. \int_C U^* \frac{\partial U}{\partial n} \, ds \qquad (35)$$

This is the basis of the uniqueness proof.

If U_1 and U_2 are two solutions satisfying the same boundary condition on C, $W = U_1 - U_2$ must also satisfy this condition, and

$$\mathrm{Im}.k^2 \iint_\Sigma |W|^2 \, d\Sigma = \mathrm{Im}. \int_C W^* \frac{\partial W}{\partial n} \, ds. \qquad (36)$$

The classical cases are now as follows.

(i) $U = 0$ on C, implying $W^* = 0$ on C. The right hand side of (36) is therefore zero, requiring that $|W| = 0$ at every point of Σ. Hence $U_1 = U_2$ and the solution is unique.

(ii) $\frac{\partial U}{\partial n} = 0$ on C, implying $\frac{\partial W}{\partial n} = 0$ on C. By the same argument, the solution is unique.

(iii) $\frac{\partial U}{\partial n} = -\alpha U$ on C for specified α continuous as a function of s. Equation (36) then gives

$$\text{Im.}k^2 \iint_\Sigma |W|^2 \, d\Sigma = -\int_C (\text{Im.}\alpha)|W|^2 \, ds,$$

and if $\text{Im.}\alpha \geq 0$ at all points of C

$$\text{Im.}k^2 \iint_\Sigma |W|^2 \, d\Sigma \leq 0.$$

Hence $|W| = 0$ throughout Σ, and a necessary condition for uniqueness is

$$\text{Im. } \alpha \geq 0. \tag{37}$$

If $\alpha = ik\eta$ or ik/η where η is a normalized surface impedance, (37) implies Re. $\eta \geq 0$ corresponding to a passive surface.

For the GIBC (31)

$$\text{Im.}k^2 \iint_\Sigma |W|^2 \, d\Sigma = -\text{Im.} \int_C W^* \left\{ \alpha W - \frac{\partial}{\partial s}\left(\beta \frac{\partial W}{\partial s}\right) + \frac{\partial^2}{\partial s^2}\left(\gamma \frac{\partial^2 W}{\partial s^2}\right) \right\} ds. \tag{38}$$

and provided α, $\beta \frac{\partial U}{\partial s}$ and $\frac{\partial}{\partial s}\left(\gamma \frac{\partial^2 U}{\partial s^2}\right)$ are continuous on C, two integrations by parts give

$$\text{Im.}k^2 \iint_\Sigma |W|^2 \, d\Sigma = -\text{Im.} \int_C \left\{ \alpha |W|^2 + \beta \left|\frac{\partial W}{\partial s}\right|^2 + \gamma \left|\frac{\partial^2 W}{\partial s^2}\right|^2 \right\} ds.$$

Then if

$$\text{Im. } \alpha, \beta, \gamma \geq 0 \tag{39}$$

the solution is unique by the same argument as before.

When there is a discontinuity in the boundary condition on C, a knowledge of the boundary condition alone is not sufficient, and additional information is required concerning the allowed behavior of the field at the discontinuity. This is hardly surprising since the boundary condition is not defined there, and a change from $U = 0$ to $\frac{\partial U}{\partial n} = 0$ at (say) $s = 0$ corresponds to an abrupt change from $\alpha = \infty$ to $\alpha = 0$ in a first order impedance boundary condition. The additional information is the so-called edge condition which ensures that the edge does not appear like a true source by demanding that the energy density be integrable in the vicinity of the edge. This restricts the maximum singularity of any field component at the edge, and in the case of a discontinuity in a plane, implies that there is no singularity greater than $\rho^{-1/2}$. Some of the consequences are

- a (total) field component parallel to the edge is finite and continuous there,
- a current component perpendicular to the edge is, in general, zero there,
- a field component perpendicular to the edge, or a current component parallel to the edge, may be infinite (as $\rho^{-1/2}$) there.

In most instances, the physically meaningful solution is the one with the maximum allowed singularity consistent with the above.

For a first order impedance boundary condition, including $U = 0$ and $\frac{\partial U}{\partial n} = 0$ as special cases, the addition of an edge condition is sufficient to ensure uniqueness, and allows us to dispense with the requirement that α be continuous. For the junction of two first order impedance half planes it is found that $U(x, 0)$ is continuous and finite at $x = 0$, but is non-zero unless one of the impedances is infinite, i.e. $\alpha = \infty$ on left or right. However, for a GIBC of higher order, additional constraints are necessary, and these have been referred to as contact conditions [Ljalinov, 1992].

