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Uniqueness criteria for GIBCs of odd and even orders

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1 Introduction

Generalized impedance boundary conditions (GIBCs) are finding increasing application as a means of simulating the scattering properties of a surface. In a recent report [Senior, 1993a] we examined the conditions under which a restricted class of even order GIBCs applied to an arbitrary cylindrical surface specify a unique solution of the boundary value problem, and derived the additional constraints that must be imposed at any (line) discontinuity in the surface properties. We now extend the treatment to a more general class of GIBC, including odd orders as well as even.

2 The Nature of GIBCs

For a planar surface $y = 0$ where x, y, z are Cartesian coordinates, a generic form of GIBC is

$$\prod_{m=1}^{M} \left(\frac{\partial}{\partial y} + ik\gamma_m \right) U = 0 \tag{1}$$

specifying the field in $y \geq 0$ where U is any Cartesian component of the field and a time factor $e^{-i\omega t}$ has been assumed. We shall concentrate on the two-dimensional problem where there is no z dependence, and in the case of an H-polarized field, U can be identified with the component H_z. The GIBC (1) is of Mth order as determined by the highest derivative of the field, and is a logical extension of the standard impedance boundary condition

$$\left(\frac{\partial}{\partial y} + ik\gamma\right) U = 0$$

for which $M = 1$. For most practical purposes it is sufficient to confine attention to GIBCs of order $M \leq 4$.

As noted by Senior and Volakis [1989], (1) can be written as

$$\sum_{m=0}^{M} \frac{b_m}{(ik)^m} \frac{\partial^m U}{\partial y^m} = 0, \tag{2}$$

and by using the two-dimensional wave equation to eliminate all even derivatives with respect to y, boundary conditions of increasing order are then as follows:

$M = 1$

$$\frac{\partial U}{\partial y} = -ik\frac{b_0}{b_1}U \tag{3}$$

$M = 2$

$$\frac{\partial U}{\partial y} = -ik\left\{\frac{b_0 + b_2}{b_1} + \frac{b_2}{b_1}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\}U \tag{4}$$

$M = 3$

$$\left\{1 + \frac{b_3}{b_1 + b_3}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\}\frac{\partial U}{\partial y} = -ik\left\{\frac{b_0 + b_2}{b_1 + b_3} + \frac{b_2}{b_1 + b_3}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\}U \tag{5}$$

$M = 4$

$$\begin{aligned}\left\{1 + \frac{b_3}{b_1 + b_3}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right\}\frac{\partial U}{\partial y} &= -ik\left\{\frac{b_0 + b_2 + b_4}{b_1 + b_3} + \frac{b_0 + 2b_4}{b_1 + b_3}\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right. \\ &\qquad \left. + \frac{b_4}{b_1 + b_3}\frac{1}{k^4}\frac{\partial^4}{\partial x^4}\right\}U, \end{aligned} \tag{6}$$

and so on. As M increases, higher order even derivatives with respect to x appear on first the right hand side and then the left, and only if $b_m = 0$ for all odd $m > 1$ are there no x derivatives on the left.

The coefficients b_m are typically obtained by expanding the fields inside the material or coating comprising the surface in terms of a small parameter ϵ, and then matching these fields to the exterior ones at the outer boundary. The order M is determined by the highest power of ϵ retained in the expansions, but to any given order in ϵ, it is found that the x derivatives on the left can be eliminated [Senior, 1993a]. The net effect is to convert any odd order GIBC (with $M > 1$) into the next lower (even) order condition having $b_m = 0$ for all odd $m > 1$. In principle it is therefore sufficient to consider only a restricted class of even order condition, but in practice the even order conditions do not always achieve the same accuracy as the odd order ones.

A possible reason for this is the nature of any GIBC having tangential derivatives on the left hand side. The simplest example is the third order condition

$$\left(1 + \frac{1}{p^2}\frac{\partial^2}{\partial x^2}\right)\frac{\partial U}{\partial y} = -\left(\alpha - \beta\frac{\partial^2}{\partial x^2}\right)U \qquad (7)$$

applied over the entire surface $y = 0$ with α, β and p constant. The most general solution of the equation

$$\left(\frac{\partial^2}{\partial x^2} + p^2\right) v(x) = w(x)$$

is

$$v(x) = -\frac{i}{2p}\left\{e^{ipx}\int_{\Delta_1}^{x} e^{-ipx'}\, w(x')\,dx' - e^{-ipx}\int_{\Delta_2}^{x} e^{ipx'}\, w(x')\,dx'\right\}$$

for arbitrary Δ_1 and Δ_2, and (7) is therefore equivalent to

$$\begin{aligned}\frac{\partial U}{\partial y} &= \frac{ip}{2}\left\{e^{ipx}\int_{\Delta_1}^{x} e^{-ipx'}\left(\alpha - \beta\frac{\partial^2}{\partial x'^2}\right)U(x')\,dx'\right.\\ &\qquad \left. - e^{-ipx}\int_{\Delta_2}^{x} e^{ipx'}\left(\alpha - \beta\frac{\partial^2}{\partial x'^2}\right)U(x')\,dx'.\right\} \qquad (8)\end{aligned}$$

