SPECTRAL DOMAIN ANALYSIS OF MICROSTRIP PATCH ANTENNA CURRENTS
AND RADIATION

by
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PREFA CE

This thesis work was performed at the University of Michigan Radiation Laboratory during my graduate studies as an AT&T Bell Labs Ph.D. Scholar. The computing facilities available to me at the University of Michigan during this time consisted of a distributed system of Apollo computers administered by the engineering college and a large main frame system, MTS, run by the University of Michigan Computer Center. I used the Apollo system almost exclusively for the numerical work of this thesis. Although this system was much slower in computational speed than MTS, I was allowed unlimited use of these machines while cpu time on MTS had to be bought. The numerical results in this thesis were computed with 128 unknowns for the total current, taking about three to four minutes to run on an Apollo DN-3000. To apply this numerical technique to the general three dimensional problem would require roughly 128 squared unknowns and several months to run on the same machine. The extreme cost to run such a problem on MTS also ruled out the use of this machine. Therefore only the two dimensional problem was programmed and examined numerically. I should also mention that I typeset this thesis on the Apollo network using TeX and the Rackham thesis macro package written by the University of Michigan Computing Center.

I use an \( e^{-i\omega t} \) time convention throughout this thesis. I chose this time convention since it is consistent with most of the work written on the spectral Wiener-Hopf technique and dyadic Green's function theory with which I am familiar. Unfortunately this sometimes leads to confusion, since the other time convention is used in circuit theory. For example, in this thesis, an inductor would have a reactance of \(-i\omega L\) and a capacitor a susceptance of \(-i\omega C\), instead of the more traditional positive values. I apologize to the reader for any trouble this may cause; however, the results presented here can always be converted
to the other time convention by simply conjugating every complex quantity. The reader should also be aware that when discussing voltages and currents I use root mean square phasor quantities, thus eliminating many factors of one half in descriptions of power or energy. Finally, in this thesis, I have chosen to define the square root to have its branch cut along the negative real axis of its argument and the sheet is chosen so that the real part of the square root is always positive or zero. Keeping this in mind and that, physically, propagation constants must have positive imaginary parts, should resolve which root to take in any of the expressions found in this thesis.

— Tom Willis
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CHAPTER I

INTRODUCTION TO PATCH ANTENNAS

§1.1 Structure of a Patch Antenna

A microstrip patch antenna is a radiating structure which consists of a dielectric substrate-covered ground plane with a thin conducting plate or 'patch' of metal layered on top of the substrate. The patch can be any shape; however, rectangular and circular shaped ones are very common. This structure is generally excited by one of two different methods. A microstrip line, consisting of the dielectric substrate, ground plane, and a thin conducting strip above the substrate, can be run into the patch at some point along the patch perimeter. Then a quasi TEM wave can be sent down the microstrip feed to excite the antenna. An alternative method consists of running a coaxial line up from underneath the ground plane. The outer shield of the line connects to the ground plane while the inner conductor passes through a hole in the ground plane and substrate and electrically connects to the patch from below it. Depending on the dielectric constant and thickness of the substrate, the shape of the patch, and how and where the patch is fed, various radiation characteristics and input impedances for the antenna can be achieved, including circularly polarized radiation.

Patch antennas have become increasingly popular in recent years due to their ease of construction and low cost to produce. In addition arrays of patch antennas can be simply etched out of a large sheet of double sided copper clad dielectric board as well as single elements. Their compatibility with microstrip circuits and advantageous aerodynamic properties have also increased the use of patch antennas. However, patch antennas are not perfect: they are rather inefficient radiators and only work over a narrow bandwidth. Also
rigorous procedures to predict the radiation properties of these antennas have yet to be
developed. Some of the approximate analysis techniques which can be applied to the patch
antenna problem will be discussed next. These techniques are also discussed in Bahl and
Bhartia [1] and Carver and Mink [2].

§1.2 Transmission Line Model for Analysis

![Diagram of a transmission line model with symbols and equations]

**Figure 1.1 — Transmission Line Model of Rectangular Patch Antenna**

Probably the simplest model of a patch antenna is the transmission line model. First
proposed by Munson [3], this model works well for rectangular patch antennas excited in a
quasi TEM mode. Work on this technique has also been published by Derneryd [4], [5] and
improved by Sengupta [6], [7]. The antenna is modeled as a microstrip transmission line
which is fed someplace in the middle of the line, as shown in figure 1.1. The transmission line
is terminated at both ends by its "radiating admittance". The exact value of this radiating
admittance is not known; however, it is small compared to the characteristic admittance of
the line since the microstrip line essentially ends in an open circuit.

The antenna can now be analyzed as a transmission line problem. If the line extends
from \(-L/2\) to \(+L/2\) in \(z\) and is fed by a current source of magnitude \(I_o\) at the point \(z_o\),
then the current on the line can be expressed as

\[
I(z) = \begin{cases} 
I_o e^{ik_1(z-z_o)} + I_R^+ e^{-i k_1(z-L/2)} & ; z > z_o \\
I_o e^{-i k_1(z-z_o)} + I_R^- e^{i k_1(z+L/2)} & ; z < z_o
\end{cases}
\]  

(1.1)
where $I^+$ and $I^-$ are the coefficients of forward and backward traveling current waves, and $k_1$ is the propagation constant of the line. The reflected wave coefficients $I^+_R$ and $I^-_R$ are related to the incident waves through the current reflection coefficient.

$$I^+_R = I^+ e^{ik_1(L/2-z_0)}$$
$$I^-_R = I^- e^{ik_1(L/2+z_0)}$$

(1.2)

The reflection coefficient is

$$\Gamma = \frac{Y_r - Y_c}{Y_r + Y_c}$$

(1.3)

where $Y_r$ is the radiating admittance and $Y_c$ is the characteristic admittance of the microstrip line. Since the current is discontinuous by amount $I_o$ at $z_o$,

$$I_o = I(x_o+) - I(x_o-)$$

$$= I^+ + I^+_R e^{ik_1(L/2-z_0)} - I^- - I^-_R e^{ik_1(L/2+z_0)}$$

(1.4)

Also since the voltage on the line must be continuous at $z = z_o$ and

$$\frac{\partial I}{\partial z} = i k_1 Y_c V(z)$$

(1.5)

this implies

$$Y_c V(z_o) = I^+ - I^+_R e^{-ik_1(z_o-L/2)}$$

$$= -I^- + I^-_R e^{ik_1(z_o+L/2)}$$

(1.6)

Hence

$$0 = -I^+ + I^+_R e^{-ik_1(z_o-L/2)} - I^- + I^-_R e^{ik_1(z_o+L/2)}$$

(1.7)

Now using the relations eq. (1.2) solve eqs. (1.4) and (1.7) simultaneously for the current wave coefficients

$$I^+ = \frac{1}{2} I_o \frac{\Gamma e^{ik_1(L/2+z_0)} - 1}{\Gamma e^{ik_1 L} + 1 (\Gamma e^{ik_1 L} - 1)}$$

$$I^- = -\frac{1}{2} I_o \frac{\Gamma e^{ik_1(L/2-z_0)} - 1}{\Gamma e^{ik_1 L} + 1 (\Gamma e^{ik_1 L} - 1)}$$

(1.8)

and so

$$I(x) = \begin{cases} 
I_o K_+ [e^{ik_1z} + \Gamma e^{ik_1(L-z)}] & ; z > z_o \\
I_o K_- [\Gamma e^{ik_1(L+z)} + e^{-ik_1z}] & ; z < z_o 
\end{cases}$$

(1.9)
where

\[
K_+ \equiv + \frac{1}{2} \frac{\Gamma e^{i k_1 (L+z_0)} - e^{-i k_1 z_0}}{(\Gamma e^{i k_1 L} + 1)(\Gamma e^{i k_1 L} - 1)}
\]

\[
K_- \equiv - \frac{1}{2} \frac{\Gamma e^{i k_1 (L-z_0)} - e^{i k_1 z_0}}{(\Gamma e^{i k_1 L} + 1)(\Gamma e^{i k_1 L} - 1)}
\]  

(1.10)

Using the relation \( \partial I/\partial z = i k_1 Y_c V(x) \) yields for the voltage on the line

\[
Y_c V(x) = \begin{cases} 
I_o K_+ \left[ e^{i k_1 z} - \Gamma e^{i k_1 (L-z)} \right] & ; x > x_o \\
I_o K_- \left[ \Gamma e^{i k_1 (L+z)} - e^{-i k_1 z} \right] & ; x < x_o 
\end{cases}
\]  

(1.11)

The input impedance of the antenna is

\[
Z_n \equiv V(x_o)/I_o
\]  

(1.12)

therefore

\[
Y_c Z_n = K_+ \left[ e^{i k_1 z_0} - \Gamma e^{i k_1 (L-z_0)} \right] = K_- \left[ \Gamma e^{i k_1 (L+z_0)} - e^{-i k_1 z_0} \right]
\]

\[
= - \frac{1}{2} \frac{\Gamma^2 e^{2i k_1 L} - 2 \Gamma e^{i k_1 L} \cos(2 k_1 x_0) + 1}{(\Gamma e^{i k_1 L} + 1)(\Gamma e^{i k_1 L} - 1)}
\]  

(1.13)

Denoting the magnitude and phase of the reflection coefficient as

\[
\eta e^{i \psi} \equiv \Gamma
\]  

(1.14)

where \( 0 \leq \eta \leq 1 \), and \( 0 \leq \psi < 2\pi \), at resonance

\[
k_1 L = 2\pi - \psi
\]  

(1.15)

so that the input impedance of the antenna will be real. Therefore once the current reflection coefficient is known, the resonant length of the antenna can be determined by the simple relation above. Furthermore, at resonance

\[
Z_n = \frac{1}{2} Z_c \frac{\eta^2 - 2 \eta \cos(2 k_1 x_0) + 1}{1 - \eta^2}
\]  

(1.16)

where

\[
Z_c \equiv 1/Y_c
\]  

(1.17)

Notice that the input impedance must lie in the following range

\[
\frac{1}{2} Z_c \frac{1 - \eta}{1 + \eta} \leq Z_n \leq \frac{1}{2} Z_c \frac{1 + \eta}{1 - \eta}
\]  

(1.18)
The input impedance will be a minimum for \( z_o = 0 \) and a maximum for \( z_o = L/2 \) providing \( k_1L < \pi \).

Now that the resonant frequency is known in terms of \( \Gamma \), an expression for the \( Q \) of the antenna can be formulated. The magnetic energy stored on the transmission line is

\[
W_M = \frac{1}{2} \frac{k_1Z_o}{\omega} \int_{-L/2}^{+L/2} |I(z)|^2 \, dz
\]  

(1.19)

and the electric energy stored on the line is

\[
W_E = \frac{1}{2} \frac{k_1Y_o}{\omega} \int_{-L/2}^{+L/2} |V(z)|^2 \, dz
\]  

(1.20)

Therefore at resonance (\( \omega = \omega_o \)) these energies are

\[
W_M = \frac{1}{8} \frac{Z_o}{\omega_o} l_o^2 \eta^2 - 2\eta \cos(2k_1z_o) + \frac{1}{(\eta^2 - 1)^2} [k_1L(\eta^2 + 1) + 2\sin(k_1L)]
\]

\[
W_E = \frac{1}{8} \frac{Z_o}{\omega_o} l_o^2 \eta^2 - 2\eta \cos(2k_1z_o) + \frac{1}{(\eta^2 - 1)^2} [k_1L(\eta^2 + 1) - 2\sin(k_1L)]
\]  

(1.21)

Note that the magnetic and electric energies on the line are not equal at resonance. This comes about since in general some energy is stored in the region of space outside the transmission line. It is the total electric and magnetic energies, of the line and region outside, that must be equal at resonance. The difference in electric and magnetic energies stored outside the line can be found with the last equation, however these energies cannot be determined individually from only the reflection coefficient. For the microstrip antenna at resonance the aperture radiating admittance has a capacitive component so \( \psi > \pi \) and \( W_E \) is smaller than \( W_M \). Therefore the electric energy stored outside the line is at least \( W_M - W_E \). Assuming that the total magnetic energy stored outside the line is negligible, the \( Q \) of the antenna can be estimated as

\[
Q \approx \frac{2W_M}{P_R/\omega_o}
\]  

(1.22)

where the real power radiated

\[
P_R \equiv l_o^2 Z_{in}
\]  

(1.23)

is evaluated at resonance. Therefore

\[
Q \approx \frac{k_1L(\eta^2 + 1) + 2\eta \sin(k_1L)}{4(1 - \eta^2)}
\]  

(1.24)

and the half power bandwidth of the antenna is given by

\[
BW = \omega_o/Q
\]  

(1.25)
§1.3 Modal Cavity Model for Analysis

A technique more general than the transmission line model is the cavity model for patch antennas. This technique can be applied to any patch antenna whose patch is conformal to some orthogonal co-ordinate system, such as a rectangular or circular patch. This technique was suggested by Deschamps [8] and first worked on by Y.T. Lo et.al. [9]–[14]. Work in this area has also been published by Van de Capelle [15], and Van Lil et.al. [16]. The patch antenna is treated as a cavity, the patch and ground plane forming perfectly conducting walls and the aperture in the dielectric around the perimeter of the patch assumed to be a magnetic wall. To solve for the fields inside this cavity, as well as the current on the patch, a modal or eigenfunction expansion of the fields is used. This modal solution for a rectangular and circular patch antenna is summarized in the next subsections.

Rectangular Patch Antenna

The rectangular patch, shown in figure 1.2, extends from \(-a/2\) to \(a/2\) in \(z\) and \(-b/2\) to \(b/2\) in \(y\). The thickness of the dielectric slab over the ground plane is \(c\). This thickness is small compared to the wavelength in the dielectric at which the antenna is operated so all field quantities in the cavity region will be assumed independent of \(z\).

![Figure 1.2 — Rectangular Patch Antenna](image)

To solve for the fields in the cavity use the Hansen vector expansion technique, which
is discussed in Stratton [17]. To do this, find the scalar eigenfunctions of the Helmholtz operator which also satisfy the necessary boundary conditions for this cavity. Hence,

$$\nabla^2 \psi + k_1^2 \psi = \lambda \psi$$  \hspace{1cm} (1.26)$$

where $k_1$ is the propagation constant in the dielectric and $\lambda$ is the eigenvalue. The boundary conditions are:

<table>
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<tr>
<th>Transverse Magnetic Wall</th>
<th>Transverse Electric Wall</th>
<th>Longitudinal Magnetic Wall</th>
<th>Longitudinal Electric Wall</th>
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<tr>
<td>TE Mode</td>
<td>$\frac{d}{dn} \psi = 0$</td>
<td>$\psi = 0$</td>
<td>$\psi = 0$</td>
</tr>
<tr>
<td>TM Mode</td>
<td>$\psi = 0$</td>
<td>$\frac{d}{dn} \psi = 0$</td>
<td>$\frac{d}{dn} \psi = 0$</td>
</tr>
</tbody>
</table>

Choosing the piloting vector as $\hat{z}$ the patch antenna cavity has transverse electric walls and longitudinal magnetic walls. Furthermore since $\psi$ is assumed to be independent of $z$, only TM modes can exist in the cavity and so,

$$\psi_{mn}(x, y) = C_{mn} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right)$$  \hspace{1cm} (1.27)$$

where

$$k_m \equiv \frac{m\pi}{a} \quad k_n \equiv \frac{n\pi}{b}$$  \hspace{1cm} (1.28)$$

and the $C_{mn}$ are arbitrary constants. When $m$ is odd use $\sin(k_m x)$, when even use $\cos(k_m x)$. Similarly for $n$, use $\sin(k_n y)$ when odd, $\cos(k_n y)$ when even. The eigenvalue for each mode can be found by

$$\lambda_{mn} = k_1^2 - k_m^2 - k_n^2$$  \hspace{1cm} (1.29)$$

Choose the $C_{mn}$ so that these eigenfunctions form a complete ortho-normal basis set. Therefore

$$\psi_{mn}(x, y) = \frac{\chi_{mn}}{\sqrt{abc}} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right)$$  \hspace{1cm} (1.30)$$

where

$$\chi_{mn} = \begin{cases} 
1 & n = 0 \text{ and } m = 0 \\
\sqrt{2} & n = 0 \text{ or } m = 0 \\
2 & n \neq 0 \text{ and } m \neq 0 
\end{cases}$$  \hspace{1cm} (1.31)$$

And so

$$\iiint_V \psi_{mn} \psi_{m'n'} dV = \begin{cases} 
1 & m = m' \text{ and } n = n' \\
0 & \text{otherwise} 
\end{cases}$$  \hspace{1cm} (1.32)$$
where \( V \) is the region of the cavity.

Now compute the magnetic field modes by taking the curl of the eigenfunctions times the piloting vector.

\[
\vec{H}_{mn} = \nabla \times \psi_{mn}(x, y) \hat{z} \\
= \pm \frac{\chi_{mn}}{\sqrt{abc}} k_m \cos(k_m x) \cos(k_n y) \hat{z} \mp \frac{\chi_{mn}}{\sqrt{abc}} k_m \sin(k_m x) \sin(k_n y) \hat{y}
\]

(1.33)

Some of the magnetic mode fields are shown in figure 1.3 for the rectangular patch antenna.

Since \( \nabla \times \vec{H} = -i \omega \varepsilon \vec{E} \), the electric field modes

\[
i \omega \varepsilon \vec{E}_{mn} = -\nabla \times \vec{H}_{mn} \\
= -\frac{\chi_{mn}}{\sqrt{abc}} (k_m^2 + k_n^2) \cos(k_m x) \sin(k_n y) \hat{z} \\
= -(k_m^2 + k_n^2) \psi_{mn}(x, y) \hat{z}
\]

(1.34)

Therefore represent the electric field in the cavity region by its modal expansion

\[
\vec{E} = \sum_m \sum_n A_{mn} \psi_{mn}(x, y) \hat{z}
\]

(1.35)

where the \( A_{mn} \) are yet to be determined. Now since \( \nabla \cdot \psi_{mn}(x, y) \hat{z} = 0 \),

\[
\nabla \times \nabla \times \psi_{mn} \hat{z} - k_1^2 \psi_{mn} \hat{z} = [-\nabla^2 + \nabla \nabla \cdot] \psi_{mn} \hat{z} - k_1^2 \psi_{mn} \hat{z} \\
= -[\nabla^2 + k_1^2] \psi_{mn} \hat{z} \\
= -\lambda_{mn} \psi_{mn} \hat{z}
\]

(1.36)

and from Maxwell's equations

\[
\nabla \times \nabla \times \vec{E} - k_1^2 \vec{E} = i \omega \mu \vec{J}
\]

(1.37)

where \( \vec{J} \) is the excitation current in the cavity. Therefore

\[
i \omega \mu \vec{J} = \sum_m \sum_n A_{mn} \left[ \nabla \times \nabla \times \psi_{mn} \hat{z} - k_1^2 \psi_{mn} \hat{z} \right] \\
= -\sum_m \sum_n A_{mn} \lambda_{mn} \psi_{mn} \hat{z}
\]

(1.38)

Compute the \( A_{mn} \) values using the orthogonal relationships for \( \psi_{mn} \)

\[
A_{mn} \lambda_{mn} = -i \omega \mu \int \int \int_\mathcal{V} \vec{J} \cdot \hat{z} \psi_{mn} dV
\]

(1.39)

and

\[
A_{mn} = \frac{i \omega \mu}{\lambda_{mn}} \int \int \int_\mathcal{V} \vec{J} \cdot \hat{z} \psi_{mn} dV
\]

(1.40)

Notice that in the resonance condition where \( a \) and \( b \) are of the correct dimensions so that \( k_m^2 + k_n^2 = k_1^2 \) the eigenvalue \( \lambda_{mn} = 0 \) and the coefficient of that mode, \( A_{mn} \), is infinite.
Figure 1.3 — Magnetic Modes in Rectangular Patch Cavity
Circular Patch Antenna

The cavity model analysis for the circular patch antenna consists of the same steps as the rectangular patch, only for a polar co-ordinate geometry. The patch antenna, shown in figure 1.3, is centered at the origin of the $zy$ plane with a radius $a$. Again the thickness of the dielectric slab ($c$) is small compared to the wavelength in the dielectric at which the antenna operates.

Figure 1.4 — Circular Patch Antenna

We use the Hansen vector expansion technique assuming the fields are independent of $z$ in the cavity. Once again only TM modes will exist. These modes are of two types, even and odd.

$$\psi_{nm}^e(\rho, \phi) = C_{nm} J_n(\mu_{nm} \rho) \cos(n\phi)$$
$$\psi_{nm}^o(\rho, \phi) = C_{nm} J_n(\mu_{nm} \rho) \sin(n\phi)$$

where the $\mu_{nm}$ are defined by the boundary condition

$$J'_n(\mu_{nm} a) = 0$$

and can be computed from the zeros of $J'_n(x)$ listed in table 1.1.

The eigenvalue can be computed from

$$\lambda_{nm} = k_1^2 - \mu_{nm}^2$$
Table 1.1 — Zeros of $J'_n(x)$

<table>
<thead>
<tr>
<th>n</th>
<th>Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>13.879 18.745 22.629 26.246 29.729 33.131 36.481 39.792 43.075</td>
</tr>
</tbody>
</table>

To choose the $C_{nm}$ use the identity

$$
\int_0^a J_n^2(\mu_{nm} \rho) \rho d\rho = \frac{\mu_{nm} a^2 - n^2}{2 \mu_{nm}^2} J_n^2(\mu_{nm} a)
$$

(1.44)

(see Abramowitz and Stegun [18] eq. 11.4.5). Therefore

$$
C_{nm} = \chi_n \sqrt{\frac{2}{\pi c (\mu_{nm}^2 a^2 - n^2)}} \mu_{nm}
$$

(1.45)

where

$$
\chi_n = \begin{cases} 
1 & n = 0 \\
\sqrt{2} & n = 1 
\end{cases}
$$

(1.46)

and so

$$
\begin{align*}
\int\int\int_V \psi_n^\rho \psi_m^{\rho'} \psi_n^{\phi'} dV &= \begin{cases} 
1 & m = m' \text{ and } n = n' \\
0 & \text{otherwise}
\end{cases} \\
\int\int\int_V \psi_n^\phi \psi_m^{\phi'} \psi_n^{\rho'} dV &= \begin{cases} 
1 & m = m' \text{ and } n = n' \\
0 & \text{otherwise}
\end{cases} \\
\int\int\int_V \psi_n^\phi \psi_n^{\phi'} \psi_m^{\rho'} dV &= 0
\end{align*}
$$

(1.47)

where $V$ is the volume of the cavity.

Take the curl of the eigenfunctions times the piloting vector to compute the magnetic modes:

$$
\overline{H}_{mn} = \nabla \times \psi_n^\rho (\rho, \phi) \hat{z}
$$

$$
= -C_{mn} \frac{n}{\rho} J_n(\mu_{nm} \rho) \sin(n \phi) \hat{\rho} - C_{nm} \mu_{nm} J'_n(\mu_{nm} \rho) \cos(n \phi) \hat{\phi}
$$

$$
\overline{H}_{mn} = \nabla \times \psi_n^\phi (\rho, \phi) \hat{z}
$$

$$
= +C_{mn} \frac{n}{\rho} J_n(\mu_{nm} \rho) \cos(n \phi) \hat{\rho} - C_{nm} \mu_{nm} J'_n(\mu_{nm} \rho) \sin(n \phi) \hat{\phi}
$$

(1.48)
Figure 1.5 — Magnetic Modes in Circular Patch Cavity
Some of the even magnetic mode fields are shown in figure 1.5. Compute the electric modes from the curl of the magnetic modes.

\[ i \omega E_{nm} = -\nabla \times H_{nm} \]
\[ = \nabla^2 C_{nm} \psi_{nm}(\rho, \phi) \hat{z} \]
\[ = -\mu_{nm}^2 \psi_{nm}(\rho, \phi) \hat{z} \]  (1.49)

As in the rectangular case, represent the electric field in the cavity by its modal expansion.

\[ \overline{E} = \sum_n \sum_m \left[ A_{nm}^e \psi_{nm}(\rho, \phi) \hat{z} + A_{nm}^o \psi_{nm}(\rho, \phi) \hat{z} \right] \]  (1.50)

Once again from Maxwell's equations

\[ i \omega \mu \mathcal{J} = -\sum_n \sum_m \lambda_{nm} \left( A_{nm}^e \psi_{nm} + A_{nm}^o \psi_{nm} \right) \hat{z} \]  (1.51)

and

\[ A_{nm}^e = \frac{i \omega \mu}{\lambda_{nm}} \int \int \int_V \mathcal{J} \cdot \hat{z} \psi_{nm} dV \]
\[ A_{nm}^o = -\frac{i \omega \mu}{\lambda_{nm}} \int \int \int_V \mathcal{J} \cdot \hat{z} \psi_{nm} dV \]  (1.52)

Notice that for the circular patch a resonance condition exists when \( a \) is of the correct dimension so that \( \mu_{nm} = k_1 \) and \( \lambda_{nm} = 0 \). If this happens the even and odd coefficients of that mode become infinite.

Once the electric field distribution is found for the cavity region of the patch antenna, the magnetic field can also be computed. The surface current on the patch can be found using the boundary condition \( \hat{n} \times \mathcal{H} = \mathcal{K} \) for a perfect conductor.

§1.4 Modal Leaky Cavity Model for Analysis

One of the great disadvantages of the cavity model is that it does not predict the true behavior of the antenna near its resonance. Since the patch antenna does not have perfect magnetic walls at its aperture, some energy radiates away from it, and the internal fields never become infinite. Unfortunately since the patch antenna is in general a poor radiator, for all practical applications these antennas are operated at their resonance. Therefore the cavity model does not work well in exactly the region where the antenna is to be operated.
To try and improve the results which can be computed with the cavity model Carver and Coffey [19], Carver [20], and Coffey [21] have proposed using an impedance boundary condition for the aperture instead of magnetic walls. This not only keeps the fields finite at resonance but also slightly shifts where the resonances occur.

**Rectangular Patch**

The analysis for the rectangular leaky cavity model assumes that the fields in the cavity region of the antenna can at least be approximated by the same modes which exist in the perfect cavity, except that the values for \( k_m \) and \( k_n \) are different and in general are complex. To find these \( k_m \) and \( k_n \) values use the impedance boundary conditions

\[
\hat{n} \times \overline{H} = \frac{c}{a} Y_y \overline{E} \quad \text{for } y = \pm b/2 \\
\hat{n} \times \overline{H} = \frac{c}{b} Y_s \overline{E} \quad \text{for } z = \pm a/2
\]

(1.53)

where \( Y_y \) and \( Y_s \) are the radiating aperture admittances. The exact values of these admittances is not known but can be approximately computed from several different theories. Use these boundary conditions to find

\[
\pm k_m \frac{\cos (k_m b)}{\sin(k_m b)} = \frac{k_m^2 + k_n^2}{i \omega \varepsilon} \frac{c}{a} Y_y \sin (k_m b) \\
\pm k_m \frac{\cos (k_m a)}{\sin(k_m a)} = \frac{k_m^2 + k_n^2}{i \omega \varepsilon} \frac{c}{b} Y_s \sin (k_m a)
\]

(1.54)

Define the quantities

\[
\delta_y \equiv -\frac{k_m^2 + k_n^2}{i \omega \varepsilon} \frac{c}{a} Y_y \quad \delta_s \equiv -\frac{k_m^2 + k_n^2}{i \omega \varepsilon} \frac{c}{b} Y_s
\]

(1.55)

so that

\[
k_m = \delta_y \cot (k_m b/2) \quad \text{or} \quad k_m = -\delta_y \tan (k_m b/2) \\
k_m = \delta_s \cot (k_m a/2) \quad \text{or} \quad k_m = -\delta_s \tan (k_m a/2)
\]

(1.56)

depending on whether \( m \) and \( n \) are even or odd. With the trigonometric identity

\[
\tan(2\theta) = \frac{2 \cot \theta}{\cot^2 \theta - 1} = \frac{2 \tan \theta}{1 - \tan^2 \theta}
\]

(1.57)

find the characteristic equations for \( k_m \) and \( k_n \)

\[
\tan(k_m b) = \frac{2k_m \delta_y}{k_m^2 - \delta_y^2} \quad \tan(k_m a) = \frac{2k_m \delta_s}{k_m^2 - \delta_s^2}
\]

(1.58)

These transcendental equations for \( k_m \) and \( k_n \) must be solved simultaneously with an iterative scheme.
Circular Patch

The leaky cavity model analysis for the circular patch also assumes that the fields can be expanded in the same modes as the perfect cavity, except that the values for \( \mu_{nm} \) are different. Use the boundary condition

\[
\hat{n} \times \vec{H} = \frac{c}{a} Y \vec{E} \tag{1.59}
\]

so that

\[
\mu_{nm} J_n'(\mu_{nm}) = \frac{\mu_{nm}^2}{i \omega \epsilon} \frac{c}{a} Y_r J_n(\mu_{nm} a) \tag{1.60}
\]

where \( Y_r \) is the radiating aperture admittance. The characteristic equation for \( \mu_{nm} \) is

\[
\frac{c}{a} Y_r \mu_{nm} = i \frac{\omega \epsilon}{J_n'(\mu_{nm} a)} \frac{J_n(\mu_{nm} a)}{J_n(\mu_{nm} a)} \tag{1.61}
\]

As in the case of the rectangular patch, this transcendental equation must be solved iteratively for \( \mu_{nm} \).

§1.5 Scope of Thesis

All of these models used in the analysis of the patch antenna have the advantage that they are relatively simple to use. However each of them does have its drawbacks. The transmission line model is only useful for rectangular patch antennas or circular patch antennas. To apply this technique to the rectangular patch antenna, it must be excited in a quasi TEM mode. For the circular patch antenna, the antenna is treated as a radial transmission line, as in Lan and Sengupta [22], with a nonuniform characteristic impedance. The radial line, cavity, and leaky cavity models can be used when the antenna is operating at some other mode, but they only work with patch geometries which are conformal to an orthogonal co-ordinate system. Therefore they can not be used with an arbitrary patch shape, such as a triangular patch. In addition the cavity model is not very accurate in the resonance region of the antenna, which is of the most interest. Furthermore both the leaky cavity model and the transmission line model require the radiating aperture admittance, a quantity which is not exactly known, to compute the fields associated with the antenna.

The radiation mechanisms are also not well explained by the cavity model, although the real part of the empirically obtained aperture admittance is used to account for radiation.
The skin depth of the patch, at the antenna operating frequency is usually much smaller than the thickness of the patch itself, hence the patch antenna has two different current distributions on the upper and lower surfaces of the patch. To demonstrate this fact consider the following example. A patch antenna with a copper patch one tenth of a millimeter thick is operated at 1 GHz. Since for copper \( \sigma = 5.8 \times 10^7 \text{S/m} \) and \( \mu = \mu_0 = 4\pi \times 10^{-7} \)

\[
\delta_\star = \sqrt{\frac{2}{\omega \mu \sigma}} = 2.1 \times 10^{-6} \text{m}
\] (1.62)

and so the thickness of the patch is much greater than \( \delta_\star \). Therefore a more fundamental limitation on the approximate model techniques discussed in this chapter is that they can only predict the current on the underside of the patch and not the total current on the patch. This is because these techniques approximately predict the fields in the cavity region of the antenna only and say nothing about the fields above the patch. To predict the input impedance of the patch antenna knowing the aperture admittance, and hence the underside current, may be adequate. However to accurately predict the far field radiation pattern of the antenna a knowledge of the total current on the patch is necessary.

Therefore the scope of this thesis is to investigate the current distributions on the underside of the patch antenna, in the cavity region, as well as the current on the top surface of the patch antenna. The far field radiation due to these currents is also investigated. In addition some techniques for predicting the aperture admittance which should be used in the leaky cavity models are examined. Chapter II discusses the Wiener-Hopf solution to a parallel plate waveguide filled with an extended dielectric substrate for such a purpose. Although the solution for the aperture admittance can not be found in a closed form from this technique, some general properties of how these quantities behave can be derived. Chapter III formulates the exact integral equation relation for the current on the patch antenna in terms of the dyadic Green's functions. This formulation is used later in Chapter V to form the spectral domain equations governing the currents on the patch. In addition the far zone fields are also solved for in terms of the currents on the patch. Chapter IV formulates the two dimensional special case of the patch problem in the spectral domain. This problem is an idealization of the general three dimensional problem, into a two dimensional problem where the geometry of the antenna, excitation, and scattered fields are
all independent of one of the spatial dimensions. The formulation of the problem here is analogous to the Jones's spectral domain solution for the Wiener-Hopf problem presented in Chapter II. However since the antenna geometry is not semi-infinite the Wiener-Hopf technique can not be directly applied. Therefore the problem is solved numerically with the conjugate gradient-FFT technique. Chapter VI discusses this technique and the other numerical methods used to solve for the currents and fields of the two dimensional problem. Computer software has been written to implement this and some results of this code are presented in Chapter VII. Conclusions with regard to the top and bottom surface currents, far zone radiated fields, and aperture admittances for the general three dimensional problem are drawn from the two dimensional results. In addition these results are compared with more conventional techniques for computing the aperture admittance and far zone radiated fields.
CHAPTER II

WIENER-HOPF SOLUTION TO PARALLEL PLATE WAVEGUIDE PROBLEM

§2.1 Jones's Spectral Method

Before proceeding to formulate the solution to the patch antenna problem, it will be instructive to examine the fields associated with a semi-infinite parallel plate waveguide filled with an extended dielectric. The solution to this problem may be used to find the reflection coefficients and therefore the radiating aperture admittance which should be used in the leaky cavity model. These results can shed some light on the general behavior of the patch antenna with a thin dielectric substrate. To compute the solution to a semi-infinite structure such as this, the Wiener-Hopf technique may be used. Books by Weinstein [23],

Figure 2.1 — Parallel Plate Structure
Noble \cite{24}, and Mittra and Lee \cite{25} discuss the Wiener-Hopf solution to semi-infinite parallel plate waveguide problems.

The problem of interest here consists of finding the fields for a dielectric filled parallel plate waveguide structure when excited with a TEM mode. The two dimensional geometry of this structure, which is shown figure 2.1, consists of an infinite dielectric substrate on a ground plane on top of which is a perfectly conducting half plane. Therefore in the region \( z < 0 \) the structure is that of a dielectric filled parallel plate waveguide. The incident field will consist of a TEM mode wave in this region propagating in the positive \( z \) direction. When this wave impinges on the discontinuity at \( z = 0 \) a TEM mode of some amplitude will be reflected back along with higher order modes of various amplitudes and some energy will radiate away into the dielectric slab and the space above it. Of particular interest to the antenna problem are the reflection coefficient of the TEM mode and the mode conversion coefficients of the higher order modes.

Due to the two dimensional nature of the radiating structure and excitation, all field quantities will be independent of \( y \). In addition only the \( y \) component of the magnetic field will exist. Also, since the electric field is proportional to the curl of the magnetic field in current free regions, the electric field will have only \( x \) and \( z \) components.

To solve this problem use Jones's spectral method Wiener-Hopf technique as presented by Mittra and Lee \cite{25}. To that end let the magnetic field,

\[
\vec{H} = H(x, z) \hat{y} = \psi'(x, z) \hat{y} + \psi(x, z) \hat{y}
\]  

(2.1)

where \( \psi'(x, z) \) is the incident field and \( \psi(x, z) \) is the scattered field. In addition use image theory about the ground plane to define,

\[
\vec{H}(x, -z) = \vec{H}(x, z)
\]

The incident field will be taken to be what would exist if the half plane were a full plane, thereby making a parallel plate waveguide. So,

\[
\psi'(x, z) = \begin{cases} 
    e^{ik_1 z} & ; \quad -c < z < c \\
    0 & ; \quad z > c \text{ or } z < -c 
\end{cases}
\]  

(2.2)

where \( k_1 = \sqrt{\varepsilon} k \) and \( k = \omega \sqrt{\mu \varepsilon} \)
Now the scattered magnetic field must satisfy Maxwell's equations so,

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + k_1^2 \right) \psi(x, z) = 0 \quad ; \quad -c < z < c
\]
\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + k^2 \right) \psi(x, z) = 0 \quad ; \quad z > c \quad \text{or} \quad z < -c
\]  
(2.3)

Introducing the Fourier transform relations:

\[
\mathcal{F}\{f(x)\} = F(\alpha) = \int_{-\infty}^{\infty} f(x)e^{i\alpha x} \, dx
\]
\[
\mathcal{F}^{-1}\{F(\alpha)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x} \, d\alpha
\]  
(2.4)

The above equations can be transformed to,

\[
\left( \frac{\partial^2}{\partial z^2} - \alpha^2 + k_1^2 \right) \Psi(\alpha, z) = 0 \quad ; \quad -c < z < c
\]
\[
\left( \frac{\partial^2}{\partial z^2} - \alpha^2 + k^2 \right) \Psi(\alpha, z) = 0 \quad ; \quad z > c \quad \text{or} \quad z < -c
\]  
(2.5)

where \( \mathcal{F}\{\psi(x, z)\} = \Psi(\alpha, z) \)

Now introduce

\[
\gamma_1 = \sqrt{\alpha^2 - k_1^2} \quad \text{and} \quad \gamma = \sqrt{\alpha^2 - k^2}
\]  
(2.6)

The square root will be defined to have its branch cut when its argument is real and negative and the sheet is chosen so that \( \Re[\sqrt{z}] \geq 0 \). This definition of the square root will be used throughout this thesis. The media is assumed to be lossy so that \( \Im[k_1] > 0 \) and \( \Im[k] > 0 \). Therefore if \( \alpha \) is real it follows that \( \Im[\gamma] < 0 \) and \( \Im[\gamma_1] < 0 \). For the problem with lossless media the limit as \( \Im[k_1] \) tends to zero and/or \( \Im[k] \) tends to zero may be taken.

