RL 892

Diffraction by half plane junctions

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March 1993

RL-892 = RL-892

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March 15, 1993

1 Introduction

A problem of some interest is the diffraction of an electromagnetic field by the junction of two half planes at which generalized impedance boundary conditions (GIBCs) are imposed. The conditions simulate a thin dielectric layer backed by a perfect electric or magnetic conductor. Their order M is specified by the highest derivative present, and the conditions are logical extensions of the first order ones corresponding to the standard impedance boundary conditions. By increasing the order, it is possible to improve the accuracy of the simulation, but when applied to a problem where there is a discontinuity in the surface properties, GIBCs in conjunction with the standard edge conditions are no longer sufficient [Senior, 1991] to ensure a unique solution if $M > 1$. Additional constraints are necessary to produce a well-posed problem, and these have been referred to as contact conditions [Ljalinov, 1992] by analogy with the similar situation that occurs in mechanics in wave propagation across the junction of two elastic solids. Without a knowledge of these constraints, the expressions for the induced electric and magnetic currents contain arbitrary constants associated with solutions of the source-free problem.

For the two-dimensional problem of an H-polarized incident field, the general form of a GIBC imposed at the surface $y = 0$ is

$$\prod_{m=1}^{M}\left(\frac{\partial}{\partial y}+ik\gamma_m\right)H_z = 0 \tag{1}$$

which can be written alternatively as

$$\sum_{m=0}^{M} \frac{b_m}{(ik)^m} \frac{\partial^m}{\partial y^m} H_z = 0. \tag{2}$$

The parameters γ_m or, equivalently, b_m are determined by the properties of the layer and are typically obtained by expanding the interior fields in terms of a small parameter δ. When this is done, it turns out that to any given order in δ it is sufficient to confine attention to even values of M if $M > 1$ with, moreover, $b_m = 0$ for all odd $m > 1$. As shown in a recent report [Senior, 1993], it is then possible to develop a uniqueness proof that specifies the additional constraints that must be imposed at any surface discontinuity.

To illustrate the application of these constraints, we consider here the diffraction of a plane wave by the union of two half planes having first, second or fourth order GIBCs imposed at them.

2 First Order Conditions

Although a first order GIBC does not require an additional constraint, it is convenient to examine this case first.

The problem considered is the plane wave

$$U^i(x, y) = e^{-ik(x\cos\phi_0 + y\sin\phi_0)} \tag{3}$$

incident on the surface $y = 0$ at which the following boundary conditions are imposed:

$$\left(\frac{\partial}{\partial y} + ik\gamma_1\right) U = 0 \qquad x < 0 \tag{4}$$

$$\left(\frac{\partial}{\partial y} + ik\gamma_1'\right) U = 0 \qquad x > 0 \tag{5}$$

where a time factor $e^{-i\omega t}$ has been assumed and suppressed. We seek the resulting field $U(x, y)$ in $y \geq 0$ and note that if $U = H_z$, then $\frac{\partial U}{\partial y} = -ikYE_x$. In accordance with the standard edge condition, it is necessary that the singularity of E_x at $x = 0$ be integrable.

If the boundary condition (4) were to apply for all x, then

$$U(x,y) = e^{-ik(x\cos\phi_0 + y\sin\phi_0)} + \Gamma e^{-ik(x\cos\phi_0 - y\sin\phi_0)} \tag{6}$$

with

$$\Gamma = -\frac{\gamma_1 - \sin\phi_0}{\gamma_1 + \sin\phi_0}. \tag{7}$$

Denoting this field by the superscript '0' we now write

$$U(x,y) = U^0(x,y) + U^s(x,y) \tag{8}$$

and represent $U^s(x,y)$ as

$$U^s(x,y) = \int_{-\infty}^{\infty} P(\xi)\, e^{i\xi x + iy\sqrt{k^2-\xi^2}}\, \frac{d\xi}{\sqrt{k^2-\xi^2}}. \tag{9}$$

On the surface $y = 0$ the boundary conditions on U^s are

$$\left(\frac{\partial}{\partial y} + ik\gamma_1\right) U^s = 0 \qquad x < 0 \tag{10}$$

$$\left(\frac{\partial}{\partial y} + ik\gamma_1'\right) U^s = M_1 e^{-i\xi_0 x} \qquad x > 0 \tag{11}$$

where $\xi_0 = k\cos\phi_0$ and

$$M_1 = 2ik(\gamma_1 - \gamma_1')\frac{\sin\phi_0}{\gamma_1 + \sin\phi_0}.$$

From the edge condition it is necessary that $|P(\xi)| \to 0$ as $|\xi| \to \infty$.

When (10) is applied to (9) we obtain

$$i\gamma_1 \int_{-\infty}^{\infty} \left(\frac{1}{\gamma_1} + \frac{k}{\sqrt{k^2-\xi^2}}\right) P(\xi)\, e^{i\xi x}\, d\xi = 0 \tag{12}$$

for $x < 0$. Let

$$\left(\frac{1}{\gamma_1} + \frac{k}{\sqrt{k^2-\xi^2}}\right)^{-1} = K_1(\xi)\, K_1(-\xi) \tag{13}$$

where

$$K_1(\xi) = K_+\left(\xi, \frac{1}{\gamma_1}\right) \tag{14}$$

is the uper half plane (split) function defined by Senior [1952]. From the expression given there or, alternatively, from Leppington [1983],

$$K_1(\pm\xi) = \sqrt{\gamma_1}\left\{1 \mp \frac{\gamma_1 k}{\pi\xi}\ln\frac{2i\xi}{k} + O(\xi^{-1})\right\} \tag{15}$$

for large $|\xi|$ provided $\gamma_1 \neq \infty$, but if $\gamma_1 = \infty$

$$K_1(\pm\xi) = \sqrt{\frac{k \pm \xi}{k}}. \tag{16}$$

On inserting (13) into (12) we have

$$i\gamma_1 \int_{-\infty}^{\infty} \frac{P(\xi)}{K_1(\xi)\,K_1(-\xi)} e^{i\xi x}\,d\xi = 0$$

for $x < 0$, and therefore

$$P(\xi) = K_1(\xi)\,L(\xi) \tag{17}$$

where $L(\xi)$ is a lower half plane function. Similarly, for $x > 0$

$$i\gamma_1' \int_{-\infty}^{\infty} \frac{P(\xi)}{K_1'(\xi)\,K_1'(-\xi)} e^{i\xi x}\,d\xi = M_1 e^{-i\xi_0 x} \tag{18}$$

where $K_1'(\xi)$ differs from $K_1(\xi)$ in having γ_1' in place of γ_1. Hence

$$P(\xi) = \frac{K_1'(-\xi)}{\xi + \xi_0}\,U(\xi) \tag{19}$$

where $U(\xi)$ is an upper half plane function, and on combining (17) and (19) we can write

$$P(\xi) = \frac{K_1(\xi)\,K_1(\xi_0)\,K_1'(-\xi)\,K_1'(-\xi_0)}{\xi + \xi_0}\,A(\xi) \tag{20}$$

where $A(\xi)$ is a function analytic everywhere. It is therefore at most a polynomial in ξ, and because of (15) and the edge condition, $A(\xi)$ is simply a constant A_1. When (20) is inserted into (18), we have

$$i\gamma_1' A_1\,K_1(\xi_0)\,K_1'(-\xi_0) \int_{-\infty}^{\infty} \frac{K_1(\xi)}{K_1'(\xi)}\,\frac{e^{i\xi x}}{\xi + \xi_0}\,d\xi = M_1 e^{-i\xi_0 x},$$

and a residue evaluation now gives

$$-2\pi\gamma_1' A_1\, K_1(\xi_0)\, K_1(-\xi_0)\, e^{-i\xi_0 x} = M_1 e^{-i\xi_0 x}.$$

But

$$K_1(\xi_0)\, K_1(-\xi_0) = \frac{\gamma_1 \sin\phi_0}{\gamma_1 + \sin\phi_0}$$

and therefore

$$A_1 = \frac{ik}{\pi}\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_1'}\right). \tag{21}$$

Hence

$$\begin{aligned} U^s(x,y) &= \frac{ik}{\pi}\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_1'}\right) K_1(\xi_0)\, K_1'(-\xi_0) \\ &\cdot \int_{-\infty}^{\infty} \frac{K_1(\xi)\, K_1'(-\xi)}{\xi + \xi_0}\, e^{i\xi x + iy\sqrt{k^2-\xi^2}} \frac{d\xi}{\sqrt{k^2-\xi^2}} \end{aligned} \tag{22}$$

and we observe that the standard edge condition has ensured a unique solution.

It is of interest to examine the behavior of $U(x,0)$ for small $|x|$. From (22)

$$U^s(x,0) = B_1 k(\gamma_1' - \gamma_1) \int_{-\infty}^{\infty} \frac{K_1(\xi)\, K_1'(-\xi)}{\sqrt{k^2-\xi^2}} \frac{e^{i\xi x}}{\xi + \xi_0}\, d\xi \tag{23}$$

where

$$B_1 = \frac{i}{\pi\gamma_1\gamma_1'} K_1(\xi_0)\, K_1'(-\xi_0), \tag{24}$$

and the first step is to additively decompose the non-exponential portion of the integrand into functions analytic in overlapping half planes. A simple analysis shows

$$\frac{K_1(\xi)\, K_1'(-\xi)}{\sqrt{k^2-\xi^2}} = \frac{1}{k(\gamma_1' - \gamma_1)}\left\{\gamma_1' \frac{K_1(\xi)}{K_1'(\xi)} - \gamma_1 \frac{K_1'(-\xi)}{K_1(-\xi)}\right\}$$

and therefore

$$U^s(x,0) = \int_{-\infty}^{\infty} T_+(\xi)\, e^{i\xi x}\, d\xi + \int_{-\infty}^{\infty} T_-(\xi)\, e^{i\xi x}\, d\xi \tag{25}$$

where

$$T_+(\xi) = B_1 \left\{ \gamma_1' \frac{K_1(\xi)}{K_1'(\xi)} - \gamma_1' \frac{K_1(-\xi_0)}{K_1'(-\xi_0)} \right\} \frac{1}{\xi + \xi_0} \tag{26}$$

is analytic in the upper half plane Im. $\sigma >$ −Im. k, and

$$T_-(\xi) = -B_1 \left\{ \gamma_1 \frac{K_1'(-\xi)}{K_1(-\xi)} - \gamma_1' \frac{K_1(-\xi_0)}{K_1'(-\xi_0)} \right\} \frac{1}{\xi + \xi_0} \tag{27}$$

is analytic in the lower half plane Im. $\xi <$ −Im. ξ_0.