If Im. $p > 0$ the particular solution appropriate to the entire range $-\infty < x < \infty$ is obtained by putting $\Delta_1 = -\infty$, $\Delta_2 = \infty$, and when the x' derivatives are eliminated using integration by parts,

$$\begin{aligned}\frac{\partial U}{\partial y} &= p^2\beta U + \frac{ip}{2}\left(\alpha + p^2\beta\right)\\ &\quad \cdot\left\{\int_{-\infty}^{x} e^{ip(x-x')}\,U(x')\,dx' + \int_{x}^{\infty} e^{-ip(x-x')}\,U(x')\,dx'\right\}\end{aligned}$$

i.e.

$$\frac{\partial U}{\partial y} = p^2\beta U + \frac{ip}{2}\left(\alpha + p^2\beta\right)\int_{-\infty}^{\infty} e^{ip|x-x'|}\,U(x')\,dx'. \tag{9}$$

We observe the convolution form and the absence of *any* tangential derivatives—features which could be advantageous for numerical purposes. A similar conversion is possible for any (general) GIBC of order $M > 2$, and the fact that the condition can be expressed as an integral shows its more global character vis-a-vis (1) or (2).

If the boundary condition (7) is applicable only over the semi-infinite range $0 < x < \infty$, the required specification of Δ_1 and Δ_2 is $\Delta_1 = 0$ and $\Delta_2 = \infty$. Integration by parts then gives rise to an endpoint contribution and

$$\frac{\partial U}{\partial y} = p^2\beta U + \frac{ip}{2}\left(\alpha + p^2\beta\right)\int_{0}^{\infty} e^{ip|x-x'|}\,U(x')\,dx' + \frac{ip\beta}{2}\left[\frac{\partial U}{\partial x} + ipU\right]_{x=0+} e^{ipx} \tag{10}$$

for $0 < x < \infty$. It can be verified that application of the differential operator $\left(1 + \frac{1}{p^2}\frac{\partial^2}{\partial x^2}\right)$ to (9) or (10) reproduces the original boundary condition (7).

For a curved cylindrical surface with generators parallel to the z axis, a fourth order GIBC can be written as

$$\frac{\partial U}{\partial n} - \frac{\partial}{\partial s}\left\{a\frac{\partial}{\partial n}\left(a\frac{\partial U}{\partial s}\right)\right\} = -\alpha U + \frac{\partial}{\partial s}\left(\beta\frac{\partial U}{\partial s}\right) - \frac{\partial^2}{\partial s^2}\left(\gamma\frac{\partial^2 U}{\partial s^2}\right) \tag{11}$$

where s and n are orthogonal curvilinear coordinates with n perpendicular to the surface $n =$ constant. The parameters α, β, γ and a are functions of the material properties of the surface, and even for a laterally homogeneous material, may also be functions of s by virtue of the dependence of the metric coefficients on s. Particular examples of (11) have been developed by Senior [1990] and Ljalinov [1992], and for a planar surface by Senior and Volakis [1989]. For maximum generality we shall henceforth consider (11), and by successively putting γ, a and β equal to zero, we can recover third, second and first order GIBCs respectively.

3 Conditions for Uniqueness

We now seek the restrictions that must be imposed to ensure a unique solution of the boundary value problem when the GIBC (11) is employed. The procedure is an extension of that used in Section 4 of Senior [1993a].