This leads to, in terms of \( \gamma \) and \( \gamma_1 \),

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma_1^2 \right) \Psi(\alpha, z) = 0 \quad ; \quad -c < z < c
\]
\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \Psi(\alpha, z) = 0 \quad ; \quad z > c \quad \text{or} \quad z < -c
\]  
(2.7)

The solution for \( \Psi(\alpha, z) \) which satisfies the radiation condition and the ground plane symmetry is of the form,

\[
\Psi(\alpha, z) = \begin{cases} 
A(\alpha)e^{-\gamma z} & ; \quad z > c \\
C(\alpha) \cosh(\gamma_1 z) & ; \quad -c < z < c \\
A(\alpha)e^{\gamma z} & ; \quad z < -c 
\end{cases}
\]  
(2.8)
where $A(\alpha)$ and $C(\alpha)$ are unknown functions of $\alpha$. Now introduce

$$
\Psi_-(\alpha, z) = \int_{-\infty}^{0} \psi(x, z)e^{\alpha x} \, dz \\
\Psi_+(\alpha, z) = \int_{0}^{\infty} \psi(x, z)e^{\alpha x} \, dz
$$

so that

$$
\Psi(\alpha, z) = \Psi_-(\alpha, z) + \Psi_+(\alpha, z)
$$

Evaluating these expressions when $z = c$ yields,

$$
\Psi(\alpha, c+) = A(\alpha)e^{-\gamma c} \\
\Psi(\alpha, c-) = C(\alpha) \cosh(\gamma_1 c) \\
\Psi'(\alpha, c+) = -\gamma A(\alpha)e^{-\gamma c} \\
\Psi'(\alpha, c-) = \gamma_1 C(\alpha) \sinh(\gamma_1 c)
$$

where $\Psi'(\alpha, z) = \frac{\partial}{\partial z} \Psi(\alpha, z)$

Since the electric field is proportional to the curl of the magnetic field, it follows that,

$$
E_z(x, z) = \frac{1}{i \omega \varepsilon} \frac{\partial}{\partial z} H_y(x, z) = \frac{1}{i \omega \varepsilon} \frac{\partial}{\partial z} \psi(x, z)
$$

To match the tangential electric field boundary condition on the top surface of the dielectric and on the surface of the half plane the following relations must hold,

$$
\Psi_-'(\alpha, c+) = \Psi_-'(\alpha, c-) = 0
$$

$$
\Psi_+'(\alpha, c+) = \frac{1}{\varepsilon_r} \Psi_+'(\alpha, c-)
$$

and since,

$$
\Psi'(\alpha, z) = \Psi_-'(\alpha, z) + \Psi_+'(\alpha, z)
$$

It follows that,

$$
\varepsilon_r \Psi'(\alpha, c+) = \Psi'(\alpha, c-) \\
-\varepsilon_r \gamma A(\alpha)e^{-\gamma c} = \gamma_1 C(\alpha) \sinh(\gamma_1 c)
$$

and so

$$
\Psi(\alpha, c+) = -\frac{\gamma_1}{\varepsilon_r \gamma} C(\alpha) \sinh(\gamma_1 c)
$$
To match the tangential magnetic field boundary condition on the surface of the dielectric the following must be true:

\[ \Psi_+(\alpha, c-) + \int_0^\infty e^{ik_1z} e^{i\alpha z} \, dz = \Psi_+(\alpha, c+) \]  
(2.18)

Now,

\[ \int_0^\infty e^{ik_1z} e^{i\alpha z} \, dz = \frac{i}{\alpha + k_1} \]  
(2.19)

and introducing the scattered component of current,

\[ \vec{K}(\alpha) \equiv \Psi_-(\alpha, c-) - \Psi_-(\alpha, c+) \]  
(2.20)

it follows that

\[ \Psi(\alpha, c+) - \Psi(\alpha, c-) = \frac{i}{\alpha + k_1} - \vec{K}(\alpha) \]  
(2.21)

But,

\[ \Psi(\alpha, c+) - \Psi(\alpha, c-) = -C(\alpha) \left[ \frac{\gamma_1}{\epsilon_r \gamma} \sinh(\gamma_1 c) + \cosh(\gamma_1 c) \right] \]

\[ = -\Psi'(\alpha, c+) \frac{\gamma_1 \sinh(\gamma_1 c) + \epsilon_r \gamma \cosh(\gamma_1 c)}{\gamma \gamma_1 \sinh(\gamma_1 c)} \]

\[ = -\frac{\Psi'(\alpha, c+)}{\gamma \gamma_1 G(\alpha)} \]  
(2.22)

where

\[ G(\alpha) \equiv \frac{\sinh(\gamma_1 c)}{\gamma_1 \sinh(\gamma_1 c) + \epsilon_r \gamma \cosh(\gamma_1 c)} \]  
(2.23)

Therefore

\[ \frac{i}{\alpha + k_1} - \vec{K}(\alpha) = -\frac{\Psi'(\alpha, c+)}{\gamma \gamma_1 G(\alpha)} \]  
(2.24)

Equation (2.24) is the desired Wiener-Hopf equation. To solve this equation the function \( G(\alpha) \) must be factored as

\[ G(\alpha) = G_+(\alpha) G_-(\alpha) \]  
(2.25)

where \( G_+(\alpha) \) is analytic and not equal to zero for \( \Im[\alpha] > -\Im[k_1] \) and \( -\Im[k] \), and \( G_-(\alpha) \) is analytic and not equal to zero for \( \Im[\alpha] < \Im[k_1] \) and \( \Im[k] \). In addition \( G_+(\alpha) \) must be of the order \( 1/\sqrt{\alpha} \) as \( \alpha \) tends to infinity in the upper half of the \( \alpha \) plane while \( G_-(\alpha) \) must be of order \( 1/\sqrt{\alpha} \) as \( \alpha \) tends to infinity in lower half of the \( \alpha \) plane. Assuming
such factors, or split functions, can be found, then the Wiener-Hopf equation can be solved for $\psi(x, z)$ as follows

$$
\left[ \frac{i}{\alpha + k_1} - \widetilde{K}(\alpha) \right] \gamma_1 G(\alpha) = -\Psi'_+ (\alpha, c+)
$$

(2.26)

since $\Psi'_- (\alpha, c+) = 0$. And so

$$
\frac{\Psi'_+ (\alpha, c+)}{\sqrt{(\alpha + k)(\alpha + k_1)} G_+ (\alpha)} = \frac{i}{\alpha + k_1} \sqrt{(\alpha - k)(\alpha - k_1) G_- (\alpha)}
$$

$$
+ \widetilde{K}(\alpha) \sqrt{(\alpha - k)(\alpha - k_1) G_- (\alpha)}
$$

(2.27)

Subtracting the pole term at $\alpha = -k_1$ from both sides of this equation leads to

$$
-\frac{\Psi'_+ (\alpha, c+)}{\sqrt{(\alpha + k)(\alpha + k_1)} G_+ (\alpha)} + \frac{i}{\alpha + k_1} \sqrt{2k_1 (k_1 + k) G_- (-k_1)}
$$

$$
= \frac{i}{\alpha + k_1} \left[ \sqrt{(\alpha - k)(\alpha - k_1)} G_- (\alpha) + \sqrt{2k_1 (k_1 + k)} G_- (-k_1) \right]
$$

$$
+ \widetilde{K}(\alpha) \sqrt{(\alpha - k)(\alpha - k_1) G_- (\alpha)}
$$

(2.28)

Notice that the left hand side of the above equation is analytic in the upper half of the complex $\alpha$ plane while the right hand side is analytic in the lower half of the complex $\alpha$ plane. Therefore the function which is represented on each side of this equation must be an entire function, analytic for all values of $\alpha$. Furthermore from Meixner's edge condition the tangential electric field at $z = c$ which is proportional to $\psi'_c (x, c+)$ may have a singularity no worse than $1/\sqrt{z}$ at $z = 0$. This implies that the asymptotic behavior of $\Psi'_+ (\alpha, c+)$ must decrease at least as rapidly as $1/\sqrt{\alpha}$ as $\alpha$ tends to infinity. See Mittra and Lee [25] chapter 1 for a discussion of Meixner's edge condition and how it determines the asymptotic behavior of field quantities in the spectral domain. Therefore the left hand side of eq. (2.28) tends to zero as $\alpha$ goes to infinity in the upper half plane, while the right hand expression tends to zero as $\alpha$ goes to infinity in the lower half plane. From Liouville's theorem, which states that a bounded entire function is a constant, both sides of eq. (2.28) must equal zero. See Churchill [26] for a discussion of Liouville's theorem. Hence,

$$
\Psi'_+ (\alpha, c+) = \frac{i}{\alpha + k_1} \sqrt{2k_1 (k_1 + k)} G_- (-k_1) \sqrt{(\alpha + k)(\alpha + k_1)} G_+ (\alpha)
$$

$$
= -\gamma A(\alpha) e^{-\gamma c}
$$

$$
= \frac{\gamma_1}{\epsilon_r} C(\alpha) \sinh (\gamma_1 c)
$$

(2.29)
So the unknowns $A(\alpha)$ and $C(\alpha)$ are given by

$$
A(\alpha) = -\sqrt{2k_1(k_1 + k)} G_-( -k_1) \sqrt{\frac{\alpha + k}{\alpha + k_1}} G_+(\alpha) e^{i\alpha z} / \gamma
$$

$$
C(\alpha) = i\varepsilon \sqrt{2k_1(k_1 + k)} G_-( -k_1) \sqrt{\frac{\alpha + k}{\alpha + k_1}} G_+(\alpha)/[\gamma_1 \sinh(\gamma_1 c)]
$$

(2.30)

and the scattered field, $\psi(x, z)$, can be found by an inverse Fourier transform of $\Psi(\alpha, z)$.

§ 2.2 Split Function for Free Space Case

The solution to the split functions $G_+(\alpha)$ and $G_-(\alpha)$ for the special case where $\varepsilon = 1$ is presented in this section. In this special case $k_1 = k$, $\gamma_1 = \gamma$, and therefore

$$
G(\alpha) = e^{-\gamma c} \frac{\sinh(\gamma c)}{\gamma}
$$

(2.31)

This factorization will be accomplished by factoring the $e^{-\gamma c}$ and the $\sinh(\gamma c)/\gamma$ terms separately.

To factor $e^{-\gamma c}$, the function $\ln[e^{-\gamma c}] = -\gamma c$ will be decomposed into the sum of two functions $S_+(\alpha)$ and $S_-(\alpha)$ which are analytic in the upper and lower halves of the $\alpha$ plane respectively. In general a function $S(\alpha)$, which is analytic on a strip around the real alpha axis, can be decomposed by transforming it back to the spatial domain and then computing

$$
S_+(\alpha) = \int_{0}^{\infty} s(x) e^{i\alpha x} dx
$$

$$
S_-(\alpha) = \int_{-\infty}^{0} s(x) e^{i\alpha x} dx
$$

(2.32)

where $s(x) = \mathcal{F}^{-1}\{S(\alpha)\}$. If the function $s(x)$ can be dominated by exponential functions at infinity, so that

$$
s(x) < K_{+} e^{-\alpha x} \text{ as } x \to +\infty
$$

$$
s(x) < K_{-} e^{+\alpha x} \text{ as } x \to -\infty
$$

(2.33)

for some constants $K_{+}$ and $K_{-}$, then $S(\alpha)$ will be analytic in the strip $\tau_{-} < \Im\alpha < \tau_{+}$. Assuming this is the case, by interchanging the order of integration in eq. (2.32) $S_+$ and $S_-$ can be found directly from $S$ with

$$
S_+(\alpha) = \frac{+1}{2\pi i} \int_{-\infty + i\tau_{-}}^{+\infty + i\tau_{-}} \frac{S(\beta)}{\beta - \alpha} d\beta
$$

$$
S_-(\alpha) = \frac{-1}{2\pi i} \int_{-\infty - i\tau_{+}}^{+\infty - i\tau_{+}} \frac{S(\beta)}{\beta - \alpha} d\beta
$$

(2.34)
If \( S(\alpha) \) tends to zero as \( \alpha \) tends to \( \pm \infty \), as it must for its inverse Fourier transform to exist, then from Cauchy’s residue theorem it follows that

\[
S(\alpha) = S_+(\alpha) + S_-(\alpha)
\]

(2.35)

for \( \alpha \) in the strip so that \( r_- < \Im\alpha < r_+ \). Furthermore \( S_+(\alpha) \) will be analytic when \( \Im\alpha > r_- \) and \( S_-(\alpha) \) will be analytic when \( \Im\alpha < r_+ \).

Now apply this general decomposition procedure to the function \( 1/\gamma \). This function is analytic in the strip \( -\Im\gamma[k] < \Im\alpha[k] < \Im\gamma[k] \). Both contour integrals in eq. (2.34) can be deformed to an integral along the line \( \beta = kt \), where \( t \) is a real parameter which ranges from \( -\infty \) to \( +\infty \). The functions \( S_+ \) and \( S_- \) can be found in terms of this integral. Two different cases must be considered depending on whether the value of \( \alpha \) is to the right or left of this deformed contour in the \( \alpha \) plane. Defining

\[
I \equiv \frac{1}{2\pi i} \int_C \frac{S(\beta)}{\beta - \alpha} d\beta
\]

(2.36)

where \( C \) is the deformed contour, the \( S_+ \) and \( S_- \) functions are:

\[
\begin{align*}
S_+(\alpha) &= +I + \frac{1}{\gamma} \quad \Re[\alpha]\Re[k] > \Im[\alpha]\Im[k] \\
S_-(\alpha) &= -I \\
S_+(\alpha) &= +I \\
S_-(\alpha) &= -I + \frac{1}{\gamma} \quad \Re[\alpha]\Re[k] < \Im[\alpha]\Im[k]
\end{align*}
\]

(2.37)

The integral, \( I \), can be evaluated with the substitutions

\[
\beta = \pm k \sec \theta
\]

(2.38)

and using the standard integral relations

\[
\begin{align*}
\int \frac{d\theta}{k - \alpha \cos \theta} &= \frac{1}{\gamma} \ln \left( \frac{\gamma \tan(\theta/2) - (\alpha - k)}{\gamma \tan(\theta/2) + (\alpha - k)} \right) \\
\int \frac{d\theta}{k + \alpha \cos \theta} &= -\frac{1}{\gamma} \ln \left( \frac{\gamma \tan(\theta/2) + (\alpha + k)}{\gamma \tan(\theta/2) - (\alpha + k)} \right)
\end{align*}
\]

(2.39)

Care must be taken when using these integral relations since the natural log function does not necessarily take on its principle value, but instead is defined on a sheet which is continuous for the contour of integration used. Keeping this in mind,

\[
\begin{align*}
I &= -\frac{i}{\pi \gamma} \ln \left( \frac{\gamma - \alpha}{k} \right) \quad \Re[\alpha]\Re[k] > \Im[\alpha]\Im[k] \\
I &= -\frac{i}{\pi \gamma} \ln \left( \frac{\alpha - \gamma}{k} \right) \quad \Re[\alpha]\Re[k] < \Im[\alpha]\Im[k]
\end{align*}
\]

(2.40)
Therefore

\[
S_+ (\alpha) = -\frac{i}{\pi \gamma} \ln \left[ \frac{\alpha - \gamma}{k} \right] \\
S_- (\alpha) = +\frac{i}{\pi \gamma} \ln \left[ \frac{\gamma - \alpha}{k} \right]
\]  

(2.41)

for all values of \( \alpha \). Examining \( S_+ (\alpha) \) near \( \alpha = k \) shows that it is indeed analytic in the upper half of the \( \alpha \) plane. Conversely, examining \( S_- (\alpha) \) near \( \alpha = -k \) shows it is analytic in the lower half plane.

Now that the decomposition of \( 1/\gamma \) is known, the decomposition of \( -\gamma c \) is simple to compute. Since

\[
-\gamma c = -\gamma^2 c/\gamma
\]

(2.42)

and \( \gamma^2 \) is an entire function it follows that \( -\gamma c \) can be decomposed as

\[
-\gamma c = \frac{i\gamma c}{\pi} \ln \left[ \frac{\alpha - \gamma}{k} \right] - \frac{i\gamma c}{\pi} \ln \left[ \frac{\gamma - \alpha}{k} \right]
\]

(2.43)

Therefore factor \( e^{-\gamma c} \) as

\[
e^{-\gamma c} = e^{+i\gamma c \ln[(\alpha - \gamma)/k]/\pi} e^{-i\gamma c \ln[(\gamma - \alpha)/k]/\pi}
\]

(2.44)

Now factor the function \( \sinh(\gamma c)/\gamma \). Notice that even though \( \gamma \) has branch points at \( \alpha = \pm k \), the function \( \sinh(\gamma c)/\gamma \) is an entire function, analytic for all values of \( \alpha \). Since \( \sinh(\gamma c)/\gamma \) is an entire function with a countable number of simple zeros, represent it as the infinite product

\[
\frac{\sinh(\gamma c)}{\gamma} = \frac{\sin(kc)}{k} \prod_{n=1}^{\infty} \left( 1 - \frac{\alpha}{\beta_n} \right) \left( 1 + \frac{\alpha}{\beta_n} \right)
\]

(2.45)

where the zeros of \( \sinh(\gamma c) \) are \( \pm \beta_n = \pm i \sqrt{\frac{n\pi}{c^2} - k^2} \). This infinite product can also be written as

\[
\frac{\sinh(\gamma c)}{\gamma} = \frac{\sin(kc)}{k} \prod_{n=1}^{\infty} \left( 1 - \frac{\alpha}{\beta_n} \right) e^{-i\alpha/n\pi} \prod_{n=1}^{\infty} \left( 1 + \frac{\alpha}{\beta_n} \right) e^{+i\alpha/n\pi}
\]

(2.46)

The addition of the cancelling phase factors in each of the two infinite products is necessary to insure the convergence of each infinite product individually. See Copson [27] section 6.83 for a discussion on the convergence of infinite products of this type, or Titchmarsh [28] chapter 1 for a discussion on infinite products in general.
Therefore the split functions are of the form

\[
G_+(\alpha) = e^{P(\alpha)} \sqrt{\frac{\sin(kc)}{k}} e^{i(\gamma c/\pi) \ln[(\alpha - \gamma)/k]} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{\beta_n}\right) e^{i\alpha c/n\pi} \\
G_-(\alpha) = e^{-P(\alpha)} \sqrt{\frac{\sin(kc)}{k}} e^{-(i\gamma c/\pi) \ln[(\gamma - \alpha)/k]} \prod_{n=1}^{\infty} \left(1 - \frac{\alpha}{\beta_n}\right) e^{-i\alpha c/n\pi}
\]

(2.47)

where \(P(\alpha)\) is an entire function and is necessary to ensure the algebraic behavior of \(G_+(\alpha)\) in the upper half and \(G_-(\alpha)\) in the lower half of the \(\alpha\) plane as \(|\alpha| \to \infty\). To determine \(P(\alpha)\) examine the asymptotic behavior of the rest of \(G_+(\alpha)\).

\[
\lim_{|\alpha| \to \infty} \gamma \approx \pm \alpha
\]

(2.48)

\[
\lim_{|\alpha| \to \infty} \frac{e^{(i\gamma c/\pi) \ln[(\alpha - \gamma)/k]}}{e^{(i\alpha c/\pi) \ln[2\alpha/k]}} \approx e^{(i\alpha c/\pi)[\Gamma - 1 + \ln(-i\alpha/\pi)]}
\]

(2.49)

Furthermore it can be shown for \(\Re m[\alpha] > 0\), see Mitra and Lee [25] chapter 1, that

\[
\lim_{|\alpha| \to \infty} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{\beta_n}\right) e^{i\alpha c/n\pi} \approx A\alpha^{-1/2} e^{(i\alpha c/\pi)[\Gamma - 1 + \ln(-i\alpha/\pi)]}
\]

(2.50)

where \(\Gamma\) is Euler's constant defined by

\[
\Gamma \equiv \lim_{M \to \infty} \sum_{m=1}^{M} \frac{1}{m} - \ln(M) \approx 0.57721566490153
\]

(2.51)

Therefore choose

\[
P(\alpha) = \frac{i\alpha c}{\pi} \left[\ln \left(\frac{2\pi}{kc}\right) - \Gamma + 1 + \frac{i\pi}{2}\right]
\]

(2.52)

So \(G_+(\alpha)\) behaves as \(\alpha^{-1/2}\) as \(|\alpha| \to \infty\) in the upper half of the \(\alpha\) plane. It also follows that \(G_-(\alpha)\) behaves as \(\alpha^{-1/2}\) as \(|\alpha| \to \infty\) in the lower half of the \(\alpha\) plane. Hence the final result is

\[
G_+(\alpha) = G_-(\alpha) = \sqrt{\frac{\sin(kc)}{k}} e^{(i\alpha c/\pi)[1 - \Gamma + \ln(2\pi/kc) + i\pi/2]} \times \\
e^{(i\gamma c/\pi) \ln[(\alpha - \gamma)/k]} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{\beta_n}\right) e^{i\alpha c/n\pi}
\]

(2.53)
§2.3 Split Function for General Case

In 1968, Bates and Mittra [29] published a paper on finding the split functions for the lossless problem where $\varepsilon_r$ is real and greater than one. Unfortunately this solution is not in a closed form since it involves integrals which in general can not be evaluated analytically. Their solution consists of finding the split functions for

$$K(\alpha) = \frac{\gamma \gamma_1 \sinh(\gamma_1 c) \sinh(\gamma \alpha)}{\gamma_1 \sinh(\gamma_1 c) \cosh(\gamma \alpha) + \varepsilon_r \gamma \cosh(\gamma_1 c) \sinh(\gamma \alpha)}$$  \hspace{1cm} (2.54)$$

which comes from a closed region problem. Then they consider the limit as $a \to \infty$ and the closed region problem becomes the open region problem of interest.

$$\lim_{a \to \infty} K(\alpha) = L(\alpha) = \frac{\gamma \gamma_1 \sinh(\gamma_1 c)}{\gamma_1 \sinh(\gamma_1 c) + \varepsilon_r \gamma \cosh(\gamma_1 c)}$$  \hspace{1cm} (2.55)$$

Notice that

$$\gamma \gamma_1 G(\alpha) = L(\alpha)$$  \hspace{1cm} (2.56)$$

The factorization of $L(\alpha)$ is somewhat involved so only the results will be stated here.

$$\gamma \gamma_1 G(\alpha) = L_+(\alpha)L_-(\alpha)$$  \hspace{1cm} (2.57)$$

where

$$L_+(\alpha) = \sqrt{\alpha + k(\alpha + k_1)M_+(\alpha)} \quad L_-(\alpha) = \sqrt{\alpha - k(\alpha - k_1)M_+(-\alpha)}$$  \hspace{1cm} (2.58)$$

Now

$$M_+(\alpha) = \sqrt{f_1} \prod_{n=1}^{\infty} \frac{1 + \alpha/\beta_n}{\prod_{n=1}^{M} (1 + \alpha/\delta_n)} e^{-\alpha/\delta_n} e^{H(\alpha)-\chi(\alpha)}$$  \hspace{1cm} (2.59)$$

where

$$f_1 = \frac{\sin(k_1 c)}{k_1 [-k_1 \sin(k_1 c) - i \varepsilon_r k \cos(k_1 c)]}$$

$$\beta_n = \sqrt{k^2 - \left(\frac{nx}{c}\right)^2}$$

$$\delta_n = \sqrt{k^2 + p_n^2}$$  \hspace{1cm} (2.60)$$

The $p_n$ correspond to the $M$ propagating surface wave modes and are the positive real solutions to the characteristic equation

$$\varepsilon_r p \cos(hc) = h \sin(hc)$$  \hspace{1cm} (2.61)$$
where \( h = \sqrt{(\varepsilon_r - 1)k^2 - p^2} \). Furthermore,

\[
\chi(\alpha) = (i\alpha c/\pi) [\Gamma + \ln(kc/2\pi)] + \frac{\alpha c}{2} + I(\alpha) + \alpha \sum_{n=1}^{M} \frac{1}{\delta_n}
\]

\[
H(\alpha) = \int_{0}^{\infty} \left\{ \ln \left[ 1 + \frac{\alpha}{\sqrt{k^2 - \omega^2}} \right] - \frac{\alpha}{\sqrt{k^2 - \omega^2}} \right\} Z(\omega) - \frac{c}{\pi} d\omega
\]

\[
I(\alpha) = -\alpha \int_{0}^{\infty} \frac{1}{\sqrt{k^2 - \omega^2}} Z(\omega) d\omega
\]

\[
Z(\omega) = \frac{c}{\pi} - \frac{\varepsilon_r \sin(\omega c) \cos(\omega c)}{\pi \left[ \varepsilon_r^2 c^2 \cos^2(\omega c) + \omega^2 \sin^2(\omega c) \right]}
\]

where \( \omega_1 = \sqrt{\omega^2 + (\varepsilon_r - 1)k^2} \).

The integrals \( H(\alpha) \) and \( I(\alpha) \) can not be analytically integrated and therefore must be treated numerically. The function \( L_+(\alpha) \) is analytic and nonzero in the upper half of the \( \alpha \) plane and \( L_-(\alpha) \) is analytic and nonzero in the lower half of the \( \alpha \) plane. In addition, the split function \( L_+(\alpha) \) behaves as \( \alpha^{1/2} \) in the upper half and \( L_-(\alpha) \) behaves as \( \alpha^{1/2} \) in the lower half of the \( \alpha \) plane as \( |\alpha| \to \infty \).

In the limit as \( \varepsilon_r \to 1 \), the function \( Z(\omega) \) becomes zero and

\[
\lim_{\varepsilon_r \to 1} L_+(\alpha) = (\alpha + k) G_+(\alpha)
\]

\[
\lim_{\varepsilon_r \to 1} L_-(\alpha) = (\alpha - k) G_-(\alpha)
\]

where \( G_+(\alpha) \) and \( G_-(\alpha) \) for \( \varepsilon_r = 1 \) were given in the last section. In general, the \( L \) split functions are related to the \( G \) split functions by

\[
L_+(\alpha) = \sqrt{(\alpha + k)(\alpha + k_1)} G_+(\alpha)
\]

\[
L_-(\alpha) = \sqrt{(\alpha - k)(\alpha - k_1)} G_-(\alpha)
\]

§2.4 Conclusions which Apply to the Antenna Problem

To find the reflection and mode conversion coefficients examine the scattered field in the waveguide region of the parallel plate problem. For \( z < c \),

\[
\Psi(\alpha, z) = C(\alpha) \cosh(\gamma z)
\]

\[
= i\varepsilon_r \sqrt{2k_1(k_1 + k)} G_+(k_1) \sqrt{\frac{\alpha + k}{\alpha + k_1}} \frac{G_+(\alpha) \cosh(\gamma z)}{\gamma_1 \sinh(\gamma_1 c)}
\]
Therefore in this region

\[ \psi(z, x) = \varepsilon_r \sqrt{2k_1(k_1+k)} G_+(k_1) \times \]
\[ \frac{i}{2\pi} \int_{-\infty}^{+\infty} \sqrt{\frac{\alpha + k}{\alpha + k_1}} G_+(\alpha) \frac{\cosh(\gamma_1 z)}{\gamma_1 \sinh(\gamma_1 c)} e^{-i\alpha x} d\alpha \quad (2.66) \]

This integrand is analytic, except for simple poles at \( \alpha = \beta_m \) and \( \alpha = k_1 \), in the upper half of the \( \alpha \) plane and for \( z < 0 \) the contour may be closed at infinity there. Having done this, evaluate the integral with Cauchy's residue theorem. We have

\[ \frac{i}{2\pi} \int_{-\infty}^{+\infty} \sqrt{\frac{\alpha + k}{\alpha + k_1}} G_+(\alpha) \frac{\cosh(\gamma_1 z)}{\gamma_1 \sinh(\gamma_1 c)} e^{-i\alpha x} d\alpha = -\sqrt{\frac{k + k_1}{2k_1}} \frac{G_+(k_1)}{2k_1 c} e^{-ik_1 x} \]
\[ - \sum_{m=1}^{\infty} \sqrt{\frac{\beta_m + k}{\beta_m + k_1}} \frac{G_+(\beta_m)}{\beta_m c} \cos \left[ \frac{m \pi}{c} (z - c) \right] e^{-i\beta_m x} \quad (2.67) \]

where \( \beta_m = \sqrt{k_1^2 - (\frac{m \pi}{c})^2} \).

Notice that each of the poles enclosed by the contour of integration corresponds to a reflected parallel plate waveguide mode which can exist for \( z < c \) and \( z < 0 \). Therefore the TEM magnetic reflection coefficient is

\[ R_0 = -\varepsilon_r \frac{k_1 + k}{2k_1 c} G_+^2(k_1) \quad (2.68) \]

and the magnetic mode conversion coefficients are

\[ R_m = -\varepsilon_r \sqrt{2k_1(k_1+k)} G_+(k_1) \sqrt{\frac{\beta_m + k}{\beta_m + k_1}} \frac{G_+(\beta_m)}{\beta_m c} \quad (2.69) \]

for \( m = 1, 2, 3, \ldots \). Unfortunately, since \( G_+(\alpha) \) can not in general be determined in a closed form, neither can these reflection and mode conversion coefficients. However some of the properties of the coefficients for thin dielectric substrates can be inferred.

Consider the behavior of these coefficients as \( c \) tends to zero as in the case of the thin patch antenna.

\[ \lim_{c \to 0} \beta_m \approx i \frac{m \pi}{c} \quad (2.70) \]

In the special case for \( \varepsilon_r = 1 \) and so \( k = k_1 \),

\[ \lim_{c \to 0} G_+(k) \approx \sqrt{c} \]
\[ \lim_{c \to 0} G_+(\beta_m) \approx \sqrt{c} \prod_{n=1}^{\infty} \left( 1 + \frac{m}{n} \right) e^{-m/n} \quad (2.71) \]
With the aid of Stirling's approximation, \( N! \approx e^{-N} N^{N} \sqrt{2\pi N} \), reduce the infinite product above.

\[
\prod_{n=1}^{\infty} \left( 1 + \frac{m}{n} \right) e^{-m/n} \approx e^{-m^2/m!}
\]  
(2.72)

Therefore for \( \epsilon_r = 1 \),

\[
\lim_{\epsilon \to 0} R_0 \approx -1 \\
\lim_{\epsilon \to 0} R_m \approx -\frac{2kc}{im\pi} e^{-m^2/m!}
\]
(2.73)

Notice that the mode conversion coefficients are all proportional to \( c \) and therefore are tending to zero. Furthermore, for a given \( c \), as \( m \) increases the mode conversion coefficients rapidly become smaller due to their \( e^{-m^2/m!} \) behavior. Similar results can be shown for the general case where \( \epsilon_r > 1 \). For this case with small \( c \), only one positive real root exits for the characteristic equation \((M = 1)\) whose value is about

\[
p_1 \approx \frac{\epsilon_r - 1}{\epsilon_r} k^2 c
\]
(2.74)

Now the TEM magnetic reflection coefficient is related to the radiating aperture admittance per unit length by

\[
R_0 = \frac{Y_r - Y_c}{Y_r + Y_c}
\]
(2.75)

The quantity \( Y_r \) is the aperture admittance per unit length and \( Y_c = \frac{1}{c} \sqrt{\epsilon_r \epsilon/\mu_0} \) is the characteristic admittance per unit length of the parallel plate guide. Therefore

\[
Y_r = Y_c \frac{1 + R_0}{1 - R_0}
\]
(2.76)

and as \( c \) becomes small \( Y_r \) tends to zero. The fact that as \( c \) tends to zero \( R_0 \to -1 \) and \( R_m \to 0 \) supports the validity of treating the thin patch antenna as a leaky cavity. If the parallel plate portion of this problem formed a perfect cavity these would be the values of the reflection and mode conversion coefficients. Therefore as the dielectric substrate becomes thinner, one should expect the patch antenna to behave more and more like a perfect cavity.
CHAPTER III

INTEGRAL EQUATION FORMULATION OF ANTENNA PROBLEM

§3.1 Dyadic Green's Function

The geometry of the problem to be solved is shown above in figure 3.1 of a bottom fed patch antenna. A patch of arbitrary shape is located on top of a dielectric substrate of thickness $c$ and dielectric constant $\varepsilon$. It is driven by a $\hat{z}$ directed line current source of magnitude $I_0$ at the position $z = z_0$ and $y = y_0$ which comes from the ground plane and terminates on the patch.

To formulate an integral equation relation for the surface current on the patch antenna, the Green's function for a ground plane covered with a dielectric substrate must be found. In
general, it is impossible to find a scalar Green’s function which will satisfy all the necessary boundary conditions at the dielectric substrate and free space interface for an arbitrarily directed source current. Therefore the general dyadic Green’s function approach will be used. The dyadic Green’s function for this problem can be found using the Ohm-Rayleigh transform technique along with the principle of scattering superposition for dyadic Green’s functions as discussed in Tai [30].

The Ohm-Rayleigh method yields the following for the electric type free space dyadic Green’s function:

$$
\overline{G}_{e0}(\overline{R} | \overline{R}') = -\frac{1}{k^2} \delta(\overline{R} - \overline{R}') \hat{z} \hat{z} \\
+ \frac{i}{8\pi^2} \left\{ \int_{-\infty}^{+\infty} \left[ \overline{N}(-h_0)\overline{N}'(-h_0) + \overline{M}(-h_0)\overline{M}'(-h_0) \right] \frac{da_x da_y}{h_0(\alpha_x^2 + \alpha_y^2)} \right\}; z < z' \\
+ \frac{i}{8\pi^2} \left\{ \int_{-\infty}^{+\infty} \left[ \overline{N}(-h_0)\overline{N}'(+h_0) + \overline{M}(-h_0)\overline{M}'(+h_0) \right] \frac{da_x da_y}{h_0(\alpha_x^2 + \alpha_y^2)} \right\}; z > z'
$$

(3.1)

where

$$
\overline{N}(h) = \frac{-\alpha_x \hat{z} - \alpha_y \hat{y} + (\alpha_x^2 + \alpha_y^2) \hat{z}}{\sqrt{h^2 + \alpha_x^2 + \alpha_y^2}} e^{-i(\alpha_x z + \alpha_y y)} e^{-ihs} \\
\overline{N}'(h) = \frac{+\alpha_x \hat{z} + \alpha_y \hat{y} + (\alpha_x^2 + \alpha_y^2) \hat{z}}{\sqrt{h^2 + \alpha_x^2 + \alpha_y^2}} e^{i(\alpha_x z' + \alpha_y y')} e^{-ihs'} \\
\overline{M}(h) = [i\alpha_x \hat{z} + i\alpha_y \hat{y}] e^{-i(\alpha_x z + \alpha_y y)} e^{-ihs} \\
\overline{M}'(h) = [+i\alpha_x \hat{z} - i\alpha_y \hat{y}] e^{i(\alpha_x z' + \alpha_y y')} e^{-ihs'}
$$

(3.2)

and

$$
h_0 = \sqrt{k^2 - \alpha_x^2 - \alpha_y^2}
$$

(3.3)

with $k$ the free space wave number. Using scattering superposition find the dyadic Green’s function for the half space above a ground plane.

$$
\overline{G}_{eH0}(\overline{R} | \overline{R}') = \overline{G}_{e0}(\overline{R} | \overline{R}') \\
+ \frac{i}{8\pi^2} \int_{-\infty}^{+\infty} \left[ p\overline{N}(-h_0)\overline{N}'(-h_0) + q\overline{M}(-h_0)\overline{M}'(-h_0) \right] \frac{da_x da_y}{h_0(\alpha_x^2 + \alpha_y^2)}
$$

(3.4)

where $p$ and $q$ are to be determined from the boundary condition

$$
\hat{z} \times \overline{G}_{eH0}(\overline{R} | \overline{R}') = 0
$$

(3.5)
at \( z = 0 \). Therefore \( p = +1 \) and \( q = -1 \) so

\[
\overline{G}_{eH0}(R \mid R') = -\frac{1}{k^2} \delta(R - R') \hat{z} \hat{z}
\]

\[
+ \frac{i}{8\pi^2} \left\{ \int_{-\infty}^{+\infty} \left[ \overline{N}(+h_0) + \overline{N}(-h_0) \right] \overline{N}'(-h_0) \right. \\
\left. + \left[ \overline{M}(+h_0) - \overline{M}(-h_0) \right] \overline{M}'(-h_0) \right\} \frac{da_x da_y}{h_0(\alpha_x^2 + \alpha_y^2)} \quad ; z < z' \\
\int_{-\infty}^{+\infty} \left[ \overline{N}(-h_0) \left[ \overline{N}'(+h_0) + \overline{N}'(-h_0) \right] \\
+ \overline{M}(+h_0) \left[ \overline{M}'(+h_0) - \overline{M}'(-h_0) \right] \right\} \frac{da_x da_y}{h_0(\alpha_x^2 + \alpha_y^2)} \quad ; z > z'
\]

(3.6)

Now use scattering superposition once again to find the dyadic Green's functions for the dielectric substrate above a ground plane for a source current in the region of the substrate.

Following the notation in Tai [30], this is given by

\[
\overline{G}_{eH1}^{(11)}(R \mid R') = \overline{G}_{eH1}(R \mid R') \\
+ \frac{i}{8\pi^2} \int_{-\infty}^{+\infty} \left\{ R_m \left[ \overline{N}(+h_1) + \overline{N}(-h_1) \right] \left[ \overline{N}'(+h_1) + \overline{N}'(-h_1) \right] \\
+ R_e \left[ \overline{M}(+h_1) - \overline{M}(-h_1) \right] \left[ \overline{M}'(+h_1) - \overline{M}'(-h_1) \right] \right\} \frac{da_x da_y}{h_1(\alpha_x^2 + \alpha_y^2)}
\]

(3.7)

\[
\overline{G}_{e}^{(21)}(R \mid R') = \frac{i}{8\pi^2} \int_{-\infty}^{+\infty} \left\{ T_m \overline{N}(-h_0) \left[ \overline{N}'(+h_1) + \overline{N}'(-h_1) \right] \\
+ T_e \overline{M}(-h_0) \left[ \overline{M}'(+h_1) - \overline{M}'(-h_1) \right] \right\} \frac{da_x da_y}{h_1(\alpha_x^2 + \alpha_y^2)}
\]

(3.8)

where

\[
h_1 = \sqrt{k_1^2 - \alpha_x^2 - \alpha_y^2}
\]

(3.9)

and \( \overline{G}_{eH1}(R \mid R') \) is computed from \( \overline{G}_{eH0}(R \mid R') \) by replacing \( h_0 \) with \( h_1 \) and \( k \) by \( k_1 \). Recall that \( k_1 = \sqrt{\varepsilon_k} k \). Solve for the quantities \( R_m, \ R_e, \ T_m, \) and \( T_e \) using the boundary conditions

\[
\hat{z} \times \overline{G}_{e}^{(11)}(R \mid R') = \hat{z} \times \overline{G}_{e}^{(21)}(R \mid R') \\
\hat{z} \times \nabla \times \overline{G}_{e}^{(11)}(R \mid R') = \hat{z} \times \nabla \times \overline{G}_{e}^{(21)}(R \mid R')
\]

(3.10)
at \( z = c \), the upper surface of the substrate. Therefore

\[
R_m = \frac{k_2^2 h_1 - k_1^2 h_0}{2 [k_1^2 h_0 \cos(h_1 c) - i k_2^2 h_1 \sin(h_1 c)]} e^{ih_1 c} \\
T_m = \frac{k k_1 h_1}{k_1^2 h_0 \cos(h_1 c) - i k_2 h_1 \sin(h_1 c)} e^{-i h_0 c} \\
R_e = \frac{h_1 - h_0}{2 [h_1 \cos(h_1 c) - i h_0 \sin(h_1 c)]} e^{ih_1 c} \\
T_e = \frac{h_1}{h_1 \cos(h_1 c) - i h_0 \sin(h_1 c)} e^{-i h_0 c}
\]

(3.11)

From the dyadic Green's function theory the electric field observed in the dielectric substrate due to a current source there is

\[
\overline{E}(\overline{R}') = i \omega \mu \iiint_V \overrightarrow{J}(\overline{R}) \cdot \overrightarrow{G}_e^{(11)}(\overline{R} | \overline{R'}) dV
\]

(3.12)

for \( 0 \leq z' \leq c \). Furthermore the electric field observed above the substrate due to a current source in the substrate is

\[
\overline{E}(\overline{R}') = i \omega \mu \iiint_V \overrightarrow{G}_e^{(21)}(\overline{R}' | \overline{R}) \cdot \overrightarrow{J}(\overline{R}) dV
\]

(3.13)

for \( z' \geq c \). Both volume integrations in the last two equations are performed over the volume of the dielectric substrate.

Now that the required dyadic Green's functions have been determined an integral equation relation for the three dimensional patch antenna can be formulated. Before proceeding to the next section to do so however, notice that these Green's functions are formulated as two dimensional Fourier inversion integrals. An examination of the behavior of these integrands as \( \alpha_x \) or \( \alpha_y \) becomes large reveals that these are only formal transform relations. The dyadic terms involving the \( \overrightarrow{N} \) vectors are of the order \( \alpha_x \) or \( \alpha_y \) as these tend to infinity and so the inverse transforms of these functions are not rigorously defined.

