

**Vector Green Functions**  
**versus**  
**Dyadic Green Functions**

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Dyadic Green functions, denoted by  $\overline{\overline{G}}$ , are introduced to integrate the vector wave equations for  $\overline{E}$  and  $\overline{H}$  to provide an integration solution of these differential equations [1]. A dyadic Green function is made of three vector Green functions, denoted by  $\overline{G}^{(i)}$  with  $i = 1, 2, 3$  such that

$$\overline{\overline{G}} = \sum_i \overline{G}^{(i)} \hat{x}_i \quad (1)$$

where  $\hat{x}_i$  with  $i = 1, 2, 3$  correspond to  $\hat{x}, \hat{y}, \hat{z}$ , the unit vectors in cartesian coordinates. In [1], the eigenfunction expansion of various  $\overline{\overline{G}}$ 's are found by dealing directly with the differential equation of  $\overline{\overline{G}}$ . In this note we are going to find the expansions for the vector Green function by means of (1). We consider the eigenfunction expansion for the vector Green function in a rectangular waveguide as an example.

Maxwell equations for the field quantities  $\overline{E}$  and  $\overline{H}$  in a region assumed to be occupied by air (vacuum) are:

$$\nabla \overline{E} = i\omega\mu_0 \overline{H} \quad (2)$$

$$\nabla \overline{H} = \overline{J} - i\omega\epsilon_0 \overline{E} \quad (3)$$

In this note we are using the new notations [1] for the divergence and the curl in writing the equations. The dot and the cross signs are used exclusively for the scalar and the vector products. For an infinitesimal current source with current moment  $c_i \hat{x}_i$  located at  $\overline{R}'$  we write

$$\overline{J} = c_i \hat{x}_i \delta(\overline{R} - \overline{R}') \quad (4)$$

We introduce the vector Green functions  $\overline{G}_e^{(i)}$  and  $\overline{G}_m^{(i)}$  such that

$$\overline{E} = \overline{G}_e^{(i)} \quad (5)$$

$$i\omega\mu_0 \overline{H} = \overline{G}_m^{(i)} \quad (6)$$

$$i\omega\mu_0 \overline{J} = i\omega\mu_0 c_i \hat{x}_i \delta(\overline{R} - \overline{R}') \quad (7)$$

and let the current moment be so normalized that  $i\omega\mu_0 c_i = 1$ , then the equations for  $\bar{G}_e^{(i)}$  and  $\bar{G}_m^{(i)}$  are:

$$\nabla \bar{G}_e^{(i)} = \bar{G}_m^{(i)} \quad (8)$$

$$\nabla \bar{G}_m^{(i)} = \hat{x}_i \delta(\bar{R} - \bar{R}') + k^2 \bar{G}_e^{(i)} \quad (9)$$

where  $k^2 = \omega^2 \mu_0 \epsilon_0$ . The functions  $\bar{G}_e^{(i)}$  and  $\bar{G}_m^{(i)}$  with  $i = 1, 2, 3$  are called, respectively, the electric and magnetic vector Green functions.

There are three functions for each set corresponding to three different orientations of the infinitesimal current source. By eliminating  $\bar{G}_e^{(i)}$  or  $\bar{G}_m^{(i)}$  between (8) and (9) we obtain the equations for  $\bar{G}_e^{(i)}$  and  $\bar{G}_m^{(i)}$ . They are

$$\nabla \nabla \bar{G}_e^{(i)} - k^2 \bar{G}_e^{(i)} = \hat{x}_i \delta(\bar{R} - \bar{R}') \quad (10)$$

$$\nabla \nabla \bar{G}_m^{(i)} - k^2 \bar{G}_m^{(i)} = \nabla [\hat{x}_i \delta(\bar{R} - \bar{R}')] \quad (11)$$