To see this, let A and B be points of discontinuity in β and/or γ on C, (see Fig. 2) with

$$\alpha = \alpha_\pm(s), \qquad \beta = \beta_\pm(s), \qquad \gamma = \gamma_\pm(s)$$

on $C_\pm$ making up C. When the right hand side of (38) is integrated by parts, we obtain

$$\begin{aligned}
&\int_C W^* \left\{ \alpha W - \frac{\partial}{\partial s}\left(\beta \frac{\partial W}{\partial s}\right) + \frac{\partial^2}{\partial s^2}\left(\gamma \frac{\partial^2 W}{\partial s^2}\right)\right\} ds \\
&= \int_C \left\{ \alpha |W|^2 + \beta \left|\frac{\partial W}{\partial s}\right|^2 + \gamma \left|\frac{\partial^2 W}{\partial s^2}\right|^2 \right\} ds \\
&\quad + \left[W^* \left\{ \beta \frac{\partial W}{\partial s} - \frac{\partial}{\partial s}\left(\gamma \frac{\partial^2 W}{\partial s^2}\right)\right\} + \frac{\partial W^*}{\partial s}\left(\gamma \frac{\partial^2 W}{\partial s^2}\right)\right]_{A-}^{A+} \\
&\quad - \left[W^* \left\{ \beta \frac{\partial W}{\partial s} - \frac{\partial}{\partial s}\left(\gamma \frac{\partial^2 W}{\partial s^2}\right)\right\} + \frac{\partial W^*}{\partial s}\left(\gamma \frac{\partial^2 W}{\partial s^2}\right)\right]_{B-}^{B+}, \qquad (40)
\end{aligned}$$

and provided U and $\frac{\partial U}{\partial s}$ are continuous and finite at a discontinuity, the additional constraints required are, in general,

$$\begin{aligned}
\left[\beta \frac{\partial U}{\partial s} - \frac{\partial}{\partial s}\left(\gamma \frac{\partial^2 U}{\partial s^2}\right)\right]_-^+ &= \Delta_1 U \\
\left[\gamma \frac{\partial^2 U}{\partial s^2}\right]_-^+ &= \Delta_2 \frac{\partial U}{\partial s} \qquad (41)
\end{aligned}$$

for specified Δ_1 and Δ_2. The right hand side of (40) then becomes

$$\begin{aligned}
\int_C \left\{ \alpha |W|^2 + \beta \left|\frac{\partial W}{\partial s}\right|^2 + \gamma \left|\frac{\partial^2 W}{\partial s^2}\right|^2 \right\} ds + \Delta_1 |W|_A^2 + \Delta_2 \left|\frac{\partial W}{\partial s}\right|_A^2 \\
+ \Delta_1 |W|_B^2 + \Delta_2 \left|\frac{\partial W}{\partial s}\right|_B^2 ,
\end{aligned}$$

and if

$$\text{Im}.\Delta_1, \Delta_2 \geq 0 \qquad (42)$$

where Δ_1 and Δ_2 are defined for passage through the discontinuity in the direction of increasing s, the solution of the boundary value problem is unique. It should be emphasized that (41) and (42) are additional to the standard edge conditions, and are necessary if β and/or γ is discontinuous. Special cases are those in which $U = 0$ at the discontinuity, eliminating the first of the constraints (41), or $\frac{\partial U}{\partial s} = 0$ at the discontinuity, eliminating the second. If both are zero, neither constraint appears.

5 Physical Interpretation of the Constraints

When applied to the junction of two uniform planar surfaces the constraints (41) become

$$\left[\beta\frac{\partial U}{\partial x} - \gamma\frac{\partial^3 U}{\partial s^3}\right]_-^+ = \Delta_1 U \tag{43}$$

$$\left[\gamma\frac{\partial^2 U}{\partial x^2}\right]_-^+ = \Delta_2\frac{\partial U}{\partial x}. \tag{44}$$

For two abutting half planes each subject to a GIBC of the form (32), it is found that prior to the imposition of the constraints, the solution involves two arbitrary constants c_1 and c_2. U, $\frac{\partial U}{\partial x}$, $\frac{\partial^2 U}{\partial x^2}$ and $\frac{\partial^3 U}{\partial x^3}$ have finite limits as $x \to \pm 0$; U and $\frac{\partial U}{\partial x}$ are continuous at $x = 0$ for all values of the constants, and are zero for a particular choice of c_1 and c_2; and $\frac{\partial^4 U}{\partial x^4}$ has a logarithmic singularity at $x = 0$. When (43) and (44) are imposed, the resulting solution is uniquely specified and satisfies the reciprocity condition for all Δ_1 and Δ_2, infinite as well as finite, provided Δ_1 and Δ_2 are independent of the incident field direction. Mathematically, $\Delta_1 = \infty$ implies $U = 0$ at $x = 0$ and $\Delta_2 = \infty$ implies $\frac{\partial U}{\partial x} = 0$ at $x = 0$. For a second order GIBC, i.e. $\gamma = 0$ in (32), the solution initially contains a single arbitrary constant c_1. U is continuous at $x = 0$, but $\frac{\partial U}{\partial x}$ has a logarithmic singularity there. Imposition of (43) specifies c_1.