The first (second) integral on the right hand side of (25) represents a function which is zero for $x > (<) 0$, and accordingly, for $x > 0$,

$$U^s(x,0) = \int_{-\infty}^{\infty} T_-(\xi)\, e^{i\xi x}\, d\xi. \tag{28}$$

When the expression (27) for $T_-(\xi)$ is inserted, the contribution of the second term can be evaluated by path closure, giving

$$U^s(x,0) = -B_1\gamma_1 \int_{-\infty}^{\infty} \frac{K_1'(-\xi)}{K_1(-\xi)} \frac{e^{i\xi x}}{\xi + \xi_0}\, d\xi + 2\pi i B_1 \gamma_1' \frac{K_1(-\xi_0)}{K_1'(-\xi_0)}\, e^{-i\xi_0 x},$$

and since

$$2\pi i B_1 \gamma_1' \frac{K_1(-\xi_0)}{K_1'(-\xi_0)}\, e^{-i\xi_0 x} = -U^0(x,0),$$

we have

$$U(x,0) = -B_1\gamma_1 \int_{-\infty}^{\infty} \frac{K_1'(-\xi)}{K_1(-\xi)} \frac{e^{i\xi x}}{\xi + \xi_0}\, d\xi. \tag{29}$$

For large $|\xi|$

$$\frac{K_1'(-\xi)}{K_1(-\xi)} \cdot \frac{1}{\xi + \xi_0} = \sqrt{\frac{\gamma_1'}{\gamma_1}} \left\{ \frac{1}{\xi} + (\gamma_1' - \gamma_1) \frac{k}{\pi \xi^2} \ln \frac{2i\xi}{k} + O(\xi^{-2}) \right\}$$

and hence, as $x \to 0+$ (see Appendix A)

$$U(x,0) = -2\pi i B_1 \sqrt{\gamma_1 \gamma_1'} \left\{ 1 - i(\gamma_1' - \gamma_1) \frac{k}{\pi} x \ln x + O(x) \right\}. \tag{30}$$

For $x < 0$

$$\begin{aligned} U^s(x,0) &= \int_{-\infty}^{\infty} T_+(\xi)\, e^{i\xi x}\, d\xi \\ &= B_1 \gamma_1' \int_{-\infty}^{\infty} \left\{ \frac{K_1(\xi)}{K_1'(\xi)} - \frac{K_1(-\xi_0)}{K_1'(-\xi_0)} \right\} \frac{e^{i\xi x}}{\xi + \xi_0}\, d\xi, \end{aligned} \tag{31}$$

and though we cannot now evaluate the contribution of the second term by path closure, when its asymptotic expansion for large $|\xi|$ is inserted, the resulting contribution for small $|\xi|$ is precisely the expansion of $-U^0(x,0)$. Thus

$$U(x,0) \sim B_1\gamma_1' \int_{-\infty}^{\infty} \frac{K_1(\xi)}{K_1'(\xi)} \frac{e^{i\xi x}}{\xi+\xi_0}\, d\xi, \tag{32}$$

and since

$$\frac{K_1(\xi)}{K_1'(\xi)} \cdot \frac{1}{\xi+\xi_0} = \sqrt{\frac{\gamma_1}{\gamma_1'}} \left\{ \frac{1}{\xi} + (\gamma_1'-\gamma_1)\frac{k}{\pi\xi^2} \ln\frac{2i\xi}{k} + O(\xi^{-2}) \right\}$$

for large $|\xi|$,

$$U(x,0) = -2\pi i B_1 \sqrt{\gamma_1\gamma_1'} \left\{ 1 - i(\gamma_1'-\gamma_1)\frac{k}{\pi} x \ln|x| + O(x) \right\} \tag{33}$$

as $x \to 0-$.

Comparison with (30) shows that (33) is also applicable as $x \to 0+$ and demonstrates that U and $\frac{\partial U}{\partial x}$ are continuous at $x = 0$. If γ_1 and/or $\gamma_1' \neq \infty$, $U(0,0)$ is finite but $\frac{\partial U}{\partial x}$ is infinite logarithmically, and as evident from the boundary conditions, $\frac{\partial U}{\partial y}$ has a finite jump discontinuity at $x = 0$ as long as $\gamma_1' \neq \gamma_1$.

3 Second Order Conditions

These are the lowest order GIBCs for which additional constraints are necessary over and beyond the standard edge conditions to ensure a unique solution of the boundary value problem. The boundary conditions imposed at the surface $y = 0$ are

$$\prod_{m=1}^{2} \left(\frac{\partial}{\partial y} + ik\gamma_m \right) U = 0 \qquad x < 0 \tag{34}$$

$$\prod_{m=1}^{2} \left(\frac{\partial}{\partial y} + ik\gamma_m' \right) U = 0 \qquad x > 0 \tag{35}$$

which we write as

$$\left(\frac{\partial^2}{\partial y^2} + ikb_1\frac{\partial}{\partial y} - k^2 b_0 \right) U = 0 \qquad x < 0 \tag{36}$$

$$\left(\frac{\partial^2}{\partial y^2} + ikb_1'\frac{\partial}{\partial y} - k^2 b_0'\right) U \;=\; 0 \qquad x > 0 \tag{37}$$

where

$$\begin{aligned} b_1 &= \gamma_1 + \gamma_2, & b_0 &= \gamma_1\gamma_2 \\ b_1' &= \gamma_1' + \gamma_2', & b_0' &= \gamma_1'\gamma_2'. \end{aligned} \tag{38}$$

If the plane wave (3) were incident on a surface $y = 0$ having the boundary condition (36) applied for all x, the total field would be as shown in (6) with

$$\Gamma = -\prod_{m=1}^{2} \frac{\gamma_m - \sin\phi_0}{\gamma_m + \sin\phi_0}. \tag{39}$$

Denoting this field by the superscript '0', we again write

$$U(x,y) = U^0(x,y) + U^s(x,y) \tag{8}$$

and represent $U^s(x,y)$ as

$$U^s(x,y) = \int_{-\infty}^{\infty} P(\xi)\, e^{i\xi x + iy\sqrt{k^2-\xi^2}} \, \frac{d\xi}{\sqrt{k^2-\xi^2}}. \tag{9}$$

We note the requirement that $|P(\xi)| \to 0$ as $|\xi| \to \infty$ from the edge condition, and on the surface $y = 0$

$$\left(\frac{\partial^2}{\partial y^2} + ikb_1\frac{\partial}{\partial y} - k^2 b_0\right) U^s \;=\; 0 \qquad x < 0 \tag{40}$$

$$\left(\frac{\partial^2}{\partial y^2} + ikb_1'\frac{\partial}{\partial y} - k^2 b_0'\right) U^s \;=\; M_2 e^{-i\xi_0 x} \quad x > 0 \tag{41}$$

where

$$M_2 = k^2\left\{\prod_{m=1}^{2}(\gamma_m' - \sin\phi_0) + \Gamma\prod_{m=1}^{2}(\gamma_m' + \sin\phi_0)\right\}. \tag{42}$$

Application of (40) to (9) gives

$$-kb_0\int_{-\infty}^{\infty} \frac{\sqrt{k^2-\xi^2}}{k}\left(\frac{1}{\gamma_1} + \frac{k}{\sqrt{k^2-\xi^2}}\right)\left(\frac{1}{\gamma_2} + \frac{k}{\sqrt{k^2-\xi^2}}\right) P(\xi)\, e^{i\xi x}\, d\xi = 0 \tag{43}$$

for $x < 0$ and hence

$$P(\xi) = \sqrt{\frac{k}{k+\xi}}\, K_2(\xi)\, L(\xi) \tag{44}$$

where $L(\xi)$ is a lower half plane function and

$$K_2(\xi) = K_+\left(\xi, \frac{1}{\gamma_1}\right) K_+\left(\xi, \frac{1}{\gamma_2}\right). \tag{45}$$

Similarly, for $x > 0$

$$-kb_0' \int_{-\infty}^{\infty} \frac{\sqrt{k^2-\xi^2}}{k} \frac{P(\xi)}{K_2'(\xi)\, K_2'(-\xi)} \frac{e^{i\xi x}}{\xi+\xi_0}\, d\xi = M_2 e^{-i\xi_0 x} \tag{46}$$

where $K_2'(\xi)$ differs from $K_2(\xi)$ is having γ_1' and γ_2' in place of γ_1 and γ_2, and therefore

$$P(\xi) = \sqrt{\frac{k}{k-\xi}}\, \frac{K_2'(-\xi)}{\xi+\xi_0}\, U(\xi) \tag{47}$$

where $U(\xi)$ is an upper half plane function. The combination of (44) and (47) gives

$$P(\xi) = \sqrt{\frac{k^2}{k^2-\xi^2}\cdot\frac{k^2}{k^2-\xi_0^2}}\, \frac{K_2(\xi)\, K_2(\xi_0)\, K_2'(-\xi)\, K_2'(-\xi_0)}{\xi+\xi_0}\, A(\xi)$$

where $A(\xi)$ is an analytic function, and since $|P(\xi)| \to 0$ as $|\xi| \to \infty$, $A(\xi)$ is at most a first order polynomial in ξ. Hence

$$P(\xi) = \sqrt{\frac{k^2}{k^2-\xi^2}\cdot\frac{k^2}{k^2-\xi_0^2}}\, K_2(\xi)\, K_2(\xi_0)\, K_2'(-\xi)\, K_2'(-\xi_0)\, \frac{c_0+c_1\xi}{\xi+\xi_0} \tag{48}$$

for some constants c_0 and c_1.

