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Studies in Radar Cross-Sections-XI

The Numerical Determination of the Radar Cross-Section of a Prolate Spheroid

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STUDIES IN RADAR CROSS-SECTIONS

I Scattering by a Prolate Spheroid, by F. V. Schultz (UMM-42, March 1950).

II The Zeros of the Associated Legendre Functions $P_n^m(\mu')$ of Non-Integral Degree, by K. M. Siegel, D. M. Brown, H. E. Hunter, H. A. Alperin, and C. W. Quillen (UMM-82, April 1951).

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NOMENCLATURE

Symbol	Definition	First Used on Page
A_{mn}	Separation constant for wave equation	19
A_n	Expansion coefficient for incident wave	28
a	Semi-major axis of spheroid, $a = F\xi_0$	32
B_{Nn}	Determinantal element	38
b	Semi-minor axis of spheroid	48
C_{Nn}	Determinantal element	39
c	$c = kF$	19
c_{mn}	Expansion coefficient for wave functions	9
D_{Nn}	Determinantal element	39
D_{uvw}	Range of variables u, v, w	6
d_k^{mn}	Spheroidal coefficients	41
E_k^m	Coefficient in recurrence relation for d_k^{mn}	42
$\underline{E}$	Electric vector of electromagnetic field	5
$\underline{e}, \underline{e}_u, \underline{e}_v$, etc.,	Unit vectors in directions of subscript variables	9

NOMENCLATURE
(Continued)

F	Semi-focal length of spheroid	17
$\underline{F}_{mn}$	General wave function	9
$\underline{F}$	Solution of vector wave equation	5
F_k^{mn}	Coefficient in recurrence relations for d_k^{mn}	42
f	Solution of scalar wave equation	6
$f_{mn}^{(\ )}$	Scalar wave function	7
G_k^m	Coefficient in recurrence relation for d_k^{mn}	42
g_k^{mn}	Expansion coefficient in series for A_{mn}	47
$\underline{H}$	Magnetic vector of electromagnetic field	5
h_k^{mn}	Expansion coefficient in series for A_{mn}	47
I_k^{Nn}	Boundary integrals of angular functions	50
i	Imaginary unit, $i = \sqrt{-1}$	5
$J_n(c\xi)$	Bessel function of first kind	47
$j_n(c\xi)$	Spherical Bessel function of first kind	41
k	Wave number, $k = \frac{2\pi}{\lambda} = \omega\sqrt{\epsilon\mu}$	4
L_u, L_v, L_w	Linear differential operators resulting from separation of wave equation	6

NOMENCLATURE
(Continued)

${}^{x}\underline{M}_{mn}^{(\ )}, {}^{y}\underline{M}_{mn}^{(\ )}, {}^{z}\underline{M}_{mn}^{(\ )}$	Vector wave functions	12
m	Characteristic value of separation constant p	7
N_{mn}	Normalization integral for angular functions	24
$P_{k+m}^{m}(\eta)$	Associated Legendre function of first kind	43
p	Separation constant for scalar wave equation	6
$Q_{k+m}^{m}(\xi)$	Associated Legendre function of second kind	43
q	Separation constant for scalar wave equation	6
$R_{mn}^{(\ )}(c,\xi)$	Radial spheroidal function	19
$\underline{r}$	Radius vector	16
r	Separation constant for scalar wave equation	6
$\underline{S}$	Poynting vector, $\underline{S} = \frac{1}{2}\,\mathrm{Re}(\underline{E} \times \underline{H}^{*})$	15
$S_{mn}^{(\ )}(c,\eta)$	Angular spheroidal function	19
t	Time variable	4
$U(u)$	Solution of equation $L_u[U(u), p, q] = 0$	6
U_{Nn}	Determinantal element	39

NOMENCLATURE
(Continued)

u	Arbitrary independent variable	6
$V(v)$	Solution of equation $L_v[V(v), p, r] = 0$	6
v	Arbitrary independent variable	6
V_{Nn}	Determinantal element	39
$W(w)$	Solution of equation $L_w[W(w), p, q, r] = 0$	6
W_{Nn}	Determinantal element	39
w	Arbitrary independent variable	6
x, y, z	Rectangular coordinates	7
α_n	Expansion coefficient in series for scattered wave	30
α'_n	Normalized expansion coefficient, $\alpha'_n = \alpha_n/E'a$	32
α_{mn}	Coefficient in recurrence relation for $R_{mn}^{(\ )}$	25
β_n	Expansion coefficient in series for scattered wave	30
β'_n	Normalized expansion coefficient, $\beta'_n = \beta_n/E'a$	32
β_{mn}	Coefficient in recurrence relation for $R_{mn}^{(\ )}$	25
$\Gamma(x)$	Gamma function	41
γ_{mn}	Coefficient in recurrence relation for $R_{mn}^{(\ )}$	25
γ_n	Expansion coefficient in series for velocity potential	35

NOMENCLATURE
(Continued)