§3.2 Integral Equation Derivation

Now that the dyadic Green's functions for a dielectric substrate on a ground plane with a source current in the substrate are known, derive an integral equation relation for the surface currents on the micro-strip patch. To that end, define the excitation vector function in terms of the position of the source current as

\[
\overrightarrow{h}(\overline{R}') \equiv - \int_0^c \overrightarrow{z} \cdot \overrightarrow{G}_e^{(11)}(\overline{R} | \overline{R'}) dz
\]

(3.14)
where $R_o = x_o \hat{z} + y_o \hat{y} + z \hat{z}$. Then the electric field in the substrate can be expressed as

$$\overline{E}(\overline{r}') = i\omega \mu \iint_S \overline{K}(\overline{r}) \cdot \overline{G}_{e}^{(11)}(\overline{r} | \overline{r}') dS - i\omega \mu I_o \overline{h}(\overline{r}')$$

(3.15)

where $\overline{K}(\overline{r})$ is the surface current on the patch and the surface integrals are performed over the patch. Furthermore if $\overline{r}'$ is restricted to a point on the patch, which is represented by $\overline{r}'$, the tangential electric field must be zero. Therefore

$$\overline{E}(\overline{r}') \cdot \hat{z} = i\omega \mu \iint_S \overline{K}(\overline{r}) \cdot \overline{G}_{e}^{(11)}(\overline{r} | \overline{r}') \cdot \hat{z} dS - i\omega \mu I_o \overline{h}(\overline{r}') \cdot \hat{z} = 0$$

$$\overline{E}(\overline{r}') \cdot \hat{y} = i\omega \mu \iint_S \overline{K}(\overline{r}) \cdot \overline{G}_{e}^{(11)}(\overline{r} | \overline{r}') \cdot \hat{y} dS - i\omega \mu I_o \overline{h}(\overline{r}') \cdot \hat{y} = 0$$

(3.16)

which leads to

$$\iint_S [K_x(\overline{r})G_{zx}(\overline{r} | \overline{r}') + K_y(\overline{r})G_{zy}(\overline{r} | \overline{r}')] dS = I_o h_x(\overline{r}')$$

$$\iint_S [K_x(\overline{r})G_{zy}(\overline{r} | \overline{r}') + K_y(\overline{r})G_{vy}(\overline{r} | \overline{r}')] dS = I_o h_y(\overline{r}')$$

(3.17)

where

$$G_{zx}(\overline{r} | \overline{r}') \equiv \hat{z} \cdot \overline{G}_{e}^{(11)}(\overline{r} | \overline{r}') \cdot \hat{z}$$

$$G_{zy}(\overline{r} | \overline{r}') \equiv \hat{z} \cdot \overline{G}_{e}^{(11)}(\overline{r} | \overline{r}') \cdot \hat{y}$$

$$G_{yx}(\overline{r} | \overline{r}') \equiv \hat{y} \cdot \overline{G}_{e}^{(11)}(\overline{r} | \overline{r}') \cdot \hat{z}$$

$$G_{yy}(\overline{r} | \overline{r}') \equiv \hat{y} \cdot \overline{G}_{e}^{(11)}(\overline{r} | \overline{r}') \cdot \hat{y}$$

$$K_x(\overline{r}) \equiv \overline{K}(\overline{r}) \cdot \hat{z}$$

$$K_y(\overline{r}) \equiv \overline{K}(\overline{r}) \cdot \hat{y}$$

$$h_x(\overline{r}') \equiv \overline{h}(\overline{r}') \cdot \hat{z}$$

$$h_y(\overline{r}') \equiv \overline{h}(\overline{r}') \cdot \hat{y}$$

(3.18)

Now simplifying these terms

$$G_{zx}(\overline{r} | \overline{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \alpha_x^2 \Gamma_m + \alpha_y^2 \Gamma_e \right] \Omega(\overline{r} - \overline{r}') \frac{d\alpha_x d\alpha_y}{h_1(\alpha_x^2 + \alpha_y^2)}$$

$$G_{zy}(\overline{r} | \overline{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \alpha_x \alpha_y \Gamma_m - \alpha_x \alpha_y \Gamma_e \right] \Omega(\overline{r} - \overline{r}') \frac{d\alpha_x d\alpha_y}{h_1(\alpha_x^2 + \alpha_y^2)}$$

$$G_{yx}(\overline{r} | \overline{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \alpha_x \alpha_y \Gamma_m - \alpha_x \alpha_y \Gamma_e \right] \Omega(\overline{r} - \overline{r}') \frac{d\alpha_x d\alpha_y}{h_1(\alpha_x^2 + \alpha_y^2)}$$

$$G_{yy}(\overline{r} | \overline{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \alpha_y^2 \Gamma_m + \alpha_x^2 \Gamma_e \right] \Omega(\overline{r} - \overline{r}') \frac{d\alpha_x d\alpha_y}{h_1(\alpha_x^2 + \alpha_y^2)}$$

(3.19)
\[ h_x(\vec{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} i\alpha_x \Gamma_m \Omega(\vec{r}_o - \vec{r}') \frac{d\alpha_x d\alpha_y}{\hbar_1^3} \]
\[ h_y(\vec{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} i\alpha_y \Gamma_m \Omega(\vec{r}_o - \vec{r}') \frac{d\alpha_x d\alpha_y}{\hbar_1^3} \] (3.20)

where
\[ \Gamma_x \equiv \frac{h_1 \sin(h_1 c)}{h_1 \cos(h_1 c) - i\hbar_0 \sin(h_1 c)} \]
\[ \Omega(\vec{r}) \equiv e^{-i\alpha_x} e^{-i\alpha_y} \]
\[ \Gamma_m \equiv \frac{1}{k^2 \epsilon_x \epsilon_0 \cos(h_1 c) - i\hbar_1 \sin(h_1 c)} \]
(3.21)

These expressions can be further reduced by converting the two dimensional inverse Fourier transform to the inverse Fourier-Bessel transform with the substitutions
\[ \alpha_x = \lambda \cos \nu \quad x - x' = \rho \cos \phi \quad x_o - x' = \rho_o \cos \phi_o \]
\[ \alpha_y = \lambda \sin \nu \quad y - y' = \rho \sin \phi \quad y_o - y' = \rho_o \sin \phi_o \] (3.22)

So that
\[ h_0 = \sqrt{k^2 - \lambda^2} \quad h_1 = \sqrt{k_1^2 - \lambda^2} \] (3.23)

The integrations in \( \alpha_x \) and \( \alpha_y \) transform into integrations in \( \lambda \) and \( \nu \). With the aid of the identity
\[ \int_0^{2\pi} e^{i(n\theta - z \sin \theta)} d\theta = 2\pi J_n(z) \] (3.24)
where \( J_n(z) \) is the \( n \)th order Bessel function of the first kind, perform the integrations in \( \nu \). Since these inversion integrals are only formally defined, the \( \lambda^2 \) factors associated with \( \Gamma_m \) should be factored out of the integrals and replaced by the appropriate differentiation operations. Therefore

\[ G_{xx}(\vec{r} | \vec{r}') = \frac{1}{2\pi} \frac{\partial^2}{\partial x^2} \int_0^{\infty} J_0(\lambda \rho) \Gamma_m \frac{d\lambda}{h_1 \lambda} \]
\[ + \frac{1}{4\pi} \int_0^{\infty} \lambda^2 \left[ J_0(\lambda \rho) + J_2(\lambda \rho) \cos(2\phi) \right] \Gamma_x \frac{d\lambda}{h_1 \lambda} \]
\[ G_{xy}(\vec{r} | \vec{r}') = \frac{1}{2\pi} \frac{\partial^2}{\partial x \partial y} \int_0^{\infty} J_0(\lambda \rho) \Gamma_m \frac{d\lambda}{h_1 \lambda} \]
\[ + \frac{1}{4\pi} \int_0^{\infty} \lambda^2 J_2(\lambda \rho) \sin(2\phi) \Gamma_x \frac{d\lambda}{h_1 \lambda} \]
\[ G_{yx}(\vec{r} | \vec{r}') = \frac{1}{2\pi} \frac{\partial^2}{\partial y \partial x} \int_0^{\infty} J_0(\lambda \rho) \Gamma_m \frac{d\lambda}{h_1 \lambda} \]
\[ + \frac{1}{4\pi} \int_0^{\infty} \lambda^2 J_2(\lambda \rho) \sin(2\phi) \Gamma_x \frac{d\lambda}{h_1 \lambda} \]
\[
G_{uv}(\vec{r} \mid \vec{r}') = \frac{1}{2\pi} \int_0^{\infty} J_0(\lambda\rho) \Gamma_m \frac{d\lambda}{h_1 \lambda} \frac{\partial^2}{\partial x' \partial y'} \int_0^{\infty} J_0(\lambda\rho) \Gamma_m \frac{d\lambda}{h_1 \lambda} + \frac{1}{4\pi} \int_0^{\infty} \lambda^2 \left[ J_0(\lambda\rho) - J_2(\lambda\rho) \cos(2\phi) \right] \Gamma_m \frac{d\lambda}{h_1 \lambda} \tag{3.25}
\]

\[
h_x(\vec{r}') = \frac{1}{2\pi} \int_0^{\infty} J_0(\lambda\rho_0) \Gamma_m \frac{d\lambda}{h_1^2} \frac{\partial}{\partial x'} \int_0^{\infty} J_0(\lambda\rho_0) \Gamma_m \frac{d\lambda}{h_1^2} \tag{3.26}
\]

The $z$ and $y$ components of the surface current on the patch can now be solved for numerically with a moment method technique using the coupled integral equations. Unfortunately this is not a numerically attractive problem since a Hankel transform integral must be performed to evaluate the Green's functions for each value of $\vec{r} - \vec{r}'$. In addition the derivatives associated with the Green's functions and the excitation functions must be computed numerically which is an ill-behaved operation.

§3.3 Decoupling of Free Space Case

The coupled integral equations for the $z$ and $y$ components of surface current on the patch can be decoupled for the special case when the dielectric constant of the substrate ($\varepsilon_r$) is one. Decoupling these integral equations is numerically attractive as well as theoretically interesting. If these equations are to be solved by the moment method with $N$ basis functions for $K_x$ and $N$ basis functions for $K_y$, decoupling these equations reduces a system of $2N$ equations into two systems of $N$ equations. Further, the two matrices for the system of $N$ equations will be shown to be equivalent and therefore the matrix decomposition will only need to be performed once. Since a matrix decomposition requires on the order of $N^3$ operations, the computation time required will roughly be 1/8th if the problem is decoupled.

To decouple the integral equations, use the fact that $\hat{z} \cdot \hat{H}(\vec{r}')$ is zero for $\vec{r}'$ on the patch. The magnetic field can be found from the curl of the electric field, therefore

\[
\hat{z} \cdot \hat{H}(\vec{r}') = \iint_S \hat{z} \cdot \nabla' \times \overline{G}_e^{(11)}(\vec{r}' \mid \vec{r}) \cdot \overline{K}(\vec{r}) dS = \iint_S \left[ K_x(\vec{r}) G_{xx}(\vec{r} \mid \vec{r}') + K_y(\vec{r}) G_{xy}(\vec{r} \mid \vec{r}') \right] dS = 0 \tag{3.27}
\]
where
\[ G_{zz}(\bar{\tau} | \bar{\tau}') = -\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{i\alpha_y}{k_1^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} \]
\[ G_{zy}(\bar{\tau} | \bar{\tau}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{i\alpha_y}{k_1^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} \]  
(3.28)

and so
\[ \iint_S K_e(\bar{\tau}) G_{zz}(\bar{\tau} | \bar{\tau}') dS = -\iint_S K_e(\bar{\tau}) G_{zy}(\bar{\tau} | \bar{\tau}') dS \]  
(3.29)

Differentiating this expression with respect to \( \tau' \) and \( \tau'' \) yields the relations
\[ \iint_S K_e(\bar{\tau}) \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{\alpha_x \alpha_y}{k_1^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} dS \]
\[ = \iint_S K_e(\bar{\tau}) \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{\alpha_x \alpha_y}{k_1^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} dS \]
\[ \iint_S K_e(\bar{\tau}) \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{\alpha_x \alpha_y}{k_1^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} dS \]
\[ = \iint_S K_e(\bar{\tau}) \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{\alpha_x \alpha_y}{k_1^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} dS \]  
(3.30)

Now for the special case where \( \epsilon_r = 1 \)
\[ k_1 = k \quad R_e = R_m = 0 \]
\[ h_1 = h_0 \quad T_e = T_m = 1 \]
\[ k^2 \Gamma_m = h_0^2 \Gamma_e = h_0^2 e^{h_0^2 \sin(h_0 c)} \]  
(3.31)

Furthermore for this case
\[ G_{zz}(\bar{\tau} | \bar{\tau}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} k^2 - \frac{\alpha_x^2}{k^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} \]
\[ G_{zy}(\bar{\tau} | \bar{\tau}') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} k^2 - \frac{\alpha_x \alpha_y}{k^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} \]
\[ G_{zy}(\bar{\tau} | \bar{\tau}') = G_{zy}(\bar{\tau} | \bar{\tau}') = -\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{\alpha_x \alpha_y}{k^2} \Gamma_e \Omega(\bar{\tau} - \bar{\tau}') \frac{d\alpha_x d\alpha_y}{h_1} \]  
(3.32)
Using the relations in eq. (3.30) reformulate the integral contributions in eq. (3.17) involving $G_{xz}(\vec{r} | \vec{r}')$ and $G_{zy}(\vec{r} | \vec{r}')$ which couple $K_x(\vec{r})$ and $K_y(\vec{r})$ together.

\[
\int_{S} \int_{S} K_x(\vec{r}) G_{yz}(\vec{r} | \vec{r}') dS = -\frac{1}{4\pi^2} \int_{S} \int_{S} K_y(\vec{r}) \int_{-\infty}^{+\infty} \frac{\alpha_x^2}{k^2} \Gamma_e \Omega(\vec{r} - \vec{r}') \frac{d\alpha_x d\alpha_y}{h_1} dS
\]

\[
\int_{S} \int_{S} K_y(\vec{r}) G_{yz}(\vec{r} | \vec{r}') dS = -\frac{1}{4\pi^2} \int_{S} \int_{S} K_x(\vec{r}) \int_{-\infty}^{+\infty} \frac{\alpha_x^2}{k^2} \Gamma_e \Omega(\vec{r} - \vec{r}') \frac{d\alpha_x d\alpha_y}{h_1} dS \tag{3.33}
\]

Therefore integral equation (3.17) decouples to

\[
\frac{1}{4\pi^2} \int_{S} \int_{S} K_x(\vec{r}) \int_{-\infty}^{+\infty} \frac{h_1}{k^2} \Gamma_e \Omega(\vec{r} - \vec{r}') d\alpha_x d\alpha_y dS = I_o h_x(\vec{r}')
\]

\[
\frac{1}{4\pi^2} \int_{S} \int_{S} K_y(\vec{r}) \int_{-\infty}^{+\infty} \frac{h_1}{k^2} \Gamma_e \Omega(\vec{r} - \vec{r}') d\alpha_x d\alpha_y dS = I_o h_y(\vec{r}') \tag{3.34}
\]

leaving one integral equation involving only $K_x(\vec{r})$ and another involving only $K_y(\vec{r})$ so that these currents may be solved for independently from one another. Notice also that the integral equations for $K_x(\vec{r})$ and $K_y(\vec{r})$ have the same kernel function. Therefore if the moment method were used to solve this equation the same matrix would be generated for both the $K_x(\vec{r})$ and $K_y(\vec{r})$ problems.

Unfortunately the general problem where $\epsilon_r \neq 1$ cannot be decoupled since only for the free space case can $\Gamma_m$ and $\Gamma_e$ be related to one another through eq. (3.31). In general with a dielectric constant other than one, $\Gamma_m$ and $\Gamma_e$ will become singular for different values of $\alpha$ corresponding to the propagation constants of the TM and TE surface modes. Therefore it is impossible to decouple these equations since the relation eq. (3.30) used for the free space case only involves $\Gamma_e$.

§3.4 Two Dimensional Special Case

Another special microstrip patch antenna problem of interest is the two dimensional case. In this problem all the field quantities and the geometry of the antenna are independent of $y$. The line source excitation current is replaced by a sheet current at $z = z_o$. In addition no current flows on the patch in the $y$ direction. Since all field quantities are independent of $y$ in the spatial domain, the spectral domain functions will behave as the
impulse function \( \delta(\alpha_y) \). Therefore the integral equation for the two dimensional case can be obtained from the general case by setting \( \alpha_y \) to zero and replacing two dimensional integrations in \( \alpha_x \) and \( \alpha_y \) with one dimensional integrations in \( \alpha \). In addition, integrations in \( z \) and \( y \) simply become integrations in \( z \). Thus \( K_y(\hat{r}) = 0 \) and,

\[
\int_{-L/2}^{+L/2} K_y(x) G(x \mid \hat{z}) dx = I_o h(\hat{z})
\]

(3.35)

where

\[
G(x \mid x') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \Gamma_m e^{-i\alpha(z-x')} \frac{d\alpha}{h_1}
\]

\[
h(x') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} i\alpha \Gamma_m e^{-i\alpha(z-x')} \frac{d\alpha}{h_1}
\]

and

\[
h_0 = \sqrt{k^2 - \alpha^2} \quad h_1 = \sqrt{k_1^2 - \alpha^2}
\]

(3.36)

(3.37)

The \( \hat{z} \) directed current on the patch is \( K_y(x) \) and the patch extends from \(-L/2\) to \(+L/2\) in \( z \).

Notice that upon making the substitutions

\[
i\gamma = h_0 \quad i\gamma_1 = h_1
\]

(3.38)

and from the definition of \( \Gamma_m \) in eq. (3.21) the spectral domain kernel function of this integral equation is given by

\[
\Gamma_m/h_1 = -\frac{1}{k^2} \gamma \gamma_1 G(\alpha)
\]

(3.39)

a function encountered in the spectral domain Wiener-Hopf problem in Chapter II. Indeed this two dimensional antenna problem can also be attacked from a spectral domain approach. Chapter IV of this thesis discusses this technique in great detail.
CHAPTER IV

SPECTRAL DOMAIN FORMULATION OF 2-D PROBLEM

§4.1 Solution of Total Strip Current

Consider the two dimensional antenna structure shown in figure 4.1. This structure consists of a ground plane covered with a dielectric slab, and on top of the slab a thin metallic strip of width $L$ and infinite length. Further associate a rectangular co-ordinate system with this geometry so that the ground plane is the $x$-$y$ plane and the strip is situated above and parallel to the $y$ axis. An infinite sheet current flows in $x$ direction from the ground plane to the metallic strip at a position $x = x_o$, where $-L/2 < x_o < +L/2$, exciting the structure. The problem is to find the total current on the strip given the magnitude of
the exciting sheet current \((I_0)\), its position \((x_0)\), the dielectric constant of the slab \((\varepsilon_r)\), the width of the strip \((L)\), and the thickness of the slab \((c)\). Once the total current on the strip is found, other parameters such as the current on the top surface of the strip, the current on the bottom surface of the strip, the equivalent radiating aperture admittance per unit length of the antenna, and the transverse far field pattern can also be computed.

Due to the two dimensional nature of the radiating structure and excitation current, all field quantities will be independent of \(y\). In addition only the \(y\) component of the magnetic field will exist, and hence the current on the patch will flow only in the \(\hat{z}\) direction. Also, since the electric field is proportional to the curl of the magnetic field in current free regions, the electric field will have no \(y\) component. These properties considerably simplify this two dimensional problem compared to the general three dimensional patch antenna problem.

To solve this problem in the spectral domain we will begin in an analogous fashion to the Jones’ method Wiener-Hopf solution as in Chapter II of this thesis. To that end let the magnetic field,

\[
\vec{H} = H(x, z)\hat{y} = \psi'(x, z)\hat{y} + \psi(x, z)\hat{y}
\]

where \(\psi'(x, z)\) is the incident field and \(\psi(x, z)\) is the scattered field. In addition use image theory about the ground plane to define

\[
\vec{H}(x, -z) = \vec{H}(x, z)
\]

The incident field will be taken to be what would exist if the width of the strip were infinite, thereby making a parallel plate waveguide. So,

\[
\psi'(x, z) = \begin{cases} 
+\frac{1}{2}I_0e^{ik_1(z-x_0)} & ; x > x_0 \quad \text{and} \quad -c < z < c \\
-\frac{1}{2}I_0e^{-ik_1(z-x_0)} & ; x < x_0 \quad \text{and} \quad -c < z < c \\
0 & ; x > c \quad \text{or} \quad x < -c
\end{cases}
\]

where \(k_1 = \sqrt{\varepsilon_r}k\) and \(k = \omega\sqrt{\mu_0\varepsilon}\). Since the scattered magnetic field must satisfy Maxwell’s equations,

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + k_1^2 \right) \psi(x, z) = 0 \quad ; -c < z < c
\]

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + k^2 \right) \psi(x, z) = 0 \quad ; x > c \quad \text{or} \quad x < -c
\]
Taking the Fourier transform of these equations yields

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma_1^2 \right) \Psi(\alpha, z) = 0 \quad ; -c < z < c \\
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \Psi(\alpha, z) = 0 \quad ; z > c \quad \text{or} \quad z < -c
\]

(4.4)

where \( \Psi(\alpha, z) = \mathcal{F}\{\psi(x, z)\} \), \( \gamma_1 = \sqrt{\alpha^2 - k_1^2} \), and \( \gamma = \sqrt{\alpha^2 - k^2} \).

Now the solution for \( \Psi(\alpha, z) \) which satisfies the radiation condition and the ground plane symmetry is of the form

\[
\Psi(\alpha, z) = \begin{cases} 
A(\alpha)e^{-\gamma z} & ; z > c \\
C(\alpha) \cosh(\gamma_1 z) & ; -c < z < c \\
A(\alpha)e^{\gamma z} & ; z < -c
\end{cases}
\]

(4.5)

where \( A(\alpha) \) and \( C(\alpha) \) are unknown functions of \( \alpha \). Now introduce

\[
\Psi_-(\alpha, z) = \int_{-\infty}^{-L/2} \psi(x, z)e^{i\alpha z} \, dz \\
\Psi_0(\alpha, z) = \int_{-L/2}^{+L/2} \psi(x, z)e^{i\alpha z} \, dz \\
\Psi_+(\alpha, z) = \int_{+L/2}^{+\infty} \psi(x, z)e^{i\alpha z} \, dz
\]

(4.6)

so that

\[
\Psi(\alpha, z) = \Psi_-(\alpha, z) + \Psi_0(\alpha, z) + \Psi_+(\alpha, z)
\]

(4.7)

Evaluating these expressions when \( z = c \) yields

\[
\Psi(\alpha, c+) = A(\alpha)e^{-\gamma c} \\
\Psi(\alpha, c-) = C(\alpha) \cosh(\gamma_1 c) \\
\Psi'(\alpha, c+) = -\gamma A(\alpha)e^{-\gamma c} \\
\Psi'(\alpha, c-) = \gamma_1 C(\alpha) \sinh(\gamma_1 c)
\]

(4.8)

where \( \Psi'(\alpha, z) = \frac{\partial}{\partial z} \Psi(\alpha, z) \)

Since the electric field is proportional to the curl of the magnetic field, it follows that,

\[
E_z(x, z) = \frac{1}{i\omega} \frac{\partial}{\partial z} H_y(x, z) = \frac{1}{i\omega} \frac{\partial}{\partial x} \psi(x, z)
\]

(4.9)
To match the tangential electric field boundary condition on the top surface of the dielectric and on the surface of the strip the following relations must hold:

\[ \Psi'_0(\alpha, c+) = \Psi'_0(\alpha, c-) = 0 \]  
(4.10)

\[ \Psi'_+(\alpha, c+) = \frac{1}{\epsilon_r} \Psi'_+(\alpha, c-) \]
\[ \Psi'_-(\alpha, c+) = \frac{1}{\epsilon_r} \Psi'_-(\alpha, c-) \]  
(4.11)

and since

\[ \Psi'(\alpha, z) = \Psi'_-(\alpha, z) + \Psi'_0(\alpha, z) + \Psi'_+(\alpha, z) \]  
(4.12)

it follows that,

\[ \epsilon_r \Psi'(\alpha, c+) = \Psi'(\alpha, c-) \]
\[ -\epsilon_r \gamma A(\alpha) e^{-\gamma c} = \gamma_1 C(\alpha) \sinh(\gamma_1 c) \]  
(4.13)

Therefore \( \Psi(\alpha, c+) \) can be determined in terms of \( C(\alpha) \),

\[ \Psi(\alpha, c+) = -\frac{\gamma_1}{\epsilon_r \gamma} C(\alpha) \sinh(\gamma_1 c) \]  
(4.14)

To match the tangential magnetic field boundary condition on the surface of the dielectric the following must be true:

\[ \Psi_+(\alpha, c-) + \frac{1}{2} I_o \int_{+L/2}^{+\infty} e^{i k_1(z-s_0)} e^{i\alpha z} dz = \Psi_+(\alpha, c+) \]
\[ \Psi_-(\alpha, c-) - \frac{1}{2} I_o \int_{-L/2}^{-\infty} e^{-i k_1(z-s_0)} e^{i\alpha z} dz = \Psi_-(\alpha, c+) \]  
(4.15)

These integrals can be computed as

\[ \int_{+L/2}^{+\infty} e^{i k_1(z-s_0)} e^{i\alpha z} dz = \frac{i}{\alpha + k_1} e^{i k_1(L/2-s_0)} e^{i\alpha L/2} \]
\[ \int_{-L/2}^{-\infty} e^{-i k_1(z-s_0)} e^{i\alpha z} dz = \frac{-i}{\alpha - k_1} e^{i k_1(L/2+s_0)} e^{-i\alpha L/2} \]  
(4.16)

and introducing the scattered component of current

\[ \vec{K}(\alpha) \equiv \Psi_0(\alpha, c-) - \Psi_0(\alpha, c+) \]  
(4.17)
it follows that

$$\Psi(\alpha, c+) - \Psi(\alpha, c-) = \frac{I_0}{2} e^{ik_1 L/2} \left[ \frac{e^{i(\alpha L/2 - k_1 z_0)}}{\alpha + k_1} + \frac{e^{i(k_1 z_0 - aL/2)}}{\alpha - k_1} \right] - \widetilde{K}(\alpha)$$  \hspace{1cm} (4.18)

Using eq. (4.8) and eq. (4.14) yields

$$\Psi(\alpha, c+) - \Psi(\alpha, c-) = -C(\alpha) \left[ \frac{\gamma_1}{\epsilon_1} \sinh(\gamma_1 c) + \cosh(\gamma_1 c) \right]$$

$$= -\Psi'(\alpha, c+) \frac{\gamma_1 \sinh(\gamma_1 c) + \epsilon_1 \gamma \cosh(\gamma_1 c)}{\gamma \gamma_1 \sinh(\gamma_1 c)}$$

$$= -\Psi'(\alpha, c+) \frac{\gamma \gamma_1 G(\alpha)}{\gamma \gamma_1 \sinh(\gamma_1 c)}$$  \hspace{1cm} (4.19)

where, as in the Wiener-Hopf problem of Chapter II,

$$G(\alpha) \equiv \frac{\sinh(\gamma_1 c)}{\gamma \gamma_1 \sinh(\gamma_1 c) + \epsilon_1 \gamma \cosh(\gamma_1 c)}$$  \hspace{1cm} (4.20)

Therefore

$$\frac{I_0}{2} e^{ik_1 L/2} \left[ \frac{e^{i(\alpha L/2 - k_1 z_0)}}{\alpha + k_1} + \frac{e^{i(k_1 z_0 - aL/2)}}{\alpha - k_1} \right] - \widetilde{K}(\alpha) = -\Psi'(\alpha, c+) \frac{\gamma \gamma_1 G(\alpha)}{\gamma \gamma_1 \sinh(\gamma_1 c)}$$  \hspace{1cm} (4.21)

Defining

$$H(\alpha) \equiv \frac{1}{2} e^{ik_1 L/2} \left[ \frac{e^{i(\alpha L/2 - k_1 z_0)}}{\alpha + k_1} + \frac{e^{i(k_1 z_0 - aL/2)}}{\alpha - k_1} \right]$$  \hspace{1cm} (4.22)

leads to

$$I_0 H(\alpha) - \widetilde{K}(\alpha) = -\Psi'(\alpha, c+) \frac{\gamma \gamma_1 G(\alpha)}{\gamma \gamma_1 \sinh(\gamma_1 c)}$$  \hspace{1cm} (4.23)

So that the total current on the strip is given by

$$\widetilde{K}'(\alpha) = \widetilde{K}(\alpha) + \frac{I_0}{2} \left[ \int_{z_0}^{L/2} e^{ik_1(z-z_0)} e^{ias} dz - \int_{-L/2}^{z_0} e^{-ik_1(z-z_0)} e^{ias} dz \right]$$  \hspace{1cm} (4.24)

But

$$\int_{z_0}^{L/2} e^{ik_1(z-z_0)} e^{ias} dz = \frac{i}{\alpha + k_1} \left[ e^{ias_0} - e^{ik_1(L/2-z_0)} e^{iasL/2} \right]$$

$$\int_{-L/2}^{z_0} e^{-ik_1(z-z_0)} e^{ias} dz = \frac{i}{\alpha - k_1} \left[ e^{ik_1(L/2+z_0)} e^{-iasL/2} - e^{ias_0} \right]$$  \hspace{1cm} (4.25)

Therefore

$$\widetilde{K}'(\alpha) = \widetilde{K}(\alpha) - \frac{I_0}{2} e^{ik_1 L/2} \left[ \frac{e^{i(\alpha L/2 - k_1 z_0)}}{\alpha + k_1} + \frac{e^{i(k_1 z_0 - aL/2)}}{\alpha - k_1} \right] +$$

$$\frac{I_0}{2} e^{ias_0} \left[ \frac{1}{\alpha + k_1} + \frac{1}{\alpha - k_1} \right]$$

$$= \widetilde{K}(\alpha) - I_0 H(\alpha) + \frac{I_0}{2} e^{ias_0} \left[ \frac{1}{\alpha + k_1} + \frac{1}{\alpha - k_1} \right]$$  \hspace{1cm} (4.26)
Furthermore defining

\[
I(\alpha) \equiv \frac{1}{2} e^{i\alpha_0} \left[ \frac{1}{\alpha + k_1} + \frac{1}{\alpha - k_1} \right] = e^{i\alpha_0} \frac{i\alpha}{\gamma_1^2}
\]  \hspace{1cm} (4.27)

implies

\[
\tilde{K}^i(\alpha) = \tilde{K}(\alpha) - I_0 H(\alpha) + I_0 I(\alpha)
\]  \hspace{1cm} (4.28)

\[
\tilde{K}(\alpha) = \tilde{K}^i(\alpha) + I_0 H(\alpha) - I_0 I(\alpha)
\]  \hspace{1cm} (4.29)

Using this expression for \(\tilde{K}(\alpha)\) in equation (4.23) yields,

\[
-\tilde{K}^i(\alpha) + I_0 I(\alpha) = -\frac{\Psi'(\alpha, c^+)}{\gamma_1 G(\alpha)}
\]  \hspace{1cm} (4.30)

\[
-\gamma_1 G(\alpha) \tilde{K}^i(\alpha) + I_0 \gamma_1 G(\alpha) I(\alpha) = -\Psi(\alpha, c^+)
\]  \hspace{1cm} (4.31)

Taking the inverse Fourier transform of the above equation leads to

\[
-\mathcal{F}^{-1} \left\{ \gamma_1 G(\alpha) \tilde{K}^i(\alpha) \right\} + I_0 \mathcal{F}^{-1} \left\{ \gamma_1 G(\alpha) I(\alpha) \right\} = -\frac{\partial}{\partial z} \psi(x, c^+)
\]  \hspace{1cm} (4.32)

Now if \(-L/2 < x < +L/2\) then \(\frac{\partial}{\partial z} \psi(x, c^+) = 0\) since the tangential electric field on the strip is zero. Therefore,

\[
\mathcal{F}^{-1} \left\{ \gamma_1 G(\alpha) \tilde{K}^i(\alpha) \right\} = I_0 \mathcal{F}^{-1} \left\{ \gamma_1 G(\alpha) I(\alpha) \right\} \quad ; -L/2 < x < +L/2
\]

\[
\mathcal{F}^{-1} \left\{ \tilde{K}^i(\alpha) \right\} = 0 \quad ; z > +L/2 \text{ or } z < -L/2
\]  \hspace{1cm} (4.33)

The above equation is solved numerically for the unknown total current in the spatial domain using the conjugate gradient–FFT technique. This technique and the computer program which implements it are discussed in Chapter VI.

§4.2 Solution of Associated Fields and Currents

Once the total current on the strip is known several other quantities of interest can also be found. This section will present the solution of many of these quantities in terms of the spectral representation of the total current.
Top Surface Current

One quantity that is of interest to examine is the surface current density which exits solely on the top side of the strip. This can be computed using the boundary condition relation that \( \hat{n} \times \vec{H} = \vec{K} \) on the surface of a perfect conductor. So on the strip, \(-L/2 < z < +L/2\), the total top surface current is

\[
K^t(x, c+) = -H(x, c+) = -\mathcal{F}^{-1}\{\Psi(\alpha, c+)\}
\]  
(4.34)

Now using the relation

\[
-\Psi'(\alpha, c+) = \gamma A(\alpha) e^{-\gamma z} = \gamma_1 G(\alpha) \left[ I_o I(\alpha) - \mathcal{H}^t(\alpha) \right]
\]  
(4.35)

compute

\[
\Psi(\alpha, c+) = A(\alpha) e^{-\gamma z} = \gamma_1 G(\alpha) \left[ I_o I(\alpha) - \mathcal{H}^t(\alpha) \right]
\]  
(4.36)

The top surface current can now be found with the inverse Fourier transform by

\[
K^t(x, c+) = -\mathcal{F}^{-1}\left\{ \gamma_1 G(\alpha) \left[ I_o I(\alpha) - \mathcal{H}^t(\alpha) \right] \right\}; -L/2 < z < +L/2
\]  
(4.37)

Of course off the strip, where \( z < -L/2 \) or \( z > +L/2 \), the top surface current is zero.

To numerically evaluate this Fourier inversion integral with the Fast Fourier Transform, the integrand must be a well behaved function for real values of \( \alpha \). The transform of the total current \( \mathcal{H}^t(\alpha) \) is the Fourier integral of that current over the finite width of the strip. Therefore, since the total current is finite, \( \mathcal{H}^t(\alpha) \) can not become infinite for any \( \alpha \). On the other hand, notice that \( I_o I(\alpha) \) becomes singular at \( \alpha = \pm k_1 \). However the total expression for the transform of the top surface current has the following limit

\[
\lim_{\alpha \to \pm k_1} -\gamma_1 G(\alpha) \left[ I_o I(\alpha) - \mathcal{H}^t(\alpha) \right] = -I_o e^{i\pi \alpha} \frac{i\alpha}{\epsilon, \gamma}
\]  
(4.38)

which is finite as long as \( \epsilon, \gamma \neq 1 \) so that as \( \alpha \to \pm k_1 \) then \( \gamma \neq 0 \). Therefore, except for the surface mode pole in \( G(\alpha) \), the expression for the top surface current is a well behaved function for real \( \alpha \) even in the limit as \( \text{Re}[k_1] \) tends to zero. Section 7.4 discusses the surface mode pole which exists in \( G(\alpha) \) in detail, while Section 6.6 describes how to numerically handle this Fourier inversion integral with the surface mode pole.
Bottom Surface Current

In a similar fashion the surface current density which exists solely on the bottom side of the strip can be found. Using the same magnetic field boundary condition relation on the strip for $-L/2 < z < +L/2$,

$$K^t(x, c--) = H(x, c--) = \mathcal{F}^{-1}\{\Psi(\alpha, c-}\} + \psi^t(x, c--) \quad (4.39)$$

With the expression

$$\frac{1}{\epsilon_r} \psi^t(\alpha, c--) = \frac{\gamma_1}{\epsilon_r} C(\alpha) \sinh(\gamma_1 c) = -\gamma\gamma_1 G(\alpha) \left[ I_o I(\alpha) - \bar{K}^t(\alpha) \right] \quad (4.40)$$

compute

$$\Psi(\alpha, c--) = C(\alpha) \cosh(\gamma_1 c) = -\epsilon_r \gamma G(\alpha) \left[ I_o I(\alpha) - \bar{K}^t(\alpha) \right] \coth(\gamma_1 c) \quad (4.41)$$

So now the bottom surface current can be computed using the inverse Fourier transform by

$$K^t(x, c--) = \mathcal{F}^{-1}\left\{-\epsilon_r \gamma G(\alpha) \left[ I_o I(\alpha) - \bar{K}^t(\alpha) \right] \coth(\gamma_1 c)\right\} + \psi^t(x, c--) \quad (4.42)$$

Now since

$$\mathcal{F}\{\psi^t(x, c--)\} = I_o I(\alpha) \quad (4.43)$$

the equation for $K^t$ on the strip can be simplified as

$$K^t(x, c--) = \mathcal{F}^{-1}\left\{-\epsilon_r \gamma G(\alpha) \left[ I_o I(\alpha) - \bar{K}^t(\alpha) \right] \coth(\gamma_1 c) + I_o I(\alpha)\right\} \quad (4.44)$$

$$= \mathcal{F}^{-1}\left\{ \left[1 - \epsilon_r \gamma G(\alpha) \coth(\gamma_1 c)\right] I_o I(\alpha) + \epsilon_r \gamma G(\alpha) \coth(\gamma_1 c) \bar{K}^t(\alpha)\right\}$$

$$= \mathcal{F}^{-1}\left\{ \gamma_1 G(\alpha) I_o I(\alpha) + \epsilon_r \gamma G(\alpha) \coth(\gamma_1 c) \bar{K}^t(\alpha)\right\} \quad (4.44)$$

Off the strip, $z < -L/2$ or $z > +L/2$, the bottom surface current must be zero.

Again notice that even though $I_o I(\alpha)$ becomes singular at $\alpha = \pm k_1$, the transform of the bottom current does not, provided $\epsilon_r \neq 1$:

$$\lim_{\alpha \to \pm k_1} \gamma_1 G(\alpha) I_o I(\alpha) + \epsilon_r \gamma G(\alpha) \coth(\gamma_1 c) \bar{K}^t(\alpha) = I_o e^{j\omega t + \frac{\imath \alpha c}{\epsilon_r \gamma}} + \bar{K}^t(\alpha) \quad (4.45)$$

Therefore the bottom surface current can be computed numerically using the FFT, provided the surface mode pole in $G(\alpha)$ is accounted for as in Section 6.6.
Electric Field in the Dielectric Slab

Another quantity of interest is the z component of the electric field in the dielectric slab. This can be computed from Maxwell’s equation for the curl of the magnetic field. So,

$$\mathbf{E} = \frac{1}{i \omega} [\mathbf{J} - \nabla \times \mathbf{H}]$$  \hspace{1cm} (4.46)

In the dielectric slab the current is

$$\mathbf{J} = I_o \delta(x - x_o) \hat{z}$$  \hspace{1cm} (4.47)

and so

$$\mathbf{E} = \frac{1}{i \omega} \left[ I_o \delta(x - x_o) \hat{z} - \frac{\partial}{\partial z} H(z, z) \hat{z} + \frac{\partial}{\partial z} H(z, z) \hat{z} \right]$$  \hspace{1cm} (4.48)

Now

$$H(z, z) = \psi(z, z) + \psi'(z, z)$$

and for $-c \leq z \leq c$

$$\frac{\partial}{\partial z} \psi'(z, z) = I_o \delta(z - x_o) + \begin{cases} \frac{I_o}{2} i k_1 e^{+i k_1 (z - z_o)} ; & x > x_o \\ 0 ; & x < x_o \end{cases} \hspace{1cm} (4.49)$$

Therefore

$$E_z(z, z) = -\frac{1}{i \omega} \left[ \frac{I_o}{2} i k_1 e^{+i k_1 (z - z_o)} - \frac{\partial}{\partial z} \psi(z, z) \right]$$  \hspace{1cm} (4.50)

Fourier transforming this equation leads to

$$\mathcal{F}\{E_z(z, z)\} = \frac{1}{i \omega} \left[ -\frac{I_o}{\gamma_1} e^{i\alpha z} k_1^2 + i \alpha \psi(z, z) \right]$$

$$= \frac{1}{i \omega} \left[ -\frac{I_o}{\gamma_1} e^{i\alpha z} k_1^2 + i \alpha \psi(z, z) \right]$$

$$+ i \alpha e_1 e^{i\alpha z} \frac{\cosh(\gamma_1 z)}{\sinh(\gamma_1 c)} \left[ I_o I(\alpha) - \tilde{K}^2(\alpha) \right]$$  \hspace{1cm} (4.51)

The transform of the $E_z$ electric field component consists of terms with $1/\gamma_1$ behavior. Therefore this spectral domain function must also be examined in the neighborhood of $\alpha = \pm k_1$. To facilitate this, algebraically manipulate the expression for $\mathcal{F}\{E_z\}$.

$$i \omega \mathcal{F}\{E_z(z, z)\} = -\frac{I_o e^{i\alpha z}}{\gamma_1^2} k_1^2 - \frac{i \alpha e_1 \gamma \cosh(\gamma_1 z)}{\gamma_1 \sinh(\gamma_1 c) + \epsilon \gamma \cosh(\gamma_1 c)} \left[ I_o e^{i\alpha z} \frac{i \alpha}{\gamma_1^2} - \tilde{K}^2(\alpha) \right]$$
\[ I_0 \frac{e^{i\alpha z_0}}{\alpha} \left\{ \alpha^2 \frac{\varepsilon_\gamma \cosh(\gamma_1 x)}{\gamma_1 \sinh(\gamma_1 z) + \varepsilon_\gamma \cosh(\gamma_1 c)} - k_1^2 \right\} \]

\[ + \frac{i \alpha \varepsilon_\gamma \cosh(\gamma_1 x)}{\gamma_1 \sinh(\gamma_1 c) + \varepsilon_\gamma \cosh(\gamma_1 c)} \hat{K}^i(\alpha) \]

\[ = I_0 \frac{e^{i\alpha z_0}}{\alpha} \left\{ \gamma_1^2 + \alpha^2 \varepsilon_\gamma \left[ \cosh(\gamma_1 x) - \cosh(\gamma_1 c) \right] - \gamma_1 \sinh(\gamma_1 c) \right\} \]

\[ + \frac{i \alpha \varepsilon_\gamma \cosh(\gamma_1 x)}{\gamma_1 \sinh(\gamma_1 c) + \varepsilon_\gamma \cosh(\gamma_1 c)} \hat{K}^i(\alpha) \]

(4.52)

Now take the limit,

\[ \lim_{\alpha \to \pm k_1} i \omega \varepsilon F \{ E_\alpha (x, z) \} = I_0 e^{i\alpha z_0} \left\{ 1 + k_1^2 \left[ \frac{1}{2} (x^2 - c^2) - \frac{c}{\varepsilon_\gamma} \right] \right\} + i \alpha \hat{K}^i(\alpha) \]

(4.53)

Fortunately, this quantity is finite as long as \( \varepsilon_\gamma \neq 1 \) and so the task of numerically computing the electric field is realizable with \( \varepsilon_\gamma \) not too near one.