When this is substituted into (46), we obtain

$$-\frac{kb_0'}{\sin\phi_0} K_2(\xi_0)\, K_2'(-\xi_0) \int_{-\infty}^{\infty} \frac{K_2(\xi)}{K_2'(\xi)}\, \frac{c_0+c_1\xi}{\xi+\xi_0}\, e^{i\xi x}\, d\xi = M_2 e^{-i\xi_0 x},$$

and a residue evaluation now gives

$$c_0 - \xi_0 c_1 = \frac{i}{2\pi k b_0 b_0'}\, \frac{(\gamma_1+\sin\phi_0)(\gamma_2+\sin\phi_0)}{\sin\phi_0}\, M_2$$

i.e.

$$c_0 - \xi_0 c_1 = \frac{i}{\pi k} \frac{b_1 - b_1'}{b_0 b_0'} (s^2 - \xi_0^2) \tag{49}$$

where

$$s^2 = k^2 \left\{ 1 + \frac{b_1 b_0' - b_0 b_1'}{b_1 - b_1'} \right\}. \tag{50}$$

From (49)

$$\frac{c_0 + \xi c_1}{\xi + \xi_0} = \frac{i}{\pi k} \frac{b_1 - b_1'}{b_0 b_0'} \left\{ \frac{s^2 - \xi_0^2}{\xi + \xi_0} + c_2 \right\}$$

where

$$c_2 = -i\pi k \frac{b_0 b_0'}{b_1 - b_1'} c_1$$

is an arbitrary constant, and thus

$$P(\xi) = \frac{i}{\pi} \frac{b_1 - b_1'}{b_0 b_0'} \frac{1}{\sin\phi_0 \sqrt{k^2 - \xi^2}} K_2(\xi) \, K_2(\xi_0) \, K_2'(-\xi) \, K_2'(-\xi_0) \left\{ \frac{s^2 - \xi_0^2}{\xi + \xi_0} + c_2 \right\}. \tag{51}$$

For large $|\xi|$, $P(\xi) = O(|\xi|^{-1})$ if $c_2 \neq 0$ and $O(|\xi|^{-2})$ if $c_2 = 0$, and the expression for $U^s(x, y)$ is

$$\begin{aligned} U^s(x,y) \;=\; & \frac{i}{\pi} \frac{b_1 - b_1'}{b_0 b_0'} \frac{K_2(\xi_0) \, K_2'(-\xi_0)}{\sin\phi_0} \int_{-\infty}^{\infty} \frac{K_2(\xi) \, K_2'(-\xi)}{k^2 - \xi^2} \\ & \cdot \left\{ \frac{s^2 - \xi_0^2}{\xi + \xi_0} + c_2 \right\} e^{i\xi x + iy\sqrt{k^2 - \xi^2}} \, d\xi. \end{aligned} \tag{52}$$

The presence of the arbitrary constant c_2 shows the need for an additional constraint to ensure a unique solution, and we discuss this later.

We now consider the behavior of $U(x, 0)$ for small $|x|$. From (52)

$$U^s(x, 0) = B_2 (b_1 - b_1') \int_{-\infty}^{\infty} \frac{K_2(\xi) \, K_2'(-\xi)}{k^2 - \xi^2} \left\{ \frac{s^2 - \xi_0^2}{\xi + \xi_0} + c_2 \right\} e^{i\xi x} \, d\xi \tag{53}$$

where

$$B_2 = \frac{i}{\pi} \frac{1}{b_0 b_0'} \frac{K_2(\xi_0) \, K_2'(-\xi_0)}{\sin\phi_0}. \tag{54}$$

Appearances to the contrary, B_2 is finite for $\phi_0 = 0, \pi$. The first step is to additively decompose the first factor in the integrand of (53), and a simple analysis shows that

$$K_2(\xi)\, K_2'(-\xi) = \frac{1}{b_1 - b_1'} \frac{k^2 - \xi^2}{\xi^2 - s^2} \left\{ b_0 b_1' \frac{K_2'(-\xi)}{K_2(-\xi)} - b_1 b_0' \frac{K_2(\xi)}{K_2'(\xi)} \right\}.$$

Hence

$$\begin{aligned} \frac{K_2(\xi)\, K_2'(-\xi)}{k^2 - \xi^2} &= \frac{1}{b_1 - b_1'} \frac{1}{\xi^2 - s^2} \left\{ b_0 b_1' \frac{K_2'(-\xi)}{K_2(-\xi)} + \alpha_1 \xi + \alpha_2 \right. \\ &\quad \left. - \left[b_1 b_0' \frac{K_2(\xi)}{K_2'(\xi)} + \alpha_1 \xi + \alpha_2 \right] \right\}, \end{aligned} \tag{55}$$

and since this is true for any α_1 and α_2, we can choose them to eliminate the poles at $\xi = -s$ $(+s)$ from the first (second) group of terms in (55). Then

$$\begin{aligned} \alpha_1 &= -\frac{1}{2s} \left\{ b_1 b_0' \frac{K_2(s)}{K_2'(s)} - b_0 b_1' \frac{K_2'(s)}{K_2(s)} \right\} \\ \alpha_2 &= -\frac{1}{2} \left\{ b_1 b_0' \frac{K_2(s)}{K_2'(s)} + b_0 b_1' \frac{K_2'(s)}{K_2(s)} \right\} \end{aligned}$$

implying

$$\alpha_2^2 - \alpha_1^2 s^2 = b_0 b_1 b_0' b_1',$$

and

$$U^s(x,0) = B_2 \int_{-\infty}^{\infty} \frac{1}{\xi^2 - s^2} \{S_-(\xi) - S_+(\xi)\} \left\{ \frac{s^2 - \xi_0^2}{\xi + \xi_0} + c_2 \right\} e^{i\xi x}\, d\xi \tag{56}$$

where

$$S_-(\xi) = b_0 b_1' \frac{K_2'(-\xi)}{K_2(-\xi)} + \alpha_1 \xi + \alpha_2 \tag{57}$$

$$S_+(\xi) = b_1 b_0' \frac{K_2(\xi)}{K_2'(\xi)} + \alpha_1 \xi + \alpha_2. \tag{58}$$

Finally, on eliminating the pole at $\xi = -\xi_0$ from the upper half plane function, we have

$$U^s(x,0) = \int_{-\infty}^{\infty} T_+(\xi)\, e^{i\xi x}\, d\xi + \int_{-\infty}^{\infty} T_-(\xi)\, e^{i\xi x}\, d\xi \tag{59}$$

where

$$T_+(\xi) = -B_2\left[\frac{S_+(\xi)}{\xi^2 - s^2}\left\{\frac{s^2 - \xi_0^2}{\xi + \xi_0} + c_2\right\} + \frac{S_+(-\xi_0)}{\xi + \xi_0}\right] \tag{60}$$

is analytic in the upper half plane Im. $\xi > -$Im. k, and

$$T_-(\xi) = B_2\left[\frac{S_-(\xi)}{\xi^2 - s^2}\left\{\frac{s^2 - \xi_0^2}{\xi + \xi_0} + c_2\right\} + \frac{S_+(-\xi_0)}{\xi + \xi_0}\right] \tag{61}$$

is analytic in the lower half plane Im. $\xi < -$Im. ξ_0.

The first integral on the right hand side of (59) represents a function which is zero for $x > 0$ and hence, for $x > 0$,

$$U^s(x, 0) = \int_{-\infty}^{\infty} T_-(\xi)\, e^{i\xi x}\, d\xi. \tag{62}$$

The term involving $S_+(-\xi_0)$ can be evaluated by path closure and the residue at $\xi = -\xi_0$ gives

$$2\pi i B_2\, S_+(-\xi_0)\, e^{-i\xi_0 x} = -U^0(x, 0) - 2\pi i B_2(\alpha_1\xi_0 - \alpha_2),$$

but instead of doing this, we introduce the asymptotic expansion of $T_-(\xi)$ in total. From (15) and (45)

$$K_2(\pm\xi) = \sqrt{b_0}\left\{1 \mp \frac{b_1 k}{\pi\xi}\ln\frac{2i\xi}{k} + O(\xi^{-1})\right\} \tag{63}$$

for large $|\xi|$, and therefore

$$\frac{S_-(\xi)}{\xi^2 - s^2} = \frac{1}{\xi}\left\{\alpha_1 + \frac{1}{\xi}\left(\alpha_2 + b_1'\sqrt{b_0 b_0'}\right) - b_1'\sqrt{b_0 b_0'}(b_1 - b_1')\frac{k}{\pi\xi^2}\ln\frac{2i\xi}{k} + O(\xi^{-2})\right\}.$$

Also

$$\frac{s^2 - \xi^2}{\xi + \xi_0} + c_2 = c_2 + \frac{1}{\xi}(s^2 - \xi_0^2) + O(\xi^{-2}) \tag{64}$$

giving

$$\begin{aligned} T_-(\xi) \;=\; & \frac{B_2}{\xi}\Bigg\{(c_2 - \xi_0)\alpha_1 + \alpha_2 + \frac{1}{\xi}\left[(c_2 - \xi_0)\alpha_2 + \alpha_1 s^2 + c_2 b_1'\sqrt{b_0 b_0'}\right] \\ & - c_2 b_1'\sqrt{b_0 b_0'}(b_1 - b_1')\frac{k}{\pi\xi^2}\ln\frac{2i\xi}{k} + O(\xi^{-2})\Bigg\} \\ & + \frac{B_2}{\xi} b_1 b_0' \frac{K_2(-\xi_0)}{K_2'(-\xi_0)}\left\{1 - \frac{\xi_0}{\xi} + O(\xi^{-2})\right\}, \end{aligned}$$

and hence (see Appendix A)

$$
\begin{aligned}
U^s(x,0) &= 2\pi i B_2\Big\{(c_2-\xi_0)\alpha_1+\alpha_2+ix\Big[(c_2-\xi_0)\alpha_2+\alpha_1 s^2+c_2 b_1'\sqrt{b_0 b_0'}\Big] \\
&\quad - c_2 b_1'\sqrt{b_0 b_0'}(b_1-b_1')\frac{k}{2\pi}x^2\ln x+O(x^2)\Big\} \\
&\quad - \frac{2b_1\sin\phi_0}{(\gamma_1+\sin\phi_0)(\gamma_2+\sin\phi_0)}\Big\{1-ix\xi_0+O(x^2)\Big\}
\end{aligned}
$$

for small x. We recognise the last term as the expansion of $U^0(x,0)$ and therefore

$$
\begin{aligned}
U(x,0) &= 2\pi i B_2\Big\{(c_2-\xi_0)\alpha_1+\alpha_2+ix\Big[(c_2-\xi_0)\alpha_2+\alpha_1 s^2+c_2 b_1'\sqrt{b_0 b_0'}\Big] \\
&\quad - c_2 b_1'\sqrt{b_0 b_0'}(b_1-b_1')\frac{k}{2\pi}x^2\ln x+O(x^2)\Big\} \qquad (65)
\end{aligned}
$$

as $x \to 0+$.