$\Delta(\xi,\eta)$	$\Delta(\xi,\eta) = \xi^2 - \eta^2$	18
$\delta_{i,j}$	Kronecker delta function	65
ϵ	Dielectric constant	4
ζ	Arbitrary independent variable	20
η	Angular spheroidal variable	17
θ	Spherical variable	51
$K_{mn}^{( )}$	Ratio of radial to angular function	21
λ	Wavelength of radiation	46
μ	Permeability	4
$\mu_{i,j}$	Parity function, $\mu_{i,j} = \frac{1}{2}[1 + (-1)^{i+j}]$	41
ξ	Radial spheroidal variable	17
π_{mn}	Coefficient in recurrence relation for $R_{mn}^{( )}$	25
ρ_{mn}	Coefficient in recurrence relation for $R_{mn}^{( )}$	25
ρ	Function of index k which tends to zero as k approaches integral values	43
$\Sigma(\eta)$	$\Sigma(\eta) = 1 - \eta^2$	18
σ	Scattering cross-section	32
σ_{mn}	Coefficient in recurrence relation for $R_{mn}^{( )}$	25

NOMENCLATURE
(Continued)

$T(\xi)$	$T(\xi) = \xi^2 - 1$	18
ϕ	Azimuthal variable	7
$X_{mn}^{(\ )}$	Spheroidal scalar wave function	59
$\chi_{mn}^{(\ )}$	Spheroidal scalar wave function, $X_{mn}^{(\ )}(x,y,z) = \chi_{mn}^{(\ )}(\eta, \xi, \phi)$	59
$\psi_{mn}^{(\ )}$	Spheroidal wave function	22
ψ	Velocity potential	34
ω	Circular frequency of radiation	4

PREFACE

This paper is the eleventh in a series of reports growing out of studies of radar cross-sections at the Willow Run Research Center of the University of Michigan. The primary aims of this program are:

(1) To show that radar cross-sections can be determined analytically.

(2) To elaborate means for computing cross-sections of objects of military interest.

(3) To demonstrate that these theoretical cross-sections are in agreement with experimentally determined values.

Intermediate objectives are:

(1) To compute the exact theoretical cross-sections of various simple bodies by solution of the appropriate boundary-value problems arising from the electromagnetic vector wave equation.

(2) To examine the various approximations possible in this problem, and determine the limits of their validity and utility.

(3) To find means of combining the simple body solutions in order to determine the cross-sections of composite bodies.

(4) To tabulate various formulas and functions necessary to enable such computations to be done quickly for arbitrary objects.

(5) To collect, summarize, and evaluate existing experimental data.

Titles of the papers already published or presently in process of publication are listed on the back of the title page.

K. M. Siegel

INTRODUCTION

In May 1951 it was found necessary to determine theoretically the nose-on radar cross-section of a prolate spheroid of fixed dimensions as the wavelength of the radiation decreased from the Rayleigh region (wavelength greater than characteristic dimension) to the first maximum in the resonance region (wavelength approximately equal to characteristic dimension). It was known that a maximum was to be expected in this "resonance" region; the exact location and width of this maximum (in terms of wavelength) and its exact height (in terms of cross-section) were in question.

The work of Mie (Ref. 28) on the sphere and of Schultz (Ref. 21) on the prolate spheroid supplied the basis for these computations. By 1 September 1952 the necessary numerical analysis had been done and the problem had been coded for the Mark III Electronic Calculator at the U. S. Naval Proving Ground, Dahlgren, Virginia; and the large scale digital computations were begun soon thereafter. By 15 November 1952, the computations described in this report were completed.

This paper presents the solution to this problem. Although it was not possible to compute a sufficient number of points to determine the entire curve of cross-section vs. wavelength, the location, width, and height of the first resonance maximum are now accurately known, as is the entire curve for greater wavelengths; for shorter wavelengths only certain general features are known. This curve is presented as Figure 3, in Section VIII.

The prolate spheroid problem can by no means be considered completed. This report contains an addition to our knowledge of exact solutions and can be considered a good start in our ability to predict the results of the scattering of sound or electromagnetic waves by a prolate spheroid. However, the numerical results obtained apply to only one spheroid (ratio of major to minor axis 10:1), to only one position of the transmitter (nose-on), and to only one direction of scattering (back-scattering).

In addition to the solution of the scattering problem, this report includes recurrence relations and other identities that illuminate the mathematical structure of the spheroidal functions used for the solution. General results about the recurrence relations among solutions of linear second-order differential equations, obtained in the course of the present work and given in detail in References 19 and 20, may be of value in other problems of mathematical physics.

The approach used to solve the problem of the spheroid is precisely that used by Mie forty-six years ago to solve the problem of the sphere. Anyone who has carried out the numerical computations necessary to determine the scattering from a sphere has probably felt that there must be a less cumbersome way of handling such problems. Until such a way is found, however, the present method must be used. When this method is applied to determine the scattering from a spheroid, there are added difficulties due to the fact that the only known representations of the spheroidal functions that are of practical applicability are representations in series of spherical or cylindrical functions. It is hoped that the theoretical results obtained for the spheroidal functions are a start towards simplifying the problem of scattering by a prolate spheroid at least so that it is no more difficult than the problem of scattering by a sphere.

Many people cooperated in the work described in this report, and to give a precise assessment of each personal contribution is quite impossible. Aside from the authors, the major collaborators were: Dr. W. Bauer, Mr. R. Beach, Dr. D. M. Brown, Mr. H. Hunter, Dr. L. M. Rauch, and Miss I. Wyman.

We wish to express our sincere appreciation to the Bureau of Ordnance, U. S. Navy, for its generous support of the numerical computation on the Mark III Electronic Calculator. We are further indebted to Mr. Ralph A. Niemann, Head of the Computation Division, Computation and Ballistics Department, U. S. Naval Proving Ground, Dahlgren, Virginia, and to Messrs. G. H. Gleissner, D. F. Eliezer, Karl Kozarsky, and A. M. Fleishman of the programming and coding staff at Dahlgren who spent many hours preparing the difficult machine program and checking the numerical results.