**Far Zone Field**

To describe points in space above the dielectric slab, introduce the polar co-ordinate variables \( \rho \) and \( \theta \) so that

\[ x = \rho \sin \theta \quad z = \rho \cos \theta + c \]

(4.54)

where \( -\pi/2 \leq \theta \leq \pi/2 \). Now proceed to find the far zone magnetic field distribution as a function of the polar angle \( \theta \). For \( z \geq c \),

\[ \Psi(\alpha, x) = A(\alpha) e^{-\gamma z} = \gamma_1 G(\alpha) \left[ I_0 I(\alpha) - \hat{K}(\alpha) \right] e^{\gamma \rho} e^{-\gamma z} \]

(4.55)

and

\[ H(x, z) = F^{-1} \{ \Psi(\alpha, x) \} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \gamma_1 G(\alpha) \left[ I_0 I(\alpha) - \hat{K}(\alpha) \right] e^{\gamma \rho} e^{-\gamma z} e^{-i\alpha z} d\alpha \]

(4.56)

For convenience define

\[ F(\alpha) \equiv \gamma_1 G(\alpha) \left[ I_0 I(\alpha) - \hat{K}(\alpha) \right] \]

(4.57)

so

\[ H(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha) e^{-\gamma \rho \cos \theta} e^{-i\alpha \sin \theta} d\alpha \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha) e^{i(\gamma - \rho \sin \theta + i\gamma \cos \theta)} d\alpha \]

(4.58)
Now $\gamma = -i\sqrt{k^2 - \alpha^2}$ and $\Re(\gamma) \geq 0$, so in the far zone limit,

$$\lim_{\rho \to \infty} H(x, z) \approx \frac{1}{2\pi} \int_{-k}^{+k} F(\alpha)e^{i[\alpha \sin \theta + \sqrt{k^2 - \alpha^2} \cos \theta]} d\alpha$$

as long as $\theta \neq \pm \pi/2$. Notice that $F(\alpha)$ is the spectral domain representation of the magnetic field at $z = c$. Since $F(\alpha)$ is also the spectral function used to find the top surface current, from that subsection it follows that $F(\alpha)$ is a finite function for real $\alpha$ provided the dielectric substrate has some loss. Therefore this integral can now be evaluated using stationary phase integration. See Papoulis [31] chapter 7 for an excellent discussion of stationary phase integration. The phase function of the integrand and its derivatives are,

$$\mu(\alpha) = -\alpha \sin \theta + \sqrt{k^2 - \alpha^2} \cos \theta$$

$$\mu'(\alpha) = -\sin \theta - \frac{\alpha}{\sqrt{k^2 - \alpha^2}} \cos \theta$$

$$\mu''(\alpha) = -\frac{k^2}{(k^2 - \alpha^2)^{3/2}} \cos \theta$$

In solving for the stationary point where $\mu'(\alpha) = 0$, two possible values for $\alpha$ are found at $\alpha = \pm k \sin \theta$. However,

$$\mu'(k \sin \theta) = -\sin \theta - \frac{k \sin \theta}{\sqrt{k^2 - k^2 \sin^2 \theta}} \cos \theta = -2 \sin \theta$$

$$\mu'(-k \sin \theta) = -\sin \theta + \frac{k \sin \theta}{\sqrt{k^2 - k^2 \sin^2 \theta}} \cos \theta = 0$$

Therefore one stationary phase point exists at $\alpha = -k \sin \theta$ in the interval of integration. Now

$$\mu(-k \sin \theta) = k \sin^2 \theta + \sqrt{k^2 - k^2 \sin^2 \theta} \cos \theta = k$$

$$\mu''(-k \sin \theta) = -\frac{k^2}{[k^2 - k^2 \sin^2 \theta]^{3/2}} \cos \theta = -\frac{1}{k \cos^2 \theta}$$

Hence the second derivative of the phase function is negative at the stationary point and so

$$\lim_{\rho \to \infty} H(x, z) = \frac{1}{2\pi} e^{-i \pi /4} e^{i \mu(-k \sin \theta)} F(-k \sin \theta) \sqrt{\frac{2\pi}{-\rho \mu''(-k \sin \theta)}}$$

$$= e^{-i \pi /4} e^{ik \rho} F(-k \sin \theta) \cos \theta \sqrt{\frac{k}{2\pi \rho}}$$

for $-\pi/2 < \theta < \pi/2$. Notice that the far zone magnetic field is an outwardly propagating cylindrical wave which diminishes as $1/\sqrt{\rho}$. Furthermore, this expression for the far field is
equivalent to what Fourier optics predicts given the magnetic field distribution across the aperture plane \( z = c \).

For the special case where \( \theta = \pm \pi/2 \) and the dielectric substrate is lossy,

\[
\lim_{z \to \pm \infty} H(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\pm \infty} F(\alpha) e^{-i\alpha z} d\alpha \quad (4.65)
\]

The phase function of the integrand

\[
\mu(\alpha) = \pm \alpha \quad (4.66)
\]

and so no stationary phase points exist. Therefore the far zone magnetic field integral must be evaluated by computing its end point contributions (see Papoulis [31]). This in turn implies that the magnetic field on the surface of the dielectric slab must diminish at least as rapidly as \( 1/z \) in the far zone, provided the substrate is not lossless.

**Aperture Admittance**

For the two dimensional antenna structure, the radiating aperture admittance per unit length can be related to the TEM magnetic field reflection coefficients at \( z = +L/2 \) or \( z = -L/2 \). These reflection coefficients can in turn be computed from the bottom surface current. Beneath the strip, the magnetic field can be expanded in terms of the two dimensional modes which can exist in a parallel plate waveguide. There is a TEM mode which originates at \( z = z_o \) and which propagates in both directions due to the source sheet current.

In addition there is a collection of modes, both TEM and higher order, which reflect back and propagate in the negative \( z \) direction from the edge of the strip at \( z = +L/2 \). There is also a collection of modes which reflect back and propagate in the positive \( z \) direction from the edge of the strip at \( z = -L/2 \). And since in the cavity region under the strip, \( H(x, c-) = K^t(x, c-) \), expand the bottom surface current as

\[
K^t(x, c-) = \psi^t(x, c-) + \sum_{m=0}^{\infty} \left[ a_m e^{+i\beta_m(z+L/2)} + b_m e^{-i\beta_m(z-L/2)} \right] \quad (4.67)
\]

where,

\[
\beta_m = \sqrt{k^2_1 - \left( \frac{mx}{c} \right)^2} \quad (4.68)
\]

and the coefficients \( a_m \) and \( b_m \) are unknown reflection and mode conversion coefficients which must be solved for. Since the bottom surface current can be found numerically,
these coefficients can also be computed with a least squares best fit type of procedure (see Section 6.6). Notice that the propagation constant $\beta_0 = k_1$ corresponds to the TEM mode, and for the thin dielectric slab condition all higher order modes are highly evanescent.

Now, compute the incident and reflected TEM magnetic fields at $x = \pm L/2$ by

$$H^+(+L/2, c-) = \frac{1}{2} I_o e^{i k_1 (L/2 - x_0)} + a_0 e^{i k_1 L}$$

$$H^-(+L/2, c-) = b_0$$  \hspace{1cm} (4.69)

$$H^-(L/2, c-) = -\frac{1}{2} I_o e^{i k_1 (L/2 + x_0)} + b_0 e^{i k_1 L}$$

$$H^+(L/2, c-) = a_0$$  \hspace{1cm} (4.70)

So the TEM magnetic reflection coefficients at $x = \pm L/2$ are

$$\Gamma_+ \equiv \frac{H^-(+L/2, c-)}{H^+(+L/2, c-)} = e^{-i k_1 L/2} \frac{b_0}{I_o/2 e^{-i k_1 x_0} + a_0 e^{i k_1 L/2}}$$

$$\Gamma_- \equiv \frac{H^+(-L/2, c-)}{H^-(L/2, c-)} = e^{-i k_1 L/2} \frac{a_0}{I_o/2 e^{-i k_1 x_0} + b_0 e^{i k_1 L/2}}$$  \hspace{1cm} (4.71)

These reflection coefficients $\Gamma_+$ and $\Gamma_-$ must have a magnitude less than or equal to one to maintain the conservation of energy. In addition for the thin dielectric case, the Wiener-Hopf solution of radiation from a semi-infinite parallel plate waveguide predicts that, these reflection coefficients should be near negative one.

Now find the radiating aperture admittance per unit length from either reflection coefficient by,

$$Y_r = \frac{1}{c} \sqrt{\varepsilon_r \varepsilon_0 / \mu_0} \frac{1 + \Gamma}{1 - \Gamma}$$  \hspace{1cm} (4.72)

The value which is computed for the aperture admittance should be independent of which reflection coefficient $\Gamma_+$ or $\Gamma_-$ is used. The reflection coefficient should also be independent of $(x_0)$ the position of the source current and $(L)$ the width of the strip.
CHAPTER V

SPECTRAL DOMAIN FORMULATION OF 3-D PROBLEM

§5.1 Solution of Total Patch Current

The general three dimensional microstrip patch antenna problem can also be formulated in the spectral domain. Notice from the integral equation relations given by eqs. (3.17) – (3.19) that the spatial variation of the Green's functions are of the form $\mathbf{r} - \mathbf{r}'$. Therefore the spatial integrations in $x$ and $y$ are in the form of two dimensional convolutions. These convolutions can be expressed as products in the spectral domain.

Introducing the two dimensional Fourier transform relations

$$\mathcal{F}_2\{f(x, y)\} = F(\alpha_x, \alpha_y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)e^{i\alpha_x x}e^{i\alpha_y y} \, dx \, dy$$

$$\mathcal{F}_2^{-1}\{F(\alpha_x, \alpha_y)\} = f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\alpha_x, \alpha_y)e^{-i\alpha_x x}e^{-i\alpha_y y} \, d\alpha_x \, d\alpha_y$$

(5.1)

define the transforms of the $x$ and $y$ components of the surface current.

$$\mathcal{K}_x(\alpha_x, \alpha_y) \equiv \mathcal{F}_2\{K_x(\mathbf{r})\} \quad \mathcal{K}_y(\alpha_x, \alpha_y) \equiv \mathcal{F}_2\{K_y(\mathbf{r})\}$$

(5.2)
Transforming the integral equations given by eq. (3.17) yields
\[
\mathcal{F}_2^{-1}\left\{ \frac{\mathcal{K}_x(\alpha_x, \alpha_y)}{\alpha_x^2 + \alpha_y^2} [\alpha_x^2 \Gamma_m + \alpha_y^2 \Gamma_e] \right\} + \mathcal{F}_2^{-1}\left\{ \frac{\mathcal{K}_y(\alpha_x, \alpha_y) \alpha_x \alpha_y}{\Gamma_m - \Gamma_e} \right\} \\
= -i \mathcal{F}_2^{-1}\left\{ e^{i \alpha_x z_0} e^{i \alpha_y y_0} \Gamma_m \frac{i \alpha_x}{\lambda^2} \right\}
\]
; \bar{\tau} \in S

\[
\mathcal{F}_2^{-1}\left\{ \frac{\mathcal{K}_x(\alpha_x, \alpha_y) \alpha_x \alpha_y}{\alpha_x^2 + \alpha_y^2} [\Gamma_m - \Gamma_e] \right\} + \mathcal{F}_2^{-1}\left\{ \frac{\mathcal{K}_y(\alpha_x, \alpha_y) \alpha_x^2 \alpha_y}{\alpha_x^2 + \alpha_y^2} [\alpha_x^2 \Gamma_m + \alpha_y^2 \Gamma_e] \right\} \\
= -i \mathcal{F}_2^{-1}\left\{ e^{i \alpha_x z_0} e^{i \alpha_y y_0} \Gamma_m \frac{i \alpha_y}{\lambda^2} \right\}
\]

\[
\mathcal{F}_2^{-1}\left\{ \mathcal{K}_x(\alpha_x, \alpha_y) \right\} = 0
\]
; \bar{\tau} \not\in S \quad (5.3)

\[
\mathcal{F}_2^{-1}\left\{ \mathcal{K}_y(\alpha_x, \alpha_y) \right\} = 0
\]

These inverse two dimensional Fourier transforms are rigorously defined if \( \mathcal{K}_x \) and \( \mathcal{K}_y \) behave as \( 1/\lambda^2 \) as \( \lambda \) tends to infinity. Recall that in terms of \( \alpha_x \) and \( \alpha_y \)

\[
\lambda = \sqrt{\alpha_x^2 + \alpha_y^2} \quad (5.4)
\]

This is an important difference between the spectral domain formulation and the integral equation approach, which necessitates differentiation to evaluate the inverse transform of the kernel functions. Since no differentiation is required in the spectral domain solution, provided the currents are sufficiently well behaved, this technique is numerically superior to the integral equation solution. The unknowns to be solved for can be made to have the proper asymptotic behavior by subtracting a known discontinuity function from the currents, as is discussed for the two dimensional case in Section 6.5. Once this is done, equation (5.3) can be solved with a conjugate gradient-FFT numerical procedure. However based on experience with the two dimensional problem, a large number of unknowns would be required and so significant computational power, both in memory and speed, necessary.
§5.2 Surface Currents

If the computing resources are available to compute the total current on the patch antenna, the top and bottom surface currents can then also be solved for with the spectral domain approach. Just as in the two dimensional antenna problem, the surface currents can be expressed as inverse Fourier transforms of functions in terms of the total current. Use the boundary condition $\hat{n} \times \vec{H} = \vec{K}$ for a perfect conductor to compute the currents from the tangential magnetic fields at the surface of the patch. The magnetic field can be found from the curl of the electric field, therefore for the magnetic field above the patch

$$\vec{H}(\vec{R}') = \iiint_V \nabla' \times \frac{\vec{G}_e^{(21)}}{\hat{r}'} \cdot \vec{J} (\vec{R}) dV$$

(5.5)

where the volume $V$ extends over the region of the substrate. Now

$$\nabla' \times \frac{\vec{G}_e^{(21)}}{\hat{r}'} (\vec{R}' | \vec{R}) = \frac{ik}{8\pi^2} \iiint_{-\infty}^{+\infty} \left\{ T_m M'(-h_0) \left[ N(+h_1) + N(-h_1) \right] + T_s N'(-h_0) \left[ M(+h_1) - M(-h_1) \right] \right\} \frac{d\alpha_x d\alpha_y}{h_1(\alpha_x^2 + \alpha_y^2)}$$

(5.6)

Furthermore restricting $\vec{R}'$ to $\vec{r}'$, a point on the top of the patch,

$$\vec{H}(\vec{r}') = \iiint_S \nabla' \times \frac{\vec{G}_e^{(21)}}{\hat{r}'} (\vec{r}' | \vec{r}) \cdot \hat{z} K_s(\vec{r}) dS + \iiint_S \nabla' \times \frac{\vec{G}_e^{(21)}}{\hat{r}'} (\vec{r}' | \vec{r}) \cdot \hat{y} K_v(\vec{r}) dS$$

$$+ I_o \int_0^\phi \nabla' \times \frac{\vec{G}_e^{(21)}}{\hat{r}'} (\vec{r}' | \vec{R}_o) \cdot \hat{z} dz$$

(5.7)

where the surface integrals are performed over the surface of the patch. Interchanging the order of the spatial and spectral integrations yields

$$\vec{H}(\vec{r}') = \frac{i}{4\pi^2} I_o \iiint_{-\infty}^{+\infty} T_m M'(-h_0) \sin(h_1 c) e^{-i(\alpha_x z_o - \alpha_y y_o)} \frac{d\alpha_x d\alpha_y}{h_1^2}$$

$$- \frac{k}{4\pi^2} \iiint_{-\infty}^{+\infty} \left\{ \frac{\alpha_x h_1}{k} T_m M'(-h_0) + i \alpha_y T_s N'(-h_0) \right\} \times$$

$$\frac{\sin(h_1 c) K_z(-\alpha_x, -\alpha_y)}{h_1(\alpha_x^2 + \alpha_y^2)} \frac{d\alpha_x d\alpha_y}{h_1^2}$$

$$- \frac{k}{4\pi^2} \iiint_{-\infty}^{+\infty} \left\{ \frac{\alpha_y h_1}{k} T_m M'(-h_0) - i \alpha_x T_s N'(-h_0) \right\} \times$$

$$\frac{\sin(h_1 c) K_v(-\alpha_x, -\alpha_y)}{h_1(\alpha_x^2 + \alpha_y^2)} \frac{d\alpha_x d\alpha_y}{h_1^2}$$

(5.8)
where

\[ \hat{K}_x(\alpha_x, \alpha_y) \equiv \mathcal{F}_2\{K_x(\vec{r})\} \quad \hat{K}_y(\alpha_x, \alpha_y) \equiv \mathcal{F}_2\{K_y(\vec{r})\} \]  \hspace{1cm} (5.9)

Now using the boundary condition

\[ \hat{K}^T(\vec{r}') = K_x^T(\vec{r}') \hat{z} + K_y^T(\vec{r}') \hat{y} = \hat{z} \times \overrightarrow{H}(\vec{r}') \]  \hspace{1cm} (5.10)

where \( \vec{r}' \in S \), find the top surface current. Substituting \(-\alpha_x\) for \(\alpha_x\) and \(-\alpha_y\) for \(\alpha_y\) in the spectral integrations yields the following for the \(z\) and \(y\) components of the top surface current:

\[ K_x^T(\vec{r}') = \mathcal{F}_2^{-1}\left\{ \frac{\alpha_x k k_0}{h_0^2} \Gamma_m e^{i \alpha_x \omega_0 e^{i \alpha_y \omega_0}} \right\} \]

\[ - \mathcal{F}_2^{-1}\left\{ \frac{\alpha_y k k_1}{h_0} \Gamma_m - i \alpha_x^2 \Gamma_0 \right\} \frac{\hat{K}_x(\alpha_x, \alpha_y)}{h_1(\alpha_x^2 + \alpha_y^2)} \]

\[ - \mathcal{F}_2^{-1}\left\{ \frac{\alpha_y k k_1}{h_0} \Gamma_m + i \alpha_x \alpha_y \Gamma_0 \right\} \frac{\hat{K}_y(\alpha_x, \alpha_y)}{h_1(\alpha_x^2 + \alpha_y^2)} \]

\[ K_y^T(\vec{r}') = \mathcal{F}_2^{-1}\left\{ \frac{\alpha_y k k_1}{h_0^2} \Gamma_m e^{i \alpha_x \omega_0 e^{i \alpha_y \omega_0}} \right\} \]

\[ - \mathcal{F}_2^{-1}\left\{ \frac{\alpha_x k k_1}{h_0} \Gamma_m + i \alpha_x \alpha_y \Gamma_0 \right\} \frac{\hat{K}_x(\alpha_x, \alpha_y)}{h_1(\alpha_x^2 + \alpha_y^2)} \]

\[ - \mathcal{F}_2^{-1}\left\{ \frac{\alpha_x k k_1}{h_0} \Gamma_m - i \alpha_x^2 \Gamma_0 \right\} \frac{\hat{K}_y(\alpha_x, \alpha_y)}{h_1(\alpha_x^2 + \alpha_y^2)} \]  \hspace{1cm} (5.11)

Once the top surface current is known the bottom surface current can be computed by

\[ \hat{K}^B(\vec{r}') = \hat{K}(\vec{r}') - \hat{K}^T(\vec{r}') \]  \hspace{1cm} (5.12)

\[ \text{§5.3 Far Zone Fields} \]

Once the total surface current on the patch antenna is known the the far zone electric fields can be computed from eq. (3.13) with two dimensional stationary phase integrations. For an arbitrary observation point in space above the dielectric substrate the electric field due to surface currents on the patch and the excitation current is

\[ \overline{E}(\vec{R}') = i \omega \mu I_o \int_0^\infty \overline{G}_e^{(21)}(\vec{R}' | \overline{R}_e) \cdot \hat{z} \, dz \]

\[ + i \omega \mu \int_S \overline{G}_e^{(21)}(\vec{R}' | \overline{r}) \cdot \hat{z} K_x(\overline{r}) \, dS \]

\[ + i \omega \mu \int_S \overline{G}_e^{(21)}(\vec{R}' | \overline{r}) \cdot \hat{y} K_y(\overline{r}) \, dS \]  \hspace{1cm} (5.13)
Interchanging the order of integration of the spatial and spectral variables yields:

\[ E(R') = \frac{i}{4\pi^2} I_0 \int_{-\infty}^{+\infty} T_m \varpi'(-h_0) \frac{\sin(h_1 c)}{k_1 h^2} e^{-i\alpha z s \omega} e^{-i\omega y o} \, da_x \, da_y \]

\[ + \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \left[ T_m \varpi'(-h_0) \frac{\alpha_x h_1}{k_1} \sin(h_1 c) + T_x \varpi'(-h_0) \alpha_x \cos(h_1 c) \right] \times \]

\[ \frac{\widetilde{K}_s(-\alpha_x, -\alpha_y)}{h_1 (\alpha_x^2 + \alpha_y^2)} \, da_x \, da_y \]

\[ + \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \left[ T_m \varpi'(-h_0) \frac{\alpha_y h_1}{k_1} \sin(h_1 c) - T_x \varpi'(-h_0) \alpha_x \cos(h_1 c) \right] \times \]

\[ \frac{\widetilde{K}_y(-\alpha_x, -\alpha_y)}{h_1 (\alpha_x^2 + \alpha_y^2)} \]  

(5.14)

where as before

\[ \widetilde{K}_s(\alpha_x, \alpha_y) \equiv \mathcal{F}_2 \{ K_s(\varpi) \} \quad \widetilde{K}_y(\alpha_x, \alpha_y) \equiv \mathcal{F}_2 \{ K_y(\varpi) \} \]  

(5.15)

Now

\[ \varpi'(-h_0) = \frac{1}{k} \left[ -\alpha_x h_0 \hat{x} - \alpha_y h_0 \hat{y} + (\alpha_x^2 + \alpha_y^2) \hat{z} \right] \vartheta(\alpha_x, \alpha_y) \]

\[ \varpi'(-h_0) = [i\alpha_x \hat{z} - i\alpha_y \hat{y}] \vartheta(\alpha_x, \alpha_y) \]  

(5.16)

where

\[ \vartheta(\alpha_x, \alpha_y) = e^{i\alpha_x s \omega} e^{i\alpha_y y o} e^{\sqrt{k^2 - \alpha_x^2 - \alpha_y^2}} \]  

(5.17)

Now define the observation point in terms of a shifted spherical co-ordinate system.

\[ x' \equiv R' \sin \theta \cos \phi \quad y' \equiv R' \sin \theta \sin \phi \quad z' \equiv R' \cos \theta + c \]  

(5.18)

where \( R' \geq 0, 0 \leq \theta < \pi/2, \) and \( 0 \leq \phi < 2\pi. \) Then, in terms of this new co-ordinate system,

\[ \vartheta(\alpha_x, \alpha_y) = e^{iR' \mu(\alpha_x, \alpha_y)} e^{i\omega_0 c} \]  

(5.19)

where

\[ \mu(\alpha_x, \alpha_y) \equiv \alpha_x \sin \theta \cos \phi + \alpha_y \sin \theta \sin \phi + \sqrt{k^2 - \alpha_x^2 - \alpha_y^2} \cos \theta \]  

(5.20)

Because of the \( \vartheta(\alpha_x, \alpha_y) \) factor in the integrands, in the far zone limit as \( R' \) tends to infinity the integrals for the electric field can be computed with the two dimensional stationary phase
technique (see Papoulis [31] chapter 7). Therefore examining the partial derivatives of the phase function \( \mu(\alpha_x, \alpha_y) \) reveals:

\[
\begin{align*}
\mu_x &\equiv \frac{\partial \mu}{\partial \alpha_x} = \sin \theta \cos \phi - \frac{\alpha_x}{\sqrt{k^2 - \alpha_x^2 - \alpha_y^2}} \cos \theta \\
\mu_y &\equiv \frac{\partial \mu}{\partial \alpha_y} = \sin \theta \sin \phi - \frac{\alpha_y}{\sqrt{k^2 - \alpha_x^2 - \alpha_y^2}} \cos \theta \\
\mu_{xy} &\equiv \frac{\partial^2 \mu}{\partial \alpha_x \partial \alpha_y} = -\frac{\alpha_x \alpha_y}{(k^2 - \alpha_x^2 - \alpha_y^2)^{3/2}} \cos \theta \\
\mu_{xx} &\equiv \frac{\partial^2 \mu}{\partial \alpha_x^2} = \frac{\alpha_x^2 - k^2}{(k^2 - \alpha_x^2 - \alpha_y^2)^{3/2}} \cos \theta \\
\mu_{yy} &\equiv \frac{\partial^2 \mu}{\partial \alpha_y^2} = \frac{\alpha_y^2 - k^2}{(k^2 - \alpha_x^2 - \alpha_y^2)^{3/2}} \cos \theta
\end{align*}
\] (5.21)

A two dimensional stationary phase point where both \( \mu_x \) and \( \mu_y \) are zero exists for \( \alpha_x = k \sin \theta \cos \phi \) and \( \alpha_y = k \sin \theta \sin \phi \). Now if \( F(\alpha_x, \alpha_y) \) is a continuous function, then the integral of \( F(\alpha_x, \alpha_y) \) with a phase function containing a stationary point

\[
\lim_{R' \to \infty} \int_{-\infty}^{+\infty} F(\alpha_x, \alpha_y) e^{iR' \mu(\alpha_x, \alpha_y)} d\alpha_x d\alpha_y = 2\pi i \frac{F(\alpha_{x0}, \alpha_{y0})}{R' \sqrt{\mu_{xx} \mu_{yy} - \mu_{xy}^2}} e^{iR' \mu(\alpha_{x0}, \alpha_{y0})} \quad (5.22)
\]

where the stationary point is at \( \alpha_x = \alpha_{x0}, \alpha_y = \alpha_{y0} \) and all the partial derivatives of \( \mu \) are evaluated there. This result assumes that \( \mu_{xx} \mu_{yy} - \mu_{xy}^2 \neq 0 \) at the stationary point. For the functions of interest in this problem,

\[
\mu = k \\
\mu_{xx} \mu_{yy} - \mu_{xy}^2 = \frac{1}{k^2 \cos^2 \theta} \quad (5.23)
\]
evaluated at the stationary point. Notice also that at the stationary point

\[
\begin{align*}
\overline{N}'(-h_0) &= -k \sin \theta \vartheta(\alpha_x, \alpha_y) \hat{\vartheta} \quad h_0 = k \cos \theta \\
\overline{M}'(-h_0) &= -ik \sin \phi \vartheta(\alpha_x, \alpha_y) \hat{\varphi} \quad h_1 = \sqrt{k_1^2 - k^2 \sin^2 \theta}
\end{align*}
\] (5.24)

Therefore in the far zone field no radial component of the electric field exists as must be true to satisfy the radiation condition. So

\[
\lim_{R' \to \infty} \overline{E}(R') = E_\vartheta(R') \hat{\vartheta} + E_\varphi(R') \hat{\varphi} \quad (5.25)
\]
Now use the stationary phase integral formula eq. (5.22) and evaluate the $\theta$ and $\phi$ components of the far field.

\[
\lim_{R' \to \infty} E_\theta(R') = \frac{1}{2\pi} T_m \cos \theta e^{ikc\cos \theta} \frac{k}{k_1} \sin(h_1 c) \frac{e^{ikR'}}{R'} \times \\
\left\{ \frac{k \sin \theta}{k_1^2 - k^2 \sin^2 \theta} e^{-ik \sin \theta (\cos \phi x_0 + \sin \phi y_0)} - i \left[ \cos \phi \tilde{K}_{x_0} + \sin \phi \tilde{K}_{y_0} \right] \right\}
\]

\[
\lim_{R' \to \infty} E_\phi(R') = \frac{1}{2\pi} T_s \cos \theta e^{ikc\cos \theta} \frac{k}{h_1} \cos(h_1 c) \frac{e^{ikR'}}{R'} \times \\
\left\{ \sin \phi \tilde{K}_{x_0} - \cos \phi \tilde{K}_{y_0} \right\}
\]

(5.26)

where

\[
\tilde{K}_{x_0} \equiv \tilde{K}_x(-k \sin \theta \cos \phi, -k \sin \theta \sin \phi)
\]

\[
\tilde{K}_{y_0} \equiv \tilde{K}_y(-k \sin \theta \cos \phi, -k \sin \theta \sin \phi)
\]

(5.27)

and $T_m$, $T_s$, and $h_1$ are evaluated at the stationary point.

Notice that both components of the electric field are outwardly radiating spherical waves attenuating as $1/R'$. These field components also have a $\cos \theta$ behavior just as in the two dimensional problem. Finally note that the contribution to $E_\theta$ due to the $\hat{z}$ directed source current, which has an $I_z$ coefficient, has the correct $\sin \theta$ behavior for such a dipole.
CHAPTER VI

NUMERICAL IMPLEMENTATION

§6.1 Introduction

The heart of the problem for the two dimensional strip antenna case is to determine the total surface current on the strip by solving equation (4.33). Unfortunately the Wiener-Hopf technique only works with semi-infinite structures and so is not applicable to this problem. Furthermore, there is no other known analytical technique which will yield a closed form solution to this type of equation. Therefore to find this current, a numerical solution will be implemented. Since this problem is already formulated in the spectral domain, a natural technique to apply is the conjugate gradient-FFT. This technique has fairly recently become popular in the area of numerical electromagnetics, and a number of papers have been published by Sarkar et.al. [32]-[37] on this topic.

This technique is applicable to any convolution type integral equation of the form,

\[ \int_a^b f(x')k(x - x')dx' = h(x) \quad ; \ a \leq x \leq b \]

\[ f(x) = 0 \quad ; \ x < a \ or \ x > b \]  \hspace{1cm} (6.1)

where \( h(x) \) is a known excitation function, \( k(x - x') \) is a known kernel function, and \( f(x) \) is the unknown to be solved for over the interval \([a, b]\). In addition \( f(x) \) will be taken to be zero outside this interval. The integral equation is solved iteratively using the conjugate gradient technique. The linear convolution integral operator is performed in the spectral domain by the use of the FFT. This allows for a very rapid evaluation of the operator on each iteration of the conjugate gradient.
The conjugate gradient will converge to the exact answer in \( N \) steps assuming no round-off errors occur, where \( N \) is the number of unknowns. Since the transform of the kernel function is performed analytically, each iteration of the conjugate gradient requires a FFT, the product of the transformed current vector by the transform of the kernel function, and an inverse FFT. Therefore the number of complex multiplications required is \( 2N \log_2(N) + N \) for each iteration. So the solution of the problem requires on the order of \( 2N^2 \log_2(N) \) multiplications. A moment method solution to a problem with \( N \) unknowns will require on the order of \( N^3/3 \) multiplications to perform a matrix decomposition. Therefore the conjugate gradient-FFT method will be much more computationally efficient than a moment method problem with the same unknowns, particularly as the number of unknowns becomes large.

§6.2 Conjugate Gradient Technique

To fully understand the conjugate gradient-FFT technique, first the conjugate gradient method as applied to a linear \( N \)-dimensional vector equation must be discussed. Hestenes and Steifel [38] published the first classical paper on this technique in 1952. In addition there have been several books on numerical computing which also discuss the conjugate gradient including works by Ralston [39], Golub and VanLoan [40], and Hestenes [41].

The conjugate gradient method is used to solve an equation of the form

\[
\overline{L}_h \{ \overline{x} \} = \overline{b}
\]  

(6.2)

where \( \overline{L}_h \) is a complex Hermitian linear operator on an \( N \)-dimensional vector which can always be expressed in terms of a complex Hermitian \( N \times N \) matrix. The complex vector \( \overline{b} \) and operator \( \overline{L}_h \) are known and \( \overline{x} \) must be found. Any linear vector operator \( \overline{L} \) maps one vector into another and satisfies the relation

\[
\overline{L} \{ a\overline{u} + b\overline{w} \} = a\overline{L} \{ \overline{u} \} + b\overline{L} \{ \overline{w} \}
\]  

(6.3)

for any complex constants \( a \) and \( b \) and vectors \( \overline{u} \) and \( \overline{w} \). A Hermitian operator also satisfies the following inner product relation

\[
\langle \overline{u} \cdot \overline{L}_h \{ \overline{w} \} \rangle = \langle \overline{w} \cdot \overline{L}_h \{ \overline{u} \} \rangle^*
\]  

(6.4)
for any vectors $\mathbf{v}$ and $\mathbf{w}$. An inner product over a vector space maps two vectors into a scalar quantity and must obey the following three conditions:

Symmetry

$$\langle \mathbf{v} \cdot \mathbf{w} \rangle = \langle \mathbf{w} \cdot \mathbf{v} \rangle^*$$

Positivity

$$\langle \mathbf{v} \cdot \mathbf{v} \rangle > 0 \quad \text{if} \quad \mathbf{v} \neq \mathbf{0}$$

Bilinearity

$$\langle (a\mathbf{v} + b\mathbf{w}) \cdot (c\mathbf{x} + d\mathbf{y}) \rangle = a^*c \langle \mathbf{v} \cdot \mathbf{x} \rangle + a^*d \langle \mathbf{v} \cdot \mathbf{y} \rangle + b^*c \langle \mathbf{w} \cdot \mathbf{x} \rangle + b^*d \langle \mathbf{w} \cdot \mathbf{y} \rangle$$

(6.5)

The inner product used here will be

$$\langle \mathbf{v} \cdot \mathbf{w} \rangle = \mathbf{v}^T \mathbf{w}$$

(6.6)

where $\mathbf{v}^T$ is the transpose conjugate of $\mathbf{v}$. Notice that for a Hermitian linear operator $\langle \mathbf{v} \cdot \mathbf{L}_h \{\mathbf{v}\} \rangle$ is a real quantity for any vector $\mathbf{v}$.