Consider the region Σ bounded internally by a cylindrical surface with boundary curve C at which (11) is applied, and externally by a cylinder of infinitely large radius whose boundary curve is C_∞. A special case is that in which C is of infinite extent, and C_∞ is then the portion of the infinite cylinder providing closure in the upper half space. From Green's theorem

$$k^2 \iint_\Sigma |U|^2 \, d\Sigma - \iint_\Sigma |\nabla U|^2 \, d\Sigma = \left(\int_C + \int_{C_\infty} \right) U^* \frac{\partial U}{\partial n} \, ds$$

where the asterisk denotes the complex conjugate, and if k has a small positive imaginary part corresponding to a slight loss in the medium,

$$\text{Im. } k^2 \iint_\Sigma |U|^2 \, d\Sigma = \text{Im. } \int_C U^* \frac{\partial U}{\partial n} \, ds. \qquad (12)$$

Similarly

$$\text{Im. } k^2 \iint_\Sigma \left| a \frac{\partial U}{\partial s} \right|^2 \, d\Sigma = \text{Im. } \int_C a^* \frac{\partial U^*}{\partial s} \frac{\partial}{\partial n} \left(a \frac{\partial U}{\partial s} \right) \, ds \qquad (13)$$

provided $a\frac{\partial U}{\partial s}$ also satisfies the scalar wave equation, i.e.

$$(\nabla^2 + k^2) a \frac{\partial U}{\partial s} = 0.$$

In the particular case of a circular cylinder with $s = \rho\phi$, it is necessary for a/ρ to be independent of ρ.

If U_1 and U_2 are two solutions satisfying (11), then $W = U_1 - U_2$ must also satisfy the same condition on C, and from (12)

$$\text{Im. } k^2 \iint_\Sigma |W|^2 \, d\Sigma = \text{Im. } \int_C W^* \frac{\partial W}{\partial n} \, ds.$$

Also, from (13),

$$\text{Im. } k^2 \iint_\Sigma \left| a \frac{\partial W}{\partial s} \right|^2 \, d\Sigma = \text{Im. } \int_C a^* \frac{\partial W^*}{\partial s} \frac{\partial}{\partial n} \left(a \frac{\partial W}{\partial s} \right) \, ds$$

and therefore

$$\text{Im. } k^2 \iint_\Sigma \left\{ |W|^2 + \left| a\frac{\partial W}{\partial s} \right|^2 \right\} d\Sigma$$
$$= \text{ Im. } \int_C \left\{ W^* \frac{\partial W}{\partial n} + a^* \frac{\partial W^*}{\partial s}\frac{\partial}{\partial n}\left(a \frac{\partial W}{\partial s} \right) \right\} ds. \qquad (14)$$

We first determine the restrictions on α, β, γ and a on the assumption that the four quantities have all necessary continuity on C. From (11)

$$\int_C W^* \frac{\partial W}{\partial n} ds = \int_C W^* \frac{\partial}{\partial s}\left\{ a \frac{\partial}{\partial n}\left(a\frac{\partial W}{\partial s}\right)\right\} ds \qquad (15)$$
$$- \int_C W^* \left\{ \alpha W - \frac{\partial}{\partial s}\left(\beta \frac{\partial W}{\partial s}\right) + \frac{\partial^2}{\partial s^2}\left(\gamma \frac{\partial^2 W}{\partial s^2}\right)\right\} ds,$$

and using integration by parts we obtain

$$\int_C W^* \frac{\partial W}{\partial n} ds = -\int_C a \frac{\partial W^*}{\partial s}\frac{\partial}{\partial n}\left(a\frac{\partial W}{\partial s}\right) ds$$
$$- \int_C \left\{ \alpha |W|^2 + \beta \left| \frac{\partial W}{\partial s}\right|^2 + \gamma \left| \frac{\partial^2 W}{\partial s^2}\right|^2 \right\} ds,$$

that is,

$$\int_C \left\{ W^* \frac{\partial W}{\partial n} + a \frac{\partial W^*}{\partial s}\frac{\partial}{\partial n}\left(a\frac{\partial W}{\partial s}\right)\right\} ds$$
$$= -\int_C \left\{ \alpha |W|^2 + \beta \left| \frac{\partial W}{\partial s}\right|^2 + \gamma \left| \frac{\partial^2 W}{\partial s^2}\right|^2 \right\} ds.$$

Hence, if a is real (so that $a^* = a$)

$$\text{Im. } k^2 \iint_\Sigma \left\{ |W|^2 + \left| a\frac{\partial W}{\partial s} \right|^2 \right\} d\Sigma$$
$$= -\text{Im. } \int_C \left\{ \alpha |W|^2 + \beta \left| \frac{\partial W}{\partial s}\right|^2 + \gamma \left| \frac{\partial^2 W}{\partial s^2}\right|^2 \right\} ds, \qquad (16)$$

and the solution is unique if

$$\text{Im. } \alpha, \beta, \gamma \geq 0 \tag{17}$$

with a real.