Section I contains a discussion of the physical problem to be solved and the mathematical procedures employed. It may be omitted by those familiar with the method of Hansen.

Section II, which is essentially independent of the remainder of the report, contains a discussion of a new theory of recurrence relations as applied to the prolate spheroidal functions. This contribution was made after work on the original problem had been completed, and the results of this section were not used in the computations.

Section III reviews the work of Schultz in obtaining an exact expression for the radar cross-section of a prolate spheroid.

Section IV carries out a similar analysis for the scattering of scalar (sound) waves by a prolate spheroid.

Section V discusses the approximate expression which was actually evaluated by machine computation, and other approximations which made a machine program possible.

Section VI describes the Mark III digital computer on which the calculations were carried out.

Section VII outlines the program for machine computation.

Section VIII lists the numerical results obtained, and compares the conclusions drawn from the data with various other results.

Those interested only in the numerical results of this investigation may turn immediately to Section VIII.

I

GENERAL THEORY

The radar cross-section of a perfectly conducting prolate spheroid is to be determined for an incident field due to a plane electromagnetic wave whose direction of propagation is parallel to the major axis of the spheroid. It is assumed that the medium exterior to the scattering body is free space, with fixed dielectric constant ϵ and permeability μ and with zero conductivity. Furthermore both the incident and the scattered field are assumed monochromatic, with fixed circular frequency ω; the solutions for polychromatic fields may subsequently be derived by Fourier series or Fourier integral methods. The mathematical problem of obtaining an expression for the scattered field from knowledge of the incident field, and deducing the radar cross-section, is first discussed in general terms.

Either the electric vector or the magnetic vector alone suffices to determine the radar cross-section. In the monochromatic case, the electric vector may be written $e^{i\omega t}\,\underline{E}$ and the magnetic vector $e^{i\omega t}\underline{H}$, where $\underline{E}$ and $\underline{H}$ are complex vector functions of position. For this form of time dependence, Maxwell's equations combine to form the system of differential equations

$$\nabla \times \nabla \times \underline{F} - k^2\underline{F} = 0, \quad \nabla \cdot \underline{F} = 0 \qquad \text{(I-1)}$$

for $\underline{F} = \underline{E}$ and for $\underline{F} = \underline{H}$, where $k^2 = \epsilon\,\mu\,\omega^2$. The method of solution to be described below is such as to ensure that the vectors obtained to satisfy (I-1) will automatically satisfy Maxwell's equations as well.

On the surface of the scattering body, it is required that the total electric vector field be normal and the total magnetic vector field be tangential. At infinity, it is required that the scattered field behave asymptotically like a diverging spherical wave with center in the scattering body. In the present report the scattered electric field will be obtained.

Mathematically, the boundary conditions on the electric field may be described as follows. Let u, v, w be the variables of a coordinate system in which $u = u_0$, a constant, is the equation of the surface of the scattering body. Denote the incident electric vector by $\underline{E}'$, the scattered electric vector by $\underline{E}$, and the total electric vector by $\underline{E}'' = \underline{E}' + \underline{E}$. On the surface of the scattering body it is required that the components $E''_v(u_0, v, w)$ and $E''_w(u_0, v, w)$ vanish, or that

$$E_v(u_0, v, w) = -E'_v(u_0, v, w),\ E_w(u_0, v, w) = -E'_w(u_0, v, w). \qquad \text{(I-2)}$$

At infinity it is required that on a sphere of radius R with center in the scattering body, and with exterior unit normal $\underline{N}$, one has

$$\lim_{R \to \infty} R\,[(\underline{N} \cdot \nabla)\underline{E} + i\,k\,\underline{E}] = 0, \qquad \text{(I-3)}$$

uniformly in all directions.[1]

Recent results of F. Rellich (Ref. 2), H. Weyl (Ref. 3 and 5), and C. Mueller (Ref. 4) show that the mathematical problem of solving the system of equations (I-1) subject to the boundary conditions (I-2) and (I-3) possesses a unique solution, provided that the scattering body is sufficiently smooth. In the present case the smoothness condition is satisfied, and to prove that the desired electromagnetic field has been obtained it is sufficient to show that the conditions (I-1), (I-2), and (I-3) have been met.

If a vector $\underline{F}$ furnishes a solution of the system (I-1), each cartesian component f(x, y, z) of the vector must be a solution of the scalar wave equation

$$\nabla^2 f + k^2 f = 0. \qquad \text{(I-4)}$$

Furthermore, if the vector $\underline{F}$ satisfies the "vector radiation condition" (I-3), each cartesian coordinate f must satisfy the scalar radiation

1. This form of the boundary condition at infinity is readily obtained, e.g., by combining the results of pages 68 and 85 of S. Silver (Ref. 1) with well-known vector identities.

condition: on a sphere of radius R, one has

$$\lim_{R \to \infty} R[\underline{N} \cdot (\nabla f) + i k f] = 0, \tag{I-5}$$

uniformly in all directions.