Furthermore assume a set of $N$ vectors $\mathbf{v}_k$ exists which are $\mathbf{L}$ – orthogonal, that is

$$\langle \mathbf{v}_j \cdot \mathbf{L}_h \{\mathbf{v}_i\} \rangle = 0 \quad \text{if} \quad i \neq j$$

(6.7)

This set of vectors $\mathbf{v}_k$ form a complete orthogonal basis of the $N$–dimensional vector space. Therefore any $N$–dimensional vector can be represented as a linear combination of the $\mathbf{v}_k$. If an estimate of $\mathbf{z}$ is designated as $\mathbf{z}_1$ then their difference can be represented as

$$\mathbf{z} - \mathbf{z}_1 = \sum_{j=1}^{N} \alpha_j \mathbf{v}_j$$

(6.8)

where

$$\alpha_j = \frac{\langle \mathbf{v}_j \cdot \mathbf{L}_h \{\mathbf{z} - \mathbf{z}_1\} \rangle}{\langle \mathbf{v}_j \cdot \mathbf{L}_h \{\mathbf{v}_j\} \rangle}$$

(6.9)

utilizing the $\mathbf{L}$ – orthogonal properties of the $\mathbf{v}_k$. To construct an $\mathbf{L}$ – orthogonal basis set $\mathbf{v}_k$ use the Gram-Schmidt procedure on a linearly independent set of $N$ vectors $\mathbf{u}_i$. Therefore

$$\mathbf{v}_1 = \mathbf{u}_1$$
\[ \bar{v}_2 = \bar{u}_2 - \frac{\langle \bar{v}_1 \cdot \bar{L}_h \{ \bar{u}_2 \} \rangle}{\langle \bar{v}_1 \cdot \bar{L}_h \{ \bar{v}_1 \} \rangle} \bar{v}_1 \]
\[ \bar{v}_3 = \bar{u}_3 - \frac{\langle \bar{v}_1 \cdot \bar{L}_h \{ \bar{u}_3 \} \rangle}{\langle \bar{v}_1 \cdot \bar{L}_h \{ \bar{v}_1 \} \rangle} \bar{v}_1 - \frac{\langle \bar{v}_2 \cdot \bar{L}_h \{ \bar{u}_3 \} \rangle}{\langle \bar{v}_2 \cdot \bar{L}_h \{ \bar{v}_2 \} \rangle} \bar{v}_2 \]
\[ \vdots \]
\[ \bar{v}_n = \bar{u}_n + \sum_{k=1}^{n-1} \beta_{nk} \bar{v}_k \quad (6.10) \]

where
\[ \beta_{nk} = -\frac{\langle \bar{v}_k \cdot \bar{L}_h \{ \bar{u}_n \} \rangle}{\langle \bar{v}_k \cdot \bar{L}_h \{ \bar{v}_k \} \rangle} \quad (6.11) \]

Notice that if \( \bar{L}_h \) is also positive definite so that
\[ \langle \bar{v} \cdot \bar{L}_h \{ \bar{v} \} \rangle > 0 \quad \text{if} \quad \bar{v} \neq \bar{0} \quad (6.12) \]

then the \( \bar{v}_h \) can always be constructed since
\[ \langle \bar{v}_k \cdot \bar{L}_h \{ \bar{v}_k \} \rangle \neq 0 \quad (6.13) \]

For the conjugate gradient method choose the \( \bar{u}_i \) as
\[ \bar{u}_i = \bar{b} - \bar{L}_h \{ \bar{x}_i \} \equiv \bar{r}_i \quad (6.14) \]

where \( \bar{r}_i \) is the residual vector and
\[ \bar{x}_i = \bar{x}_1 + \sum_{j=1}^{i-1} \alpha_j \bar{v}_j \quad (6.15) \]

Therefore
\[ \bar{x} = \bar{x}_{N+1} \quad (6.16) \]

If the \( \bar{u}_i \) had been chosen as the unitary vectors of this vector space this same procedure yields the Gaussian elimination solution for \( \bar{x} \). In any case for this choice of \( \bar{u}_i \)

\[ \bar{z}_{i+1} = \bar{z}_i + \alpha_i \bar{u}_i \]
\[ \bar{r}_{i+1} = \bar{r}_i - \alpha_i \bar{L}_h \{ \bar{u}_i \} = \bar{r}_i - \sum_{k=1}^{i} \alpha_k \bar{L}_h \{ \bar{v}_k \} \quad (6.17) \]
Now
\[ \langle \vec{v}_k \cdot \vec{f}_k \rangle = \langle \vec{v}_k \cdot \vec{f}_1 \rangle - \sum_{m=1}^{k-1} \alpha_m \langle \vec{v}_k \cdot \overline{L}_h \{ \vec{v}_m \} \rangle = \langle \vec{v}_k \cdot \vec{f}_1 \rangle \] (6.18)

Therefore from eq. (6.9)
\[ \alpha_k = \frac{\langle \vec{v}_k \cdot \left[ \vec{b} - \overline{L}_h \{ \vec{x}_1 \} \right] \rangle}{\langle \vec{v}_k \cdot \overline{L}_h \{ \vec{v}_k \} \rangle} = \frac{\langle \vec{v}_k \cdot \vec{f}_1 \rangle}{\langle \vec{v}_k \cdot \overline{L}_h \{ \vec{v}_k \} \rangle} = \frac{\langle \vec{v}_k \cdot \vec{f}_k \rangle}{\langle \vec{v}_k \cdot \overline{L}_h \{ \vec{v}_k \} \rangle} \] (6.19)

Also since \( \vec{f}_n = \vec{u}_n \)
\[ \vec{v}_n = \vec{f}_n + \sum_{k=1}^{n-1} \beta_{nk} \vec{v}_k \] (6.20)

and so
\[ \vec{f}_n = \vec{v}_n - \sum_{k=1}^{n-1} \beta_{nk} \vec{v}_k \] (6.21)

Furthermore from eq. (6.17)
\[ \vec{f}_j = \vec{f}_1 - \sum_{k=1}^{j-1} \alpha_k \overline{L}_h \{ \vec{v}_k \} \] (6.22)

Now using eqs. (6.19) thru (6.22) derive the following relations.

for \( j > 1 \):
\[ \langle \vec{f}_j \cdot \vec{f}_1 \rangle = \langle \vec{f}_j \cdot \vec{v}_1 \rangle = \left\langle \left[ \vec{f}_1 - \sum_{k=1}^{j-1} \alpha_k \overline{L}_h \{ \vec{v}_k \} \right] \cdot \vec{v}_1 \right\rangle \\
= \langle \vec{f}_1 \cdot \vec{v}_1 \rangle - \sum_{k=1}^{j-1} \left\langle \alpha_k \overline{L}_h \{ \vec{v}_k \} \cdot \vec{v}_1 \right\rangle = \langle \vec{f}_1 \cdot \vec{v}_1 \rangle - \sum_{k=1}^{j-1} \left\langle \overline{L}_h \{ \vec{v}_k \} \cdot \vec{v}_1 \right\rangle \\
= \langle \vec{f}_1 \cdot \vec{v}_1 \rangle - \alpha_1 \langle \vec{v}_1 \cdot \overline{L}_h \{ \vec{v}_1 \} \rangle = (\vec{f}_1 \cdot \vec{v}_1) - (\vec{v}_1 \cdot \vec{f}_1)^* = 0 \] (6.23)

Therefore
\[ \langle \vec{f}_j \cdot \vec{f}_1 \rangle = 0 \quad \text{and} \quad \langle \vec{f}_j \cdot \vec{v}_1 \rangle = 0 \quad j > 1 \] (6.24)

for \( j > 2 \):
\[ \langle \vec{f}_j \cdot \vec{f}_2 \rangle = \langle \vec{f}_j \cdot [\vec{v}_2 - \beta_{21} \vec{v}_1] \rangle = \langle \vec{f}_j \cdot \vec{v}_2 \rangle - \beta_{21} \langle \vec{f}_j \cdot \vec{v}_1 \rangle \\
= \langle \vec{f}_j \cdot \vec{v}_2 \rangle = \left\langle \left[ \vec{f}_1 - \sum_{k=1}^{j-1} \alpha_k \overline{L}_h \{ \vec{v}_k \} \right] \cdot \vec{v}_2 \right\rangle \\
= \langle \vec{f}_1 \cdot \vec{v}_2 \rangle - \alpha_2 \langle \vec{v}_2 \cdot \overline{L}_h \{ \vec{v}_2 \} \rangle = (\vec{f}_1 \cdot \vec{v}_2) - (\vec{v}_2 \cdot \vec{f}_1)^* = 0 \] (6.25)

Therefore
\[ \langle \vec{f}_j \cdot \vec{f}_2 \rangle = 0 \quad \text{and} \quad \langle \vec{f}_j \cdot \vec{v}_2 \rangle = 0 \quad j > 2 \] (6.26)
for \( j > 3 \):

\[
\langle \bar{\varphi}_j \cdot \bar{\varphi}_3 \rangle = \langle \bar{\varphi}_j \cdot [\bar{v}_3 - \beta_{31}\bar{v}_1 - \beta_{32}\bar{v}_2] \rangle = \langle \bar{\varphi}_j \cdot \bar{v}_3 \rangle - \beta_{31} \langle \bar{\varphi}_j \cdot \bar{v}_1 \rangle - \beta_{32} \langle \bar{\varphi}_j \cdot \bar{v}_2 \rangle \\
= \langle \bar{\varphi}_j \cdot \bar{v}_3 \rangle = \left( \bar{\varphi}_1 - \sum_{k=1}^{i-1} \alpha_k \bar{L}_h \{\bar{v}_k\} \right) \cdot \bar{v}_3 \\
= \langle \bar{\varphi}_1 \cdot \bar{v}_3 \rangle - \alpha_3^* \langle \bar{v}_3 \cdot \bar{L}_h \{\bar{v}_3\} \rangle^* = \langle \bar{\varphi}_1 \cdot \bar{v}_3 \rangle - \langle \bar{v}_3 \cdot \bar{\varphi}_1 \rangle^* = 0
\tag{6.27}
\]

Therefore

\[
\langle \bar{\varphi}_j \cdot \bar{\varphi}_3 \rangle = 0 \quad \text{and} \quad \langle \bar{\varphi}_j \cdot \bar{v}_3 \rangle = 0 \quad j > 3
\tag{6.28}
\]

in general for \( j > i \):

\[
\langle \bar{\varphi}_j \cdot \bar{\varphi}_i \rangle = \left( \bar{\varphi}_j \cdot \left[ \bar{v}_i - \sum_{k=1}^{i-1} \beta_{ik} \bar{v}_k \right] \right) = \langle \bar{\varphi}_j \cdot \bar{v}_i \rangle - \sum_{k=1}^{i-1} \beta_{ik} \langle \bar{\varphi}_j \cdot \bar{v}_k \rangle \\
= \langle \bar{\varphi}_j \cdot \bar{v}_i \rangle = \left( \bar{\varphi}_1 - \sum_{k=1}^{i-1} \alpha_k \bar{L}_h \{\bar{v}_k\} \right) \cdot \bar{v}_i \\
= \langle \bar{\varphi}_1 \cdot \bar{v}_i \rangle - \alpha_i^* \langle \bar{v}_i \cdot \bar{L}_h \{\bar{v}_i\} \rangle^* = \langle \bar{\varphi}_1 \cdot \bar{v}_i \rangle - \langle \bar{v}_i \cdot \bar{\varphi}_1 \rangle^* = 0
\tag{6.29}
\]

Therefore

\[
\langle \bar{\varphi}_j \cdot \bar{\varphi}_i \rangle = 0 \quad \text{and} \quad \langle \bar{\varphi}_j \cdot \bar{v}_i \rangle = 0 \quad j > i
\tag{6.30}
\]

and since \( \langle \bar{\varphi}_j \cdot \bar{\varphi}_i \rangle = \langle \bar{\varphi}_i \cdot \bar{\varphi}_j \rangle^* \) it follows that

\[
\langle \bar{\varphi}_i \cdot \bar{\varphi}_j \rangle = 0 \quad i \neq j
\tag{6.31}
\]

Also notice that

\[
\langle \bar{v}_n \cdot \bar{\varphi}_n \rangle = \left( \bar{\varphi}_n + \sum_{k=1}^{n-1} \beta_{nk} \bar{v}_k \right) \cdot \bar{\varphi}_n \\
= \langle \bar{\varphi}_n \cdot \bar{\varphi}_n \rangle + \sum_{k=1}^{n-1} \beta_{nk} \langle \bar{\varphi}_n \cdot \bar{v}_k \rangle \\
= \langle \bar{\varphi}_n \cdot \bar{\varphi}_n \rangle + \sum_{k=1}^{n-1} \beta_{nk} \langle \bar{\varphi}_n \cdot \bar{v}_k \rangle^* \\
= \langle \bar{\varphi}_n \cdot \bar{\varphi}_n \rangle
\tag{6.32}
\]

Furthermore from eq. (6.17)

\[
\bar{L}_h \{\bar{v}_k\} = \frac{1}{\alpha_k} \left[ \bar{\varphi}_k - \bar{\varphi}_{k+1} \right]
\tag{6.33}
\]
Now using eq. (6.30) thru (6.33)

\[
\beta_{nk} = -\frac{\langle \bar{u}_k \cdot \bar{L}_h \{ \bar{r}_n \} \rangle}{\langle \bar{u}_k \cdot \bar{L}_h \{ \bar{v}_k \} \rangle} = -\frac{\langle \bar{r}_n \cdot \bar{L}_h \{ \bar{u}_k \} \rangle^*}{\langle \bar{v}_k \cdot \bar{L}_h \{ \bar{v}_k \} \rangle} = -\frac{\langle (\bar{r}_k - \bar{r}_{k+1}) \cdot \bar{r}_n \rangle}{\langle \bar{v}_k \cdot \bar{L}_h \{ \bar{v}_k \} \rangle} = \alpha_n \frac{\langle (\bar{r}_{k+1} - \bar{r}_k) \cdot \bar{r}_n \rangle}{\langle \bar{r}_k \cdot \bar{r}_k \rangle} \tag{6.34}
\]

Therefore

\[
\beta_{nk} = 0 \quad \text{if} \quad k < n - 1 \tag{6.35}
\]

\[
\beta_{n,n-1} = \frac{\langle \bar{r}_n \cdot \bar{r}_n \rangle}{\langle \bar{r}_{n-1} \cdot \bar{r}_{n-1} \rangle} \tag{6.36}
\]

and so

\[
\bar{v}_n = \bar{r}_n + \beta_{n,n-1} \bar{v}_{n-1} \tag{6.37}
\]

Also

\[
\alpha_n = \frac{\langle \bar{r}_n \cdot \bar{r}_n \rangle}{\langle \bar{v}_n \cdot \bar{L}_h \{ \bar{v}_n \} \rangle} \tag{6.38}
\]

Notice that both \( \beta_{n,n-1} \) and \( \alpha_n \) are real quantities.

Therefore the conjugate gradient algorithm takes on the following form. First set \( \bar{x}_1 \)
equal to an initial estimate of \( \bar{x} \). If no estimate of \( \bar{x} \) can be made set \( \bar{x}_1 \) to \( \bar{0} \). Then compute \( \bar{r}_1 = \bar{b} - \bar{L}_h \{ \bar{x}_1 \} \) and set \( \bar{v}_1 = \bar{r}_1 \). Then iterate on the following steps, starting at \( i = 1 \), until \( \bar{x}_i \) converges to within some error tolerance.

\[
\alpha_i = \frac{\langle \bar{r}_i \cdot \bar{r}_i \rangle}{\langle \bar{v}_i \cdot \bar{L}_h \{ \bar{v}_i \} \rangle}
\]

\[
\bar{x}_{i+1} = \bar{x}_i + \alpha_i \bar{v}_i
\]

\[
\bar{r}_{i+1} = \bar{r}_i - \alpha_i \bar{L}_h \{ \bar{v}_i \}
\]

\[
\beta_{i+1,i} = \frac{\langle \bar{r}_{i+1} \cdot \bar{r}_{i+1} \rangle}{\langle \bar{r}_i \cdot \bar{r}_i \rangle}
\]

\[
\bar{v}_{i+1} = \bar{r}_{i+1} + \beta_{i+1,i} \bar{v}_i \tag{6.39}
\]
§6.3 Modified Conjugate Gradient Technique

Unfortunately when formulating electromagnetic scattering and antenna problems, the problem usually involves a complex symmetric operator and not a complex Hermitian operator. This is due to the reciprocal nature of the Green's function. This section presents an extension of the conjugate gradient technique which can be used on complex symmetric instead of Hermitian operators. The phrase "modified conjugate gradient technique" has been coined to describe this method. This technique is equivalent to a special case of the biconjugate gradient technique. Jacobs discusses the biconjugate gradient in the book edited by Duff [42] and in Jacobs [43].

The modified conjugate gradient method is used to solve an equation of the form

$$\overline{L}_o \{\overline{z}\} = \overline{b}$$  \hspace{1cm} (6.40)

where $\overline{L}_o$ is a linear complex symmetric operator on an $N$-dimensional vector which can be expressed in terms of a complex symmetric $N \times N$ matrix. A complex symmetric operator obeys the following inner product relation

$$\left< \overline{w}^* \cdot \overline{L}_o \{\overline{w}\} \right> = \left< \overline{w}^* \cdot \overline{L}_o \{\overline{v}\} \right>$$  \hspace{1cm} (6.41)

for any complex vectors $\overline{v}$ and $\overline{w}$.

Furthermore assume a set of $N$ vectors $\overline{v}_k$ exists which are conjugate $\overline{L} -$ orthogonal, that is

$$\left< \overline{v}_j^* \cdot \overline{L}_o \{\overline{v}_i\} \right> = 0 \quad \text{if} \quad i \neq j$$  \hspace{1cm} (6.42)

This set of vectors $\overline{v}_k$ form a complete orthogonal basis of the $N$-dimensional vector space. Therefore any $N$-dimensional vector can be represented as a linear combination of the $\overline{v}_k$. If an estimate of $\overline{z}$ is designated as $\overline{z}_1$ then their difference can be represented as

$$\overline{z} - \overline{z}_1 = \sum_{j=1}^{N} \alpha_j \overline{v}_j$$  \hspace{1cm} (6.43)

the $\alpha_j$ can be found from

$$\alpha_j = \frac{\left< \overline{v}_j^* \cdot \overline{L}_o \{\overline{z} - \overline{z}_1\} \right>}{\left< \overline{v}_j^* \cdot \overline{L}_o \{\overline{v}_j\} \right>}$$  \hspace{1cm} (6.44)
To construct a conjugate $\overline{L}$-orthogonal basis set $\overline{u}_k$ use the Gram-Schmidt procedure on a linearly independent set of $N$ vectors $\overline{u}_i$. Therefore

$$\overline{v}_1 = \overline{u}_1$$
$$\overline{v}_2 = \overline{u}_2 - \frac{\left< \overline{v}_1 \cdot \overline{L}_e \left\{ \overline{u}_2 \right\} \right>}{\left< \overline{v}_1 \cdot \overline{L}_e \left\{ \overline{v}_1 \right\} \right>} \overline{v}_1$$
$$\overline{v}_3 = \overline{u}_3 - \frac{\left< \overline{v}_1 \cdot \overline{L}_e \left\{ \overline{u}_3 \right\} \right>}{\left< \overline{v}_1 \cdot \overline{L}_e \left\{ \overline{v}_1 \right\} \right>} \overline{v}_1 - \frac{\left< \overline{v}_2 \cdot \overline{L}_e \left\{ \overline{u}_3 \right\} \right>}{\left< \overline{v}_2 \cdot \overline{L}_e \left\{ \overline{v}_2 \right\} \right>} \overline{v}_2$$

$$\vdots$$

$$\overline{v}_n = \overline{u}_n + \sum_{k=1}^{n-1} \beta_{nk} \overline{v}_k$$ (6.45)

where

$$\beta_{nk} = -\frac{\left< \overline{v}_k \cdot \overline{L}_e \left\{ \overline{u}_n \right\} \right>}{\left< \overline{v}_k \cdot \overline{L}_e \left\{ \overline{v}_k \right\} \right>}$$ (6.46)

For the modified conjugate gradient method again choose the $\overline{u}_i$ as

$$\overline{u}_i = \overline{\delta} - \overline{L}_e \left\{ \overline{x}_i \right\} \equiv \overline{r}_i$$ (6.47)

where $\overline{r}_i$ is the residual vector and

$$\overline{x}_i \equiv \overline{x}_1 + \sum_{j=1}^{i-1} \alpha_j \overline{v}_j$$ (6.48)

Therefore

$$\overline{x} = \overline{x}_{N+1}$$ (6.49)

For this choice of $\overline{u}_i$

$$\overline{x}_{i+1} = \overline{x}_i + \alpha_i \overline{v}_i$$

$$\overline{r}_{i+1} = \overline{r}_i - \alpha_i \overline{L}_e \left\{ \overline{v}_i \right\} = \overline{r}_i - \sum_{k=1}^{i} \alpha_k \overline{L}_e \left\{ \overline{u}_k \right\}$$ (6.50)

Now

$$\left< \overline{v}_k \cdot \overline{r}_k \right> = \left< \overline{v}_k \cdot \overline{r}_1 \right> - \sum_{m=1}^{k-1} \alpha_m \left< \overline{v}_m \cdot \overline{L}_e \left\{ \overline{u}_m \right\} \right> = \left< \overline{v}_k \cdot \overline{r}_1 \right>$$ (6.51)

Therefore from eq. (6.44)

$$\alpha_k = \frac{\left< \overline{v}_k \cdot \left[ \overline{\delta} - \overline{L}_e \left\{ \overline{x}_1 \right\} \right] \right>}{\left< \overline{v}_k \cdot \overline{L}_e \left\{ \overline{v}_k \right\} \right>} = \frac{\left< \overline{v}_k \cdot \overline{r}_1 \right>}{\left< \overline{v}_k \cdot \overline{L}_e \left\{ \overline{u}_k \right\} \right>} = \frac{\left< \overline{v}_k \cdot \overline{r}_k \right>}{\left< \overline{v}_k \cdot \overline{L}_e \left\{ \overline{u}_k \right\} \right>}$$ (6.52)
Also since \( \bar{r}_n = \bar{u}_n \)

\[
\bar{v}_n = \bar{r}_n + \sum_{k=1}^{n-1} \beta_{nk} \bar{v}_k
\]

and so

\[
\bar{r}_n = \bar{v}_n - \sum_{k=1}^{n-1} \beta_{nk} \bar{v}_k
\]

Furthermore from eq. (6.50)

\[
\bar{r}_j = \bar{r}_1 - \sum_{k=1}^{j-1} \alpha_k \bar{L}_s \{ \bar{v}_k \}
\]

Now using eqs. (6.52) thru (6.55) the following relations are derived just as in the standard conjugate gradient case.

\[
\langle \bar{r}_j \cdot \bar{r}_i \rangle = 0 \quad \text{and} \quad \langle \bar{r}_j \cdot \bar{v}_i \rangle = 0 \quad j > i
\]  \( (6.56) \)

and since \( \langle \bar{r}_j \cdot \bar{r}_i \rangle = \langle \bar{r}_i \cdot \bar{r}_j \rangle \) it follows that

\[
\langle \bar{r}_i \cdot \bar{r}_j \rangle = \langle \bar{r}_j \cdot \bar{r}_j \rangle = 0 \quad i \neq j
\]  \( (6.57) \)

Also

\[
\langle \bar{v}_n \cdot \bar{r}_n \rangle = \langle \bar{r}_n \cdot \bar{r}_n \rangle
\]

Furthermore from eq. (6.50)

\[
\bar{L}_s \{ \bar{v}_k \} = \frac{1}{\alpha_k} [\bar{r}_k - \bar{r}_{k+1}]
\]  \( (6.59) \)

Now using eq. (6.56) thru (6.59)

\[
\beta_{nk} = -\frac{\langle \bar{v}_k \cdot \bar{L}_s \{ \bar{r}_n \} \rangle}{\langle \bar{v}_k \cdot \bar{L}_s \{ \bar{v}_k \} \rangle} = -\frac{\langle \bar{r}_n \cdot \bar{L}_s \{ \bar{v}_k \} \rangle}{\langle \bar{v}_k \cdot \bar{L}_s \{ \bar{v}_k \} \rangle} = -\frac{\langle \bar{r}_n \cdot (\bar{r}_k - \bar{r}_{k+1}) \rangle}{\langle \bar{v}_k \cdot \bar{r}_k \rangle} = \frac{\langle \bar{r}_n \cdot (\bar{r}_{k+1} - \bar{r}_k) \rangle}{\langle \bar{r}_k \cdot \bar{r}_k \rangle}
\]

Therefore

\[
\beta_{nk} = 0 \quad \text{if} \quad k < n - 1
\]  \( (6.61) \)

\[
\beta_{n,n-1} = \frac{\langle \bar{r}_n \cdot \bar{r}_n \rangle}{\langle \bar{r}_{n-1} \cdot \bar{r}_{n-1} \rangle}
\]  \( (6.62) \)
and so
\[ \bar{v}_n = \bar{r}_n + \beta_{n,n-1} \bar{v}_{n-1} \]  
(6.63)

Also
\[ \alpha_n = \frac{\langle \bar{r}_n \cdot \bar{r}_n \rangle}{\langle \bar{v}_n \cdot L_4 \{\bar{v}_n\} \rangle} \]
(6.64)

Notice that both \( \beta_{n,n-1} \) and \( \alpha_n \) are complex quantities.

Therefore the modified conjugate gradient algorithm takes on the following form. First set \( \bar{z}_1 \) equal to an initial estimate of \( \bar{z} \). If no estimate of \( \bar{z} \) can be made set \( \bar{z}_1 \) to \( \bar{u} \). Then compute \( \bar{r}_1 = \bar{b} - L_4 \{\bar{z}_1\} \) and set \( \bar{v}_1 = \bar{r}_1 \). Then iterate on the following steps, starting at \( i = 1 \), until \( \bar{z}_i \) converges to within some error tolerance.

\[ \alpha_i = \frac{\langle \bar{r}_i \cdot \bar{r}_i \rangle}{\langle \bar{v}_i \cdot L_4 \{\bar{v}_i\} \rangle} \]
\[ \bar{z}_{i+1} = \bar{z}_i + \alpha_i \bar{v}_i \]
\[ \bar{r}_{i+1} = \bar{r}_i - \alpha_i L_4 \{\bar{v}_i\} \]
\[ \beta_{i+1,i} = \frac{\langle \bar{r}_{i+1} \cdot \bar{r}_{i+1} \rangle}{\langle \bar{r}_i \cdot \bar{r}_i \rangle} \]
\[ \bar{v}_{i+1} = \bar{r}_{i+1} + \beta_{i+1,i} \bar{v}_i \]
(6.65)

§6.4 The Fast Fourier Transform

The Fast Fourier Transform, or FFT, is an efficient algorithm to perform the discrete Fourier transform and was first proposed by Cooley and Tukey [44] in 1965. Several books have also been written which discuss this numerical technique including those by Oppenheim and Schafer [45], Brigham [46], and Ramirez [47]. The discrete Fourier and discrete inverse Fourier transform relations are given by

\[ F_k = \sum_{n=0}^{N-1} f_n e^{i(2\pi/N)nk} \]
\[ f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{-i(2\pi/N)nk} \]  
(6.66)

Notice that by their definition both \( f_n \) and \( F_k \) must be periodic sequences. A continuous Fourier transform can be approximated by the discrete transform if the bandwidth in both
domains is taken large enough so that the continuous functions are essentially zero outside these bands. Since

$$\mathcal{F}\{f(x)\} \equiv F(\alpha) = \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx$$  \hspace{1cm} (6.67)$$

Then

$$F(k\Delta \alpha) = \int_{-\infty}^{+\infty} f(x) e^{i k \Delta \alpha x} dx \approx \Delta x \sum_{n=-N/2}^{N/2-1} f(n\Delta x) e^{i k \alpha n \Delta x}$$  \hspace{1cm} (6.68)$$

where

$$\Delta x = L / N \quad \Delta \alpha = W / N$$  \hspace{1cm} (6.69)$$

The bandwidth in the spatial domain being $L$ and the bandwidth in the spectral domain $W$. If the bandwidth product relation

$$LW = 2\pi N \quad \text{and} \quad \Delta x \Delta \alpha = 2\pi / N$$  \hspace{1cm} (6.70)$$

is also imposed and denoting $f(n\Delta x)$ as $f_n^k$, then the approximation to the Fourier transform can be written as

$$F(k\Delta \alpha) \approx \Delta x \sum_{n=-N/2}^{N/2-1} f_n^k e^{i(2\pi / N)nk}$$  \hspace{1cm} (6.71)$$

For convenience define the $f_n^k$ outside the interval of the sum in a periodic fashion,

$$f_{n \pm mN}^k = f_n^k \quad -N/2 \leq n \leq N/2 - 1$$  \hspace{1cm} (6.72)$$

And so

$$F(k\Delta \alpha) \approx \Delta x \sum_{n=0}^{N-1} f_n^k e^{i(2\pi / N)nk} \equiv \Delta x F_k$$  \hspace{1cm} (6.73)$$

The sequence $F_k$ is just the discrete Fourier transform of the sequence $f_n^k$. In a similar fashion the continuous inverse Fourier transform can be approximated by the inverse discrete Fourier transform

$$f(n\Delta x) \approx \frac{\Delta \alpha}{2\pi} \sum_{k=0}^{N-1} F_k e^{-i(2\pi / N)nk} \equiv \frac{N}{2\pi} \Delta \alpha f_n$$  \hspace{1cm} (6.74)$$

where

$$F_k^l = F_k^l_{\pm mN} = F(k\Delta \alpha) \quad -N/2 \leq k \leq N/2 - 1$$  \hspace{1cm} (6.75)$$

and the sequence $f_n$ is the discrete inverse Fourier transform of the sequence $F_k^l$. So the values of the continuous functions at regular samples are related to the discrete sequences by

$$F_k^l \approx \Delta x F_k^l \quad \text{and} \quad f_n^l \approx \frac{1}{\Delta x} f_n^l$$  \hspace{1cm} (6.76)$$
The FFT algorithm is a computationally fast method to compute the discrete Fourier transform. If the number of points \( N \) in the sequence to be transformed is some power of two, so \( N = 2^\nu \), then the number of multiplications needed to perform the FFT is \( N \log_2(N) \). To see how this comes about consider

\[
F_k = \sum_{n=0}^{N-1} f_n e^{i(2\pi/N)nk} \\
= \sum_{r=0}^{N/2-1} f_{2r} e^{i(2\pi/N)2rk} + \sum_{r=0}^{N/2-1} f_{2r+1} e^{i(2\pi/N)(2r+1)k} \\
= \sum_{r=0}^{n/2-1} f_{2r} e^{i(2\pi/N)rk} + e^{i(2\pi/N)k} \sum_{r=0}^{N/2-1} f_{2r+1} e^{i(2\pi/N)rk} \\
= G_k + e^{i(2\pi/N)k} H_k
\]  

(6.77)

where

\[
G_k = \sum_{r=0}^{N/2-1} f_{2r} e^{i(2\pi/N)rk} \quad H_k = \sum_{r=0}^{N/2-1} f_{2r+1} e^{i(2\pi/N)rk}
\]  

(6.78)

The sequences \( G_k \) and \( H_k \) are merely the discrete Fourier transform of the even and odd elements of the \( f_n \) sequence. Therefore the discrete Fourier transform of the \( N \) valued sequence \( f_n \) can be computed by the above equation from both discrete Fourier transforms of the \( N/2 \) valued odd and even sequences of \( f_n \). These discrete Fourier transforms of the \( N/2 \) valued sequences can in turn be computed with four discrete Fourier transforms of \( N/4 \) valued sequences, and so on. Therefore in general, \( 2^m \) discrete Fourier transforms of \( N/2^m \) valued sequences will need to be computed for \( m = \nu - 1, \nu - 2, \nu - 3, \ldots, 0 \). For each value of \( m \), \( N \) multiplications are required and \( m \) takes on \( \log_2(N) \) different values. Therefore the entire computation requires \( N \log_2(N) \) multiplications to complete.

§6.5 Numerical Procedure for Total Current

The actual solution of the two dimensional antenna problem for the total current on the patch still remains to be discussed. Recall that the equation

\[
\mathcal{F}^{-1}\{\gamma\gamma_1 G(\alpha) K^t(\alpha)\} = L \mathcal{F}^{-1}\{\gamma\gamma_1 G(\alpha) F(\alpha)\} \quad ; -L/2 < z < +L/2
\]

\[
\mathcal{F}^{-1}\{K^t(\alpha)\} = 0 \quad ; z > +L/2 \text{ or } z < -L/2
\]  

(6.79)
is to be solved numerically using the conjugate gradient–FFT. This leads to fairly straightforward numerical code except for a few things which must be handled specially. A listing of the computer program STRIP which has been programmed to solve this problem is in the appendix of this thesis.

First the quantities \( \gamma \) and \( \gamma_1 \) which are computed by taking square roots must be on the proper sheet. Care has been taken to ensure that their real parts are positive or zero and that their imaginary parts are zero or negative. Second, since \( I(\alpha) \) becomes singular when \( \alpha = \pm k_1 \), the computer code can not directly evaluate this function there. Instead to evaluate the right hand side of the above expression the following limit is used.

\[
\lim_{\alpha \to \pm k_1} \gamma \gamma_1 G(\alpha) I(\alpha) = \frac{ie^{ix_0}}{\epsilon_r} \quad (6.80)
\]

Finally the most significant numerical problem involved with the implementation of this solution involves the spectral bandwidth of the functions involved. Consider the behavior of the following functions as \( \alpha \) tends to \( \pm \infty \).

\[
\begin{align*}
\lim_{\alpha \to \pm \infty} \gamma(\alpha) &= \pm \alpha \\
\lim_{\alpha \to \pm \infty} \gamma_1(\alpha) &= \pm \alpha \\
\lim_{\alpha \to \pm \infty} G(\alpha) &= \pm \frac{1}{\epsilon_r + 1} \\
\lim_{\alpha \to \pm \infty} I(\alpha) &= i e^{ix_0} \frac{1}{\alpha}
\end{align*} \quad (6.81)
\]

So the expression on the right hand side of the equation behaves as

\[
\lim_{\alpha \to \pm \infty} \gamma \gamma_1 G(\alpha) I(\alpha) = \pm \frac{ie^{ix_0}}{\epsilon_r + 1} \quad (6.82)
\]

Unfortunately this expression does not go to zero and so to evaluate its inverse transform requires infinite bandwidth. This poor behavior in the spectral domain is due to the discontinuity in the spatial domain of \( I(\alpha) \) at \( x_0 \). The total current in spatial domain has the same discontinuity at the feed point \( x_0 \) as well. Therefore the solution is to subtract out, in the spatial domain, some function which has this discontinuity behavior. This discontinuity function must also be zero for \( x > +L/2 \) and \( x < -L/2 \) and smoothly varying everywhere.
except at $x_o$. The function used is given by

$$
\varsigma(x) = \begin{cases} 
-1/2 \left( \frac{x + L/2}{\Delta} \right)^2 \left[ 3 - 2 \left( \frac{x + L/2}{\Delta} \right) \right] & -L/2 \leq x \leq -L/2 + \Delta \\
-1/2 & -L/2 + \Delta \leq x < x_o \\
+1/2 & x_o < x \leq +L/2 - \Delta \\
+1/2 \left( \frac{x - L/2}{\Delta} \right)^2 \left[ 3 + 2 \left( \frac{x - L/2}{\Delta} \right) \right] & +L/2 - \Delta \leq x \leq +L/2 
\end{cases}
$$

(6.83)

To determine the quantity $\Delta$ a compromise between two contending considerations must be found. Since the closest the feed point $x_o$ may come to the edge of the strip is $\Delta$, it should be kept as small as possible. On the other hand the Fourier transform of the discontinuity function $\varsigma(x)$ spreads out more in the spectral domain thus requiring more bandwidth as $\Delta$ becomes smaller. The computer code uses the value of $L/20$ for $\Delta$. The Fourier transform of this discontinuity function is

$$\mathcal{F}(\varsigma(x)) \equiv \eta(\alpha)$$

$$= \frac{i}{\alpha} e^{i\alpha x_o} - \Delta \left\{ \frac{12i}{(\alpha \Delta)^3} \left[ \sin(\alpha \Delta - \alpha L/2) + \sin(\alpha L/2) \right] - \frac{6i}{(\alpha \Delta)^2} \left[ \cos(\alpha L/2) + \cos(\alpha \Delta - \alpha L/2) \right] \right\}$$

(6.84)

So now subtracting $L_0 \mathcal{F}^{-1} \{ \gamma \gamma_1 G(\alpha) \eta(\alpha) \}$ from both sides of eq. (6.79) the following results:

$$\mathcal{F}^{-1} \{ \gamma \gamma_1 G(\alpha) \left[ K^s(\alpha) - L_0 \eta(\alpha) \right] \} = L_0 \mathcal{F}^{-1} \{ \gamma \gamma_1 G(\alpha) \left[ I(\alpha) - \eta(\alpha) \right] \}$$

(6.85)

where $-L/2 < x < +L/2$. Now use the conjugate gradient–FFT to solve for $K^s(\alpha) - \varsigma(x)$.

Notice that this quantity is also zero for $x > +L/2$ or $x < -L/2$. Now consider the behavior of $I(\alpha) - \eta(\alpha)$ as $\alpha$ tends to $\pm \infty$.

$$\lim_{\alpha \to \pm \infty} I(\alpha) - \eta(\alpha) = \left\{ ik^2 e^{i\alpha x_o} + \frac{6i}{\Delta^2} \left[ \cos(\alpha L/2) + \cos(\alpha \Delta - \alpha L/2) \right] \right\} \frac{1}{\alpha^3}$$

(6.86)

So the right hand side of eq. (6.85) is of order $1/\alpha^2$ as $\alpha$ tends to $\pm \infty$ and there is no problem with taking its inverse Fourier transform over a finite bandwidth.

One final note about about the $\eta(\alpha)$ function must be mentioned. This function is very ill-behaved for $\alpha$ near zero due to the catastrophic cancellation of terms in its expression.
To overcome this problem $\eta(\alpha)$ is evaluated by the following first few terms of its Taylor series expansion for small $\alpha$.

$$
\eta(\alpha) \approx -z_\alpha + \left( \frac{3}{20} \Delta^2 - \frac{1}{4} \Delta L + \frac{1}{8} L^2 - \frac{1}{2} z_\alpha^2 \right) i\alpha + \frac{1}{6} z_\alpha^2 \alpha^2
- \left( \frac{1}{168} \Delta^4 - \frac{1}{60} \Delta^3 L + \frac{3}{160} \Delta^2 L^2 - \frac{1}{96} \Delta L^3 + \frac{1}{384} L^4 - \frac{1}{24} z_\alpha^4 \right) i\alpha^3
- \frac{1}{120} z_\alpha^5 \alpha^4
$$

(6.87)

§6.6 Numerical Procedure for Associated Fields and Currents

To compute the far zone fields the function $\gamma_1 G(\alpha)[I_0 I(\alpha) - K^t(\alpha)]$ must be evaluated for various values of $\alpha$. This poses no problem for $\gamma_1$, $G(\alpha)$, and $I(\alpha)$, but the quantity $K^t(\alpha)$ is not known as an analytic function. The $K^t(\alpha)$ function is only known at discrete values of $\alpha$ which were solved for numerically. However, in the spatial domain this current must be zero for $x < -L/2$ or $x > +L/2$. Therefore from Nyquist’s sampling theorem $K^t(\alpha)$ can be reconstructed from the sampled values as long as the sampling rate is great enough. To sample at or above this Nyquist sampling rate in the spectral domain a pad of zeroes at least as wide as the strip is necessary in the spatial domain when taking the FFT. This is always the case since this size pad is required to avoid aliasing in the circular convolution (see Oppenheim and Schafer [44] chapter 3). Therefore $K^t(\alpha)$ can be found at any value of $\alpha$ within the bandwidth computed by performing a sinc function expansion interpolation of the computed discrete values.

The numerical implementation of the solution for the top surface current, the bottom surface current, and electric field in the dielectric slab is somewhat involved. The various limits discussed in Chapter IV for these spectral functions as $\alpha$ tends to $\pm k_1$ must be used since the function $I(\alpha)$ can not be directly evaluated at these values. In addition, when finding the top and bottom surface currents, special care must be taken in computing the inverse Fourier transform integrals when $\alpha$ is near $\pm \alpha_1$, the lowest order TM surface mode pole. This is necessary since $G(\alpha)$ is a rapidly varying function near this pole which is close to the real axis when the loss in the dielectric is small. In the interval $-2\alpha_{1R} < \alpha < +2\alpha_{1R}$, where $\alpha_{1R} = \Re[\alpha_1]$, these integrals are computed using Simpson’s rule under the following
transformations:

\[
\alpha = -\alpha_{1R} \sec \theta \quad \text{for} \quad -2\alpha_{1R} < \alpha < -\alpha_{1R}
\]
\[
\alpha = -\alpha_{1R} \cos \theta \quad \text{for} \quad -\alpha_{1R} < \alpha < +\alpha_{1R}
\]
\[
\alpha = +\alpha_{1R} \sec \theta \quad \text{for} \quad +\alpha_{1R} < \alpha < +2\alpha_{1R}
\] (6.88)

These transformations provide a numerically well behaved integral which can be accurately computed without requiring an excessive number of function evaluations. The contributions to the inverse Fourier transform for \( \alpha > +2\alpha_{1R} \) and \( \alpha < -2\alpha_{1R} \) are performed with the standard inverse FFT.

Finally to compute the aperture admittance recall that the unknown complex \( a_m \) and \( b_m \) coefficients must be determined from the following relation.

\[
K_x^i(x, c-) - \psi^i(x, c-) \equiv R(x) = \sum_{m=0}^{\infty} a_m A_m(x) + b_m B_m(x)
\] (6.89)

where

\[
A_m(x) \equiv e^{+i\beta_m(x+L/2)} \quad B_m(x) \equiv e^{-i\beta_m(x-L/2)}
\] (6.90)

and

\[
\beta_m = \sqrt{k_i^2 - \left( \frac{m\pi}{c} \right)^2}
\] (6.91)

The quantity \( R(x) \) is approximated by a finite number of modes and so an error term can be defined as

\[
\epsilon \equiv \sum_{n=0}^{N-1} \left| R(x_n) - \sum_{m=0}^{M} [a_m A_m(x_n) + b_m B_m(x_n)] \right|^2
\]

\[
= \sum_{n=0}^{N-1} \left\{ R(x_n) R^*(x_n) - R(x_n) \sum_{m=0}^{M} [a_m^* A_m^*(x_n) + b_m^* B_m^*(x_n)]
\right.

\left. - R^*(x_n) \sum_{m=0}^{M} [a_m^* A_m^*(x_n) + b_m^* B_m^*(x_n)]
\right.

\left. + \sum_{m=0}^{M} [a_m A_m(x_n) + b_m B_m(x_n)] \sum_{j=0}^{M} [a_j^* A_j^*(x_n) + b_j^* B_j^*(x_n)] \right\}
\] (6.92)

where the sums over \( n \) include all the points in \( x \) where \( K_x^i(x) \) has been computed. Denoting the real and imaginary parts of \( a_m \) and \( b_m \) by

\[
a_m = a_m' + ia_m'' \quad b_m = b_m' + ib_m''
\] (6.93)
find the minimum for \( \varepsilon \) as a function of \( a_m', a_m'', b_m', b_m'' \). So solve for these quantities which make the partial derivatives of \( \varepsilon \) equal zero.

\[
\frac{\partial \varepsilon}{\partial a_i'} = \sum_{n=0}^{N-1} \left\{ -R(x_n)A_i^*(x_n) - R^*(x_n)A_i(x_n) + A_i(x_n) \sum_{m=0}^{M} [a_m A_m^*(x_n) + b_m B_m^*(x_n)] \\
+ A_i^*(x_n) \sum_{m=0}^{M} [a_m A_m(x_n) + b_m B_m(x_n)] \right\}
\]

\[
\frac{\partial \varepsilon}{\partial a_i''} = \sum_{n=0}^{N-1} \left\{ R(x_n)A_i^*(x_n) - R^*(x_n)A_i(x_n) + A_i(x_n) \sum_{m=0}^{M} [a_m A_m^*(x_n) + b_m B_m^*(x_n)] \\
- A_i^*(x_n) \sum_{m=0}^{M} [a_m A_m(x_n) + b_m B_m(x_n)] \right\}
\] (6.94)

Setting these partial derivatives to zero implies

\[
0 = \Re \left\{ - \sum_{n=0}^{N-1} R(x_n)A_i^*(x_n) \\
+ \sum_{m=0}^{M} a_m \sum_{n=0}^{N-1} A_i^*(x_n)A_m(x_n) + \sum_{m=0}^{M} b_m \sum_{n=0}^{N-1} A_i^*(x_n)B_m(x_n) \right\}
\]

\[
0 = \Im \left\{ - \sum_{n=0}^{N-1} R(x_n)A_i^*(x_n) \\
+ \sum_{m=0}^{M} a_m \sum_{n=0}^{N-1} A_i^*(x_n)A_m(x_n) + \sum_{m=0}^{M} b_m \sum_{n=0}^{N-1} A_i^*(x_n)B_m(x_n) \right\}
\] (6.95)

Therefore

\[
\sum_{n=0}^{N-1} R(x_n)A_i^*(x_n) = \sum_{m=0}^{M} a_m \sum_{n=0}^{N-1} A_i^*(x_n)A_m(x_n) + \sum_{m=0}^{M} b_m \sum_{n=0}^{N-1} A_i^*(x_n)B_m(x_n)
\] (6.96)

Taking partial derivatives with respect to \( b_i' \) and \( b_i'' \) yields in a similar fashion

\[
\sum_{n=0}^{N-1} R(x_n)B_i^*(x_n) = \sum_{m=0}^{M} a_m \sum_{n=0}^{N-1} B_i^*(x_n)A_m(x_n) + \sum_{m=0}^{M} b_m \sum_{n=0}^{N-1} B_i^*(x_n)B_m(x_n)
\] (6.97)

Solving the two above equations with \( I = 1, 2, 3, \ldots, M \) for \( a_m \) and \( b_m \) entails solving a linear system of equations with \( 2(M + 1) \) unknowns. The matrix which results from this system of equations is complex Hermitian and positive definite. Therefore this system can be solved with a Cholesky decomposition and back substitution. The computer code uses LINPACK subroutines to perform this.
CHAPTER VII

RESULTS AND CONCLUSIONS

§7.1 Currents on the Patch Antenna

This section presents computed plots of the total, top, and bottom surface currents on the patch antenna. These plots show typical currents on the microstrip antenna for various parameters of strip width, excitation position, substrate thickness and dielectric constant. The thicknesses used for the dielectric substrate correspond to commercially available materials at 10 GHz. The values of dielectric constant were taken in the range of 2 to 10, once again to cover the range of typical materials used. These plots display current per unit length vs. position across the antenna measured in free space wavelengths.