Similarly, for $x<0$,

$$
U^s(x,0)=\int_{-\infty}^{\infty} T_+(\xi)\, e^{i\xi x}\, d\xi, \qquad (66)
$$

and for large $|\xi|$

$$
\begin{aligned}
T_+(\xi) &= -\frac{B_2}{\xi}\Big\{(c_2-\xi_0)\alpha_1+\alpha_2+\frac{1}{\xi}\Big[(c_2-\xi_0)\alpha_2+\alpha_1 s^2+c_2 b_1\sqrt{b_0 b_0'}\Big] \\
&\quad - c_2 b_1\sqrt{b_0 b_0'}(b_1-b_1')\frac{k}{\pi\xi^2}\ln\frac{2i\xi}{k}+O(\xi^{-2})\Big\} \\
&\quad - \frac{B_2}{\xi}b_1 b_0'\frac{K_2(-\xi_0)}{K_2'(-\xi_0)}\Big\{1-\frac{\xi_0}{\xi}+O(\xi^{-2})\Big\},
\end{aligned}
$$

implying

$$
\begin{aligned}
U(x,0) &= 2\pi i B_2\Big\{(c_2-\xi_0)\alpha_1+\alpha_2+ix\Big[(c_2-\xi_0)\alpha_2+\alpha_1 s^2+c_2 b_1\sqrt{b_0 b_0'}\Big] \\
&\quad - c_2 b_1\sqrt{b_0 b_0'}(b_1-b_1')\frac{k}{2\pi}x^2\ln|x|+O(x^2)\Big\} \qquad (67)
\end{aligned}
$$

as $x \to 0-$. Comparison of (65) and (67) shows that

(i) $U(x,0)$ is finite and continuous at $x=0$ for all (finite) c_2,

(ii) $U(0,0)=0$ if $c_2=\xi_0-\alpha_2/\alpha_1$,

(iii) $\frac{\partial U}{\partial x}$ has a finite jump discontinuity at $x=0$ for all (finite) c_2,

(iv) $\lim_{x\to 0-}\frac{1}{b_1}\frac{\partial U}{\partial x}=\lim_{x\to 0+}\frac{1}{b_1'}\frac{\partial U}{\partial x}$ if $c_2=\xi_0-\alpha_1 s^2/\alpha_2$, and

(v) from the boundary conditions, $\frac{\partial U}{\partial y}$ has a logarithmic singularity at $x=0$.

The boundary conditions (36) and (37) are particular examples of the second order GIBC

$$\frac{\partial U}{\partial y}=-\left(\alpha-\beta\frac{\partial^2}{\partial x^2}\right)U \tag{68}$$

discussed by Senior [1993] with

$$\alpha=ik\frac{1+b_0}{b_1},\qquad \beta=-\frac{i}{kb_1}\qquad (x<0)$$

$$\alpha=ik\frac{1+b_0'}{b_1'},\qquad \beta=-\frac{i}{kb_1'}\qquad (x>0).$$

As shown there, the solution of the boundary value problem is unique if

$$\text{Im. }\alpha,\ \text{Im. }\beta\geq 0$$

and, in addition to the standard edge condition,

$$\left[U\beta\frac{\partial U}{\partial x}\right]_-^+=0 \tag{69}$$

The added constraint is therefore

$$U(0,0)=0\qquad\text{or}\qquad\left[\beta\frac{\partial U}{\partial x}\right]_-^+=0 \tag{70}$$

and which is appropriate depends on the physical problem.

If $U=H_z$ and the boundary condition specifies the electric current $\mathbf{J}$ induced in the surface, then $\mathbf{J}(x)=\hat{x}\,U(x,0)$, and since the electric current is perpendicular to the edge, the required constraint is $U(0,0) = 0$, demanding (see (ii) above)

$$c_2 = \xi_0 - \alpha_2/\alpha_1. \tag{71}$$

On the other hand, if the condition specifies the magnetic current $\mathbf{J}^*$, U is not required to be zero at the edge, and the appropriate constraint is then the second of (70), demanding (see (iv) above)

$$c_2 = \xi_0 - \alpha_1 s^2/\alpha_2. \tag{72}$$

In both cases it can be verified that the resulting solution is in accordance with the reciprocity condition.

4 Fourth Order Conditions

For many layered structures, fourth order GIBCs are capable of providing a very accurate simulation of the scattring properties, and we now consider the diffraction of the plane wave (3) by the junction of two half planes each subject to the special form of fourth order GIBCs discussed by Senior [1993]. The boundary conditions imposed at the surface $y = 0$ are

$$\prod_{m=1}^{4}\left(\frac{\partial}{\partial y} + ik\gamma_m\right)U = 0 \qquad x < 0 \tag{73}$$

$$\prod_{m=1}^{4}\left(\frac{\partial}{\partial y} + ik\gamma'_m\right)U = 0 \qquad x > 0 \tag{74}$$

with

$$\sum_{m=1}^{4}\gamma_m = 0 = \sum_{m=1}^{4}\gamma'_m, \tag{75}$$

and these can be written as

$$\sum_{m=0}^{4}\frac{b_m}{(ik)^m}\frac{\partial^m U}{\partial y^m} = 0 \qquad x < 0 \tag{76}$$

$$\sum_{m=0}^{4}\frac{b'_m}{(ik)^m}\frac{\partial^m U}{\partial y^m} = 0 \qquad x > 0 \tag{77}$$

where

$$b_4 = 1, \qquad b_3 = 0, \qquad b_2 = \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_1\gamma_4 + \gamma_2\gamma_3 + \gamma_2\gamma_4 + \gamma_3\gamma_4$$
$$b_1 = \gamma_1\gamma_2\gamma_3 + \gamma_1\gamma_2\gamma_4 + \gamma_1\gamma_3\gamma_4 + \gamma_2\gamma_3\gamma_4, \qquad b_0 = \gamma_1\gamma_2\gamma_3\gamma_4$$

with analogous definitions of the b'_m.

If the plane wave were incident on a surface having the boundary condition (73) for all x, $-\infty < x < \infty$, the total field would be as shown in (6) with

$$\Gamma = -\prod_{m=1}^{4} \frac{\gamma_m - \sin\phi_0}{\gamma_m + \sin\phi_0}. \tag{78}$$

Denoting this field by the superscript '0', we again write

$$U(x,y) = U^0(x,y) + U^s(x,y) \tag{8}$$

with

$$U^s(x,y) = \int_{-\infty}^{\infty} P(\xi)\, e^{i\xi x + iy\sqrt{k^2-\xi^2}} \frac{d\xi}{\sqrt{k^2-\xi^2}}. \tag{9}$$

where $|P(\xi)| \to 0$ as $|\xi| \to \infty$. The boundary conditions on U^s are then

$$\prod_{m=1}^{4}\left(\frac{\partial}{\partial y} + ik\gamma_m\right) U^s \;=\; 0 \qquad\qquad x < 0 \tag{79}$$

$$\prod_{m=1}^{4}\left(\frac{\partial}{\partial y} + ik\gamma'_m\right) U^s \;=\; M_4 e^{-i\xi_0 x} \qquad x > 0 \tag{80}$$

where

$$M_4 = -k^2 \left\{ \prod_{m=1}^{4} (\gamma'_m - \sin\phi_0) + \Gamma \prod_{m=1}^{4} (\gamma'_m + \sin\phi_0) \right\}. \tag{81}$$

When (79) is applied to (9), we obtain

$$k^3 b_0 \int_{-\infty}^{\infty} \left(\frac{\sqrt{k^2-\xi^2}}{k}\right)^3 \prod_{m=1}^{4}\left(\frac{1}{\gamma_m} + \frac{k}{\sqrt{k^2-\xi^2}}\right) P(\xi)\, e^{i\xi x}\, d\xi = 0 \tag{82}$$

for $x < 0$ and hence

$$P(\xi) = \left(\frac{k}{k+\xi}\right)^{3/2} K_4(\xi)\, L(\xi) \tag{83}$$

where $L(\xi)$ is a lower half plane function and

$$K_4(\xi) = K_+\left(\xi, \frac{1}{\gamma_1}\right) K_+\left(\xi, \frac{1}{\gamma_2}\right) K_+\left(\xi, \frac{1}{\gamma_3}\right) K_+\left(\xi, \frac{1}{\gamma_4}\right). \tag{84}$$