Suppose that the scalar wave equation separates in the u, v, w coordinate system, i.e., possesses solutions of the form

$$f = U(u)\, V(v)\, W(w),$$

where each of the functions U, V, W satisfies an ordinary linear differential equation of the second order. In this case powerful tools are furnished by the methods of functional analysis, as applied, for example, by P. M. Morse and H. Feshbach (Ref. 6) and by B. Friedman (Ref. 7). Corresponding to each solution f there are three separation constants, p, q, r (only two of which are independent), which appear in the separated differential equations. With L_u, L_v, L_w denoting the appropriate linear differential operators, one finds that U, V, and W respectively satisfy the equations

$$L_u[U(u), p, q] = 0, \; L_v[V(v), p, r] = 0, \text{ and } L_w[W(w), p, q, r] = 0.$$

In the case to be considered (axial symmetry), L_w depends on p alone, and the notation is to follow this assumption.

The further condition that, for each fixed set p, q, r, the scalar wave equation possess one solution which is finite and single-valued in the entire range D_{uvw} of the variables u, v, and w determines sequences of separation constants p_m, q_{mn}, and r_{mn}, and corresponding sequences of solutions $U_{mn}(u)$, $V_{mn}(v)$, and $W_m(w)$ of the equations

$$L_u[U_{mn}, p_m, q_{mn}] = 0, \; L_v[V_{mn}, p_m, r_{mn}] = 0, \; L_w[W_m, p_m] = 0,$$

respectively (m, n = 0, 1, 2, ...). In the most frequently occurring case of an axially symmetric coordinate system, the variable $w = \phi$ is the azimuthal angle about the axis of symmetry. In this case one has

$$L_w[W_m, p_m] \equiv L_\phi[W_m, p_m] \equiv \frac{d^2 W_m}{d\phi^2} + p_m W_m(\phi) = 0,$$

and the condition that the solution be single-valued determines the separation constants $p_m = m^2$ and the functions $W_0 = a_0$, a constant, and $W_m = a_m \cos m\phi + b_m \sin m\phi$, a_m and b_m constants ($m = 0, 1, 2, \ldots$). The functions $f^{(1)}_{mn} = U^{(1)}_{mn}(u)\, V^{(1)}_{mn}(v)\, W_m(w)$ finite in the entire range D_{uvw} are called scalar wave functions of the first kind. A theorem found to be very useful is that every solution $g(x, y, z)$ of the equation (I-4) which is finite and single-valued in the region D_{uvw} can be expressed as a double series

$$g = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} f^{(1)}_{mn}(x, y, z), \tag{I-6}$$

and that the series is uniformly convergent in any closed, bounded subregion of D_{uvw}.[2]

If the surface $u = u_0$ of the scattering body encloses the finite singular points of the differential equation $L_u = 0$, a more general series expansion appertains. Let $U^{(2)}_{mn}(u)$ be a set of solutions of $L_u[U_{mn}, p_m, q_{mn}] = 0$ each of which is linearly independent of the function of the first kind with the same subscripts m, n. These are called functions of the second kind, and so are the wave functions $f^{(2)}_{mn} = U^{(2)}_{mn} V^{(1)}_{mn} W_m$. In general the variable u will be infinite in range, and the functions of the second kind can be so chosen that of the two combinations

$$f^{(3)}_{mn} = f^{(1)}_{mn} + i f^{(2)}_{mn}, \quad f^{(4)}_{mn} = f^{(1)}_{mn} - i f^{(2)}_{mn} \tag{I-7}$$

one satisfies the radiation condition (I-5) while the other satisfies the "absorption condition" obtained from (I-5) if one replaces the imaginary

2. This theorem is easily deduced from the results about separable equations given in Reference 6 (1953), especially in Section 6.3, in conjunction with well-known theorems about Fourier series.

unit i by its negative. In the present work, the functions of the fourth kind satisfy the radiation condition. In the region D_{u_0vw} exterior to the scattering body, say described by $u \geq u_0$, the functions of the third and fourth kinds are finite and single-valued, and therefore any solution $g'(x, y, z)$ having the same behavior in D_{u_0vw} may be expanded in a double series of wave functions of the third and fourth kinds, which converges uniformly in any bounded, closed sub-region of the region. By choosing functions of the fourth kind one is certain that g' will itself satisfy the radiation condition, and conversely that any function satisfying the radiation condition will have such an expansion.

In the usual case both $V_{mn}^{(1)}(v)$ and $W_m(w)$ form orthogonal sets over their respective ranges D_v and D_w, that is

$$\int_{D_v} \int_{D_w} f_{mn} \, V_{m'n'}^{(1)}(v) \, W_{m'}(w) \, dvdw = 0 \qquad \text{(I-8)}$$

unless both $m = m'$ and $n = n'$ (or there is a similar relation with a weight factor under the integral sign). The standard Fourier series methods now enable one to determine the coefficients in the series for a function g or g' from the values of the function on $u = u_0$ alone.

The solution of the vector wave equation system (I-1) proceeds in two steps. First one must obtain an expression for the incident electric vector, and then given this vector one must satisfy the boundary conditions (I-2) on the surface of the scattering body by a vector function that is a solution of (I-1) and satisfies the radiation condition (I-3) at infinity. To this end a variety of methods have been devised for generating sets $\underline{F}_{mn}$ of vector functions that satisfy (I-1) from sets f_{mn} of scalar functions that satisfy the scalar wave equation (I-4). Two properties of these sets are of importance: there must be enough functions to permit the expression of an arbitrary wave function in terms of them (completeness) and it must be possible to satisfy the boundary conditions (I-2). Previous treatments of such problems have generally assumed the first property without proof, and have concentrated on the objective of making the boundary conditions easy to satisfy. A brief discussion of these previously suggested procedures will be followed by a sketch of a completeness proof for the method of the present report, and this proof with minor modifications can be extended to any method in which differentiation operators applied to solutions of the scalar wave equation are used to obtain solutions of the vector wave equation.