Figures 7.1 - 7.3 demonstrate the behavior of the currents across the antenna when it is smaller than, at, and larger than the resonant length. The substrate has a dielectric constant and thickness of \( \varepsilon_r = 2.33 + i 0.0035 \) and \( k_c = 0.1660 \), the amplitude of the excitation current is \( I_o = 1.0 \), and the feed point is at \( z_o = -0.2L \). The resonant length for this antenna is \( L = 0.292\lambda_o \), where \( \lambda_o \) is the free space wavelength. This length is slightly less than half the wavelength in the dielectric, as is typical for a patch antenna due to the capacitive component of the aperture admittance. Notice at resonance the real part of the total current is very small while the imaginary part is at a maximum. When the length of the antenna is less than the resonance value the real part is positive and the imaginary part is smaller. When the length of the antenna is greater than at resonance the real part of the total current is negative and again the imaginary part is smaller than at resonance. Figure 7.2(b) shows the amplitude and phase of the resonant currents for comparison with the real and imaginary parts shown in figure 7.2(a).
Figure 7.1 — Antenna Currents $L = 0.278\lambda_o$
Figure 7.2 — (a) Antenna Currents $L = 0.292\lambda_o$
real and imaginary part
Figure 7.2 — (b) Antenna Currents $L = 0.292\lambda_0$

amplitude and phase
Figure 7.3 — Antenna Currents $L = 0.306\lambda_0$. 
The following set of plots, figures 7.4 - 7.6, shows the real and imaginary resonant current distributions computed for the two dimensional antenna problem as a function of feed position. The substrate has a dielectric constant of $\varepsilon_r = 2.33+i0.0035$ and a normalized thickness $kc = 0.1660$. The excitation amplitude ($I_o$) is one. As stated before, the resonant length for this structure is $L = 0.292\lambda_0$. Notice that as the current source excitation moves away from the center of the patch antenna it couples more energy into the resonant mode of the antenna. This behavior is predicted by the transmission line model discussed in Section 1.2. From the analysis of this model, at resonance

$$\Gamma e^{i\kappa L} = \eta$$

(7.1)

where $\eta$, a positive number less than one, is the magnitude of $\Gamma$, the current reflection coefficient. Therefore at resonance the current coefficients given by eq. (1.10) are

$$K_+ = \frac{1}{2(\eta^2 - 1)} [\eta e^{ik_1x_0} - e^{-ik_1x_0}]$$

$$K_- = -\frac{1}{2(\eta^2 - 1)} [\eta e^{-ik_1x_0} - e^{ik_1x_0}]$$

(7.2)

So the current on the transmission line is

$$I(x) = \begin{cases} 
\frac{1}{2} I_o + iI_o \frac{\eta}{\eta^2 - 1} \sin(2k_1x_o) & ; x > x_o \\
-\frac{1}{2} I_o + iI_o \frac{\eta}{\eta^2 - 1} \sin(2k_1x_o) & ; x < x_o 
\end{cases}$$

(7.3)

and the imaginary part of the current is proportional to $\sin(2k_1x_o)$. Also notice that as the total current becomes larger the top surface current does as well, even though it is always considerably smaller than the total.
Total Current

Bottom Current

Top Current

\[ \frac{x}{\lambda_0} \]

\[ \frac{x}{\lambda_0} \]

Figure 7.4 — Resonant Currents \( x_0 = 0.0 \)
Figure 7.5 — Resonant Currents $z_0 = -0.2L$
Figure 7.6 — Resonant Currents $x_0 = -0.4L$
The next set of plots, figures 7.7 – 7.11, displays the behavior of the currents on the patch antenna as a function of dielectric constant for the substrate. All plots are for currents at resonance. Again the feed point is taken at \( z_o = -0.2L \) and the excitation amplitude \( I_o = 1.0 \). The normalized thickness of the substrate is \( k\epsilon = 0.1660 \). The loss tangent for the dielectric substrate in all cases is 0.0015. The figure below displays peak imaginary total current versus the substrate permittivity. Notice that the magnitude of the total resonant currents versus the dielectric constant varies in a peculiar fluctuating fashion. This current magnitude assumes a maximum around \( \Re[\epsilon_r] \approx 4 \) and rapidly decreases for smaller dielectric constants. This type of behavior may have some practical implications on the efficiency of a patch antenna for a given substrate thickness.

![Plot of \( \Re[\epsilon_r] \) against \( \epsilon_r \)](image)

**Peak Imaginary Total Current vs. Permittivity**
Figure 7.7 — Resonant Currents $\epsilon_r = 2.00 + i 0.0030 \quad L = 0.314 \lambda_o$
Figure 7.8 — Resonant Currents  \( \epsilon_r = 2.33 + i.0035 \quad L = 0.292\lambda_0 \)
Figure 7.9 — Resonant Currents \( \varepsilon = 3.80 + i 0.0057 \) \( L = 0.219\lambda_o \)
Figure 7.10 — Resonant Currents  \( \epsilon_r = 6.15 + i0.092 \)  \( L = 0.172\lambda_0 \)
Figure 7.11 — Resonant Currents  \( \varepsilon_r = 10.00 + i.0015 \quad L = 0.132 \lambda_0 \)
The final set of plots in this section, figures 7.12 – 7.17, shows the behavior of the currents on the antenna as the thickness of the substrate is varied. All these plots are at resonance. The dielectric constant of the substrate \( \varepsilon_r = 2.33 + i0.035 \), the feed point \( x_o = -0.2L \), and the excitation amplitude \( I_o = 1.0 \). As the plot below of total peak imaginary current versus substrate thickness shows, the resonant current strength of the antenna diminishes as the thickness of the substrate increases. Therefore these currents verify the conclusion which was drawn in Chapter II from the Wiener-Hopf analysis that the antenna should act more like a perfect cavity as the thickness of the substrate decreased. Notice also that as the substrate thickness increases, the total current at resonance decreases dramatically, and the top surface current actually increases slightly.

![Peak Imaginary Total Current vs. Substrate Thickness](image-url)
Figure 7.12 — Resonant Currents  \( kc = 0.0266 \quad L = 0.321 \lambda_0 \)
Figure 7.13 — Resonant Currents  \( kc = 0.0532 \quad L = 0.315\lambda_o \)
Figure 7.14 — Resonant Currents  \( k_c = 0.0831 \)  \( L = 0.308\lambda_o \)
Figure 7.15 — Resonant Currents  \( kc = 0.1660 \)  \( L = 0.292\lambda_o \)
Figure 7.16 — Resonant Currents

\[ kc = 0.2490 \quad L = 0.278 \lambda_o \]
Figure 7.17 — Resonant Currents $k_c = 0.3320$ $L = 0.265\lambda_o$
In conclusion, these numerical results show that at resonance the total current on the patch antenna becomes large and is in phase quadrature with the excitation. In addition for a fixed excitation current magnitude, larger currents will be induced on the antenna as the source point is moved towards the edge of the antenna. The resonance effect is also more pronounced for thinner dielectric substrates. The top surface current is typically considerably smaller than the total current for the thin substrate antenna, however at resonance when the total current becomes large the top surface current also increases. With respect to the total current, the top surface current becomes more significant as the thickness of the dielectric substrate increases.

§7.2 Radiation from Patch Antenna

Once the total current is known on the patch antenna the far zone fields can be computed from the results presented in Section 4.2 for the two dimensional problem and the results in Section 5.3 for the three dimensional problem. These formulas give the exact far field radiation for the patch antenna in terms of the currents on the antenna and are summarized below.

For the two dimensional problem:

$$\lim_{\rho \to \infty} H_y = e^{-ikx} e^{ik\phi} \sqrt{\frac{k}{2\pi \rho} \frac{h_1 \sin(h_1 c) \cos \theta}{h_1 \sin(h_1 c) + i\epsilon, k \cos \theta \cos(h_1 c)}} \times \left[ I_o \frac{ik \sin \theta}{h_1^2} e^{-ikx \sin \theta} - \tilde{K}^t(-k \sin \theta) \right]$$

(7.4)

where

$$h_1 \equiv \sqrt{k_1^2 - k^2 \sin^2 \theta}$$

(7.5)

and $\tilde{K}^t(\alpha)$ is the Fourier transform of the current on the patch.

For the three dimensional problem:

$$\lim_{R \to \infty} E_\theta = e^{ikR} \frac{h_1 \sin(h_1 c) \cos \theta}{2\pi R h_1 \sin(h_1 c) + i\epsilon, k \cos \theta \cos(h_1 c)} \times \left[ I_o \frac{ik \sin \theta}{h_1^2} e^{-ik \sin \theta (\cos \phi x_0 + \sin \phi y_0)} + \cos \phi \tilde{K}_{x_0} + \sin \phi \tilde{K}_{y_0} \right]$$

$$\lim_{R \to \infty} E_\phi = e^{ikR} \frac{k \cos(h_1 c) \cos \theta}{2\pi R h_1 \cos(h_1 c) - ik \cos \theta \sin(h_1 c)} \left[ \sin \phi \tilde{K}_{x_0} - \cos \phi \tilde{K}_{y_0} \right]$$

(7.6)
where

\[
\begin{align*}
\vec{K}_x &= \vec{K}_x(-k \sin \theta \cos \phi, -k \sin \theta \sin \phi) \\
\vec{K}_y &= \vec{K}_y(-k \sin \theta \cos \phi, -k \sin \theta \sin \phi)
\end{align*}
\] (7.7)

and \(\vec{K}_x(\alpha_x, \alpha_y)\) and \(\vec{K}_y(\alpha_x, \alpha_y)\) are the two dimensional Fourier transforms of the \(x\) and \(y\) components of current on the patch antenna. The magnitude of the excitation current sheet for the two dimensional case or the excitation line current for the three dimensional case is \(I_o\). The excitation current is located at \(x_o\) for the two dimensional case and at \((x_o, y_o)\) for the three dimensional problem.

The traditional method for predicting the far zone fields for a microstrip patch antenna consists of modeling the radiation from slots at the edge of the antenna. This approximate technique is presented in many standard modern texts on antenna theory such as Balanis [48] or Johnson and Jasik [49]. This technique makes several approximate assumptions however, and completely ignores the radiation due to the top surface current on the patch. Since the formulas presented in this section are exact, they should be used instead of the radiating slot model which is not. As can be seen from the plots of current in Section 7.1, the top surface current on the thin patch antenna is very small in comparison to the total current. Therefore, for the thin patch antenna the far zone fields may be computed fairly accurately from the bottom surface current alone. This is fortunate since the bottom surface current may be obtained from the simple leaky cavity or transmission line model. However, if the substrate is thicker, the top surface current becomes significant. To find the far zone radiated fields for such an antenna, the total current on the patch must be known and the leaky cavity or transmission line models will no longer work.

The magnitude of the far zone fields due to the top surface current and the excitation current, along with the total far zone fields are shown in figure 7.18 for a typical thin two dimensional patch antenna. The dielectric constant of the substrate is \(\varepsilon_r = 2.33 + i0.0035\), its normalized thickness \(k_e = 0.1660\), the length of the patch \(L = 0.292\lambda_o\), excitation magnitude \(I_o = 1.0\), and position \(x_o = -0.2L\). The far zone fields for a patch antenna with a thicker substrate where the top surface current begins to become important are shown in figure 7.19. Figure 7.19 has a substrate dielectric constant \(\varepsilon_r = 10.0 + i0.0150\), normalized
Figure 7.18 — Far Zone Pattern in the Transverse Plane for Thin Antenna
Figure 7.19 — Far Zone Pattern in the Transverse Plane for Thicker Antenna
substrate thickness $k_c = 0.3320$, patch length of $L = .105\lambda_0$, excitation magnitude of $I_o = 1.0$, and position $x_o = -0.2L$.

The far field pattern of the two dimensional patch antenna will be symmetric if the phase of the total current on the patch antenna is constant. For the thin patch antenna at resonance this will be approximately true. The bottom surface current will be large compared to the top surface current and it will essentially be in phase quadrature with the excitation. The total far field patterns for these antennas are slightly asymmetric since the excitation current is not located at the center of the antenna and the total current on the antenna is not exactly in phase. In general, the two dimensional patch antennas will radiate slightly more energy off to the side of the antenna closest to the excitation current. This asymmetric radiation becomes more pronounced as the thickness of the dielectric substrate increases and the phase variation of the total current becomes larger. As can be seen in figure 7.19 the radiation due to the top surface current is beginning to become significant for the thicker substrate as well. Therefore, if the substrate of the patch antenna is thin enough the far zone fields will be symmetric and can be computed from the bottom surface current alone, which may be found with the leaky cavity or transmission line model. However if the thickness becomes greater the top surface current must also be known to predict the far zone fields which in general will be asymmetric. In either case, the radiation formulas presented here should be used in preference to the radiating slot model.

§7.3 Aperture Admittance, Resonant Length, and $Q$

This section presents numerical values found for the reflection coefficient, resonant length, and $Q$ of the idealized two dimensional model of the patch antenna. The TEM magnetic reflection coefficient can be related to the radiating aperture admittance per unit length by

$$Y_r = Y_e \frac{1 + R_o}{1 - R_o}$$

(7.8)

where $Y_e = \frac{1}{2} \sqrt{\varepsilon_r \varepsilon_0 / \mu_0}$ is the characteristic admittance per unit length of the antenna and $R_o = \eta e^{i\phi}$ is the reflection coefficient. Therefore only the reflection coefficient will be tabulated here. From the reflection coefficient the resonant length of the antenna and its $Q$ can be computed from the transmission line model as outlined in Chapter I. Tables 7.1 – 7.5
list the magnitude and phase of the reflection coefficient, the resonant length of the antenna, and its $Q$ which were computed numerically for various thicknesses and dielectric constants of the substrate.

| Table 7.1 — Resonance Parameters $\epsilon_r = 2.00 + i0.0030$ |
|---|---|---|---|---|
| $kc$ | $\eta$ | $\psi$ | Length | $Q$ |
| 0.0266 | 0.9840 | 3.206 | 0.346$\lambda_0$ | 48.68 |
| 0.0532 | 0.9686 | 3.269 | 0.339$\lambda_0$ | 24.60 |
| 0.0831 | 0.9518 | 3.332 | 0.332$\lambda_0$ | 15.69 |
| 0.1660 | 0.9066 | 3.490 | 0.314$\lambda_0$ | 8.01 |
| 0.2490 | 0.8627 | 3.631 | 0.298$\lambda_0$ | 5.32 |
| 0.3320 | 0.8207 | 3.765 | 0.283$\lambda_0$ | 3.96 |

| Table 7.2 — Resonance Parameters $\epsilon_r = 2.33 + i0.0035$ |
|---|---|---|---|---|
| $kc$ | $\eta$ | $\psi$ | Length | $Q$ |
| 0.0266 | 0.9754 | 3.204 | 0.321$\lambda_0$ | 31.52 |
| 0.0532 | 0.9522 | 3.265 | 0.315$\lambda_0$ | 16.04 |
| 0.0831 | 0.9274 | 3.327 | 0.308$\lambda_0$ | 10.43 |
| 0.1660 | 0.8637 | 3.479 | 0.292$\lambda_0$ | 5.38 |
| 0.2490 | 0.8049 | 3.614 | 0.278$\lambda_0$ | 3.64 |
| 0.3320 | 0.7500 | 3.743 | 0.265$\lambda_0$ | 2.75 |

| Table 7.3 — Resonance Parameters $\epsilon_r = 3.80 + i0.0057$ |
|---|---|---|---|---|
| $kc$ | $\eta$ | $\psi$ | Length | $Q$ |
| 0.0266 | 0.9832 | 3.215 | 0.251$\lambda_0$ | 46.43 |
| 0.0532 | 0.9694 | 3.288 | 0.246$\lambda_0$ | 25.27 |
| 0.0831 | 0.9564 | 3.366 | 0.238$\lambda_0$ | 17.61 |
| 0.1660 | 0.9290 | 3.571 | 0.221$\lambda_0$ | 10.64 |
| 0.2490 | 0.9100 | 3.768 | 0.205$\lambda_0$ | 8.24 |
| 0.3320 | 0.9009 | 3.966 | 0.189$\lambda_0$ | 7.33 |

| Table 7.4 — Resonance Parameters $\epsilon_r = 6.15 + i0.0092$ |
|---|---|---|---|---|
| $kc$ | $\eta$ | $\psi$ | Length | $Q$ |
| 0.0266 | 0.9823 | 3.219 | 0.197$\lambda_0$ | 44.07 |
| 0.0532 | 0.9631 | 3.286 | 0.192$\lambda_0$ | 24.20 |
| 0.0831 | 0.9546 | 3.380 | 0.186$\lambda_0$ | 16.90 |
| 0.1660 | 0.9240 | 3.598 | 0.172$\lambda_0$ | 9.90 |
| 0.2490 | 0.8954 | 3.809 | 0.159$\lambda_0$ | 7.02 |
| 0.3320 | 0.8672 | 4.019 | 0.145$\lambda_0$ | 5.34 |
Table 7.5 — Resonance Parameters \( \epsilon_r = 10.00 + i0.0150 \)

<table>
<thead>
<tr>
<th>( kc )</th>
<th>( \eta )</th>
<th>( \psi )</th>
<th>Length</th>
<th>( Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0266</td>
<td>0.9458</td>
<td>3.246</td>
<td>0.153( \lambda_0 )</td>
<td>14.10</td>
</tr>
<tr>
<td>0.0532</td>
<td>0.9330</td>
<td>3.328</td>
<td>0.149( \lambda_0 )</td>
<td>11.35</td>
</tr>
<tr>
<td>0.0831</td>
<td>0.9231</td>
<td>3.422</td>
<td>0.144( \lambda_0 )</td>
<td>9.82</td>
</tr>
<tr>
<td>0.1660</td>
<td>0.8956</td>
<td>3.680</td>
<td>0.131( \lambda_0 )</td>
<td>7.08</td>
</tr>
<tr>
<td>0.2490</td>
<td>0.8537</td>
<td>3.934</td>
<td>0.118( \lambda_0 )</td>
<td>4.86</td>
</tr>
<tr>
<td>0.3320</td>
<td>0.7895</td>
<td>4.189</td>
<td>0.105( \lambda_0 )</td>
<td>3.16</td>
</tr>
</tbody>
</table>

Other approximate expressions exist for these reflection coefficients which have been derived from the Wiener-Hopf analysis method. In particular Kuster, Johnk, and Chang [50] and Chang and Kuester [51] have published work in this area. The expressions proposed by these authors tend to be somewhat complicated, however, since the evaluation of the integrals involved with this method is difficult. Sengupta [7] has proposed a simple approximate expression for the aperture admittance based on physical concepts.

The expression for the magnetic mode reflection coefficient for this problem given by Chang and Kuester is

\[
R_0 = -\epsilon^{i\chi(0)}
\]

(7.9)

where

\[
\chi(0) = \frac{2k_1c}{\pi} \left\{ \frac{1}{\epsilon_r} [1 - \Gamma - \ln(-ikc)] + \ln(2\pi) - 2Q_o(-\delta_c) \right\}
\]

(7.10)

and

\[
Q_o(\varepsilon) \equiv \sum_{m=1}^{\infty} \varepsilon^m \ln(m) \quad \delta_c \equiv \frac{\epsilon_r - 1}{\epsilon_r + 1}
\]

(7.11)

with Euler’s number \( \Gamma \approx 0.57721566490153 \). This approximate expression should be valid as long as \( k_1c \ll 1 \).

Sengupta’s expression for the normalized radiating aperture admittance per unit length for the two dimensional problem of consideration in this thesis is given by

\[
Y_r = Y_e(G - iB)
\]

(7.12)

where

\[
G = \frac{k_1c}{2\epsilon_r}
\]

\[
B = \frac{k_1c}{\pi \epsilon_r} \left[ 1 - \Gamma + \ln \left( \frac{2\pi}{kc} \right) \right]
\]

(7.13)
and \( \Gamma \) is Euler's number once again. Therefore the reflection coefficient is given by

\[
R_o = \frac{G - iB - 1}{G - iB + 1}
\]  

(7.14)

Sengupta's formula should also be valid for the thin patch antenna where \( k_c c \ll 1 \).

Comparisons of the numerical results, Sengupta's approximate formula, and Chang and Kuester's approximate formula are presented in tables 7.6 - 7.10 for substrates with various dielectric constants and thicknesses.

**Table 7.6 — Reflection Coefficient  \( \epsilon_r = 2.00 + i0.0030 \)**

<table>
<thead>
<tr>
<th>( k_c )</th>
<th>Numerical</th>
<th>Chang &amp; Kuester</th>
<th>Sengupta</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>( \psi )</td>
<td>( \eta )</td>
<td>( \psi )</td>
</tr>
<tr>
<td>0.0266</td>
<td>0.9840</td>
<td>3.206</td>
<td>0.9814</td>
</tr>
<tr>
<td>0.0332</td>
<td>0.9668</td>
<td>3.269</td>
<td>0.9631</td>
</tr>
<tr>
<td>0.0331</td>
<td>0.9518</td>
<td>3.332</td>
<td>0.9429</td>
</tr>
<tr>
<td>0.1660</td>
<td>0.9066</td>
<td>3.490</td>
<td>0.8892</td>
</tr>
<tr>
<td>0.2490</td>
<td>0.8627</td>
<td>3.631</td>
<td>0.8386</td>
</tr>
<tr>
<td>0.3320</td>
<td>0.8207</td>
<td>3.765</td>
<td>0.7908</td>
</tr>
</tbody>
</table>

**Table 7.7 — Reflection Coefficient  \( \epsilon_r = 2.33 + i0.0035 \)**

<table>
<thead>
<tr>
<th>( k_c )</th>
<th>Numerical</th>
<th>Chang &amp; Kuester</th>
<th>Sengupta</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>( \psi )</td>
<td>( \eta )</td>
<td>( \psi )</td>
</tr>
<tr>
<td>0.0266</td>
<td>0.9754</td>
<td>3.204</td>
<td>0.9827</td>
</tr>
<tr>
<td>0.0332</td>
<td>0.9522</td>
<td>3.265</td>
<td>0.9657</td>
</tr>
<tr>
<td>0.0331</td>
<td>0.9274</td>
<td>3.327</td>
<td>0.9470</td>
</tr>
<tr>
<td>0.1660</td>
<td>0.8637</td>
<td>3.479</td>
<td>0.8970</td>
</tr>
<tr>
<td>0.2490</td>
<td>0.8049</td>
<td>3.614</td>
<td>0.8495</td>
</tr>
<tr>
<td>0.3320</td>
<td>0.7500</td>
<td>3.743</td>
<td>0.8045</td>
</tr>
</tbody>
</table>

**Table 7.8 — Reflection Coefficient  \( \epsilon_r = 3.80 + i0.0057 \)**

<table>
<thead>
<tr>
<th>( k_c )</th>
<th>Numerical</th>
<th>Chang &amp; Kuester</th>
<th>Sengupta</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>( \psi )</td>
<td>( \eta )</td>
<td>( \psi )</td>
</tr>
<tr>
<td>0.0266</td>
<td>0.9832</td>
<td>3.215</td>
<td>0.9864</td>
</tr>
<tr>
<td>0.0332</td>
<td>0.9694</td>
<td>3.258</td>
<td>0.9731</td>
</tr>
<tr>
<td>0.0331</td>
<td>0.9564</td>
<td>3.306</td>
<td>0.9583</td>
</tr>
<tr>
<td>0.1660</td>
<td>0.9260</td>
<td>3.571</td>
<td>0.9154</td>
</tr>
<tr>
<td>0.2490</td>
<td>0.9100</td>
<td>3.768</td>
<td>0.8801</td>
</tr>
<tr>
<td>0.3320</td>
<td>0.9009</td>
<td>3.966</td>
<td>0.8434</td>
</tr>
</tbody>
</table>
Table 7.9 — Reflection Coefficient \( \varepsilon_r = 6.15 + i0.0092 \)

<table>
<thead>
<tr>
<th>kc</th>
<th>Numerical</th>
<th>Chang &amp; Kuster</th>
<th>Sengupta</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>( \psi )</td>
<td>( \eta )</td>
<td>( \psi )</td>
</tr>
<tr>
<td>0.0266</td>
<td>0.9223</td>
<td>3.219</td>
<td>0.9893</td>
</tr>
<tr>
<td>0.0532</td>
<td>0.9681</td>
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<td>0.9788</td>
</tr>
<tr>
<td>0.0831</td>
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<td>0.9670</td>
</tr>
<tr>
<td>0.1660</td>
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<td>0.9353</td>
</tr>
<tr>
<td>0.2490</td>
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<td>3.809</td>
<td>0.9045</td>
</tr>
<tr>
<td>0.3320</td>
<td>0.8672</td>
<td>4.019</td>
<td>0.8747</td>
</tr>
</tbody>
</table>

Table 7.10 — Reflection Coefficient \( \varepsilon_r = 10.00 + i0.0150 \)

<table>
<thead>
<tr>
<th>kc</th>
<th>Numerical</th>
<th>Chang &amp; Kuster</th>
<th>Sengupta</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>( \psi )</td>
<td>( \eta )</td>
<td>( \psi )</td>
</tr>
<tr>
<td>0.0266</td>
<td>0.9458</td>
<td>3.246</td>
<td>0.9916</td>
</tr>
<tr>
<td>0.0532</td>
<td>0.9330</td>
<td>3.328</td>
<td>0.9833</td>
</tr>
<tr>
<td>0.0831</td>
<td>0.9231</td>
<td>3.422</td>
<td>0.9741</td>
</tr>
<tr>
<td>0.1660</td>
<td>0.8956</td>
<td>3.680</td>
<td>0.9489</td>
</tr>
<tr>
<td>0.2490</td>
<td>0.8537</td>
<td>3.934</td>
<td>0.9243</td>
</tr>
<tr>
<td>0.3320</td>
<td>0.7895</td>
<td>4.189</td>
<td>0.9003</td>
</tr>
</tbody>
</table>

These approximate expressions are in good agreement with the numerical results presented here, except for the largest dielectric constants and thicknesses of the substrate. For these cases the numerical results yield reflection coefficients of smaller magnitude and greater phase angles than those predicted by Sengupta. The results of Chang and Kuster agree with the numerical results quite well in phase for all the values presented here. However, they tend to predict reflection coefficient magnitudes somewhat larger than the numerical results for the thickest substrates with a dielectric constant of 10. Overall however, the numerical results seem to agree with the approximate expressions, with somewhat better agreement to Chang and Kuster.

§7.4 Surface Wave Modes

Consider the two dimensional surface wave modes which can exist in a lossless dielectric substrate over a ground plane. Two types of surface wave mode can exist, the TM and TE modes. The electric and magnetic fields for the TE and TM modes are given by
where \( \psi(x, z) = \tilde{\psi}(z)e^{\iota \alpha z} \). Since, for the lossless dielectric substrate, the surface wave should not attenuate as it propagates, the quantity \( \alpha \) must be real. To satisfy Maxwell's equations in the free space above the dielectric substrate when \( z > c \),

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \psi(x, z) + k^2 \psi(x, z) = 0 \quad ; \quad z > c
\]

(7.16)

and so

\[
\frac{\partial^2}{\partial x^2} \tilde{\psi}(z) + (k^2 - \alpha^2) \tilde{\psi}(z) = 0 \quad ; \quad z > c
\]

(7.17)

Therefore the solution for \( \tilde{\psi}(z) \) is of the form

\[
\tilde{\psi}(z) = C_1 e^{-\iota \alpha z} \quad ; \quad z > c
\]

(7.18)

where \( \Re{[p]} > 0 \) so that the energy in the surface wave is concentrated in the region of the dielectric substrate, \( C_1 \) is some constant and

\[
k^2 - \alpha^2 + p^2 = 0
\]

(7.19)

Since \( k \) and \( \alpha \) are real quantities and \( \Re{[p]} > 0 \), it follows that \( p \) is a real positive quantity.

To satisfy Maxwell's equations in the dielectric substrate where \( 0 < z < c \),

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \psi(x, z) + k_1^2 \psi(x, z) = 0 \quad ; \quad 0 < z < c
\]

(7.20)

so

\[
\frac{\partial^2}{\partial x^2} \tilde{\psi}(z) + (k_1^2 - \alpha^2) \tilde{\psi}(z) = 0 \quad ; \quad 0 < z < c
\]

(7.21)

Therefore the solution for \( \tilde{\psi}(z) \) is of the form

\[
\tilde{\psi}(z) = C_2 e^{i\theta z} + C_3 e^{-i\theta z} \quad ; \quad 0 < z < c
\]

(7.22)

where \( C_1 \) and \( C_2 \) are some constants and

\[
k_1^2 - \alpha^2 - k^2 = 0
\]

(7.23)
Since $k_1^2 = \varepsilon, k^2$ and for a lossless dielectric $\varepsilon$, it follows that $h^2$ is real. Furthermore

$$ h^2 + p^2 = k^2(\varepsilon - 1) > 0 \quad (7.24) $$

Consider now the case of TE modes.

$$ \overrightarrow{E} = -i\alpha\overrightarrow{\psi}(z)e^{i\alpha z} \hat{y} $$

$$ i\omega\mu\overrightarrow{H} = \left[ i\alpha\frac{\partial}{\partial z}\overrightarrow{\psi}(z)\hat{z} + \alpha^2\overrightarrow{\psi}(z)\hat{z} \right] e^{i\alpha z} \quad (7.25) $$

Now satisfy the various boundary conditions. At $z = 0$ the electric field $\overrightarrow{E} \cdot \hat{y} = 0$, therefore

$$ C_3 = -C_2 \quad (7.26) $$

For $\overrightarrow{E} \cdot \hat{y}$ and $\overrightarrow{H} \cdot \hat{z}$ to be continuous at $z = c$

$$ C_1 e^{-pc} = C_2 e^{i\hbar c} + C_3 e^{-i\hbar c} = 2i \sin(\hbar c) \quad (7.27) $$

For $\overrightarrow{H} \cdot \hat{z}$ to be continuous at $z = c$

$$ 2i\hbar C_2 \cos(\hbar c) = -C_1 p e^{-pc} \quad (7.28) $$

Therefore

$$ 2i\hbar C_2 \cos(\hbar c) = -2iC_2 p \sin(\hbar c) $$

$$ \hbar \cos(\hbar c) = -p \sin(\hbar c) \quad (7.29) $$

Rearranging these relations yields the characteristic equations

$$ (\hbar c)^2 + (pc)^2 = (k\hbar)^2(\varepsilon - 1) $$

$$ pc = -\hbar c \cot(\hbar c) \quad (7.30) $$

which must be solved simultaneously for $pc$ and $\hbar c$. Since $h^2$ must be real, $h$ can be either imaginary or real. Consider the case where $h = i\eta$ an imaginary quantity. Then

$$ pc = -\eta \hbar c \coth(\eta c) \quad (7.31) $$

where $\eta$ is some real number. Notice that no value for $\eta$ will yield a positive $p$ and therefore no solutions for an imaginary $h$ exist. Now consider the case when $h$ is real. Since the
function $hc \cot(hc)$ is an even function, the negative of any solution for $h$ is also a solution. These two solutions physically correspond to the same surface wave mode, therefore examine only the positive real solutions of $h$. The positive real solutions for $p$ and $h$ to the characteristic equations have the graphical solution shown in figure 7.20. They are the intersections of the curves describing each characteristic equation in the first quadrant of the $hc - pc$ plane. Notice that if $kc\sqrt{\varepsilon_r - 1} < \pi/2$ no solutions exist. Therefore for the thin patch antenna problem the TE modes will not exist.

![Figure 7.20 — TE Surface Mode Chart](image)

Now consider the TM modes.

\[
\bar{H} = -i\alpha \tilde{\psi}(x)e^{i\alpha z}\hat{y} \\
-i \omega \varepsilon \bar{E} = \left[ i\alpha \frac{\partial}{\partial z} \tilde{\psi}(x)\hat{z} + \alpha^2 \tilde{\psi}(x)\hat{z} \right] e^{i\alpha z}
\]  
(7.32)
Again, satisfy the boundary conditions. At $z = 0$ the electric field $\mathbf{E} \cdot \hat{z} = 0$ therefore

$$C_2 = C_3 \quad (7.33)$$

For $\mathbf{H} \cdot \hat{y}$ and $\epsilon \mathbf{E} \cdot \hat{z}$ to be continuous at $z = c$

$$C_1 e^{-pc} = C_2 e^{i\eta c} + C_3 e^{-i\eta c} = 2C_2 \cos(\eta c) \quad (7.34)$$

For $\mathbf{E} \cdot \hat{z}$ to be continuous at $z = c$

$$-\epsilon_r pC_1 e^{-pc} = -2C_2 \eta \sin(\eta c)$$

$$\epsilon_r p \cos(\eta c) = \eta \sin(\eta c) \quad (7.35)$$

This is the same characteristic equation as given in Chapter II eq. (2.61). Rearranging these relations yields the characteristic equations

$$(hc)^2 + (pc)^2 = (k_c)^2 (\epsilon_r - 1)$$

$$pc = \frac{1}{\epsilon_r} hc \tan(\eta c) \quad (7.36)$$

which must be solved for $hc$ and $pc$. If $\eta = i\eta$ an imaginary quantity,

$$pc = -\frac{1}{\epsilon_r} \eta c \tanh(\eta c) \quad (7.37)$$

Once again no imaginary solutions for $h$ exist which yield a positive value for $p$. Also, as for the TE modes, only the positive real solutions for $h$ need be considered. The graphical solution to the characteristic equations for the TM modes is shown in figure 7.21. Notice that if $k_c \sqrt{\epsilon_r - 1} < \pi$ only one solution exits. Therefore for the thin patch antenna problem only the lowest order TM mode can exist. The exact values for $p$ and $h$ of this mode can be found with Newton's method. However with an electrically thin dielectric substrate so $k_c \sqrt{\epsilon_r - 1} \ll 1$ the solution for $p$ is approximately given by

$$p_1 \approx \frac{\epsilon_r - 1}{\epsilon_r} k^2 c \quad (7.38)$$

and the corresponding propagation constant of the surface wave is

$$\alpha_1 = \sqrt{k^2 + p_1^2} \quad (7.39)$$
This propagation constant must be in the range

\[ k < \alpha_1 < k_1 \]  \hspace{1cm} (7.40)

Now consider how this lowest order TM surface mode affects the behavior of the two dimensional patch antenna. For field quantities in the spectral domain, a surface mode corresponds to a pole in the complex \( \alpha \)-plane at the propagation constant for that surface mode. In general, with a lossy dielectric substrate, these surface mode poles will occur in pairs, \( \pm \alpha_n \), one pole in the upper half of the \( \alpha \) plane and the other in the lower half. The lower half plane poles corresponding to surface modes propagating in the positive \( z \) direction, upper half plane poles corresponding to negative \( z \) propagating modes. The contributions of these poles to the fields can be computed, when performing the Fourier inversion integral, with Cauchy's residue theorem. The contour of integration along the real
\( \alpha \) axis can be closed at infinity in the upper half plane for \( z < -L/2 \) or the lower half plane for \( z > +L/2 \). In the limit as the losses in the dielectric go to zero, the surface mode poles migrate to the real \( \alpha \) axis, so that these modes will not attenuate with propagation.

The total magnetic field in the spectral domain is given by

\[
\Psi^s(\alpha, z) = A(\alpha)e^{-\gamma z}
\]

\[
= \gamma_1 G(\alpha) \left[ I_0 I(\alpha) - \tilde{K}^4(\alpha) \right] e^{-\gamma(z-c)} \quad ; \quad z > c
\]

\[
\Psi^s(\alpha, z) = I_0 I(\alpha) + C(\alpha) \cosh(\gamma_1 z)
\]

(7.41)

\[
= I_0 I(\alpha) - \epsilon_\gamma G(\alpha) \left[ I_0 I(\alpha) - \tilde{K}^4(\alpha) \right] \frac{\cosh(\gamma_1 z)}{\sinh(\gamma_1 c)} \quad ; \quad 0 < z < c
\]

For a lossless dielectric substrate, these functions are finite for real \( \alpha \) except at the poles \( \alpha = \pm \alpha_1 \). Therefore use Cauchy's residue theorem to compute the surface mode component of the total magnetic field.

for \( z < -L/2 \);

\[
H_s(z, z) = T_0 \left[ I_0 I(\alpha_1) - \tilde{K}^4(\alpha_1) \right] e^{-\gamma(z-c)} e^{-i\alpha_1 z} \quad ; \quad z > c
\]

(7.42)

\[
H_s(z, z) = +T_1 \left[ I_0 I(\alpha_1) - \tilde{K}^4(\alpha_1) \right] \cosh(\gamma_1 z) e^{-i\alpha_1 z} \quad ; \quad 0 < z < c
\]

for \( z > +L/2 \);

\[
H_s(z, z) = -T_0 \left[ I_0 I(-\alpha_1) - \tilde{K}^4(-\alpha_1) \right] e^{-\gamma(z-c)} e^{i\alpha_1 z} \quad ; \quad z > c
\]

(7.43)

\[
H_s(z, z) = -T_1 \left[ I_0 I(-\alpha_1) - \tilde{K}^4(-\alpha_1) \right] \cosh(\gamma_1 z) e^{i\alpha_1 z} \quad ; \quad 0 < z < c
\]

where the surface wave coefficients are defined as

\[
T_0 \equiv i\gamma_1 R \quad T_1 \equiv -i\epsilon_\gamma R / \sinh(\gamma_1 c)
\]

(7.44)

and the quantities \( \gamma \) and \( \gamma_1 \) are evaluated at \( \alpha_1 \). The residue of \( G(\alpha) \) evaluated at \( \alpha_1 \) is \( R \) so

\[
R \equiv \lim_{\alpha \to \alpha_1} (\alpha - \alpha_1) G(\alpha)
\]

(7.45)
Notice that the surface wave coefficients ($T_0$ and $T_1$) are independent of the excitation current magnitude ($I_0$), its position ($x_0$), and width of the strip ($L$); they just depend on $\varepsilon_r$ and the thickness of the substrate ($c$). For the thin patch antenna with typical dielectric constants in the range of $1 < \varepsilon_r < 10$, the surface wave coefficients are typically small and so these surface wave modes are not of great importance in the near field. This is so since $G(\alpha)$ becomes zero at $\alpha = k_1$ and the surface mode pole is near $k_1$, particularly for small dielectric constants and thin substrates, therefore the residue $R$ tends to be small. In the limit as $\varepsilon_r$ tends to 1, $\alpha_1 = k_1$ and no surface mode pole exists. The following tables list the pole locations, residues, and surface wave coefficients for various typical patch antenna parameters of $kc$ and $\varepsilon_r$.