To find  $\bar{G}_e^{(i)}$  it is more convenient to find  $\bar{G}_m^{(i)}$  first because  $\bar{G}_m^{(i)}$ 's are solenoidal while  $\bar{G}_e^{(i)}$ 's are not. The eigenfunction expansion of  $\bar{G}_m^{(i)}$  requires only the solenoidal vector wave functions. Once  $\bar{G}_m^{(i)}$ 's have been found we can use (9) to find  $\bar{G}_e^{(i)}$ 's. The vector wave functions appropriate for the rectangular waveguide are

$$\bar{M}_{e_{\sigma mn}}(h) = \nabla \times \begin{bmatrix} C_x & C_y \\ S_x & S_y \\ e^{ihz} \hat{z} \end{bmatrix} \quad (12)$$

and

$$\bar{N}_{e_{\sigma mn}}(h) = \frac{1}{\kappa} \nabla \times \bar{M}_{e_{\sigma mn}}(h) \quad (13)$$

where

$$\begin{aligned}
C_x &= \cos \frac{m\pi}{a} x & S_x &= \sin \frac{m\pi}{a} x \\
C_y &= \cos \frac{n\pi}{b} y & S_y &= \sin \frac{n\pi}{b} y \\
\kappa &= (h^2 + k_c^2)^{1/2} \\
k_c^2 &= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \\
a, b &= \text{width and height of guide}
\end{aligned}$$

To solve (11) by the Ohm-Rayleigh method we let

$$\nabla [\hat{x}_i \delta (\bar{R} - \bar{R}')] = \int_{-\infty}^{\infty} dh \sum_{m,n} [\bar{N}_e(h) A^{(i)}(h) + \bar{M}_o(h) B^{(i)}(h)] \quad (14)$$

where  $\bar{N}_e(h)$  and  $\bar{M}_o(h)$  represent, respectively,  $\bar{N}_{emn}(h)$  and  $\bar{M}_{omn}(h)$ , and  $A^{(i)}$  and  $B^{(i)}$  are two scalar coefficients to be determined. The reason that we place these coefficients after the two vector functions will become clear later. The orthogonal properties of the vector wave functions are discussed in detail in [1, pp. 102-103]. By taking the scalar products between (14) and  $\bar{N}_{em'n'}(-h')$  and  $\bar{M}_{em'n'}(-h')$ , respectively, and integrating throughout the entire volume of the waveguide we can determine the coefficients  $A^{(i)}$  and  $B^{(i)}$  they are

$$A^{(i)}(h) = \frac{(2 - \delta_0) \kappa}{\pi a b k_c^2} \bar{M}'_e(-h) \cdot \hat{x}^{(i)} \quad (15)$$

$$B^{(i)}(h) = \frac{(2 - \delta_0) \kappa}{\pi a b k_c^2} \bar{N}'_o(-h) \cdot \hat{x}^{(i)} \quad (16)$$

where  $\delta_0 = 1$  when  $m$  or  $n$  equal to one and zero for other integers.

The primed functions in (15) and (16) are defined with respect to primed variables  $\hat{x}'_i$  associated with  $\bar{R}'$ , the position vector of the infinitesimal source. The eigenfunction expansion of  $\nabla [\hat{x}_i \delta (\bar{R} - \bar{R}')]$  is therefore given by

$$\begin{aligned} \nabla[\hat{x}_i; \delta(\bar{R} - \bar{R}')] &= \int_{-\infty}^{\infty} dh \sum_{m,n} \frac{(2 - \delta_0) \kappa}{\pi abk_c^2} [\bar{N}_e(h) \bar{M}'_e(-h) \cdot \hat{x}^{(i)} \\ &\quad + \bar{M}_o(h) \bar{N}'_o(-h) \cdot \hat{x}^{(i)}] \end{aligned} \quad (17)$$

We let  $\bar{G}_m^{(i)}$  have a similar expansion with unknown coefficients  $a_e$  and  $b_e$  attached to the eigenfunctions. By means of (11) we find

$$a_e = b_e = \frac{1}{\kappa^2 - k^2}$$

thus

$$\begin{aligned} \bar{G}_m^{(i)} &= \int_{-\infty}^{\infty} dh \sum_{m,n} \frac{(2 - \delta_0) \kappa}{\pi abk_c^2 (\kappa^2 - k^2)} \cdot \\ &\quad [\bar{N}_e(h) \bar{M}'_e(-h) \cdot \hat{x}^{(i)} + \bar{M}_o(h) \bar{N}'_o(-h) \cdot \hat{x}^{(i)}] \end{aligned} \quad (18)$$

The integration with respect to  $h$  can be evaluated by means of a contour integration that yields

$$\bar{G}_m^{(i)} = \bar{G}_m^{(i)\pm} = \sum_{m,n} C_{mn} k [\bar{N}_e(\pm k_g) \bar{M}'_e(\mp k_g) \cdot \hat{x}^{(i)} + \bar{M}_o(\pm k_g) \bar{N}'_o(\mp k_g) \cdot \hat{x}^{(i)}] \quad z \gtrless z' \quad (19)$$

The top line applies to  $\bar{G}_m^{(i)+}$  for  $z > z'$  and the bottom line to  $\bar{G}_m^{(i)-}$  for  $z < z'$ , and  $k_g = (k^2 - k_c^2)^{1/2}$ .