If $\underline{F}_{mn}$ (u,v,w) is the given set of wave functions, and if the incident and the scattered fields are respectively assumed to have expansions $E' = \sum \sum c'_{mn} \underline{F}_{mn}$ and $\underline{E} = \sum \sum c_{mn} \underline{F}_{mn}$, where the coefficients c'_{mn} are known, then the boundary conditions (I-2) give

$$\begin{aligned}
&\sum \sum c_{mn} \underline{F}_{mn} (u_0,v,w) \cdot \underline{e}_v (u_0,v,w) \\
&= - \sum \sum c'_{mn} \underline{F}_{mn} (u_0,v,w) \cdot \underline{e}_v (u_0,v,w), \\
&\sum \sum c_{mn} \underline{F}_{mn} (u_0,v,w) \cdot \underline{e}_w (u_0,v,w) \\
&= - \sum \sum c'_{mn} \underline{F}_{mn} (u_0,v,w) \cdot \underline{e}_w (u_0,v,w),
\end{aligned} \tag{I-9}$$

where $\underline{e}_v$ and $\underline{e}_w$ are unit vectors in the directions of increasing v and increasing w, respectively. For the plane, the sphere, the cone, the paraboloid, and the wedge, wave functions can be found that permit evaluation of the coefficients c_{mn} individually. For other scattering bodies it seems that an infinite system of equations for the infinite set c_{mn} of unknowns is required.

Some sets of wave functions that have been used may be briefly reviewed. Let $f_{mn}(x,y,z)$ denote throughout the set of solutions of the scalar wave equation (I-4) obtained by separation in the appropriate system of coordinates. In the more general case when div $\underline{E} \neq 0$, i.e., when the region surrounding the scattering body has a non-zero charge density, Maxwell's equations combine to form the wave equation

$$\nabla \times \nabla \times \underline{F} - \nabla(\nabla \cdot \underline{F}) - k^2 \underline{F} = 0 \tag{I-10}$$

for $\underline{F} = \underline{E}$ or $\underline{F} = \underline{H}$, and the additional boundary condition div $\underline{E} = G(v,w)$, a given function, on the surface of the body, is needed to determine a unique solution. For this more general problem the plane wave functions

$$f_{mn}\,\underline{a},\ f_{mn}\,\underline{a} \times \underline{b},\ f_{mn}\,(\underline{a} \times \underline{b}) \times \underline{a}$$

may be used, cf. Reference 8, p. 395, where $\underline{a}$ is a vector in the direction of propagation of the wave and $\underline{b}$ is another constant vector. If $\underline{a}$ is replaced by the radial unit vector $\underline{e}_r$, and if any one of these functions is multiplied by an amplitude factor that is a function of direction and integrated over the unit sphere of all possible directions, the result is a more general wave function.

For a sphere, the method of P. Debye (Ref. 9) has been shown by A. Sommerfeld in Reference 10, Ch. XX, to be based on the use of the vector functions $f_{mn}\,\underline{e}_r$ alone. Since these have a zero component in the direction of increasing azimuthal angle ϕ, in general both electric and magnetic vectors of this type are required for the construction of a field, see Reference 10. For axially symmetric bodies, Morse and Feshbach (Ref. 6) suggest the functions $U_{ln}(u)\,V_{ln}(v)\,\underline{e}_\phi$, and the curl of each of the functions, where U_{ln} and V_{ln} are the functions obtained by separation of variables as discussed above and $\underline{e}_\phi$ is the unit vector in the direction of increasing azimuthal angle. However, the applicability of these vector wave functions is limited by the fact that the field expressed by them must have its component in the direction of $\underline{e}_\phi$ independent of ϕ. For such a case they would greatly facilitate the satisfaction of boundary conditions on an axially symmetric body.

A very general method can be obtained by extending the idea of W. W. Hansen (Ref. 11), described in detail in Reference 8. This method was originally devised for the solution of problems in spherical and circular-cylindrical coordinate systems. For the spherical case the sets

$$\nabla f_{mn},\quad \nabla \times \underline{e}_r\, f_{mn},\quad \nabla \times (\,\nabla \times \underline{e}_r\, f_{mn}),$$

and for the cylindrical case the sets

$$\nabla f_{mn},\quad \nabla \times \underline{a}\, f_{mn},\quad \nabla \times (\,\nabla \times \underline{a}\, f_{mn})$$

are appropriate, where f_{mn} denotes the solution of the scalar wave equation in the particular coordinate system and where $\underline{a}$ denotes a vector along the axis of the cylinder. The gradient vectors have zero curl but non-zero divergence, and it is expected that they would have to be employed for the solution of the more general equation (I-10). The other two sets have zero divergence. If the expansion of an arbitrary solution of the first equation in (I-1) in terms of functions from these two sets can be carried out, in other words if completeness of one or both sets can be established, the solution will automatically satisfy the second equation div $\underline{F} = 0$, and the problem is solved. The more general method to be described includes, besides the spherical and circular-cylindrical cases, all scattering problems such that the scattering surface is a contour surface of a coordinate system in which the scalar wave equation is separable. Equations (I-9) will not, in general, reduce to a finite system, but completeness of the sets of wave functions employed will be an immediate consequence of the completeness of the set of scalar wave functions. As has been pointed out, a vector function which has been shown to solve the vector wave equation and to satisfy the boundary conditions on the body and at infinity is the unique solution (Refs. 2-5).