Table 7.11 — Surface Mode Parameters $\varepsilon_r = 2.00$

<table>
<thead>
<tr>
<th>$kc$</th>
<th>$\alpha_1/k$</th>
<th>$R$</th>
<th>$T_0/k$</th>
<th>$T_1/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0266</td>
<td>1.000088</td>
<td>i0.0001768</td>
<td>i0.0001768</td>
<td>i0.0001769</td>
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<tr>
<td>0.0533</td>
<td>1.000354</td>
<td>i0.0007069</td>
<td>i0.0007067</td>
<td>i0.0007077</td>
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<tr>
<td>0.0831</td>
<td>1.000864</td>
<td>i0.0017244</td>
<td>i0.0017209</td>
<td>i0.0017268</td>
</tr>
<tr>
<td>0.1660</td>
<td>1.003454</td>
<td>i0.0068241</td>
<td>i0.0068004</td>
<td>i0.0068946</td>
</tr>
<tr>
<td>0.2490</td>
<td>1.007795</td>
<td>i0.0151614</td>
<td>i0.0150423</td>
<td>i0.0155132</td>
</tr>
<tr>
<td>0.3320</td>
<td>1.013910</td>
<td>i0.0264423</td>
<td>i0.0260693</td>
<td>i0.0275310</td>
</tr>
</tbody>
</table>

Table 7.12 — Surface Mode Parameters $\varepsilon_r = 2.33$

<table>
<thead>
<tr>
<th>$kc$</th>
<th>$\alpha_1/k$</th>
<th>$R$</th>
<th>$T_0/k$</th>
<th>$T_1/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0266</td>
<td>1.001115</td>
<td>i0.0001999</td>
<td>i0.0002305</td>
<td>i0.0002306</td>
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<tr>
<td>0.0532</td>
<td>1.000461</td>
<td>i0.0007993</td>
<td>i0.0009215</td>
<td>i0.0009232</td>
</tr>
<tr>
<td>0.0831</td>
<td>1.001127</td>
<td>i0.0019492</td>
<td>i0.0022460</td>
<td>i0.0022564</td>
</tr>
<tr>
<td>0.1660</td>
<td>1.004628</td>
<td>i0.0077533</td>
<td>i0.0091110</td>
<td>i0.0090757</td>
</tr>
<tr>
<td>0.2490</td>
<td>1.010291</td>
<td>i0.0173331</td>
<td>i0.0198334</td>
<td>i0.0206686</td>
</tr>
<tr>
<td>0.3320</td>
<td>1.018540</td>
<td>i0.0304646</td>
<td>i0.0346356</td>
<td>i0.0372584</td>
</tr>
</tbody>
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Table 7.13 — Surface Mode Parameters $\varepsilon_r = 3.80$

<table>
<thead>
<tr>
<th>$kc$</th>
<th>$\alpha_1/k$</th>
<th>$R$</th>
<th>$T_0/k$</th>
<th>$T_1/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0266</td>
<td>1.000192</td>
<td>i0.0002297</td>
<td>i0.0003844</td>
<td>i0.0003848</td>
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<tr>
<td>0.0532</td>
<td>1.000771</td>
<td>i0.0009207</td>
<td>i0.0015402</td>
<td>i0.0015464</td>
</tr>
<tr>
<td>0.0831</td>
<td>1.001892</td>
<td>i0.0022547</td>
<td>i0.0037704</td>
<td>i0.0038070</td>
</tr>
<tr>
<td>0.1660</td>
<td>1.007763</td>
<td>i0.0091633</td>
<td>i0.0152903</td>
<td>i0.0158968</td>
</tr>
<tr>
<td>0.2490</td>
<td>1.018294</td>
<td>i0.0212117</td>
<td>i0.0352592</td>
<td>i0.0385112</td>
</tr>
<tr>
<td>0.3320</td>
<td>1.034685</td>
<td>i0.0390320</td>
<td>i0.0644847</td>
<td>i0.0755701</td>
</tr>
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Table 7.14 — Surface Mode Parameters $\epsilon_r = 6.15$

<table>
<thead>
<tr>
<th>$kc$</th>
<th>$\alpha_1/k$</th>
<th>$R$</th>
<th>$T_0/k$</th>
<th>$T_1/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0266</td>
<td>1.000249</td>
<td>0.0002190</td>
<td>0.0004970</td>
<td>0.0004979</td>
</tr>
<tr>
<td>0.0532</td>
<td>1.001001</td>
<td>0.0008810</td>
<td>0.0019989</td>
<td>0.0020136</td>
</tr>
<tr>
<td>0.0831</td>
<td>1.002472</td>
<td>0.0021726</td>
<td>0.0049282</td>
<td>0.0050170</td>
</tr>
<tr>
<td>0.1660</td>
<td>1.010508</td>
<td>0.0031525</td>
<td>0.0062727</td>
<td>0.0063239</td>
</tr>
<tr>
<td>0.2490</td>
<td>1.026386</td>
<td>0.0022540</td>
<td>0.0050794</td>
<td>0.0051000</td>
</tr>
<tr>
<td>0.3320</td>
<td>1.055139</td>
<td>0.0453105</td>
<td>0.1016884</td>
<td>0.1383468</td>
</tr>
</tbody>
</table>

Table 7.15 — Surface Mode Parameters $\epsilon_r = 10.0$

<table>
<thead>
<tr>
<th>$kc$</th>
<th>$\alpha_1/k$</th>
<th>$R$</th>
<th>$T_0/k$</th>
<th>$T_1/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0266</td>
<td>1.000288</td>
<td>0.0001917</td>
<td>0.0005752</td>
<td>0.0005770</td>
</tr>
<tr>
<td>0.0532</td>
<td>1.001165</td>
<td>0.0007757</td>
<td>0.0023268</td>
<td>0.0023567</td>
</tr>
<tr>
<td>0.0831</td>
<td>1.002909</td>
<td>0.0019345</td>
<td>0.0058015</td>
<td>0.0058655</td>
</tr>
<tr>
<td>0.1660</td>
<td>1.013131</td>
<td>0.0086546</td>
<td>0.0259255</td>
<td>0.0294981</td>
</tr>
<tr>
<td>0.2490</td>
<td>1.037096</td>
<td>0.0238865</td>
<td>0.0713580</td>
<td>0.0969719</td>
</tr>
<tr>
<td>0.3320</td>
<td>1.095356</td>
<td>0.0574011</td>
<td>0.1702809</td>
<td>0.3079435</td>
</tr>
</tbody>
</table>

Since the normalized surface wave coefficients presented in these tables are typically much less than one, the surface wave mode is essentially insignificant in the near field. Of course for the thicker substrates with higher permittivity the surface wave mode becomes more important. If, as in the case of the thin patch antenna, the surface wave is negligible then an actual patch antenna with a finite substrate and ground plane should behave in a similar fashion to the infinite model analyzed in this thesis.

§7.5 Conclusions

The solution to the arbitrarily shaped patch antenna problem has been formulated in the spectral domain. This formulation is arrived at by transforming the integral equation relations for the $x$ and $y$ components of current on the patch antenna which are derived using the dyadic Green’s function for a dielectric substrate on a ground plane. The spectral domain equations governing these currents are in a form suitable for numerical solution with the conjugate gradient-FFT technique. This numerical procedure is attractive since it requires on the order of $N^2 \log_2(N)$ multiplications to compute the solution and the problems associated with derivatives in the integral equation can be handled in the spectral domain. The two dimensional antenna problem has been programmed and computer code is available to perform parametric studies of this thin patch antenna. Such quantities as
the substrate thickness and its dielectric constant, current excitation feed point, and size of 
the patch can be varied while the effects on the total, top, and bottom surface currents as 
well as the far zone radiation pattern are observed.

The results presented in this thesis demonstrate that as the thickness of the substrate 
decreases the patch antenna behaves more like an ideal cavity; the aperture admittance 
decreasing and the $Q$ increasing. The magnitude of the bottom surface current at resonance 
also increases, thereby making the antenna a more efficient radiator, however with reduced 
bandwidth. As the thickness of the substrate increases, at resonance the top surface current 
increases slightly while the bottom surface current decreases dramatically. Therefore the 
relative importance of the top surface current becomes significant as the substrate thickness 
increases. The resonant length of the two dimensional antenna is always somewhat less than 
one half the operating frequency wavelength in the dielectric substrate. This is due to the 
capacitive component of the radiating aperture admittance. The amount of this capacitance 
increases as the substrate thickness increases, and therefore the resonant length of the 
antenna decreases.

Once the current on the patch antenna is found, the far zone fields can be computed 
with the simple formulas presented in this thesis. These relations should be used instead 
of the traditional radiating slots model to predict the far zone fields. These formulas are 
exact expressions for the far zone fields in terms of the currents on the antenna and do 
not require the approximate assumptions made with the slot model. For the thin patch 
antenna the far zone field can be found from the bottom surface current alone. For a patch 
which is conformal to some orthogonal co-ordinate system, such as circular or rectangular, 
this current can be computed with the leaky cavity modal approach or transmission line 
model. However, as the thickness of the substrate increases, the top surface current becomes 
significant and these approximate approaches can no longer be used to predict the far zone 
fields.

Numerical results for the radiating aperture admittance have also been presented and 
compared with other approximate techniques. Fairly good agreement was found between 
the numerical results and approximate formulas. The numerical results tended to deviate 
from the approximate formulas when the electrical thickness of the substrate became large.
This is reasonable since the approximate formulas are not valid for thick substrates. An estimate of the magnitude and phase of the reflection coefficient for a typical thin patch antenna can be interpolated from the data presented.

This thesis also demonstrates that a typical thin patch antenna does not couple much energy into the surface wave mode. In the near field this surface wave mode can essentially be neglected. Since the surface wave mode is small and the other fields due to the antenna decay at least as rapidly as \(1/z\) for the two dimensional problem, an actual patch antenna with a finite dielectric substrate and ground plane should behave as the infinite antenna provided the ground plane and substrate are somewhat larger than the patch. Once again however, if the substrate is electrically thick, the surface wave mode becomes important and its effect on a patch antenna with finite substrate and ground plane must be considered.

§7.6 Practical Implications

Conclusions on the behavior of a rectangular patch antenna are the most immediate practical results which can be drawn from this two dimensional idealized patch antenna study. For a rectangular patch antenna operated in its usual lowest order resonant TEM mode will behave very much like the two dimensional antenna. This is true since the fields associated with the lowest order resonant TEM mode are independent of one of the spatial directions, as in the two dimensional problem. Therefore both the top and bottom surface currents on a rectangular patch antenna can be predicted based on the the two dimensional analysis. In addition the far zone radiated field pattern of a rectangular patch antenna in the transverse plane should be the same as the far field pattern for the two dimensional problem.

Probably the most useful conclusions which can be drawn from this thesis involve assessing the validity of the approximate techniques for analyzing patch antennas as a function of dielectric substrate thickness and relative permittivity. The most significant errors in the approximate techniques are due to errors in the radiating aperture admittance, or alternatively, to errors in the mode reflection coefficients for the antenna. Although the approximate formulas for these reflection coefficients are more accurate for thinner substrates, unfortunately, patch antennas become more resonant for thinner substrates as
well. Even a very slight error in the reflection coefficient for a highly resonant antenna can cause the approximate methods to predict currents on the antenna that are considerably in error.

For antennas with thicker substrates and not as resonant nor sensitive to these errors, the approximate formulas for the reflection coefficients are not as accurate. These two complementary difficulties are the major weaknesses of the approximate analysis techniques. Be that as it may, assuming that the correct values are known for the aperture admittance so that the bottom surface current can be predicted accurately, it is of interest to know how much in error the approximate techniques will be by ignoring the top surface current. The numerical analysis of the two dimensional antenna problem allows for a quantitative estimate of this error. Table 7.16 shows the per cent error in the total current due to ignoring the top surface current as a function of dielectric substrate thickness and relative permittivity.

<table>
<thead>
<tr>
<th>$\Re[\varepsilon]$</th>
<th>2.00</th>
<th>2.33</th>
<th>3.80</th>
<th>6.15</th>
<th>10.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k\varepsilon$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0266</td>
<td>1.9%</td>
<td>1.8%</td>
<td>1.6%</td>
<td>1.5%</td>
<td>1.3%</td>
</tr>
<tr>
<td>0.0332</td>
<td>3.6%</td>
<td>3.5%</td>
<td>3.2%</td>
<td>2.9%</td>
<td>2.6%</td>
</tr>
<tr>
<td>0.0531</td>
<td>5.4%</td>
<td>5.3%</td>
<td>4.8%</td>
<td>4.2%</td>
<td>3.7%</td>
</tr>
<tr>
<td>0.1680</td>
<td>9.8%</td>
<td>9.4%</td>
<td>8.5%</td>
<td>7.6%</td>
<td>6.7%</td>
</tr>
<tr>
<td>0.2490</td>
<td>13.4%</td>
<td>12.9%</td>
<td>12.1%</td>
<td>10.9%</td>
<td>10.5%</td>
</tr>
<tr>
<td>0.3320</td>
<td>16.8%</td>
<td>16.0%</td>
<td>15.3%</td>
<td>15.1%</td>
<td>15.3%</td>
</tr>
</tbody>
</table>

Loss tangent of dielectric substrate 0.0015 in all cases

This error is defined as:

$$
\varepsilon = \sqrt{\int_{-L/2}^{+L/2} |K^\dagger(x) - K(x)|^2 dx/\int_{-L/2}^{+L/2} |K(x)|^2 dx} \times 100% \quad (7.46)
$$

where $K(x)$ is the actual total current on the antenna and $K^\dagger(x)$ is the current predicted by the transmission line model. Although the data shown in these tables is for a feed position of $x_0 = 0.3L$, this error is essentially independent of feed position. The per cent error in the total currents are shown at resonance for all cases. As can be seen from this table, the per cent error is primarily a function of substrate thickness ranging from roughly 2% to 18% for the thicknesses shown.
This thesis has also shown that the bandwidth of a patch antenna can be increased by using a thicker substrate, however increasing the bandwidth in this manner may reduce the radiation effectiveness of the antenna. The two dimensional analysis allows a quantitative prediction of how the radiation effectiveness and bandwidth are related for different substrate thicknesses. The radiation resistance of an antenna is a quantitative measure of how effective a radiator that antenna is; the larger the radiation resistance, the more effective. For the two dimensional antenna the radiation resistance is given by

\[ R = \frac{P_r}{I_o^2} \]  \hspace{1cm} (47)

where \( P_r \) is the radiated power per unit length and \( I_o \) is the magnitude of the excitation current per unit length. Since \( P_r \) has dimensions of power per length and \( I_o \) has dimensions of current per length, it follows that \( R \) has dimensions of resistance length. In general the radiation resistance will vary as a function of the feed point of the antenna, which is usually determined by impedance matching considerations. For purposes of comparison, the feed point was set to 0.2L, 0.3L, and 0.4L for the data in the tables 7.17 - 7.21 which show per cent bandwidth and normalized radiation resistance as functions of substrate thickness and dielectric constant. The radiation resistance is shown at resonance in all cases. The quantity \( Y_o = \sqrt{\varepsilon_o/\mu_o} \) is the characteristic admittance of free space. As can be seen from these tables, the bandwidth and radiation effectiveness can be improved by increasing the substrate thickness up to a point. Beyond that point, the bandwidth can be improved at the expense of the antenna radiation effectiveness. For this reason extremely thin substrate patch antennas should only be used when a very narrow bandwidth of operation is desired.

<table>
<thead>
<tr>
<th>( k_c )</th>
<th>Bandwidth</th>
<th>( kY_oR )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( z_o = 0.2L )</td>
<td>( z_o = 0.3L )</td>
</tr>
<tr>
<td>0.0266</td>
<td>2.1%</td>
<td>0.238</td>
</tr>
<tr>
<td>0.0532</td>
<td>4.1%</td>
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</tr>
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<td>0.0831</td>
<td>6.3%</td>
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<td>0.1660</td>
<td>12.5%</td>
<td>0.211</td>
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<td>0.2490</td>
<td>18.8%</td>
<td>0.183</td>
</tr>
<tr>
<td>0.3320</td>
<td>25.2%</td>
<td>0.159</td>
</tr>
</tbody>
</table>
Table 7.18 — Per Cent Bandwidth and Radiation Resistance  
\( \varepsilon_r = 2.33 + i0.0035 \)

<table>
<thead>
<tr>
<th>(kc)</th>
<th>Bandwidth</th>
<th>( kY_oR )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_o = 0.2L )</td>
<td>( x_o = 0.3L )</td>
</tr>
<tr>
<td>0.0266</td>
<td>3.2%</td>
<td>0.238</td>
</tr>
<tr>
<td>0.0532</td>
<td>6.2%</td>
<td>0.250</td>
</tr>
<tr>
<td>0.0831</td>
<td>9.6%</td>
<td>0.243</td>
</tr>
<tr>
<td>0.1660</td>
<td>18.6%</td>
<td>0.211</td>
</tr>
<tr>
<td>0.2490</td>
<td>27.4%</td>
<td>0.183</td>
</tr>
<tr>
<td>0.3320</td>
<td>36.4%</td>
<td>0.159</td>
</tr>
</tbody>
</table>

Table 7.19 — Per Cent Bandwidth and Radiation Resistance  
\( \varepsilon_r = 3.80 + i0.0057 \)

<table>
<thead>
<tr>
<th>(kc)</th>
<th>Bandwidth</th>
<th>( kY_oR )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_o = 0.2L )</td>
<td>( x_o = 0.3L )</td>
</tr>
<tr>
<td>0.0266</td>
<td>2.3%</td>
<td>0.009</td>
</tr>
<tr>
<td>0.0532</td>
<td>4.1%</td>
<td>0.023</td>
</tr>
<tr>
<td>0.0831</td>
<td>5.7%</td>
<td>0.164</td>
</tr>
<tr>
<td>0.1660</td>
<td>9.4%</td>
<td>0.528</td>
</tr>
<tr>
<td>0.2490</td>
<td>12.1%</td>
<td>0.574</td>
</tr>
<tr>
<td>0.3320</td>
<td>13.6%</td>
<td>0.373</td>
</tr>
</tbody>
</table>

Table 7.20 — Per Cent Bandwidth and Radiation Resistance  
\( \varepsilon_r = 6.15 + i0.0092 \)

<table>
<thead>
<tr>
<th>(kc)</th>
<th>Bandwidth</th>
<th>( kY_oR )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_o = 0.2L )</td>
<td>( x_o = 0.3L )</td>
</tr>
<tr>
<td>0.0266</td>
<td>2.3%</td>
<td>0.409</td>
</tr>
<tr>
<td>0.0532</td>
<td>4.1%</td>
<td>0.491</td>
</tr>
<tr>
<td>0.0831</td>
<td>5.7%</td>
<td>0.569</td>
</tr>
<tr>
<td>0.1660</td>
<td>10.1%</td>
<td>0.437</td>
</tr>
<tr>
<td>0.2490</td>
<td>14.2%</td>
<td>0.505</td>
</tr>
<tr>
<td>0.3320</td>
<td>18.7%</td>
<td>0.150</td>
</tr>
</tbody>
</table>

Table 7.21 — Per Cent Bandwidth and Radiation Resistance  
\( \varepsilon_r = 10.00 + i0.0150 \)

<table>
<thead>
<tr>
<th>(kc)</th>
<th>Bandwidth</th>
<th>( kY_oR )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_o = 0.2L )</td>
<td>( x_o = 0.3L )</td>
</tr>
<tr>
<td>0.0266</td>
<td>7.1%</td>
<td>0.087</td>
</tr>
<tr>
<td>0.0532</td>
<td>8.8%</td>
<td>0.117</td>
</tr>
<tr>
<td>0.0831</td>
<td>10.2%</td>
<td>0.155</td>
</tr>
<tr>
<td>0.1660</td>
<td>14.1%</td>
<td>0.237</td>
</tr>
<tr>
<td>0.2490</td>
<td>20.6%</td>
<td>0.139</td>
</tr>
<tr>
<td>0.3320</td>
<td>31.6%</td>
<td>0.069</td>
</tr>
</tbody>
</table>

The results of this thesis also suggest how some improvement to the approximate tech-
niques could be made. This is of interest since the numerical solution of the general three dimensional problem is computationally costly. The most significant problem with the approximate techniques is getting accurate values for the aperture admittance or reflection coefficients. Values for the reflection coefficients can be computed numerically with the technique discussed in this thesis and many such values have been tabulated in Section 7.3. These values should be particularly useful for antennas with thicker substrates since the approximate formulas for reflection coefficients are most accurate for thin substrates, while there is no such limitation on the numerically computed values. Beyond getting better values for reflection coefficients, a first order improvement in the approximate methods could be made if the average value of the top surface current were known. Assuming a constant top surface current of this value would correct for the magnitude of the far zone field at its maximum, in the broad side direction, where it is of the most interest. Table 7.22 shows the ratio of the average top surface current to the average bottom surface current as a function substrate thickness and dielectric constant. This ratio is fairly independent of the excitation feed point position. These values should be useful in making a first order improvement of the transmission line model for a rectangular patch antenna.

Table 7.22 — Average Top to Bottom Surface Current Ratio

<table>
<thead>
<tr>
<th>Re[$\epsilon_r$]</th>
<th>2.00</th>
<th>2.33</th>
<th>3.80</th>
<th>6.15</th>
<th>10.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>kc</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0260</td>
<td>-0.002-0.019</td>
<td>-0.003-0.019</td>
<td>-0.006-0.016</td>
<td>-0.007-0.014</td>
<td>-0.008-0.011</td>
</tr>
<tr>
<td>0.0532</td>
<td>-0.003-0.036</td>
<td>-0.006-0.035</td>
<td>-0.009-0.031</td>
<td>-0.014-0.028</td>
<td>-0.015-0.022</td>
</tr>
<tr>
<td>0.0831</td>
<td>-0.004-0.055</td>
<td>-0.008-0.053</td>
<td>-0.014-0.047</td>
<td>-0.020-0.039</td>
<td>-0.022-0.032</td>
</tr>
<tr>
<td>0.1660</td>
<td>-0.004-0.100</td>
<td>-0.010-0.095</td>
<td>-0.023-0.083</td>
<td>-0.035-0.068</td>
<td>-0.040-0.055</td>
</tr>
<tr>
<td>0.2490</td>
<td>-0.001-0.139</td>
<td>-0.009-0.132</td>
<td>-0.027-0.120</td>
<td>-0.043-0.100</td>
<td>0.002-0.111</td>
</tr>
<tr>
<td>0.3320</td>
<td>-0.009-0.177</td>
<td>-0.006-0.168</td>
<td>-0.022-0.168</td>
<td>-0.042-0.159</td>
<td>0.011-0.203</td>
</tr>
</tbody>
</table>

Loss tangent of dielectric substrate 0.0015 in all cases

§7.7 Areas for Future Work

Several areas for future research which can be performed are suggested by this thesis. The most obvious is the numerical implementation and solution to the general three dimensional arbitrarily shaped patch antenna using the conjugate gradient-FFT technique. Of course, substantial computer resources would have to be available for such a project. The
numerical technique presented here can also be generalized to the thick substrate patch antenna problem by assuming an excitation current which varies as a function of z and taking into consideration the higher order surface wave mode poles which will exist. Experimental confirmation of the top surface currents predicted in this thesis could also be carried out for a wide rectangular patch antenna with surface field measurements. The effects of coupling between antennas can be studied by extending the two dimensional analysis to two antennas next to one another. The results from this study would be directly applicable to the coupling effects of rectangular patch antenna elements in one direction of an array. Another possible extension of this work would be to examine a patch antenna which is excited by a microstrip transmission line instead of the coaxial line bottom fed antenna.

Another area of future research might involve the analysis of a patch antenna which is made of a lossy, possibly even a resistively tapered, patch instead of a perfectly conducting one. Experimentation with the resistive taper of such an antenna might lead to improved bandwidth without too much degradation in efficiency. In addition, since the top surface current for a thin substrate patch antenna is small compared to the bottom surface, an absorbing material coating could be placed on the top side of the patch without much effect to its behavior. This may be useful for reducing the radar cross section of such an antenna.
COMPUTER PROGRAM STRIP

The program Strip numerically solves for and plots various quantities involved with the two dimensional microstrip antenna. The problem is solved using the conjugate gradient FFT technique. See the University of Michigan Ph.D. Thesis "Spectral Domain Analysis of Microstrip Patch Antenna Currents and Radiation" by Thomas Willis for a detailed description of this problem and the numerical technique used to solve it.

Logical Device Assignment

2: File containing integer flags indicating what should be computed on a given run of this program
5: Formatted input file of antenna parameters
6: Formatted output from program
7: Unformatted record file of program output

Input Format (logical device 5) format(15,6e12.6,1x,a25)

logN: Base two log of number of points be used in representing the currents on the strip
xOn: Position of source current on the strip in terms of the strip length L (-.45 < xOn < .45)
capLn: Width of the strip in terms of free space wavelengths
c: The thickness of the substrate times k, the free space wave number (k=2*pi/wavelength)
epsilon: Complex relative permittivity of the dielectric substrate
rerr: Allowable relative error in the solution of the total current on the antenna
plotf: Character name of file to which unformatted record
output should be written for later use with the
program Graph

common /param/ epsilon, refrin, capL, c, x0, delta, M, logN
common /opattr/ xpad, Fx, Gkern, Wkn, Mp, LN
common /plottr/ nxdvis, nydivs
common /cgrad/ re, sv1, sv2, sv3
common /mode/ z, residue
Complex epsilon, refrin
Complex Wkn(2049), Gkern(2049), r(1024)
Complex Kx(1025), Kxtop(1025), Kxbot(1025)
Complex res(1024), sv1(1024), sv2(1024), sv3(1024)
Complex xpad(4096), Fx(4096), Hfar(181), Htop(181), Hbot(181)
Complex Z, Residue
Complex Rainus, Rplus
Character=25 Title, plotf, Append

nxdvis=10
nydivs=10

Open(5, file='sdat')
Open(2, file='flags')
read(2, 2100) iflag1, iflag2, iflag3
Close(2)
10 continue
write(6, 1000)
read(6, 2000, end=990) logN, x0n, capLn, c, epsilon, resid, plotf
plotf=Append(plotf)
LN=logN
M=4
Mp=M
pi=3.141592653
N=2**logN
Nstop=N/N-1
Nstop2=Nstop+2
Nhalf=N/(2*N)
N2i=N/2+1

Normalize capL and x0 and compute other constants

capL=2.*pi*capLn
x0=capL*x0n
dalpha=2.*pi/(float(N)*capL)
dx=float(N)*capL/float(N)
delta=capL/20.
refrin=csqrt(epsilon)
If (aimag(refrin).lt.0.) refrin=cmplx(real(refrin), 0.)

Compute surface mode pole and residue of G(alpha)

Call Pole(Z, Residue)
write(6, 14000) Z
Initialize Wkn for FFT subroutines

call Winit(Wkn,logN)
c write(6,8210)
c call Kernel(dalpha,logN,Gkern)
c write(6,8220)
ccc Compute excitation vector r
c call Rvctr(Wkn,Gkern,r)
ccc Solve for the total current on the strip
c call Solver(R,dx,rerr,Nstop,Kx)
ccc Shift current vector and add zeroes at endpoints

Do 400 i=1,Nstop
  index=Nstop2-i
  kndx=index-1
  Kx(index)=Kx(kndx)
  continue
400

Kx(1)=cmplx(0.,0.)
Kx(Nstop2)=cmplx(0.,0.)
c write(6,8230)
ccc Compute the top and bottom surface currents on strip
c If (Iflag1.ne.0 .or. Iflag2.ne.0) Call & SurfK(Wkn,Z,Kx,Kxtop,Kxbot)
   write(6,8240)
ccc Compute Aperture Admittance for this two dimensional
ccc
ccc Compute far field pattern
cc
ccc Add the Xeta function back to the total current

Do 500 index=1,Nstop2
  kndx=-Nhlf-1+index
  x=dx*float(kndx)
  Kx(index)=Kx(index)+cmplx(Xeta(x),0.)
  continue
500

Call SurfH(Wkn,Kx,Fx)
ccc Write data out to plot record file
Open (7, file=plotf, access='direct', RECL=8200)
Write(7, rec=1) Hfar,Htop,Hbot,
& logn,M,x0n,ca0n,c,epsion,rerr,Rplus,Rminus
write(7, rec=2) Kx
write(7, rec=3) Kawtop
write(7, rec=4) Kawbot
Close(7)
goto 10

990 stop

c
c
cccc formats

c
1000 format(/3x,'Enter logN, normalized x0, L, c, ',
& ' epsilon_r, rerr, and file name'/)
1200 format(2x,'Cholesky decomposition fails at level ',i12,','
1260 format(/3x,'Normalized error = ',e14.7,')
1400 format(/3x,'Surface mode pole found at ',f10.7, ', +i',f10.7,'')
2000 format(16,6e12.6,1x,a25)
2100 format(316)
2200 format(' Total Power = ',e14.7,5x,' Normalized Current Error = ',e14.7,5x,
& ' Normalized Aperture Error = ',e14.7)
2400 format(' Normalized input impedance-length = ',e14.7, ', +j',e14.7)
7000 format(' Enter alpha')
7100 format(e12.6)
8000 format(3x,i2,5x,e14.7, ', +j',e14.7,5x,e14.7, ', +j',e14.7)
8010 format(/3x,'Reciprocal condition number = ',e14.7)
8100 format(3x,e14.7, ', +j',e14.7)
8110 format(2(3x,e14.7, ', +j',e14.7))
8120 format(/3x, 'm = ',i2, '/)
8200 format(3x,i2,5x,e14.7, ', +j',e14.7)
8210 format('/' ' Returned from Winit')
8220 format('/' ' Returned from Kernel')
8230 format('/' ' Found total current')
8240 format('/' ' Top and bottom surface currents found')
8310 format(3x,i3,4x,i3,5x,e14.7, ', +j',e14.7)

end

C

C*******************************************************************************/

C Subroutine Solver(R,dx,xerr,Nstop,Kx)

C*******************************************************************************/

C The subroutine Solver is called by the main program to solve for the unknown
C total current using the conjugate gradient FFT technique. If the current
C fails to converge this subroutine will use the best estimate of the current
C found so far as an initial guess and try again up to six
C times before giving up.
C
C common /cgrad/ res,s1,s2,s3
complex R(1),Kx(1)
complex res(1024),s1(1024),s2(1024),s3(1024)
character*25 Title
c
erre=xerr

ccc 
Normalize R so analytic Fourier Transform and FFT compare

  c
  Do 100 indx=1,Nstop
  R(indx)=R(indx)/dx
100   continue

ccc 
Set initial guess of Kx to zero

c
  Do 200 indx=1,Nstop
  Kx(indx)=cmplx(0.,0.)
200   continue

  Do 300 Itry=1,6
  call Lopr(Kx,Nstop,sv1)
  call Error(R,sv1,Nstop,Enorm)
  erre=xerr/Enorm
  Irc=1
  call Cgsopr(R,Nstop,erre,缣, erreur,Kx,res,sv1,sv2,sv3,Irc)
  If (Irc.eq.0) goto 400
  If (Irc.eq.1) write(6,1000)
  If (Irc.eq.2) write(6,1100)
  If (Irc.eq.2) goto 400
300   continue

  write(6,1300)

c
400 continue
  call Lopr(Kx,Nstop,sv1)

c
  call Error(R,sv1,Nstop,Enorm)
  write(6,1200) Enorm

cccc 
Normalize L{Kx} back to true value

c
  Do 320 indx=1,Nstop
  svi(indx)=sv1(indx)*dx
320   continue

  return

cccc 
Formats

c
1000 format(/3x,'Conjugate gradient method fails to converge!/','/
1100 format(/3x,'Answer found to machine precision!',)/
1200 format(/3x,'Normalized error = ',e14.7,)/
1300 format(/3x,'I can not solve this hideous problem!',/ 
   & ' ... you will have to be satisfied with my best guess'/) 
   end

C Subroutine Kernel(dalpha,logN,Gkern)
The subroutine Kernel generates the Green's function kernel and stores the values in the vector Gkern. The Green's function kernel is \( \text{Gamma} \cdot \text{Gamma1} \cdot G(\alpha) \). The kernel is computed for positive \( \alpha \) only since it is an even function.

```c
complex Gkern(1)
complex Gamma, Gamma1, Gamma, Gamma1, Green

N = 2^{**\log N}
N2 = N/2
N21 = N2 + 1

ccc Compute the Green's kernel function for positive frequency
ccc
Do 100 indx = 1, N21
   alpha = dalpha * float(indx - 1)
   Gamma = Gamma(alpha)
   Gamma1 = Gamma1(alpha)
   Gkern(indx) = Gamma * Gamma1 * Green(Gamma, Gamma1)
write(6, 8000) alpha, Gamma, Gamma1, Green(Gamma, Gamma1)
   continue
100 return
ccc formats
ccc
8000 format(3x, f9.4, 3x, e14.7, ' +j', e14.7))
end
```

Subroutine Rvctr(w, Gkern, R)

The subroutine Rvctr generates the \( R \) vector needed for the excitation vector in solving for the total current on the strip using the CG-FFT technique.

```c
common /param/ epsilon, refrin, capL, c, x0, delta, M, logN
Complex Gkern(1), R(1)
Complex xig(4096), IxG(4096), w(2048)
Complex Ifunc, Eta, Halp, Phi0, IfG, Phitld, Gamma1
Complex refrin, epsilon

pi = 3.141592653
N = 2^{**\log N}
N2 = N/2
Nhlf = N/(2*M)
Nhlf1 = Nhlf - 1
dalpha = 2. * pi/(float(M) * capL)
```
\(dx = \text{float}(M) \times \text{capL}/\text{float}(N)\)

```
cccc Generate I times Green function for positive alpha
c
   Do 200 indx=1,N2
   alpha=dalpha*float(indx-1)
   Halpha=Eta(alpha)
   If (cabs(Gamn(alpha)) .le. 1.e-6) goto 100

cccc compute in straight forward manner
   IG(indx)=(Ifunc(alpha)-Halpha)*Gkern(indx)
goto 200

cccc alpha equals k1 and Ifunc becomes singular,
   use limit of Gkern*Ifunc
   100  continue
      ax=alpha*x0
      IfG=alpha*c*cmplx(-sin(ax),cos(ax))/epsilon
      IG(indx)=IfG-Halpha*Gkern(indx)
   c
   200  continue

cccc Generate I times Green function for negative alpha
   c
   Do 400 i=1,N2
      indx=N+1-i
      kndx=i+1
      alpha=-dalpha*float(i)
      Halpha=Eta(alpha)
      If (cabs(Gamn(alpha)) .le. 1.e-6) goto 300

cccc compute in straight forward manner
   IG(indx)=(Ifunc(alpha)-Halpha)*Gkern(kndx)
goto 400

cccc alpha equals -k1 and Ifunc becomes singular,
   use limit of Gkern*Ifunc
   300  continue
      ax=alpha*x0
      IfG=alpha*c*cmplx(-sin(ax),cos(ax))/epsilon
      IG(indx)=IfG-Halpha*Gkern(kndx)
   c
   400  continue

ccc Take care of averaging the special point at end of spectrum
    c
    alpha=dalpha*N2
    Halpha=Eta(alpha)
    id=N2+1
    IG(id)=0.5*(IG(id)+(Ifunc(alpha)-Halpha)*Gkern(id))
   c
```
Inverse FFT the IG vector back to the spatial domain

call Icfft(xig,w,logn,IG)

Set R equal to the points over the strip

Do 600 indx=1,Nhalf
   i=Nhalf1+indx
   R(i)=xig(indx)
600 continue

indx=(2*M-1)*Nhalf+1

Do 800 i=1,Nhalf1
   indx=indx+1
   R(i)=xig(indx)
800 continue

return

End

C
C******************************************************************************C
C
C Subroutine Error(R,LKx,N,Enorm)
C
C******************************************************************************C
C
The subroutine Error computes the normalized error of the linear operator applied to the solution for the current compared to the excitation vector which it should equal.

The excitation vector is R while the result of the linear operator on the current is LKx.

Complex R(1),LKx(1)
Complex Err,Rndx

Esum=0.
rsum=0.

Do 1000 indx=1,N
   Rndx=R(indx)
   Rr=real(Rndx)
   Ri=imag(Rndx)
   Rsum=Rsum+Rr*Rr+Ri*Ri
   Err=Rndx-LKx(indx)
   Er=real(Err)
   Ei=imag(Err)
   Esum=Esum+Er*Er+Ei*Ei
1000 continue

Enorm=sqrt(Esum/Rsum)
return

C
C formats
c
8000 format(2(6x,e14.7,' +j',e14.7))
end
C

********************************************************************************

C Subroutine SurfK(W,Pole,Ktotal,Kxtop,Kxbot)
C
********************************************************************************

C The subroutine SurfK computes the top surface current and the bottom surface current on the strip from the total current on the strip.

C common /param/ epsilon,refr,in,capL,c,x0,delta,M,logN
complex epsilon,refr
Complex xpad(4096),Fktot(4096),Fktop(4096),Fkbot(4096)
Complex Ktotal(1025),Kxtop(1025),Kxbot(1025)
complex Tkern(300),Bkern(300)
complex Ctemp,Fkt0,Fkt,Fkb,Fphase,Sbot,Stop,pole

C pi=3.141592653
N=2**logN
N2=N/2
N4=N/4
N8=N/8
N81=N8+1
Nhlf=N/(2*M)
Nhlf1=Nhlf+1
Nstop=N/M+1
rtpi=1./(2.*pi)
dx=float(M)*capL/float(N)
dalpha=2.*pi/(float(M)*capL)
arl=real(pole)

ccc Compute the Fourier Transform of the total current
ccc
Do 20 indx=1,N
xpad(indx)=complx(0.,0.)
20 continue

ccc Load the padded x vector up
ccc
Do 40 indx=1,Nhlf
i=Nhlf+indx
xpad(indx)=Ktotal(i)
40 continue

indx=(2*M-1)*Nhlf

Do 80 i=1,Nhlf
indx=indx+1
xpad(indx)=Ktotal(i)
80 continue
Call Cfft(xpad,w,logN,Fktot)

Swap around elements of Fktot to get negative frequencies first

Do 90 indx=1,N2
   kndx=N2+indx
   Ctemp=Fktot(indx)*dx
   Fktot(indx)=Fktot(kndx)*dx
   Fktot(kndx)=Ctemp
90 continue

Compute Low frequency contributions with integration under a conformal transformation

Na=Ifix(2.*ar1/dalpha)
Alim=dalpha*(float(Na)+.5)
Nedge=75
Ncntr=149

Compute the Fourier transform top and bottom current kernels

Theta0=Acos(ar1/Alim)
kndx=0
dTheta=Theta0/float(Nedge-1)

Do 100 indx=1,Nedge
   kndx=kndx+1
   Cofac=float(4.-2.*Mod(indx,2))
   If (indx.eq.1 .or. indx.eq.Nedge) Cofac=1.0
   Theta=dTheta*float(indx-Nedge)
   sct=ar1/float(Theta)
   alpha=-sct
   wght=-sct*tan(Theta)
   anorm=alpha/dalpha
   Call Sfit(Fktot,N,anorm,Fkt0)
   Call FourK(alpha,Fkt0,Fkt,Fkb)
   write(6,8100) alpha,Fkt0,Fkt,Fkb
   Tkern(kndx)=Fkt*Cofac*wght
   Bkern(kndx)=Fkb*Cofac*wght
100 continue

dTheta=pi/float(Ncntr-1)

Do 120 indx=1,Ncntr
   kndx=kndx+1
   Cofac=float(4.-2.*Mod(indx,2))
   If (indx.eq.1 .or. indx.eq.Ncntr) Cofac=0.0
   Theta=dTheta*float(indx-1)
   alpha=ar1*cos(theta)
   wght=ar1*sin(theta)
   anorm=alpha/dalpha
   Call Sfit(Fktot,N,anorm,Fkt0)
   Call FourK(alpha,Fkt0,Fkt,Fkb)
   write(6,8100) alpha,Fkt0,Fkt,Fkb
   Tkern(kndx)=Fkt*Cofac*wght
   Bkern(kndx)=Fkb*Cofac*wght
Bkern(kndx)=Fkb*cofac*wght
120  continue

c  dTheta=Theta0/float(Nedge-1)

c  Do 140 indx=1,Nedge
   kndx=kndx+1
   Cofac= float(4-2*Mod(indx,2))
   If (indx.eq.1 .or. indx.eq.Nedge) Cofac=1.0
   Theta=dTheta/float(indx-1)
   sct=ari/cos(Theta)
   alpha=sct
   wght=sct*tan(Theta)
   anorm=alpha/dalpha
   Call Sfit(Fktot,N,anorm,Fkt0)
   Call FourK(alpha,Fkt0,Fkt,Fkb)
   write(6,S100) alpha,Fkt0,Fkt,Fkb
   Tkern(kndx)=Fkt*cofac*wght
   Bkern(kndx)=Fkb*cofac*wght
140  continue

c  Now perform Fourier inversion integrations for the current on
ccc  each point of the strip using Simpson's rule.

c  Do 260 indx=1,Nstop
   x=dx*float(indx-Nhalf)
   kndx=0
   Stop=cmplx(0.,0.)
   Sbot=cmplx(0.,0.)
   dTheta=Theta0/float(Nedge-1)

c  Do 200 jndx=1,Nedge
   kndx=kndx+1
   Theta=dTheta/float(jndx-Nedge)
   alpha=-ari/cos(Theta)
   ax=alpha*x
   Fphase=cmplx(cos(ax),-sin(ax))
   Stop=Stop+Fphase*Tkern(kndx)
   Sbot=Sbot+Fphase*Bkern(kndx)
200  continue

c  Kxtop(indx)=Stop*dTheta/3.0
  Kxbot(indx)=Sbot*dTheta/3.0
  Stop=cmplx(0.,0.)
  Sbot=cmplx(0.,0.)
  dTheta=pi/float(Ncntr-1)

c  Do 220 jndx=1,Ncntr
   kndx=kndx+1
   Theta=dTheta/float(jndx-1)
   alpha=-ari*cos(Theta)
   ax=alpha*x
   Fphase=cmplx(cos(ax),-sin(ax))
   Stop=Stop+Fphase*Tkern(kndx)
   Sbot=Sbot+Fphase*Bkern(kndx)
continue

\[
\begin{align*}
\text{Kxtop}(\text{indx}) &= \text{Kxtop}(\text{indx}) + \text{Stop} \cdot \text{dTheta} / 3.0 \\
\text{Kxbot}(\text{indx}) &= \text{Kxbot}(\text{indx}) + \text{Sbot} \cdot \text{dTheta} / 3.0 \\
\text{Stop} &= \text{single}(0.0, 0.0) \\
\text{Sbot} &= \text{single}(0.0, 0.0) \\
\text{dTheta} &= \text{Theta} / \text{float}(\text{Nedge} - 1)
\end{align*}
\]

Do 240 jindx = 1, Nedge
kndx = kndx + 1
Theta = dTheta \cdot \text{float}(jindx - 1)
alpha = \text{ar}1 / \cos(\text{Theta})
ax = alpha \cdot x
Fphase = \text{cmplx}(\cos(ax), -\sin(ax))
\text{Stop} = \text{Stop} + \text{Fphase} \cdot \text{Tkern}(\text{kndx})
\text{Sbot} = \text{Sbot} + \text{Fphase} \cdot \text{Bkern}(\text{kndx})
continue

Kxtop(\text{indx}) = \text{Kxtop}(\text{indx}) + \text{Stop} \cdot \text{dTheta} / 3.0
Kxbot(\text{indx}) = \text{Kxbot}(\text{indx}) + \text{Sbot} \cdot \text{dTheta} / 3.0
continue

Compute high frequency contributions to currents with an FFT

Do 280 index = 1, N
Ipos = index - 1 - N2
alpha = dalpha \cdot \text{float}(Ipos)
\text{FktO} = \text{Fktot}(\text{index})
\text{Call FourK}(\text{alpha}, \text{FktO}, \text{Fkt}, \text{Fkb})
\text{Fktop}(\text{index}) = \text{Fkt}
\text{Fkbot}(\text{index}) = \text{Fkb}
\text{If} (\text{Iabs}(\text{Ipos}) \leq \text{Na}) \text{Fktop}(\text{index}) = \text{cmplx}(0.0, 0.0)
\text{If} (\text{Iabs}(\text{Ipos}) \leq \text{Na}) \text{Fkbot}(\text{index}) = \text{cmplx}(0.0, 0.0)
continue

Swap around elements of Fktop and Fkbot for inverse FFT

Do 300 index = 1, N2
kndx = N2 + index
\text{Ctemp} = \text{Fktop}(\text{index})
\text{Fktop}(\text{index}) = \text{Fktop}(\text{kndx})
\text{Fktop}(\text{kndx}) = \text{Ctemp}
\text{Ctemp} = \text{Fkbot}(\text{index})
\text{Fkbot}(\text{index}) = \text{Fkbot}(\text{kndx})
\text{Fkbot}(\text{kndx}) = \text{Ctemp}
continue

Get Top current

\text{Call ICFFT}(\text{xp}, \text{w}, \logN, \text{Fktop})

Swap around elements of xp to get negative x values first

Do 480 index = 1, N2
kndx = N2 + index
Ctemp=xpad indx
xpad indx=xpad kndx
xpad kndx=Ctemp

480 continue

ccc Take values over strip

c
Do 500 indx=1,Nstop
x=dx*float indx-Nhlf1
kndx=N2-Nhlf+indx
Kxtop indx=-(Kxtop indx)+xpad kndx dalpha rtpi
500 continue

c ccc Get Bottom current

c Call Icfft xpad, w, logN, Fkbot

c ccc Swap around elements of xpad to get negative x values first

c
Do 580 indx=1,N2
kndx=N2-Nhlf+indx
Ctemp=xpad indx
xpad indx=xpad kndx
xpad kndx=Ctemp
580 continue

c ccc Take values over strip and add Xeta function back

c
Do 600 indx=1,Nstop
x=dx*float indx-Nhlf1
kndx=N2-Nhlf+indx
Xxbot indx=(Xxbot indx)+xpad kndx dalpha rtpi Xeta x
600 continue

return

8100 format 3x 'alpha = ', d9.5, 3x, 'Fk-total = ', e14.7, +i, e14.7,
& 5x, 'Fk-top = ', e14.7, +i, e14.7, 5x, 'Fk-bottom = ',
& e14.7, +i, e14.7)

end

C
C***************************************************************
C
C Subroutine FourK(alpha,Fktot,Fktop,Fkbot)
C
C***************************************************************
C
C Subroutine FourK is called by SurfK to compute the top and bottom current functions in the spectral domain given a value of alpha and the spectral domain value of the total current for that alpha.