From the condition for $x > 0$

$$k^3 b_0' \int_{-\infty}^{\infty} \left(\frac{\sqrt{k^2-\xi^2}}{k}\right)^3 \prod_{m=1}^{4} \left(\frac{1}{\gamma_m'} + \frac{k}{\sqrt{k^2-\xi^2}}\right) P(\xi)\, e^{i\xi x}\, d\xi = M_4 e^{-i\xi_0 x} \tag{85}$$

implying

$$P(\xi) = \left(\frac{k}{k-\xi}\right)^{3/2} \frac{K_4'(-\xi)}{\xi+\xi_0} U(\xi) \tag{86}$$

where $U(\xi)$ is an upper half plane function and $K_4'(\xi)$ differs from $K_4(\xi)$ in having γ_m' in place of γ_m. The combination of (83) and (86) shows

$$P(\xi) = \left(\frac{k^2}{k^2-\xi^2}\right)^{3/2} \frac{K_4(\xi)\, K_4'(-\xi)}{\xi+\xi_0} A(\xi)$$

where $A(\xi)$ is an analytic function, and the allowed behavior of $P(\xi)$ as $|\xi| \to \infty$ limits $A(\xi)$ to at most a second degree polynomial in ξ. We can therefore write

$$P(\xi) = \left(\frac{k^2}{k^2-\xi^2} \cdot \frac{k^2}{k^2-\xi_0^2}\right)^{3/2} K_4(\xi)\, K_4(\xi_0)\, K_4'(-\xi)\, K_4'(-\xi_0) \frac{c_0 + c_1\xi + c_2\xi^2}{\xi+\xi_0} \tag{87}$$

for some constants c_0, c_1 and c_2. When this is substituted into (85) we obtain

$$\frac{k^3 b_0'}{\sin^3\phi_0} K_4(\xi_0)\, K_4'(-\xi_0) \int_{-\infty}^{\infty} \frac{K_4(\xi)}{K_4'(\xi)} \frac{c_0 + c_1\xi + c_2\xi^2}{\xi+\xi_0} e^{i\xi x}\, d\xi = M_4 e^{-i\xi_0 x},$$

and a residue evaluation now gives

$$2\pi i \frac{k^3 b_0'}{\sin^3\phi_0} K_4(\xi_0)\, K_4(-\xi_0) \left(c_0 - c_1\xi_0 + c_2\xi_0^2\right) = M_4.$$

But

$$K_4(\xi_0)\, K_4(-\xi_0) = b_0 \sin^4\phi_0 \left\{\prod_{m=1}^{4} (\gamma_m + \sin\phi_0)\right\}^{-1}$$

and thus

$$c_0 - c_1\xi_0 + c_2\xi_0^2 = \frac{ik}{2\pi b_0 b_0' \sin\phi_0}\left\{\prod_{m=1}^{4}(\gamma_m + \sin\phi_0)(\gamma_m' - \sin\phi_0) - \prod_{m=1}^{4}(\gamma_m - \sin\phi_0)(\gamma_m' + \sin\phi_0)\right\}.$$

Moreover

$$\left\{\quad\right\} = \frac{2}{k^4}(b_1 - b_1')\sin\phi_0(s_1^2 - x_0^2)(s_2^2 - x_0^2)$$

where

$$s_1^2 + s_2^2 = k^2\left\{2 + \frac{b_1 b_2' - b_2 b_1'}{b_1 - b_1'}\right\} \tag{88}$$

$$s_1^2 s_2^2 = k^4\left\{1 + \frac{b_1(b_0' + b_2') - b_1'(b_0 + b_2)}{b_1 - b_1'}\right\} \tag{89}$$

so that

$$c_0 = c_1\xi_0 - c_2\xi_0^2 + \frac{i}{\pi k^3}\frac{b_1 - b_1'}{b_0 b_0'}(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2).$$

Hence

$$\frac{c_0 + c_1\xi + c_2\xi^2}{\xi + \xi_0} = c_1 + c_2(\xi - \xi_0) + \frac{i}{\pi k^3}\frac{b_1 - b_1'}{b_0 b_0'}\frac{(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2)}{\xi + \xi_0}$$

which can be written as

$$\frac{c_0 + c_1\xi + c_2\xi^2}{\xi + \xi_0} = \frac{i}{\pi k^3}\frac{b_1 - b_1'}{b_0 b_0'}\left\{\frac{(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2)}{\xi + \xi_0} + c_3 + c_4(\xi - \xi_0)\right\}$$

for some constants c_3 and c_4, at present arbitrary, and the resulting expression for $P(\xi)$ is

$$P(\xi) = \frac{i}{\pi}k^3\frac{b_1 - b_1'}{b_0 b_0'}\frac{K_4(\xi)\,K_4(\xi_0)\,K_4'(-\xi)\,K_4'(-\xi_0)}{\{(k^2 - \xi^2)(k^2 - \xi_0^2)\}^{3/2}} \cdot\left\{\frac{(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2)}{\xi + \xi_0} + c_3 + c_4(\xi - \xi_0)\right\} \tag{90}$$

giving

$$U^s(x,y) = \frac{i}{\pi}\frac{b_1-b_1'}{b_0b_0'}\frac{K_4(\xi_0)\,K_4'(-\xi_0)}{\sin^3\phi_0}\int_{-\infty}^{\infty}\frac{K_4(\xi)\,K_4'(-\xi)}{(k^2-\xi^2)^2} \cdot\left\{\frac{(s_1^2-\xi_0^2)(s_2^2-\xi_0^2)}{\xi+\xi_0}+c_3+c_4(\xi-\xi_0)\right\}e^{i\xi x+iy\sqrt{k^2-\xi^2}}\,d\xi. \tag{91}$$

We note the similarity to (52), and the presence of two undetermined constants indicates the need for two constraints.

On the surface $y=0$

$$U^s(x,0) = B_4(b_1-b_1')\int_{-\infty}^{\infty}\frac{K_4(\xi)\,K_4'(-\xi)}{(k^2-\xi^2)^2} \cdot\left\{\frac{(s_1^2-\xi_0^2)(s_2^2-\xi_0^2)}{\xi+\xi_0}+c_3+c_4(\xi-\xi_0)\right\}e^{i\xi x}\,d\xi \tag{92}$$

where

$$B_4 = \frac{i}{\pi}\frac{1}{b_0b_0'}\frac{K_4(\xi_0)\,K_4'(-\xi_0)}{\sin^3\phi_0}. \tag{93}$$

To find the behavior for small $|x|$, the first step is to additively decompose the first factor in the integrand of (92), and a straightforward but tedious analysis shows

$$K_4(\xi)\,K_4'(-\xi) = \frac{1}{b_1-b_1'}\frac{(k^2-\xi^2)^2}{(\xi^2-s_1^2)(\xi^2-s_2^2)}\left\{b_1b_0'\frac{K_4(\xi)}{K_4'(\xi)}-b_0b_1'\frac{K_4'(-\xi)}{K_4(-\xi)}\right\}. \tag{94}$$

Hence

$$\frac{K_4(\xi)\,K_4'(-\xi)}{(k^2-\xi^2)^2} = \frac{1}{b_1-b_1'}\frac{1}{(\xi^2-s_1^2)(\xi^2-s_2^2)}\left\{S_+(\xi)-S_-(\xi)\right\} \tag{95}$$

where

$$S_+(\xi) = b_1b_0'\frac{K_4(\xi)}{K_4'(\xi)}+\beta_1\xi^3+\beta_2\xi^2+\beta_3\xi+\beta_4 \tag{96}$$

$$S_-(\xi) = b_0b_1'\frac{K_4'(-\xi)}{K_4(-\xi)}+\beta_1\xi^3+\beta_2\xi^2+\beta_3\xi+\beta_4. \tag{97}$$

Since (95) is valid for any $\beta_i,\ i=1,2,3,4$, we can choose the β_i to eliminate the poles at $\xi=s_1,s_2$, and $-s_1,-s_2$ from the factors involving $S_+(\xi)$

and $S_-(\xi)$ respectively. The resulting expressions for the β_i are given in Appendix B. The final step is to eliminate the pole at $\xi = -\xi_0$ from the upper half plane function, and then

$$U^s(x,0) = \int_{-\infty}^{\infty} T_+(\xi)\, e^{i\xi x}\, d\xi + \int_{-\infty}^{\infty} T_-(\xi)\, e^{i\xi x}\, d\xi \tag{98}$$

where

$$\begin{aligned} T_+(\xi) &= B_4 \left[\frac{S_+(\xi)}{(\xi^2 - s_1^2)(\xi^2 - s_2^2)} \left\{ \frac{(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2)}{\xi + \xi_0} + c_3 + c_4(\xi - \xi_0) \right\} \right. \\ &\qquad \left. - \frac{S_+(-\xi_0)}{\xi + \xi_0} - c_4\beta_1 \right] &(99) \\ T_-(\xi) &= -B_4 \left[\frac{S_-(\xi)}{(\xi^2 - s_1^2)(\xi^2 - s_2^2)} \left\{ \frac{(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2)}{\xi + \xi_0} + c_3 + c_4(\xi - \xi_0) \right\} \right. \\ &\qquad \left. - \frac{S_+(-\xi_0)}{\xi + \xi_0} - c_4\beta_1 \right] &(100) \end{aligned}$$

and the term $c_4\beta_1$ has been subtracted to make each integral converge.