To find  $\bar{G}_e^{(i)}$ , we make use of (9). The function  $\bar{G}_m^{(i)}$ , however, is discontinuous at  $z = z'$ . If we write

$$\bar{G}_m^{(i)} = \bar{G}_m^{(i)+} U(z - z') + \bar{G}_m^{(i)-} U(z' - z) \quad (20)$$

where  $U(z - z')$  and  $U(z' - z)$  are two step functions defined at  $z = z'$ , then

$$\begin{aligned}
\nabla \bar{G}_m^{(i)} &= [\nabla \bar{G}_m^{(i)+}] U(z-z') + \hat{z} \delta(z-z') \times \bar{G}_m^{(i)+} \\
&+ [\nabla \bar{G}_m^{(i)-}] U(z'-z) - \hat{z} \delta(z-z') \times \bar{G}_m^{(i)-} \\
&= [\nabla \bar{G}_m^{(i)+}] U(z-z') + [\nabla \bar{G}_m^{(i)-}] U(z'-z) \\
&\quad + \hat{z} \times [\bar{G}_m^{(i)+} - \bar{G}_m^{(i)-}] \delta(z-z')
\end{aligned} \tag{21}$$

Equation (21) can be simplified by applying the boundary condition for the  $\bar{G}_m^{(i)}$ 's. We start with

$$\hat{z} \times [\bar{H}^+ - \bar{H}^-] = \bar{J}_s^{(i)} \tag{22}$$

where  $\bar{J}_s^{(i)}$  denotes the surface current density at  $z = z'$ . For an infinitesimal source (22) can be converted to the form

$$\hat{z} \times [\bar{G}_m^{(i)+} - \bar{G}_m^{(i)-}] = \hat{x}^{(i)} \delta(\bar{R} - \bar{R}') \tag{23}$$

for  $i = 1, 2$  and

$$\hat{z} \times [\bar{G}_m^{(3)+} - \bar{G}_m^{(3)-}] = 0 \tag{24}$$

for  $i = 3$ . With the aid of (9),(21),(23), and (24) we find

$$\bar{G}_e^{(i)} = \bar{S}_e^{(i)} = \sum_{m,n} C_{mn} [\bar{M}_e(\pm k_g) \bar{M}_e(\mp k_g) + \bar{N}_o(\pm k_g) \bar{N}_o(\mp k_g)] \cdot \hat{x}_i, \quad z \geq z' \tag{25}$$

for  $i = 1, 2$  and

$$\bar{G}_e^{(3)} = -\frac{1}{k^2} \hat{z} \delta(\bar{R} - \bar{R}') + \bar{S}_e^{(3)} \tag{26}$$

for  $i = 3$  where

$$C_{mn} = \frac{i(2 - \delta_0)}{abk_c^2 k_g}$$

To find the dyadic Green function  $\bar{\bar{G}}_e$  we use (1) that yields

$$\bar{\bar{G}}_e = -\frac{1}{k^2} \hat{z} \hat{z} \delta(\bar{R} - \bar{R}') + \bar{\bar{S}}_e \quad (27)$$

where

$$\bar{\bar{S}}_e = \sum_{m,n} C_{mn} [\bar{M}_e(\pm k_g) \bar{M}'_e(\mp k_g) + \bar{N}_o(\pm k_g) \bar{N}'_o(\mp k_g)], \quad z \geq z'$$

By comparing the formulation used in [1] and the present formulation it is seen that the dyadic Green function approach is more direct and perhaps simpler than the vector Green function method even though dyadic analysis is being used therein.

## References

- [1] C.T. Tai, *Dyadic Green Functions in Electromagnetic Theory*, Second Edition, IEEE Press, Piscataway, NY. 1994.
- [2] C.T. Tai, *Generalized Vector and Dyadic analysis*, IEEE Press, Piscataway, NJ. 1992.