C common /param/ epsilon, refrin, capL, x0, delta, M, logN
common /mode/ pole, residue
complex epsilon, refrin, pole, residue
Complex Fktot, Fktop, Fkbot
complex Ifunc, Ia, Eta, Ea
complex Gamp, Gamm1, Gamma, Gammal, Green, G
complex Ccoth, Tkern, Bkern, LmTerm, Lambda, R1, phase

c
Ea=Eta(alpha)
Gamma=Gamm(alpha)
Gammal=Gamm1(alpha)
G=Green(Gamma, Gammal)
If (cabs(Gammal) .le. 1.e-6) goto 100

c
Compute Transform of currents in straight forward manner

c
Ia=Ifunc(alpha)
Tkern=Gammal*G
Bkern=epsilon*Gamma*G*Ccoth(Gammal*c)
Fxtop = Tkern*(Ia-(Ea+Fktot))
Fxbot = Tkern*Ia+Bkern*(Ea+Fktot)-Ea
return

c
alpha equals +/- k1 so use limit of I(alpha) times Tkern:

100 continue
ax=alpha*x0
LmTerm=alpha*c*complx(-sin(ax), cos(ax))/(Gamma*epsilon)
If (aimag(R1).gt.0.) R1=-R1
Tkern=complx(0.,0.)
Bkern=complx(1.,0.)
Fxtop = LmTerm-Tkern*(Ea+Fktot)
Fxbot = LmTerm-Bkern*(Ea+Fktot)-Ea
return
end

C
C*******************************************************************************
C
C
C Subroutine Aupture(Kxbot, Mbasis, RCN, Error, Rminus, Rplus)

C*******************************************************************************
C
C The subroutine Aupture computes the radiating aperture admittance of the two dimensional antenna from the E-z component and H-y component at the edge of the strip.

C
C common /param/ epsilon, refrin, capL, c, x0, delta, M, logN
complex epsilon, refrin
complex Kxbot(1025), f(10, 1025), Iv(10), A(10, 10), sv1(10)
complex Rimin, Rimpl, Yamin, Yapl
complex Psi, Beta, a0, b0, Rminus, Rplus, Rv, Rdiff
complex cexp, csqrt, conjg
real k1r, k1i

pi=3.141592653
N=2**logN
N2=N/2
N4=N/4
N8=N/8
N81=N8+1
Natop=N/N+1
Nhlf=N/(2*M)
Nhlf1=Nhlf+1
Mtop=Mbasis+2
Mtop1=Mstop-1
Mfunc=Mbasis-1
dx=float(N)*capL/float(N)
hlfL=0.5*capL
k1=real(refrin)
k2=imag(refrin)

ccc generate the basis functions for different values :

If (Mfunc.eq.0) goto 250

Do 200 mndx=1,Mfunc
    mi=2*mndx-1
    m2=2*mndx
    pimc=(pi*float(mndx))/c
    Beta=-csqrt(pimc*pimc-epsilon)
    Do 100 indx=1,Natop
        x=dx*float(indx-Nhlf1)
        f(mi,indx)=cexp(Beta*(x+hlfL))
        f(m2,indx)=cexp(-Beta*(x-hlfL))
    100    continue
200    continue
250    continue

Beta=cmplx(-k1,k1)

Do 300 indx=1,Nstop
    x=dx*float(indx-Nhlf1)
    f(Mtop1,indx)=cexp(Beta*(x+hlfL))
    f(Mstop,indx)=cexp(-Beta*(x-hlfL))
300    continue

ccc Generate the positive-definate Hermitian matrix [A

Do 600 im=1,Nstop
    A(im,im)=complx(0.,0.)
    Do 500 jm=1,Nstop
        f(im,jm)=A(im,jm)*conjg(f(im,indx))*f(jm,indx)
    500        continue
600    continue

write(6,8200) im,jm,A(im,jm)

continue
600    continue
ccc  Generate the excitation vector
   c
   Do 800 im=1,Mstop
      Iv(im)=cmplx(0.,0.)
   c
   Do 700 indx=1,Natop
      x=dx*float(indx-Nhlf1)
      Iv(im)=Iv(im)+(Kxbot(indx)-Psi(x))*cong(f(im,indx))
     700  continue
   c
   write(6,8300) im,Iv(im)
   800  continue
ccc  Solve system of equations for basis function co-efficients
   c
   Call CPOCO (A,10,Mstop,RCN,sv1,ier)
   Call CPUSL (A,10,Mstop,Iv)
   c
   If (ier.ne.0) write(6,1200) ier
   If (ier.ne.0) return
   write(6,1000) RCN
ccc  Compute the normalized error in the fit
   c
   Sum=0.
   Diff=0.
   c
   Do 900 indx=1,Natop
      x=dx*float(indx-Nhlf1)
      Rv=Kxbot(indx)-Psi(x)
      Rdiff=Rv
     900  continue
   c
   If (Mfunc.eq.0) goto 840
   c
   Do 820 mndx=1,Mfunc
      m1=2*mndx-1
      m2=2*mndx
      pimc=(pi*float(mndx))/c
      Beta=-csqrt(pimc*pimc-epslon)
      Rdiff=Rdiff-Iv(m1)*cexp(Beta*(x+hlfL))
      Rdiff=Rdiff-Iv(m2)*cexp(-Beta*(x-hlfL))
     820  continue
   c
   840 continue
   Beta=cmplx(-k11,k1r)
   c
      Rdiff=Rdiff-Iv(Mstop1)*cexp(Beta*(x+hlfL))
      Rdiff=Rdiff-Iv(Mstop)*cexp(-Beta*(x-hlfL))
   c
   Sum=Sum+cabs(Rv)*cabs(Rv)
   Diff=Diff+cabs(Rdiff)*cabs(Rdiff)
   900  continue
   c
   Error=sqrt(Diff/Sum)
   write(6,8100) Error
Compute the current reflection co-efficients

```c
a0=Iv(Matop1)  
b0=Iv(Matop)    
Beta=cmplx(-k11,k11) 
Rplus =b0/(a0+cexp(Beta*capL)+0.5*cexp(Beta*(half-x0))) 
Rminus=a0/(b0+cexp(Beta*capL)-0.5*cexp(Beta*(half+x0)))
```

```c
write(6,8000) a0,b0  
write(6,2000) Rplus,Rminus
```

```c
1000 format(/' Aperture problem reciprocal condition number = ', e14.7,/)  
1200 format(/' Cholesky decomposition fails at level ',i3,'!')  
2000 format(/' R+ = ',e14.7,' +j',e14.7,/)    
2000 format(/' R- = ',e14.7,' +j',    
8000 format(/' a0 = ',e14.7,' +j',e14.7,5x,'b0 = ',e14.7,' +j',    
8100 format(/' Normalized aperture error = ',e14.7)  
8200 format(/' i = ',i2,3x,'j = ',i2,5x,'A(i,j) = ',e14.7,' +j',    
8300 format(/' i = ',i2,5x,'Iv(i) = ',e14.7,' +j',e14.7)
```
dx=float(M)*capL/float(N)
dalpha=2.*pi/(float(M)*capL)

Do 20 indx=1,N
   xpad(indx)=cmplx(0.,0.)
  20   continue

Load the padded x vector up

Do 40 indx=1,Nhlf
   i=Nhlf+indx
   xpad(indx)=Ktotal(i)
  40   continue

indx=(2*m-1)*Nhlf

Do 80 i=1,Nhlf
   indx=indx+1
   xpad(indx)=Ktotal(i)
  80   continue

Take the FFT of the total current

Call Cfft(xpad,w,logN,Fktot)

Load the xpad vector with zeros

Do 120 indx=1,N
   xpad(indx)=cmplx(0.,0.)
  120   continue

Load the padded x vector up the top surface current

Do 140 indx=1,Nhlf
   i=Nhlf+indx
   xpad(indx)=Ktop(i)
  140   continue

indx=(2*m-1)*Nhlf

Do 180 i=1,Nhlf
   indx=indx+1
   xpad(indx)=Ktop(i)
  180   continue

Take the FFT of the top current

Call Cfft(xpad,w,logN,Fktop)

Swap around elements of Fktot and Fktop to get negative frequencies

Do 500 indx=1,N2
   kndx=N2+indx
   Ctemp=Fktot(indx)
Fktot(index)=Fktot(kndx)
Fktot(kndx)=Ctemp
Ctemp=Fktot(index)
Fktot(index)=Fktot(kndx)
Fktot(kndx)=Ctemp

500 continue

Compute far field pattern from top current for full 180 degrees

Hfar(1)=cmplx(0.,0.)
Hfar(181)=cmplx(0.,0.)
Htop(1)=cmplx(0.,0.)
Htop(181)=cmplx(0.,0.)
Hbot(1)=cmplx(0.,0.)
Hbot(181)=cmplx(0.,0.)
phase=cmplx(1./sqrt(2.),-1./sqrt(2.))

Do 600 idgree=1,179
  index=idgree+1
  theta=rad*float(idgree-90)
  cost=cos(theta)
  alpha=-sin(theta)

Interpolate to find Fourier transform of total and top current

anorm=alpha/dalpha
Call Sfit(Fktot,N,anorm,Fktot0)
Call Sfit(Fktot,N,anorm,Fktot0)

Fktot0=Fktot0*dx
Fktot0=Fktot0*dx
Ea=Eta(alpha)
Gamma=Gamma(alpha)
Gamma1=Gammai(alpha)
G=Green(Gamma,Gamma1)
Ia=Ifunc(alpha)
Tkern=Gamma1*G

Falpha=Tkern*(Ia-(Ea+Fktot0))
Fkbot0=Falpha-Fktot0

Hfar(index)=Tkern*(Ia-(Ea+Fktot0))*cost*phase
Htop(index)=Tkern*(-Fktot0)*cost*phase
Hbot(index)=Tkern*(Ia)*cost*phase

600 continue

return

formats

8000 format(f6.3,4(5x,e14.7,' +j',e14.7))
end
Subroutine Sfit(F,N,znorm,FO)

The subroutine Sfit interpolates the smoothly varying function F by Sinc function expansion.

complex F(4096),FO

N2=N/2

Interpolate function with Sinc expansion

FO=complx(0.,0.)

Do 100 indx=1,N
   ndx=-N2+1+indx
   arg=znorm-float(ndx)
   FO=FO+Sinc(arg)*F(indx)

100 continue

return
end

Subroutine Pole(z,res)

Subroutine Pole finds the location and residue of \( S(z) \) for the lowest order T.M. surface wave mode pole.

common /param/ epslon,refrin,capL,c,x0,delta,M,loN
complex epslon,refrin
complex Ccosh,Csinh,Csqrt,Complx
complex gamma,gamma1,D,Dpr,sgc,cgc,z,dz,res
logical wrong

wrong=.false.

Make initial estimate for pole location

gamma=c*(epsilon-complx(1.,0.))/epsilon
z=Csqrt(gamma*gamma+complx(1.,0.))

Use Newton's method to find actual value of pole

Do 100 indx=1,100
   gamma=Csqrt(z*z-complx(1.,0.))
   gamma1=Csqrt(z*epsilon)
   sgc=Csinh(gamma1*c)
   cgc=Ccosh(gamma1*c)
   D=gamma1*sgc+epsilon*gamma*cgc
Dpr=\(z/\text{gamma1}\ast\text{sgc}\ast(\text{cmplx}(1.,0.))+\text{epslon}\ast\text{gamma}\ast c\) \\
& \quad +z\ast\text{cgc}\ast(\text{c+epslon}/\text{gamma}) \\
dz=-D/Dpr \\
z=z+dz \\
c write(6,8000) index,dz,z \\
If (\text{cabs}(dz/z).lt.1.e-7) goto 200 \\
100 continue \\
c cccc Location of pole failed to converge, stop \\
c write(6,2000) \\
stop \\
c ccc Location of pole converged, compute residue \\
c 200 continue \\
gamma=C\text{sqr}(z+z-cmplx}(1.,0.)) \\
gamma1=C\text{sqr}(z+z-\text{epslon}) \\
If (\text{aimag}(\text{gamma1}).gt.0.) \text{gamma1}=-\text{gamma1} \\
sgc=C\text{sinh}(\text{gamma1}\ast c) \\
cgc=C\text{cosh}(\text{gamma1}\ast c) \\
Dpr=\text{z}/\text{gamma1}\ast\text{sgc}\ast(\text{cmplx}(1.,0.))+\text{epslon}\ast\text{gamma}\ast c \\
& \quad +z\ast\text{cgc}\ast(\text{c+epslon}/\text{gamma}) \\
res=sgc/Dpr \\
If (\text{aimag}(\text{epslon}).eq.0.) z=\text{cmplx}(\text{Real}(z),0.) \\
c If (\text{Real}(z).lt.0.) \text{wrong}=.\text{true.} \\
c If (\text{wrong}) z=-z \\
c If (\text{wrong}) res=-res \\
return \\
c c ccc formats \\
c 2000 format(//,' Error!! ','/ \\
& \quad 5x,'Failed to find pole in Greens function!','//) \\
8000 \quad \text{format}(i3,3x,'dz = ','e12.6,' +j',e12.6,3x, \\
& \quad \quad \quad \quad \quad \quad \quad ' z = ','e14.7,' +j',e14.7) \\
end \\
C C
C******************************************************************************C
C******************************************************************************C
C Function Gamn(alpha) 
C******************************************************************************C
C The function Gamn computes the value of Gamma for a given 
C argument of alpha. 
C
C Complex Gamn 
C
a2=alpha*alpha \\
If (a2 .gt. 1.) Gamn=cmplx(sqrt(a2-1.),0.) \\
If (a2 .eq. 1.) Gamn=cmplx(0.,0.) \\
If (a2 .lt. 1.) Gamn=cmplx(0.,-sqrt(1.-a2)) 
return 
end 
C C******************************************************************************C
Function Gamm1(alpha)

C**************************************************************C
C
C The function Gamm1 computes the value of Gamma_1 for a C
C given argument of alpha.
C
C common /param/ epsilon, refrin, capL, c, x0, delta, M, logN
Complex Gamm1
Complex epsilon, refrin
Complex csqrt, cmplx

C Gamm1=csqrt(alpha*alpha-epsilon)

C Check if Gamm1 is in the fourth quadrant of the complex plane
C
C Gr=real(Gamm1)
Gii=imag(Gamm1)
If (Gr.ge.0. .and. Gii.le.0.) return

C Gamm1 is in wrong quadrant: epsilon must be real
C
If (abs(Gr).gt.abs(Gii)) goto 200

C Real part of Gamm1 should be zero (epsilon is real)
C
eps=real(epsilon)
Gamm1=cmplx(0.-sqrt(eps-alpha*alpha))
return

C Imaginary part of Gamm1 should be zero (epsilon is real)
C
200 continue
eps=real(epsilon)
Gamm1=cmplx(sqrt(alpha*alpha-eps),0.)
return

C
C**************************************************************C
C Function Green(Gamma,Gamma1)
C**************************************************************C
C
C The function Green computes G(alpha), the Fourier transform C
C of the Green's function, given the values of Gamma and C
C Gamma_1.
C
C common /param/ epsilon, refrin, capL, c, x0, delta, Mstop, logN
complex Green, Gamma, Gamm1
complex epsilon, refrin
complex SGc, CGc, Csinh, Ccosh
C
Gr=real(Gamma)
Gii=imag(Gamma)
Gir=real(Gamm1)
Gii=imag(Gamm1)
Check for when Gamma is large with caution

\[ \text{If (Gr.eq.0 .and. (1.0e05 .le. Gr .and. Gr .le. 1.0e10))} \text{ goto 100} \]

Check for a large value of Gamma

\[ \text{If (G1r .gt. 12.0) goto 100} \]

Compute the Green function in the usual way

\[ \text{SGc=Csinh(Gamma*c)} \]
\[ \text{CGc=Cosh(Gamma*c)} \]
\[ \text{Green=SGc/(Gamma*SGc+Csinh(Gamma*CGc))} \]

return

Compute Green function for large Gamma

100 continue
\[ \text{Green=cmplx(c,0.)/eps} \]

return

Compute the Green function for large Gamma

200 continue
\[ \text{Green=1./(Gamma+eps)} \]

return

end

C******************************************************************** C

Function Cosh(z)
C********************************************************************

C The function Cosh computes the complex hyperbolic cosine

complex Cosh,z

x=real(z)
y=imag(z)

Cosh=cmplx(cosh(x)*cos(y) sinh(x)*sin(y))

return

end

C********************************************************************

Function Csinh(z)
C********************************************************************

C The function Csinh computes the complex hyperbolic sine

complex Csinh,z
C
x=real(z)
y=imag(z)

C sinh=cplx(sinh(x)*cos(y),cosh(x)*sin(y))
C
return
end

C******************************************************************************C
Function Ccoth(x)
C******************************************************************************C
C The function Ccoth computes the complex hyperbolic C
cotangent of a complex argument.

C complex Ccoth,Ccosh,Csinh,z
C
zr=real(z)
If (zr.gt. 12.0) Ccoth=cplx(1.,0.)
If (zr.ge.-12.0 .and. zr.le. 12.0) Ccoth=Ccosh(z)/Csinh(z)
If (zr.lt.-12.0) Ccoth=cplx(-1.,0.)
C
return
end

C******************************************************************************C
Function Sinc(x)
C******************************************************************************C
C The function Sinc computes the sinc function of x.

C Pi=3.141592653
C
Sinc=1.
pix=Pi*x
If (pix.ne.0.) Sinc=Sin(pix)/pix
C
return
end

C******************************************************************************C
Function Ifunc(alpha)
C******************************************************************************C
C The function Ifunc computes the value I(alpha), the Fourier C
c transform of the assumed incident field, for a given C
c argument of alpha.

C common /param/ epsilon,refrin,capL,c,x0,delta,M,logN
Complex Ifunc
complex epsilon,refrin
Complex term
C
ax=alpha*x0
term=cmplx(-sin(ax),cos(ax))

c
Ifunc=term*alpha/((alpha-refrin)*(alpha+refrin))

c
return
end

C
C*****************************************************************************
Function Eta(alpha)
C*****************************************************************************
C
C The function Eta computes Eta for a given argument of alpha. The Eta function is the Fourier transform of the numerical solution of the total current to keep its spectral representation well behaved. This is necessary for a good numerical solution since the current must be approximated by a finite bandwidth spectrum.

C
C common /param/ epsilon, refrin, capL, c, x0, delta, M, logN
complex Eta, Eta0
complex epsilon, refrin
logical small

C
ahl=0.5*alpha*capL
small=(abs(ahl) .le. 1.2)
If (small) Eta=Eta0(alpha)
If (small) return

C
ax0=alpha*x0
ad=alpha*delta
ad2=ad*ad
ad3=ad*ad2
ad4=ad2*ad2
term=2.*(sin(ahl)-sin(ahl-ad))/ad4-(cos(ahl)+cos(ahl-ad))/ad3
Eta=cmplx(-sin(ax0),cos(ax0))/alpha-cmplx(0.,8.*delta*term)
return
end

C
C*****************************************************************************
Function Eta0(a)
C*****************************************************************************
C
C The function Eta0 computes the Eta function for small arguments of alpha. This is a Taylor series expansion about alpha equals zero. Terms up to alpha to the fourth are kept in this expansion.

C
C common /param/ epsilon, refrin, L, c, x0, D, M, logN
complex Eta0
complex epsilon, refrin
real L, L2, L3, L4

c
D2=D*D
D3=D*D2
D4=D2*D2
L2=L*L
L3=L*L2
L4=L2*L2
x02=x0*x0
x03=x0*x02
x04=x02*x02
x05=x02*x03
a2=a*a
a3=a2*a2
a4=a2*a2
c
c0=-x0
c1=3./20.*D2-.25*D*L+.125*L2-.5*x02
c2=-1./6.*x03
c4=-x05/120.
c
Eta0=cmplx(c0-c2*a2+c4*a4,c1*a-c3*a3)
c
return
cend
C*****************************************************************************C
C Function Xeta(x)
C*****************************************************************************C
C The function Xeta computes the spatial domain value of the C
C discontinuity function subtracted from the total current C
C for a given value of x. C
C common /param/ epsilon,refrin,capL,c,x0,delta,M,logN
complex epsilon,refrin
C
halfL=0.5*capL
C
If (x.gt.x0) goto 400
If (x.lt.x0) goto 100
C
ccc x equals x0
C
xeta=0.
return
C
ccc x less than x0
C
100 continue
If (x.lt.delta-halfL) goto 200
xeta=-0.5
return
C
200 continue
xnu=(x+halfL)/delta
xeta=-0.5*xnu*xnu*(3.-2.*xnu)
return
x greater than x0

400 continue
   If (x.gt.halfL-delta) goto 500
   Xeta=0.5
   return

500 continue
   xnu=(x-halfL)/delta
   Xeta=0.5*xnu*xnu*(3.+2.*xnu)
   return

end

C***************************************************************************
Function Psi(x)
C***************************************************************************

common /param/ epslon,refrin,capL,c,x0,delta,M,logN
complex epslon,refrin
complex Psi,k1x,arg

Psi=complx(0.,0.)
k1x=refrin*(x-x0)
arg=complx(-aimag(k1x),real(k1x))
If (x.gt.x0) Psi=0.5*cexp(arg)
If (x.lt.x0) Psi=-0.5*cexp(-arg)

return
end

C***************************************************************************
Function Append(x)
C***************************************************************************
The function append adds the suffix '.rec' to the string x.

Character=25 Append,x

Do 100 i=1,21
   j=i+3
   If (x(i:j).eq.' ') Then
      k=i-1
      Append=x(1:k)//'.rec'
      return
   Else
      continue
   Endif
100 continue

Append=x(1:21)//'.rec'
return
end
**Subroutine Winit(W, logN)**

The subroutine Winit initializes the W kernel vector used by the subroutines CFFT and ICFFT to perform Fast Fourier Transforms and Inverse Fast Fourier Transforms.

Complex W(1)

\[ W(1) = 2 \cdot \pi \cdot \text{Float}(N) \]

\[ \text{Pi} = 3.141592653 \]

\[ N = 2^{\text{logN}} \]

\[ N/2 = \text{N/2} \]

\[ \text{Delta} = 2 \cdot \pi / \text{Float}(N) \]

Do 10 I=1,N2
\[ \text{Omega} = \text{Delta} \cdot \text{Float}(I-1) \]
\[ W(I) = \text{Cmplx(Cos(Omega), -Sin(Omega))} \]
10 Continue

Return
End

**SUBROUTINE ICFFT(X, W, LOGN, F)**

The subroutine ICFFT computes the inverse complex Fast Fourier Transform, x(n) of a complex sequence F(k) which is defined as:

\[ x(n) = \sum \{ F(k) \cdot W^{*-k\cdot n} \} \text{ for } n = 0 \text{ to } N-1, \]

where \( W = \text{Exp}[j \cdot 2 \cdot \pi / N] \)

In the calling program the vectors X, F, and W should be dimensioned as follows: X(N), F(N), and W(N/2). The vector W should be initialized as follows before being passed into this routine.

REAL PI, DELTA, OMEGA
\[ \text{Pi} = 3.14159265358979324 \]
\[ N = 2^{\text{LOGN}} \]
\[ N/2 = \text{N/2} \]
\[ \text{Delta} = 2 \cdot \text{DO} \cdot \pi / \text{DFLOAT}(N) \]

Do 10 I=1,N2
\[ \text{Omega} = \text{Delta} \cdot \text{DFLOAT}(I-1) \]
\[ W(I) = \text{Cmplx(Cos(Omega), -Sin(Omega))} \]
10 Continue
IMPLICIT REAL(A-H,O-Z)
COMPLEX X(I),F(I),W(I)
COMPLEX XTOP,XBOT

C
N=2**LOGN
N2=N/2

C
CCC INITIALIZE VECTOR X] WITH REVERSE BINARY ORDERING OF F]

C
DO 100 I=1,N
ID=I-1
IV=N2
INDEX=I

C
DO 20 IBIT=1,LOGN
IF (MOD(ID,2).EQ.1) INDEX=INDEX+IV
ID=ID/2
IV=IV/2
20 CONTINUE

C
X(I)=F(INDEX)
100 CONTINUE

C
IV=N2
IU=1

C
DO 1000 IS=1,LOGN

C
CCC WORK ON EACH BLOCK OF BUTTERFLY OPERATIONS

C
DO 800 I=1,IV
KN=1
IROW1=2*IU+(I-1)
IROW2=IROW1+IU

C
CCC COMPUTE BUTTERFLY OPERATIONS

C
DO 200 J=1,IU
IROW1=IROW1+1
IROW2=IROW2+1
XTOP=X(IROW1)
XBOT=X(IROW2)*W(KN)
X(IROW1)=XTOP+XBOT
X(IROW2)=XTOP-XBOT
KN=KN+IV
200 CONTINUE

C
800 CONTINUE

C
IV=IV/2
IU=IU+2
1000 CONTINUE

C
RETURN
END
SUBROUTINE CFFT(X,W,LOGN,F)

The subroutine CFFT computes the Complex Fast Fourier Transform, a complex sequence $x(n)$, of the sequence $F(k)$ which is defined as:

$$F(k) = \sum x(n)W^{*(k*n)} \text{ for } k=0 \text{ to } N-1,$$

where $W = \exp\left( j \times 2 \pi / N \right)$

The values of $F(k)$ are complex so that there are $N$ independent values of $F$ for $N$ values of $x$.
In the calling program the vectors $X$, $F$, and $W$ should be dimensioned as follows: $X(N)$, $F(N)$, and $W(N/2)$. The vector $W$ should be initialized as follows before being passed into this routine.

```plaintext
REAL PI,DELTA,OMEGA
PI = 3.14159265358979324D0
N = 2**LOGN
N2 = N/2
DELTA = 2.0D0*PI/FLOAT(N)
```

```plaintext
DO 10 I = 1, N2
  OMEGA = DELTA*FLOAT(I-1)
  W(I) = CMPLX(COS(OMEGA), -SIN(OMEGA))
10 CONTINUE
```

IMPLICIT REAL(A-H,O-Z)
COMPLEX X(1), F(1), W(1)
COMPLEX XTOP, XBOT, XTEMP

LOGN1 = LOGN - 1
N = 2**LOGN
N2 = N/2
IU = N2
IV = 1

LOAD F] WITH CONTENTS OF X]

```plaintext
DO 100 INDX = 1, N
  F(INDX) = X(INDX)
100 CONTINUE
```

DO 1000 IS = 1, LOGN

WORK ON EACH BLOCK OF BUTTERFLY OPERATIONS

```plaintext
DO 800 I = 1, IV
  KN = 1
800 CONTINUE
```
IROW1 = 2 * IU * (I - 1)
IROW2 = IROW1 + IU

C
CCC COMPUTE BUTTERFLY OPERATIONS
C
DO 200 J = 1, IU
IROW1 = IROW1 + 1
IROW2 = IROW2 + 1
XTOP = F(IROW1)
XBOT = F(IROW2)
F(IROW1) = XTOP + XBOT
F(IROW2) = (XTOP - XBOT) * CONJG(W(KN))
KN = KN + IV
200 CONTINUE
C
800 CONTINUE
C
IV = IV * 2
IU = IU / 2
1000 CONTINUE
C
CCC GET F(k) BY DECOMPOSING VECTOR F] IN REVERSE BINARY ORDER
C
DO 1200 I = 1, N
ID = I - 1
IV = N2
INDEX = 1
C
DO 1100 IBIT = 1, LOGN
IF (MOD(ID, 2).EQ. 1) INDEX = INDEX + IV
ID = ID / 2
IV = IV / 2
1100 CONTINUE
C
IF (INDEX.LE.1) GOTO 1200
XTEMP = F(I)
F(I) = F(INDEX)
F(INDEX) = XTEMP
1200 CONTINUE
C
RETURN
END
C
C**************************************************************
C
C Subroutine Lopr(x,Nstop,Lx)
C
C**************************************************************
C
C The subroutine Lopr performs the linear operator used
C with the conjugate gradient-FFT method on the vector x
C and returns the results in Lx.
C
C common /opratr/ xpad,Fx,Gkern,w,M,logN
Complex x(1),Lx(1)
Complex xpad(4096), Fx(4096), Gkern(2049), w(2048)

N=2**logN
N2=N/2
Nhlf=N/(2*N)
Nhlf1=Nhlf-1
N8=N/8
N81=N8-1

Do 20 indx=1,N
   xpad(indx)=cmplx(0.,0.)
20   continue

Load the padded x vector up

Do 40 indx=1,Nhlf
   i=Nhlf1+indx
   xpad(indx)=x(i)
40   continue

indx=(2*M-1)*Nhlf1

Do 80 i=1,Nhlf1
   indx=indx+1
   xpad(indx)=x(i)
80   continue

Call Cfft(xpad, w, logN, Fx)

write(6,8000) (Fx(i), i=1,N)

Do 100 indx=1,N2
   Fx(indx)=Fx(indx)*Gkern(indx)
100  continue

Do 200 i=1,N2
   kndx=i+1
   indx=N+1-i
   Fx(indx)=Fx(indx)*Gkern(kndx)
200  continue

Call Icfft(xpad, w, logN, Fx)

Take values over strip

Do 400 indx=1,Nhlf
   i=Nhlf1+indx
   Lx(i)=xpad(indx)
400  continue

indx=(2*M-1)*Nhlf1

Do 600 i=1,Nhlf1
   indx=indx+1
   Lx(i)=xpad(indx)
500      continue
C      write(6,8000) (Ax(i), i=1,Nstop)
C      return
C
8000   format(6x,e14.7,' +j',e14.7)
C
end
C
C***********************************************************************
C
C Subroutine Cgsopr(b,N,xerr,rerr,xbest,r,x,v,Av,Irc)
C
C***********************************************************************
C
C The subroutine Cgs solves a complex Hermitian positive definite system of linear equations using the conjugate gradient technique. The matrix is stored in column major packed form in the array 'apackd'. The system of equations takes the form:
C
          [A] x] = b]
C
where [A] and b] are known and we wish to solve for x].
C
The input and output variables:
C
b      The complex known resultant vector b].
N      The order of the system of equations.
C
xerr   The relative error in x at which iterative solution will stop. (See note below)
C
rerr   The relative error in the resultant at which the solution x will begin to be checked for convergence.
C
xbest   On return the optimum answer for the complex unknown array x] which was found. On input the initial guess of what x] is. If there is no initial guess, the elements of xbest should be set to zero.
C
r      The complex final residual which was computed.
Irc   Integer return code
C
    on input:
    Irc = 0  do not print residual norm
    Irc = 1  print residual norm on each iteration
C
    on return:
    Irc = 0  answer found to requested accuracy
    Irc = 1  answer found to machine precision
    Irc = 2  conjugate gradient fails
C
The following are dummy arrays passed into this subroutine to allow for variable dimensioning.
C
x      The complex final value of x] computed.
C
v      The complex final search direction vector.
C
Av     The complex final value of [A times v].
All arrays should be dimensioned to at least N in the calling program.

Note: The conjugate gradient technique will converge to the exact solution after N iterations assuming no numerical round-off errors. Alternatively this method can be terminated, to save computing time, once the norm of the residual has gotten small enough, and x] has converged. The residual r] is defined as:

\[ r] = b] - [A] x] \]

When the norm of the residual divided by the norm of the initial residual falls below the input variable rerr, this routine will begin to test x] for convergence. When the norm of the difference of x] with the best estimate of x] divided by the norm of the best estimate of x] falls below rerr this routine will terminate. Also if r] ever becomes zero this routine will terminate. If convergence is not achieved after N+10 iterations the program ends with an error flag being set.

This subroutine calls:
- Eqatv, Lopr, Sumcv, Sumv, vdot, and Svdot

Software by Tom Willis Dec. 1986

complex b(1),xbest(1),r(1)
complex x(1),v(1),Av(1)
complex vAv,Svdot,beta,alpha,r2,r2next
logical new,print

Set Irr to value of zero meaning successful execution

print=.false.
If (Irr.ne.0) print=.true.
Irr=0

Machine dependent minimum constant.
Eps is largest number such that 1. + eps = 1.

eps=0.53844180E-06
eps2=eps*eps

Initialize r and v

call Eqatv(xbest,N,x)
call Lopr(x,N,Av)
call Sumv(b,-1.,Av,N,r)
call Eqatv(r,N,v)

Begin iterative solution

nstop=N+5
rfirst = vdot(r, r, N)
prev2 = Svdot(r, r, N)
rmn = rfirst
errr2 = errr * errr
xerr2 = xerr * xerr

If (print) write(6, 8100) rfirst

Do 1000 index = 1, nstop
new = .false.
r2 = r2next
call Lopr(v, N, Av)
vAv = Svdot(v, Av, N)
write(6, 8200) vAv
alpha = r2 / vAv
Call Sumcv(x, alpha, v, N, x)
Call Sumcv(r, -alpha, Av, N, r)
r2next = Svdot(r, r, N)
rnorm = vdot(r, r, N)
If (print) write(6, 8100) rnorm

If the residual is smallest so far save x] vector as best
if (rnorm .gt. rmn) goto 800
call Eqatv(x, N, xbest)
rmn = rnorm
if (rmn .le. eps2) goto 1100
new = .true.
800 continue

If (index .eq. nstop) goto 1000
if (rnorm / rfirst .gt. errr2 .or. new) goto 900

Test to see if x] has converged
Call Sumv(xbest, -1., x, N, Av)
errv = vdot(Av, Av, N) / vdot(xbest, xbest, N)
if (errv .lt. xerr2) goto 1200

continue
beta = r2next / r2
Call Sumcv(r, beta, v, N, v)
1000 continue

Answer fails to converge to the desired accuracy
Irc = 1
return

Answer found to as much precision as machine can represent
1100 continue
Irc = 2
return
Answer found to within allowable precision

1200 continue
   Irc=0
   return

Formats for error detecting

8100 format(3x,'r2 = ',e14.7)
8200 format('vAv = ',e14.7)
end

Subroutine Sumcv(a, scale, b, N, c)

The subroutine Sumv adds two vectors together, weighting the second one by some complex scaling factor. The result is returned in the array c.

complex a(1), b(1), c(1)
complex temp, scale

Do 100 indx=1,N
   temp=a(indx)*scale*b(indx)
   c(indx)=temp
100 continue

return
end

Subroutine Sumv(a, scale, b, N, c)

The subroutine Sumv adds two vectors together, weighting the second one by some real scaling factor. The result is returned in the array c.

complex a(1), b(1), c(1)
complex temp

Do 100 indx=1,N
   temp=a(indx)*scale*b(indx)
   c(indx)=temp
100 continue

return
end
Subroutine Eqav(r,N,v)

The subroutine Eqav sets the array v equal to the array r.

complex r(1),v(1)

Do 100 indx=1,N
  v(indx)=r(indx)
100  continue

return
end

Function Vdot(a,b,N)

The function Vdot takes the conjugate dot product of two vectors a and b. It is *not* assumed that a and b are such that this product will be a real quantity.

Complex Vdot
complex a(1),b(1)

Vdot=complx(0.,0.)

Do 100 indx=1,N
  Vdot=Vdot+conjg(a(indx))*b(indx)
100  continue

return
end

Function Vdot(a,b,N)

The function Vdot takes the conjugate dot product of two vectors a and b. It is *not* assumed that a and b are such that this product will be a real quantity.

Complex Vdot
complex a(1),b(1)

Vdot=complx(0.,0.)

Do 100 indx=1,N
  Vdot=Vdot+a(indx)*b(indx)
100  continue

return
end
Function Vdot(a,b,N)

The function Vdot takes the conjugate dot product of two vectors a and b. It is assumed that a and b are such that this product will be a real quantity.

complex a(1),b(1)

Vdot=0.

Do 100 indx=1,N
   Vdot=Vdot+Real(conjg(a(indx))*b(indx))
100   continue

return
end
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