The first (second) term on the right hand side of (98) represents a function which is zero for $x > (<)0$ and hence, for $x > 0$,

$$U^s(x,0) = \int_{-\infty}^{\infty} T_-(\xi)\, e^{i\xi x}\, d\xi. \tag{101}$$

Once again, the term involving $S_+(-\xi_0)$ can be evaluated by path closure in the upper half plane, and the residue at $\xi = -\xi_0$ gives, in part, $U^0(x,0)$. To isolate this contribution we write

$$T_-(\xi) = T'_-(\xi) + B_4 b_1 b'_0 \frac{K_4(-\xi_0)}{K'_4(-\xi_0)} \frac{1}{\xi + \xi_0}$$

where

$$\begin{aligned} T'_-(\xi) &= -B_4 \left[\frac{S_-(\xi)}{(\xi^2 - s_1^2)(\xi^2 - s_2^2)} \left\{ \frac{(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2)}{\xi + \xi_0} + c_3 + c_4(\xi - \xi_0) \right\} \right. \\ &\qquad \left. + \frac{C}{\xi + \xi_0} - c_4\beta_1 \right] &(102) \end{aligned}$$

with

$$C = \beta_1 \xi_0^3 - \beta_2 \xi_0^2 + \beta_3 \xi_0 - \beta_4. \tag{103}$$

Then

$$U^s(x,0) = \int_{-\infty}^{\infty} T'_-(\xi)\, e^{i\xi x}\, d\xi + 2\pi i B_4 b_1 b'_0 \frac{K_4(-\xi_0)}{K'_4(-\xi_0)} e^{i\xi_0 x}$$

and since

$$2\pi i B_4 b_1 b'_0 \frac{K_4(-\xi_0)}{K'_4(-\xi_0)} e^{i\xi_0 x} = -\frac{2b_1}{b_0} \frac{K_4(\xi_0)\, K_4(-\xi_0)}{\sin^3 \phi_0} = -U^0(x,0),$$

we have

$$U(x,0) = \int_{-\infty}^{\infty} T'_-(\xi)\, e^{i\xi x}\, d\xi. \tag{104}$$

We seek the behavior of $T'_-(\xi)$ for large $|\xi|$ and, hence the behavior of $U(x,0)$ for small $x > 0$.

From the definition of $K_4(\xi)$,

$$\begin{aligned} \frac{1}{K_4(\xi)\, K_4(-\xi)} &= \frac{1}{b_0(k^2-\xi^2)^2} \prod_{m=1}^{4} \left(\sqrt{k^2-\xi^2} + k\gamma_m\right) \\ &= \frac{1}{b_0(k^2-\xi^2)} \left\{ k^2(1+b_2) - \xi^2 + \frac{k^3 b_1}{\sqrt{k^2-\xi^2}} + \frac{k^4 b_0}{k^2-\xi^2} \right\} \end{aligned}$$

since $b_3 = 0$, and therefore

$$K_4(\pm\xi) = \sqrt{b_0}\, \frac{k \pm \xi}{k\sqrt{1+b_2} \pm \xi} \left\{1 + O(\xi^{-2} \ln \xi)\right\}$$

for large $|\xi|$, implying

$$K_4(\pm\xi) = \sqrt{b_0} \left\{ 1 \pm \frac{k}{\xi} \left(1 - \sqrt{1+b_2}\right) + O(\xi^{-2} \ln \xi) \right\}. \tag{105}$$

Then

$$\begin{aligned} S_-(\xi) &= \xi^3 \left\{ \beta_1 + \frac{\beta_2}{\xi} + \frac{\beta_3}{\xi^2} + \frac{1}{\xi^3} \left(\beta_4 + b'_1 \sqrt{b_0 b'_0}\right) \right. \\ &\qquad \left. - \frac{k}{\xi^4} b'_1 \sqrt{b_0 b'_0} \left(\sqrt{1+b_2} - \sqrt{1+b'_2}\right) + O(\xi^{-5} \ln \xi) \right\}, \end{aligned}$$

and since

$$\frac{1}{(\xi^2-s_1^2)(\xi^2-s_2^2)} = \frac{1}{\xi^4}\left\{1+\frac{1}{\xi^2}(s_1^2+s_2^2)+\frac{1}{\xi^4}(s_1^4+s_1^2s_2^2+s_2^4)+O(\xi^{-6})\right\},$$

$$\begin{aligned}\frac{S_-(\xi)}{(\xi^2-s_1^2)(\xi^2-s_2^2)} &= \frac{1}{\xi}\left\{\beta_1+\frac{\beta_2}{\xi}+\frac{1}{\xi^2}[\beta_3+\beta_1(s_1^2+s_2^2)]\right.\\ &\quad+\frac{1}{\xi^3}\left[\beta_4+\beta_2(s_1^2+s_2^2)+b_1'\sqrt{b_0b_0'}\right]\\ &\quad+\frac{1}{\xi^4}\left[\beta_3(s_1^2+s_2^2)+\beta_1(s_1^4+s_1^2s_2^2+s_2^4)\right.\\ &\quad\left.\left.-kb_1'\sqrt{b_0b_0'}\left(\sqrt{1+b_2}-\sqrt{1+b_2'}\right)\right]+O(\xi^{-5}\ln\xi)\right\}.\end{aligned}$$

Writing

$$(s_1^2-\xi_0^2)(s_2^2-\xi_0^2)=A,$$

we have

$$\frac{A}{\xi+\xi_0}+c_3+c_4(\xi-\xi_0)=\xi\left\{c_4+\frac{1}{\xi}(c_3-c_4\xi_0)+\frac{A}{\xi^2}-\frac{Ax_0}{\xi^3}+\frac{Ax_0^2}{\xi^4}+O(\xi^{-5})\right\},$$

and therefore

$$\begin{aligned}&\frac{S_-(\xi)}{(\xi^2-s_1^2)(\xi^2-s_2^2)}\left\{\frac{(s_1^2-\xi_0^2)(s_2^2-\xi_0^2)}{\xi+\xi_0}+c_3+c_4(\xi-\xi_0)\right\}=\\ &\quad c_4\beta_1+\frac{1}{\xi}\left\{c_3\beta_1+c_4(\beta_2-\beta_1\xi_0)\right\}\\ &\quad+\frac{1}{\xi^2}\left\{c_3\beta_2+c_4\left[\beta_3+\beta_1(s_1^2+s_2^2)-\beta_2\xi_0\right]+\beta_1A\right\}\\ &\quad+\frac{1}{\xi^3}\left\{c_3\left[\beta_3+\beta_1(s_1^2+s_2^2)\right]\right.\\ &\quad\left.+c_4\left[\beta_4+\beta_2(s_1^2+s_2^2)-\beta_3\xi_0-\beta_1\xi_0(s_1^2+s_2^2)+b_1'\sqrt{b_0b_0'}\right]+A(\beta_2-\beta_1\xi_0)\right\}\\ &\quad+\frac{1}{\xi^4}\left\{c_3\left[\beta_4+\beta_2(s_1^2+s_2^2)+b_1'\sqrt{b_0b_0'}\right]\right.\\ &\qquad+c_4\left[\beta_3(s_1^2+s_2^2)+\beta_1(s_1^4+s_1^2s_2^2+s_2^4)-\beta_4\xi_0-\beta_2\xi_0(s_1^2+s_2^2)\right.\end{aligned}$$

$$
\begin{aligned}
& - b_1' \sqrt{b_0 b_0'} \left\{ k \left(\sqrt{1+b_2} - \sqrt{1+b_2'} \right) + \xi_0 \right\} \Big] \\
& + A[\beta_3 + \beta_1 (s_1^2 + s_2^2) + \beta_1 \xi_0^2 - \beta_2 \xi_0] \Big\} + O(\xi^{-5} \ln \xi).
\end{aligned}
$$

Finally, since

$$
\frac{C}{\xi + \xi_0} - c_4 \beta_1 = -c_4 \beta_1 + \frac{C}{\xi} - \frac{C\xi_0}{\xi^2} + \frac{C\xi_0^2}{\xi^3} - \frac{C\xi_0^3}{\xi^4} + O(\xi^{-5}),
$$

the expansion of $T'_-(\xi)$ is

$$
\begin{aligned}
T'_-(\xi) \;=\; & -\frac{B_4}{\xi}\Gamma_0 - \frac{B_4}{\xi^2}\Gamma_1 - \frac{B_4}{\xi^3}\left\{ \Gamma_2 + c_4 b_1' \sqrt{b_0 b_0'} \right\} \\
& - \frac{B_4}{\xi^4}\left\{ \Gamma_3 + b_1' \sqrt{b_0 b_0'} \left[c_3 - c_4 k \left(\sqrt{1+b_2} - \sqrt{1+b_2'} \right) \right] \right\} \\
& + O(\xi^{-5} \ln \xi) \qquad (106)
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_0 &= c_3 \beta_1 + c_4 (\beta_2 - \beta_1 \xi_0) + C \\
&= (c_4 - \xi_0^2)(\beta_2 - \beta_1 \xi_0) + c_3 \beta_1 + \beta_3 \xi_0 - \beta_4 \qquad (107) \\
\Gamma_1 &= c_3 \beta_2 + c_4 \left[\beta_3 + \beta_1 (s_1^2 + s_2^2) - \beta_2 \xi_0 \right] + \beta_1 A - C\xi_0 \\
&= (c_4 - \xi_0^2) \left[\beta_3 + \beta_1 (s_1^2 + s_2^2) - \beta_2 \xi_0 \right] + c_3 \beta_2 + \beta_4 \xi_0 + \beta_1 s_1^2 s_2^2 \qquad (108) \\
\Gamma_2 &= c_3 \left[\beta_3 + \beta_1 (s_1^2 + s_2^2) \right] + c_4 \left[\beta_4 + \beta_2 (s_1^2 + s_2^2) - \beta_3 \xi_0 - \beta_1 \xi_0 (s_1^2 + s_2^2) \right] \\
&\qquad + A(\beta_2 - \beta_1 \xi_0) + C\xi_0^2 \\
&= (c_4 - \xi_0^2) \left[\beta_4 + \beta_2 (s_1^2 + s_2^2) - \beta_3 \xi_0 - \beta_1 \xi_0 (s_1^2 + s_2^2) \right] \\
&\qquad + c_3 \left[\beta_3 + \beta_1 (s_1^2 + s_2^2) \right] + (\beta_2 - \beta_1 \xi_0) s_1^2 s_2^2 \qquad (109) \\
\Gamma_3 &= c_3 \left[\beta_4 + \beta_2 (s_1^2 + s_2^2) \right] \\
&\qquad + c_4 \left[\beta_3 (s_1^2 + s_2^2) + \beta_1 (s_1^4 + s_1^2 s_2^2 + s_2^4) - \beta_4 \xi_0 - \beta_2 \xi_0 (s_1^2 + s_2^2) \right] \\
&\qquad + A \left[\beta_3 + \beta_1 (s_1^2 + s_2^2) + \beta_1 \xi_0^2 - \beta_2 \xi_0 \right] - C\xi_0^3 \\
&= (c_4 - \xi_0^2) \left[\beta_1 (s_1^4 + s_1^2 s_2^2 + s_2^4) + (\beta_3 - \beta_2 \xi_0)(s_1^2 + s_2^2) - \beta_4 \xi_0 \right] \\
&\qquad + c_3 \left[\beta_4 + \beta_2 (s_1^2 + s_2^2) \right] + \left[\beta_3 + \beta_1 (s_1^2 + s_2^2) - \beta_2 \xi_0 \right] s_1^2 s_2^2 \qquad (110)
\end{aligned}
$$

and hence (see Appendix A)

$$\begin{aligned} U(x,0) &= -2\pi i B_4 \Big\{ \Gamma_0 + ix\Gamma_1 - \frac{1}{2}x^2 \left[\Gamma_2 + c_4 b_1' \sqrt{b_0 b_0'} \right] - \frac{i}{6}x^3 \Big[\Gamma_3 \\ &\quad + b_1' \sqrt{b_0 b_0'} \left\{ c_3 - c_4 k \left(\sqrt{1+b_2} - \sqrt{1+b_2'} \right) \right\} \Big] \\ &\quad + O(x^4 \ln x) \Big\} \end{aligned} \tag{111}$$

for small $x > 0$.