The incident wave and the scattered wave are expanded by essentially the same method. The most general incident wave to be considered is a plane elliptically polarized wave. Both the electric and the magnetic vector may be obtained from the superposition of two linearly polarized waves lying along the axes of the ellipse, and it is intended to give an expansion for only one of these linearly polarized waves. This wave has electric and magnetic vectors of the forms

$$\underline{E}' = \underline{E}_0' e^{ik(\underline{r} \cdot \underline{a})} \text{ and } \underline{H}' = \underline{H}_0' e^{ik(\underline{r} \cdot \underline{a})},$$

respectively, where $\underline{a}$ is a unit vector in the direction of propagation and where $\underline{E}_0'$ and $\underline{H}_0'$ are constant, complex vectors whose amplitude factors are real, mutually perpendicular vectors lying in the plane normal to $\underline{a}$. First, in terms of the scalar wave functions f_{mn} for the given coordinate system, write

$$e^{ik(\underline{r} \cdot \underline{a})} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c'_{mn} f_{mn}^{(1)}.$$

The coefficients c'_{mn} are determined from the relations

$$c'_{mn} \int_{R_v} \int_{R_w} f_{mn}^{(1)} V_{mn}^{(1)}(v)\, W_m(w)\, dv\, dw$$
$$= \int_{R_v} \int_{R_w} e^{\, ik(\underline{r} \cdot \underline{a})}\, V_{mn}^{(1)}\, W_m\, dv\, dw.$$

This scalar expansion leads immediately to an expansion of the incident wave in terms of the vector wave functions of the first kind

$${}^x\underline{M}_{mn}^{(1)} = \nabla \times \underline{e}_x f_{mn}^{(1)},\ {}^y\underline{M}_{mn}^{(1)} = \nabla \times \underline{e}_y f_{mn}^{(1)},\ {}^z\underline{M}_{mn}^{(1)} = \nabla \times \underline{e}_z f_{mn}^{(1)},$$

obtained from the second set of Hansen circular cylinder functions if one chooses the constant vector as a cartesian unit vector.

The incident electric and magnetic vectors have the forms

$$\underline{E}' = \underline{E}_0' \sum \sum c'_{mn} f_{mn}^{(1)},\ \underline{H}' = \underline{H}_0' \sum \sum c'_{mn} f_{mn}^{(1)}. \qquad \text{(I-11)}$$

The two vectors together form an electromagnetic field, and so Maxwell's equations

$$\underline{E} = -\frac{i}{k}\sqrt{\frac{\mu}{\epsilon}}\, \nabla \times \underline{H},\ \underline{H} = \frac{i}{k}\sqrt{\frac{\epsilon}{\mu}}\, \nabla \times \underline{E} \qquad \text{(I-12)}$$

must hold for $\underline{E}=\underline{E}'$, $\underline{H}=\underline{H}'$. Applying (I-12) to (I-11), one obtains

$$\underline{E}' = -\frac{i}{k}\sqrt{\frac{\mu}{\epsilon}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c'_{mn}\left[H'_{ox}\, {}^x\underline{M}_{mn}^{(1)} + H'_{oy}\, {}^y\underline{M}_{mn}^{(1)} + H'_{oz}\, {}^z\underline{M}_{mn}^{(1)}\right],$$
$$\underline{H}' = \frac{i}{k}\sqrt{\frac{\epsilon}{\mu}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c'_{mn}\left[E'_{ox}\, {}^x\underline{M}_{mn}^{(1)} + E'_{oy}\, {}^y\underline{M}_{mn}^{(1)} + E'_{oz}\, {}^z\underline{M}_{mn}^{(1)}\right], \qquad \text{(I-13)}$$

where

$$\underline{E}_0' = E'_{ox}\, \underline{e}_x + E'_{oy}\, \underline{e}_y + E'_{oz}\, \underline{e}_z,\ \underline{H}_0' = H'_{ox}\, \underline{e}_x + H'_{oy}\, \underline{e}_y + H'_{oz}\, \underline{e}_z.$$

The series (I-13) may be rewritten as series in the functions $\partial f_{mn}^{(1)}/\partial x$, $\partial f_{mn}^{(1)}/\partial y$, and $\partial f_{mn}^{(1)}/\partial z$, with coefficients that are precisely the coefficients of $f_{mn}^{(1)}$ in the expansions (I-11) of the cartesian components of $\underline{E}'$ and $\underline{H}'$. In the analogous expansion in vector wave functions $\underline{M}_{mn}$ of the scattered field $\underline{E}$, $\underline{H}$ (or of any electromagnetic field), again the components of the gradient of $f_{mn}^{(1)}$ would appear, and the coefficients would come from the expansions of the cartesian components of $\underline{E}$ and $\underline{H}$ in terms of the scalar wave functions $f_{mn}^{(1)}$. (Of course functions of the first kind must be used for fields finite at the singularities of the separated functions, while functions of the fourth kind must be used for fields finite in more restricted regions but possibly having singularities outside those regions.) These expansions in terms of scalar wave functions are, like (I-11), absolutely and uniformly convergent in any bounded, closed subregion of space containing no singularities of the field. The derived series (I-13) or their analogs have the same convergence properties in all cases where the derivatives of $f_{mn}^{(1)}$, at a fixed distance from the origin, vanish asymptotically for suitably large m and n to at least as high an order as do the functions $f_{mn}^{(1)}$ themselves. A brief computation will show that the spherical, the circular-cylindrical, and the spheroidal scalar wave functions will have this property. It is to be expected, in fact, that all wave functions M_{mn} derived by the Hansen method have the same sort of asymptotic behavior.