When there is a discontinuity in the boundary condition on C, a knowledge of the condition alone is not sufficient to ensure a unique solution, and additional information is required concerning the behavior of the field at the discontinuity. The standard edge conditions are not enough, and further constraints are necessary. To simplify the discussion, consider a surface of infinite extent with a single discontinuity in properties at $s = 0$. From (15) on integrating by parts,

$$\begin{aligned}\int W^* \frac{\partial W}{\partial n}\, ds &= \left[W^* \left\{ \beta \frac{\partial W}{\partial s} + a \frac{\partial}{\partial n} \left(a \frac{\partial W}{\partial s} \right) - \frac{\partial}{\partial s} \left(\gamma \frac{\partial^2 W}{\partial s^2} \right) \right\} \right]_-^+ \\ &\quad + \left[\frac{\partial W^*}{\partial s} \gamma \frac{\partial^2 W}{\partial s^2} \right]_-^+ - \int_C a \frac{\partial W^*}{\partial s} \frac{\partial}{\partial n} \left(a \frac{\partial W}{\partial s} \right) ds \\ &\quad - \int_C \left\{ \alpha |W|^2 + \beta \left| \frac{\partial W}{\partial s} \right|^2 + \gamma \left| \frac{\partial^2 W}{\partial s^2} \right|^2 \right\} ds\end{aligned}$$

where $|_-^+$ denotes the discontinuity at $s = 0$, and when this is substituted into (14) we obtain

$$\begin{aligned}&\text{Im. } k^2 \iint_\Sigma \left\{ |W|^2 + \left| a \frac{\partial W}{\partial s} \right|^2 \right\} d\Sigma \\ &= \text{Im. } \left[W^* \left\{ \beta \frac{\partial W}{\partial s} + a \frac{\partial}{\partial n} \left(a \frac{\partial W}{\partial s} \right) - \frac{\partial}{\partial s} \left(\gamma \frac{\partial^2 W}{\partial s^2} \right) \right\} \right]_-^+ \\ &\quad + \left[\frac{\partial W^*}{\partial s} \gamma \frac{\partial^2 W}{\partial s^2} \right]_-^+ - \int_C \left\{ \alpha |W|^2 + \beta \left| \frac{\partial W}{\partial s} \right|^2 + \gamma \left| \frac{\partial^2 W}{\partial s^2} \right|^2 \right\} ds. \quad (18)\end{aligned}$$

The additional constraints are then

$$\left[W^* \left\{ \beta \frac{\partial W}{\partial s} + a \frac{\partial}{\partial n} \left(a \frac{\partial W}{\partial s} \right) - \frac{\partial}{\partial s} \left(\gamma \frac{\partial^2 W}{\partial s^2} \right) \right\} \right]_-^+ = 0 \tag{19}$$

i.e.

$$U^*(0) = 0 \qquad \text{implying} \qquad U(0) = 0 \tag{20}$$

or

$$\left[\beta\frac{\partial U}{\partial s}+a\frac{\partial}{\partial n}\left(a\frac{\partial U}{\partial s}\right)-\frac{\partial}{\partial s}\left(\gamma\frac{\partial^2 U}{\partial s^2}\right)\right]_-^+=0, \tag{21}$$

and

$$\left[\frac{\partial W^*}{\partial s}\gamma\frac{\partial^2 W}{\partial s^2}\right]_-^+=0 \tag{22}$$

i.e.

$$\left.\frac{\partial U^*}{\partial s}\right|_{s=0}=0 \quad \text{implying} \quad \left.\frac{\partial U}{\partial s}\right|_{s=0}=0 \tag{23}$$

or

$$\left[\gamma\frac{\partial^2 U}{\partial s^2}\right]_-^+=0. \tag{24}$$

These are sufficient conditions. In the particular case of a third order GIBC applied to a sectionally uniform planar surface $y=$ constant, we have $\gamma=0$, $s=x$ and $n=y$ with β and a independent of x. Only the constraint (19) remains, and (21) reduces to

$$\left[\frac{\partial}{\partial x}\left(\beta U+a^2\frac{\partial U}{\partial y}\right)\right]_-^+=0. \tag{25}$$

Comparison of (5) and (11) shows that this can be written as

$$\left[\frac{1}{b_1+b_3}\frac{\partial}{\partial x}\left(b_2U+\frac{b_3}{ik}\frac{\partial U}{\partial y}\right)\right]_-^+=0, \tag{26}$$

and this is identical to the constraint employed by Senior [1993b].

4 Conclusions

Using an extension of the method described by Senior [1993a] we have determined the conditions under which a general fourth order GIBC specifies a unique solution of the boundary value problem when applied to an arbitrary cylindrical surface. In particular, the added constraints that must be satisfied at any line discontinuity in the surface properties have been obtained, and these are necessary for a well-posed problem. The extension to a GIBC of any order, odd as well as even, is obvious.

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