For $x < 0$

$$U^s(x,0) = \int_{-\infty}^{\infty} T_+(\xi)\, e^{i\xi x}\, d\xi, \tag{112}$$

and since the analysis is similar to the above, we will omit the details. It is found that

$$\begin{aligned} U(x,0) &= -2\pi i B_4 \Big\{ \Gamma_0 + ix\Gamma_1 - \frac{1}{2}x^2 \left[\Gamma_2 + c_4 b_1 \sqrt{b_0 b_0'} \right] - \frac{i}{6}x^3 \Big[\Gamma_3 \\ &\quad + b_1 \sqrt{b_0 b_0'} \left\{ c_3 - c_4 k \left(\sqrt{1+b_2} - \sqrt{1+b_2'} \right) \right\} \Big] \\ &\quad + O(x^4 \ln |x|) \Big\} \end{aligned} \tag{113}$$

for small $x < 0$. Comparison of (111) and (112) now shows:

(i) U and $\frac{\partial U}{\partial x}$ are finite and continuous at $x = 0$ for all (finite) c_3 and c_4,

(ii) $U(0,0) = 0$ if c_3 and c_4 are such that $\Gamma_0 = 0$,

(iii) $\frac{\partial U}{\partial x} = 0$ at $x = 0$ if c_3 and c_4 are such that $\Gamma_1 = 0$,

(iv) $\frac{\partial^2 U}{\partial x^2}$ and $\frac{\partial^3 U}{\partial x^3}$ have finite jump discontinuities at $x = 0$ if c_3 and $c_4 \neq 0$,

(v) $\lim_{x\to 0-} \frac{1}{b_1}\frac{\partial^2 U}{\partial x^2} = \lim_{x\to 0+} \frac{1}{b_1'}\frac{\partial^2 U}{\partial x^2}$ if c_3 and c_4 are such that $\Gamma_2 = 0$,

(vi) $\lim_{x\to 0-} \frac{1}{b_1}\frac{\partial^3 U}{\partial x^3} = \lim_{x\to 0+} \frac{1}{b_1'}\frac{\partial^3 U}{\partial x^3}$ if c_3 and c_4 are such that $\Gamma_3 = 0$,

(vii) $\frac{\partial^4 U}{\partial x^4}$ and (from the boundary conditions) $\frac{\partial U}{\partial y}$ have logarithmic singularities at $x = 0$.

The boundary conditions (73) and (74) with (75) are particular examples of the fourth order GIBC

$$\frac{\partial U}{\partial y} = -\left(\alpha - \beta\frac{\partial^2}{\partial x^2} + \gamma\frac{\partial^4}{\partial x^4}\right)U \tag{114}$$

discussed by Senior [1993] with

$$\alpha = ik^3\frac{1+b_0+b_2}{b_1}, \qquad \beta = -ik\frac{2+b_2}{b_1}, \qquad \gamma = \frac{i}{kb_1} \qquad (x<0)$$

$$\alpha = ik^3\frac{1+b_0'+b_2'}{b_1'}, \qquad \beta = -ik\frac{2+b_2'}{b_1'}, \qquad \gamma = \frac{i}{kb_1'}. \qquad (x>0)$$

The additional constraints necessary for a unique solution of the boundary value problem are

$$\left[U\left\{\beta\frac{\partial U}{\partial x} - \gamma\frac{\partial^3 U}{\partial x^3}\right\}\right]_-^+ = 0, \qquad \left[\frac{\partial U}{\partial x}\gamma\frac{\partial^2 U}{\partial x^2}\right]_-^+ = 0,$$

that is,

$$U = 0 \quad \text{at } x = 0 \quad \text{or} \quad \left[\beta\frac{\partial U}{\partial x} - \gamma\frac{\partial^3 U}{\partial x^3}\right]_-^+ = 0 \tag{115}$$

and

$$\frac{\partial U}{\partial x} = 0 \quad \text{at } x = 0 \quad \text{or} \quad \left[\gamma\frac{\partial^2 U}{\partial x^2}\right]_-^+ = 0. \tag{116}$$

On inserting the expressions for β and γ and using (88), the constraints reduce to

$$\Gamma_0 = 0 \qquad \text{or} \qquad \Gamma_3 = (s_1^2 + s_2^2)\Gamma_1 \tag{117}$$

and

$$\Gamma_1 = 0 \qquad \text{or} \qquad \Gamma_2 = 0. \tag{118}$$

As was true for second order GIBCs, the appropriate constraints depend on the physical problem simulated, and this is discussed by Senior [1993]. In particular, if $U = H_z$ and the conditions specify the induced electric current, then

$$\Gamma_0 = 0, \qquad \Gamma_2 = 0 \tag{119}$$

whereas for the induced magnetic current

$$\Gamma_3 = (s_1^2 + s_2^2)\Gamma_1, \qquad \Gamma_2 = 0. \tag{120}$$

The corresponding values of the constants c_3 and c_4 are derived in Appendix C, and the resulting expressions for $P(\xi)$ are in accordance with reciprocity.

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Appendix A: Initial Value Relations

Let

$$f(x) = f_1(x) + f_2(x)$$

where

$$f_1(x) = f(x)\,u(x), \qquad f_2(x) = f(x)\,\{1 - u(x)\}$$

and $u(x)$ is the unit step function. Then $f_1(x) = 0$ for $x < 0$ and $f_2(x) = 0$ for $x > 0$. Consistent with the representation (9) we define the Fourier transform pair as

$$\begin{aligned} F(\xi) = \mathcal{F}\{f(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\xi x}\,dx \\ f(x) = \mathcal{F}^{-1}\{F(\xi)\} &= \int_{-\infty}^{\infty} F(\xi)\,e^{i\xi x}\,d\xi, \end{aligned}$$

so that

$$F_1(\xi) = \frac{1}{2\pi}\int_0^{\infty} f_1(x)\,e^{-i\xi x}\,dx, \qquad F_2(\xi) = \frac{1}{2\pi}\int_0^{\infty} f_2(x)\,e^{-i\xi x}\,dx.$$

We seek to connect the behavior of $F_i(\xi)$ as $\xi \to \infty$ with the behavior of $f_i(x)$ as $|x| \to 0$, $i = 1, 2$, and vice versa.

If $f_1(x) \sim x^\alpha$ as $x \to 0+$, integration by parts shows that

$$F_1(\xi) \sim \frac{1}{2\pi}\frac{\alpha!}{(i\xi)^{\alpha+1}} \quad \text{as } \xi \to \infty.$$

We now differentiate with respect to α. Since

$$\frac{d}{d\alpha}\alpha! = \alpha!\,\psi(\alpha+1)$$

where $\psi(\alpha)$ is the digamma function [Abramowitz and Stegun, 1964], we have

$$\frac{\partial}{\partial\alpha}x^\alpha = \frac{\partial}{\partial\alpha}\left(e^{\alpha\ln x}\right) = x^\alpha \ln x$$

and

$$\frac{\partial}{\partial\alpha}\frac{\alpha!}{(i\xi)^{\alpha+1}} = -\frac{\alpha!}{(i\xi)^{\alpha+1}}\{\ln i\xi - \psi(\alpha+1)\}.$$

In the particular case when α is a non-negative integer $n-1$, we have the following results:

$F_1(\xi)$ as $\xi \to \infty$	$f_1(x)$ as $x \to 0+$
$\dfrac{1}{\xi^n}$	$2\pi i \dfrac{(ix)^{n-1}}{(n-1)!}$
$\dfrac{1}{\xi^n} \ln i\xi$	$-2\pi i \dfrac{(ix)^{n-1}}{(n-1)!} \{\ln x + O(1)\}$

For the function $f_2(x)$ it is convenient to write $y = -x$ so that

$$F_2(\xi) = \frac{1}{2\pi} \int_0^\infty f_2(-y)\, e^{i\xi y}\, dy.$$

Then if $f_2(-y) \sim y^\alpha$ as $y \to 0+$

$$F_2(\xi) \sim \frac{1}{2\pi} \frac{\alpha!}{(-i\xi)^{\alpha+1}} \quad \text{as } \xi \to \infty$$

and if $f_2(-y) \sim y^\alpha \ln y$,

$$\begin{aligned} F_2(\xi) &\sim -\frac{\alpha!}{2\pi(-i\xi)^{\alpha+1}} \{\ln(-i\xi) - \psi(\alpha+1)\} \\ &= -\frac{\alpha!}{2\pi(-i\xi)^{\alpha+1}} \{\ln i\xi + O(1)\}. \end{aligned}$$

In the particular case when α is a non-negative integer $n-1$, the results are as follows:

$F_2(\xi)$ as $\xi \to \infty$	$f_2(x)$ as $x \to 0-$
$\dfrac{1}{\xi^n}$	$-2\pi i \dfrac{(ix)^{n-1}}{(n-1)!}$
$\dfrac{1}{\xi^n} \ln i\xi$	$2\pi i \dfrac{(ix)^{n-1}}{(n-1)!} \{\ln \lvert x\rvert + O(1)\}$

Appendix B: The Constants β_i

In (96) and (97) the functions $S_+(\xi)$ and $S_-(\xi)$ are defined for the fourth order conditions, and for these to be analytic in the upper and lower half planes respectively, it is necessary to choose the constants β_i, $i = 1, 2, 3, 4$, to eliminate the poles at $\xi = s_1$ and s_2 from $S_+(\xi)$ and the poles at $\xi = -s_1$ and $-s_2$ from $S_-(\xi)$. When this is done, the equations that result are

$$\begin{aligned} p(s_1) + \beta_1 s_1^3 + \beta_2 s_1^2 + \beta_3 s_1 + \beta_4 &= 0 \\ p(s_2) + \beta_1 s_2^3 + \beta_2 s_2^2 + \beta_3 s_2 + \beta_4 &= 0 \\ q(s_1) - \beta_1 s_1^3 + \beta_2 s_1^2 - \beta_3 s_1 + \beta_4 &= 0 \\ q(s_2) - \beta_1 s_2^3 + \beta_2 s_2^2 - \beta_3 s_2 + \beta_4 &= 0 \end{aligned} \tag{B-1}$$

where

$$p(s_i) = b_1 b_0' \frac{K_4(s_i)}{K_4'(s_i)}, \qquad q(s_i) = b_0 b_1' \frac{K_4'(s_i)}{K_4(s_i)} \tag{B-2}$$

implying

$$p(s_i)\, q(s_i) = b_0 b_1 b_0' b_1'$$

for $i = 1, 2$. The solutions of (B-1) are

$$\begin{aligned} \beta_1 &= \frac{1}{2(s_1^2 - s_2^2)} \left\{ \frac{1}{s_2}[p(s_2) - q(s_2)] - \frac{1}{s_1}[p(s_1) - q(s_1)] \right\} \\ \beta_2 &= \frac{1}{2(s_1^2 - s_2^2)} \left\{ [p(s_2) + q(s_2)] - [p(s_1) + q(s_1)] \right\} \\ \beta_3 &= \frac{1}{2(s_1^2 - s_2^2)} \left\{ \frac{s_2^2}{s_1}[p(s_1) - q(s_1)] - \frac{s_1^2}{s_2}[p(s_2) - q(s_2)] \right\} \\ \beta_4 &= \frac{1}{2(s_1^2 - s_2^2)} \left\{ s_2^2[p(s_1) + q(s_1)] - s_1^2[p(s_2) - q(s_2)] \right\} \end{aligned} \tag{B-3}$$

There is a relation connecting the β_i that turns out to be important. From (B-1)

$$\begin{aligned} p(s_i) &= -2(\beta_2 s_i^2 + \beta_4) - q(s_i) \\ p(s_i) &= -2s_i(\beta_1 s_i^2 + \beta_3) + q(s_i) \end{aligned}$$

and hence, on using (B-2),

$$\begin{aligned} \{p(s_i)\}^2 &= -2(\beta_2 s_i^2 + \beta_4)\, p(s_i) - b_0 b_1 b_0' b_1' \\ \{p(s_i)\}^2 &= -2s_i(\beta_1 s_i^2 + \beta_3)\, p(s_i) + b_0 b_1 b_0' b_1'. \end{aligned}$$

When $p(s_i)$ is eliminated from these we obtain

$$(\beta_2 s_i^2 + \beta_4)^2 - s_i^2(\beta_1 s_i^2 + \beta_3)^2 = b_0 b_1 b_0' b_1',$$

and since this is true for $i = 1, 2$,

$$(\beta_2 s_1^2 + \beta_4)^2 - (\beta_2 s_2^2 + \beta_4)^2 = s_1^2(\beta_1 s_1^2 + \beta_3)^2 - s_2^2(\beta_1 s_2^2 + \beta_3)^2$$

which simplifies to give

$$(s_1^4 + s_1^2 s_2^2 + s_2^4)\beta_1^2 = (s_1^2 + s_2^2)(\beta_2^2 - 2\beta_1\beta_3) - (\beta_3^2 - 2\beta_2\beta_4). \tag{B–4}$$

Appendix C: Admissible Expressions for c_3 and c_4

The reciprocity condition concerning the interchange of the transmitter and receiver requires that $P(\xi)$ be symmetrical in ξ and ξ_0, i.e. unchanged under the transformation $\xi \leftrightarrow \xi_0$. This restricts the admissible expressions for the constants c_3 and c_4 in (90).

A simple analysis shows that the most general expressions for c_3 and c_4 consistent with reciprocity are

$$\begin{aligned} c_3 &= a + \{2b + (s_1^2 + s_2^2)\}\,\xi_0 + c\xi_0^2 \\ c_4 &= b + c\xi_0 + \xi_0^2 \end{aligned} \tag{C–1}$$

for any a, b and c independent of ξ_0, and then

$$\begin{aligned} \frac{(s_1^2 - \xi_0^2)(s_2^2 - \xi_0^2)}{\xi + \xi_0} + c_3 + c_4(\xi - \xi_0) &= \frac{1}{\xi + \xi_0}\left\{s_1^2 s_2^2 + (s_1^2 + s_2^2)\xi\xi_0 + \xi^2\xi_0^2\right\} \\ &\quad + a + b(\xi + \xi_0) + c\xi\xi_0. \end{aligned} \tag{C–2}$$

Application of either of the constraints (115) and either of the constraints (116) specifies c_3 and c_4 in accordance with (C–1), and the values of a, b and c obtained are as follows:

$\underline{\Gamma_0 = 0,\ \Gamma_1 = 0}$

$$a = \frac{\beta_3\beta_4 + \beta_1\beta_2 s_1^2 s_2^2 + \beta_1\beta_4(s_1^2 + s_2^2)}{\beta_1\beta_3 - \beta_2^2 + \beta_1^2(s_1^2 + s_2^2)}$$

$$
\begin{aligned}
b &= -\frac{\beta_2\beta_4 + \beta_1^2 s_1^2 s_2^2}{\beta_1\beta_3 - \beta_2^2 + \beta_1^2(s_1^2+s_2^2)} \\
c &= -\frac{\beta_1\beta_4 - \beta_2\beta_3}{\beta_1\beta_3 - \beta_2^2 + \beta_1^2(s_1^2+s_2^2)}
\end{aligned}
$$

$\underline{\Gamma_0 = 0,\ \ \Gamma_2 = 0}$

$$
\begin{aligned}
a &= \frac{\beta_4^2 + \beta_2^2 s_1^2 s_2^2 + \beta_2\beta_4(s_1^2+s_2^2)}{\beta_1\beta_4 - \beta_2\beta_3} \\
b &= -\frac{\beta_3\beta_4 + \beta_1\beta_2 s_1^2 s_2^2 + \beta_1\beta_4(s_1^2+s_2^2)}{\beta_1\beta_4 - \beta_2\beta_3} \\
c &= \frac{\beta_3^2 + \beta_1^2 s_1^2 s_2^2 + \beta_1\beta_3(s_1^2+s_2^2)}{\beta_1\beta_4 - \beta_2\beta_3}
\end{aligned}
$$

$\underline{\Gamma_3 = (s_1^2+s_2^2)\Gamma_1,\ \ \Gamma_1 = 0}$

$$
\begin{aligned}
a &= -\frac{\{\beta_3^2 + \beta_1^2 s_1^2 s_2^2 + \beta_1\beta_3(s_1^2+s_2^2)\}\, s_1^2 s_2^2}{\beta_3\beta_4 + \beta_1\beta_2 s_1^2 s_2^2 + \beta_1\beta_4(s_1^2+s_2^2)} \\
b &= \frac{(\beta_2\beta_3 - \beta_1\beta_4) s_1^2 s_2^2}{\beta_3\beta_4 + \beta_1\beta_2 s_1^2 s_2^2 + \beta_1\beta_4(s_1^2+s_2^2)} \\
c &= -\frac{\beta_4^2 + \beta_2^2 s_1^2 s_2^2 + \beta_2\beta_4(s_1^2+s_2^2)}{\beta_3\beta_4 + \beta_1\beta_2 s_1^2 s_2^2 + \beta_1\beta_4(s_1^2+s_2^2)}
\end{aligned}
$$

$\underline{\Gamma_3 = (s_1^2+s_2^2)\Gamma_1,\ \ \Gamma_2 = 0}$

$$
\begin{aligned}
a &= -\frac{\{\beta_3\beta_4 + \beta_1\beta_2 s_1^2 s_2^2 + \beta_2\beta_3(s_1^2+s_2^2)\}\, s_1^2 s_2^2}{\beta_4^2 + \beta_1\beta_3 s_1^2 s_2^2 + \{\beta_2\beta_4 + \beta_1^2 s_1^2 s_2^2\}\,(s_1^2+s_2^2)} \\
b &= \frac{\{\beta_3^2 - \beta_2\beta_4 + \beta_1\beta_3(s_1^2+s_2^2)\}\, s_1^2 s_2^2}{\beta_4^2 + \beta_1\beta_3 s_1^2 s_2^2 + \{\beta_2\beta_4 + \beta_1^2 s_1^2 s_2^2\}\,(s_1^2+s_2^2)} \\
c &= -\frac{\beta_1\beta_4(s_1^4 + s_1^2 s_2^2 + s_2^4) + \beta_3\beta_2 s_1^2 s_2^2 + (\beta_3\beta_4 + \beta_1\beta_2 s_1^2 s_2^2)(s_1^2+s_2^2)}{\beta_4^2 + \beta_1\beta_3 s_1^2 s_2^2 + \{\beta_2\beta_4 + \beta_1^2 s_1^2 s_2^2\}\,(s_1^2+s_2^2)}
\end{aligned}
$$

In some of these cases, the verification that the expressions for c_3 and c_4 do have the form (C–1) requires the use of (B–4).