It follows that the vectors ${}^{x}M_{mn}$, ${}^{y}M_{mn}$, ${}^{z}M_{mn}$ form a <u>complete set of vector wave functions</u>, if by completeness is meant the property that any solution of the system (I-1) has a convergent series expansion in these vector functions. For the spherical or the circular-cylindrical scalar wave functions $f_{mn}^{(1)}$, the components of the gradient can be explicitly expressed as finite linear combinations or as absolutely and uniformly convergent series in the wave functions $f_{mn}^{(1)}$ by use of the recurrence relations for the functions $U_{mn}(u)$ and $V_{mn}(v)$, and for spheroidal functions analogous expressions are given on page 79 of Appendix A. Substitution of these expressions into the series (I-13) and application of well-known theorems on the convergence of double series is an alternative method for proving the completeness of the vector wave functions.

The radiation condition (I-3) at infinity is automatically satisfied since functions of the fourth kind are used for the expansions. To satisfy the boundary conditions (I-2) on the surface of the scattering body, one must solve the equations similar to (I-9) obtained by setting equal to zero the v and w components of the total field. In the most general case it will be necessary, in addition, to utilize the boundary condition for the total magnetic vector:

$$H''_u(u_0, v, w) = 0 \text{ or}$$

$$H_u(u_0, v, w) = -H'_u(u_0, v, w).$$

The previously detailed theory shows that the magnetic vector $\underline{H}$ of the scattered field has an expansion

$$\underline{H} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [\underline{c}^x_{mn}\, \underline{e}_x + \underline{c}^y_{mn}\, \underline{e}_y + \underline{c}^z_{mn}\, \underline{e}_z]\, f^{(4)}_{mn}. \qquad \text{(I-15)}$$

The first equation in (I-12) consequently gives

$$E = -\frac{i}{k}\sqrt{\frac{\mu}{\epsilon}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [c^x_{mn}\, {}^x\underline{M}^{(4)}_{mn} + c^y_{mn}\, {}^y\underline{M}^{(4)}_{mn} + c^z_{mn}\, {}^z\underline{M}^{(4)}_{mn}]. \qquad \text{(I-16)}$$

When all the functions in (I-13), (I-15), and (I-16) are expressed in terms of the unit vectors $\underline{e}_u$, $\underline{e}_v$, and $\underline{e}_w$, and substituted into the boundary conditions (I-2) and (I-14), there result three sets of infinitely many equations in the infinitely many unknowns c^x_{mn}, c^y_{mn}, and c^z_{mn}. Another form of the solution may be obtained by first expanding the scattered electric field as in (I-15), then deriving the scattered magnetic field as in (I-16) by the second equation in (I-12). This procedure leads to three sets of equations for the three sets of coefficients in the expansion of the electric field.

In all attempts to produce numerical results from solutions of this type, certain simplifying assumptions are made. If either the incident electric or the incident magnetic vector is linearly polarized so that the line of polarization lies in a cartesian coordinate plane, one set of wave functions and the corresponding set of coefficients c_{mn} fall out entirely. If the incident wave lies along the axis of an axially symmetric scattering body, say the z-axis, the dependence on the azimuth angle ϕ

becomes very simple. Consequently the double sums in (I-13), (I-15), and (I-16) become sums over n alone, with only one value of m appearing in each sum and with one set of wave functions eliminated as above. There remain relations equivalent to two sets of infinitely many equations in infinitely many unknowns. An approximate solution of this system of equations is obtained in this report and applied to give an approximate expression for the electric vector of the field due to the scattering of a plane wave by a perfectly conducting spheroid.

From the electric vector $\underline{E}$ of the scattered field, the radar cross-section σ is derived as follows. Consider an electromagnetic field whose electric and magnetic vectors are given by $e^{i\omega t}\,\underline{E}$ and $e^{i\omega t}\,\underline{H}$, respectively, where $\underline{E}$ and $\underline{H}$ are vector functions of position. With $\underline{E}^*$ and $\underline{H}^*$ denoting the complex conjugates of $\underline{E}$ and $\underline{H}$, respectively, define the real numbers

$$E_0 = (\underline{E}\cdot\underline{E}^*)^{\frac{1}{2}}, \quad H_0 = (\underline{H}\cdot\underline{H}^*)^{\frac{1}{2}}.$$

These "magnitudes" are in the ratio

$$H_0 = \sqrt{\epsilon/\mu}\; E_0.$$

For a monochromatic, harmonic field of this sort, the mean (time average) intensity of energy flow per unit area, or mean power density, is described by the real vector[3]

$$\underline{S} = \frac{1}{2}\,\mathrm{Re}(\underline{E} \times \underline{H}^*), \tag{I-17}$$

that is, the direction of $\underline{S}$ is the direction of energy flow and the magnitude S of the vector is equal to the mean energy per unit time crossing a unit area whose normal is oriented like $\underline{S}$. A substitution in (I-17) and a simple computation show that