The description of (41) as contact conditions is taken from mechanics. In the study of wave propagation in elastic solids, the conditions correspond to the vanishing of successive moments at the junction of two structures, and there is a similar interpretation in our problem. Let

$$M_m = \int_{-\delta}^{\delta} s^m \frac{\partial U}{\partial n}\, ds \qquad (m = 1, 2, \ldots) \tag{45}$$

for small $\delta > 0$. From (31) with $H_z = U$

$$
\begin{aligned}
M_1 &= \int_{-\delta}^{\delta} s\left\{-\alpha U + \frac{\partial}{\partial s}\left(\beta\frac{\partial U}{\partial s}\right) - \frac{\partial^2}{\partial s^2}\left(\gamma\frac{\partial^2 U}{\partial s^2}\right)\right\} ds \\
&= \left[s\left\{\beta\frac{\partial U}{\partial s} - \frac{\partial}{\partial s}\left(\gamma\frac{\partial^2 U}{\partial s^2}\right)\right\}\right]_{-\delta}^{\delta} - \int_{-\delta}^{\delta}\left\{s\alpha U + \beta\frac{\partial U}{\partial s} - \frac{\partial}{\partial s}\left(\gamma\frac{\partial^2 U}{\partial s^2}\right)\right\} ds.
\end{aligned}
$$

Since U, $\frac{\partial U}{\partial s}$ and $\frac{\partial^2 U}{\partial s^2}$ are all finite at $s = \pm 0$, the first term on the right hand side vanishes as $\delta \to 0$, as does the first term in the integral. From the first of the constraints (41) it now follows that

$$\lim_{\delta\to 0} M_1 = 0 \tag{46}$$

and similarly

$$\lim_{\delta\to 0} M_2 = 0. \tag{47}$$

It is interesting to note that these do not involve Δ_1 and Δ_2.

We now revert to the case of a planar structure and consider the application of the constraints (43) and (44); in particular, the specification of Δ_1 and Δ_2. It is natural to expect that these will depend on the geometry being simulated, especially the nature of the contact between the two surfaces, and will differ for the symmetric and antisymmetric components of the field. As we shall show, however, there is no flexibility in the choice of Δ_1 and Δ_2.

The additional constraints are (43) or $U = 0$ at $x = 0$ and (44) or $\frac{\partial U}{\partial x} = 0$ at $x = 0$. From the boundary condition (32)

$$\frac{\partial U}{\partial y} + \alpha U = \beta\frac{\partial^2 U}{\partial x^2} - \gamma\frac{\partial^4 U}{\partial x^4} \tag{48}$$

and hence, for small $\delta > 0$,

$$\int_{-\delta}^{\delta}\left(\frac{\partial U}{\partial y} + \alpha U\right) dx = \left[\beta\frac{\partial U}{\partial x} - \gamma\frac{\partial^3 U}{\partial x^3}\right]_{-\delta}^{\delta}.$$

Since $\frac{\partial^4 U}{\partial x^4}$ has a logarithmic singularity at $x = 0$, $\frac{\partial U}{\partial y}$ also has, implying

$$\int_{-\delta}^{\delta}\frac{\partial U}{\partial y} dx \propto \delta(\ln\delta - 1) \to 0 \quad \text{as } \delta \to 0,$$

and since U is finite and continuous at $x = 0$,

$$\int_{-\delta}^{\delta} \alpha U \, dx \propto \delta \to 0 \quad \text{as } \delta \to 0.$$

It follows that

$$\left[\beta \frac{\partial U}{\partial x} - \gamma \frac{\partial^3 U}{\partial x^3}\right]_-^+ = 0 \tag{49}$$

showing that the only allowed value of Δ_1 is zero, and (49) can be written alternatively as

$$\left[\gamma \frac{\partial^3 U}{\partial x^3}\right]_-^+ = [\beta]_-^+ \frac{\partial U}{\partial x}. \tag{50}$$

The above process implies

$$\int^x \left(\frac{\partial U}{\partial y} + \alpha U\right) dx' - \beta \frac{\partial U}{\partial x} = -\gamma \frac{\partial^3 U}{\partial x^3} \tag{51}$$

and a further integration then gives

$$\int_{-\delta}^{\delta} \left\{ \int^x \left(\frac{\partial U}{\partial y} + \alpha U\right) dx' - \beta \frac{\partial U}{\partial x} \right\} dx = -\left[\gamma \frac{\partial^2 U}{\partial x^2}\right]_{-\delta}^{\delta}.$$

Since $\frac{\partial U}{\partial x}$ is finite and continuous at $x = 0$, the same argument as before shows

$$\left[\gamma \frac{\partial^2 U}{\partial x^2}\right]_-^+ = 0 \tag{52}$$

and thus Δ_2 is also zero.

The extension to GIBCs of still higher (even) orders is evident.

There are now four possible pairs of constraints:

(i) $U = 0$ at $x = 0$ or (50)

and

(ii) $\dfrac{\partial U}{\partial x} = 0$ at $x = 0$ or (52).

In each case, the solution of the boundary value problem is unique, and can be shown to satisfy the reciprocity condition concerning the interchange of transmitter and receiver [Senior, 1993].

To see when each combination applies, we recall that higher order GIBCs are used to improve the simulation provided by conditions of zeroth and first orders. In the case of the symmetric field with $U = H_z$, a first order condition models the magnetic current attributable to the magnetic properties of the layer material, but a higher order condition is necessary to include the effect of the normal (y) component of the electric polarization current. The magnetic current is parallel to the edge, and in the absence of the layer, would be infinite there. Accordingly, (50) is the appropriate constraint from group (i), and since $\frac{\partial U}{\partial x} = 0$ at $x = 0$ would imply that the field did not have the maximum singularity allowed by the edge condition, this is not acceptable. The required constraints are therefore (50) and (52). For the antisymmetric field with $U = H_z$, the current is an electric one perpendicular to the edge, i.e. $\mathbf{J} = \hat{x}U$, and in the absence of the layer, $U = 0$ at $x = 0$. Accordingly, $U = 0$ at $x = 0$ is now the appropriate constraint from group (i), and by the same argument as before, (52) must be chosen from group (ii). The required constraints are therefore $U(0) = 0$ and (52).

In the simpler example of a second order GIBC ($\gamma = 0$) the constraints reduce to

(a)

$$U = 0 \text{ at } x = 0$$

or

(b)

$$\left[\beta \frac{\partial U}{\partial x}\right]_{-}^{+} = 0. \tag{53}$$

For two abutting layers without any gap or insert between them, these are identical to the constraints employed by Senior [1992], who developed (53) by requiring continuity of the normal component of the electric polarization current across the junction. The condition (53) is also the same as that derived by Rojas et al. [1991] (see also Leppington [1983]) for homogeneous layers by matching the (exterior) field to a postulated low frequency expansion of the field inside the material.

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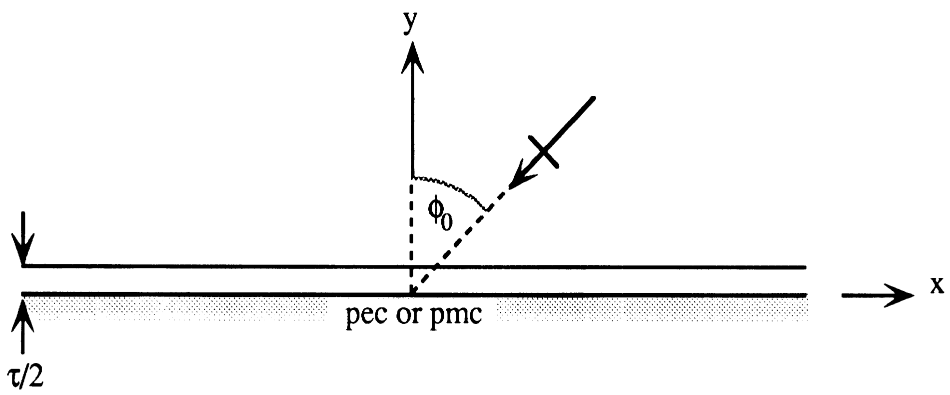

Fig. 1 : Geometry for a layer

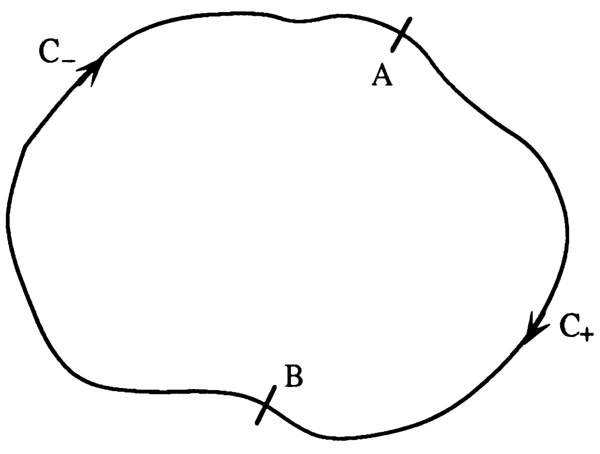

Fig. 2 : Geometry for discontinuities