$$S = \frac{1}{2} E_0 H_0 = \frac{1}{2}\sqrt{\epsilon/\mu}\; E_0^2. \tag{I-18}$$

3. A proof of this fact appears in Reference 8, page 137.

As before, let $\underline{E}'$ and $\underline{H}'$ denote the complex, time-separated electric and magnetic vectors of the incident field, and let $\underline{E}$ and $\underline{H}$ denote the complex, time-separated electric and magnetic vectors of the scattered field, respectively. Besides $E_0 = (\underline{E}\cdot\underline{E}^*)^{\frac{1}{2}}$ and $H_0 = (\underline{H}\cdot\underline{H}^*)^{\frac{1}{2}}$, introduced above, define $E_0' = (\underline{E}'\cdot\underline{E}'^*)^{\frac{1}{2}}$, $H_0' = (\underline{H}'\cdot\underline{H}'^*)^{\frac{1}{2}}$ (the symbol * again denoting the complex conjugate vector). From (I-18), the mean power density for the incident field is equal to $S' = \frac{1}{2}\sqrt{\epsilon/\mu}\,(E_0')^2$, while for the scattered field it is equal to $S = \frac{1}{2}\sqrt{\epsilon/\mu}\,(E_0)^2$. For the definition of the radar cross-section, one introduces a fictitious isotropically scattering body. The total mean power intercepted by this body is equal to the product of mean power density by the projected area of the body. If such a scatterer of cross-sectional area σ is placed at the origin, the total mean power scattered is equal to $\sigma\, S'$, and the mean power density observed at a distance r from the origin is equal to $\sigma\, S'/4\pi r^2$.

The electromagnetic wave actually scattered from the given non-isotropic body behaves asymptotically like a diverging spherical wave, cf. (I-3). Consequently the mean power density actually observed at a point far from the scattering body depends (asymptotically as $r\rightarrow\infty$) on the direction of the ray from the origin to the point of observation. In terms of a spherical coordinate system, let $\underline{r} = r\underline{e}_r$ denote such a ray. The radar cross-section σ of a scattering body is a function of the direction of $\underline{e}_r$, defined as the cross-section σ of an isotropic body of the type described which scatters the incident wave with a power density in the direction of $\underline{e}_r$ asymptotically equal to the power density of the wave actually scattered by the given body. In a formula, the radar cross-section is that value of σ for which $\sigma\, S'/4\pi r^2$ and S are asymptotically equal for arbitrarily large values of r, or

$$\sigma = \lim_{r\rightarrow\infty} 4\pi\, r^2 (E_0)^2/(E_0')^2. \qquad \text{(I-19)}$$

II

PROLATE SPHEROIDAL FUNCTIONS AND RECURRENCE RELATIONS

The scalar wave equation separates in the prolate spheroidal coordinate system, in which the coordinate surfaces are confocal ellipsoids and hyperboloids of revolution, and halfplanes through the axis of the system, say the z-axis. Denote by ϕ the azimuthal angle about this axis. The foci of the system are placed on the axis, a distance F to either side of the origin. In each halfplane ϕ = const., the curves η = const. are portions of hyperbolas, while the curves ξ = const. are portions of ellipses (Figure 1). It is customary to call η the "angular"

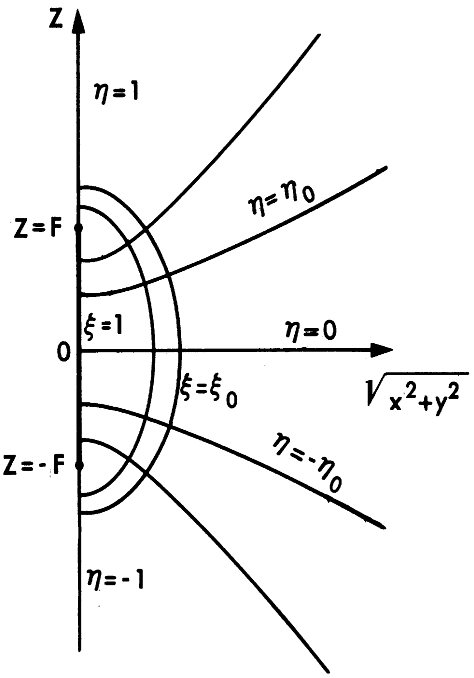

FIG. 1 PROLATE SPHEROIDAL COORDINATES

and ξ the "radial" variable, by analogy to the spherical case, since $\xi \rightarrow r/F$, $\eta \rightarrow \cos\,\theta$, as $\xi \rightarrow \infty$.

The prolate spheroidal variables are defined by the formulas

$$x = F(1 - \eta^2)^{1/2} (\xi^2 - 1)^{1/2} \cos \phi,$$
$$y = F(1 - \eta^2)^{1/2} (\xi^2 - 1)^{1/2} \sin \phi, \qquad \text{(II-1)}$$
$$z = F\eta \xi$$

where $-1 \leq \eta \leq 1$, $1 \leq \xi < \infty$, and $0 \leq \phi < 2\pi$. The partial derivatives of the spheroidal variables with respect to x, y, and z are

$$\frac{\partial \eta}{\partial x} = -\frac{\eta \Sigma^{\frac{1}{2}} T^{\frac{1}{2}} \cos \phi}{F \Delta}, \quad \frac{\partial \eta}{\partial y} = -\frac{\eta \Sigma^{\frac{1}{2}} T^{\frac{1}{2}} \sin \phi}{F \Delta}, \quad \frac{\partial \eta}{\partial z} = \frac{\xi \Sigma}{F \Delta},$$

$$\frac{\partial \xi}{\partial x} = \frac{\xi \Sigma^{\frac{1}{2}} T^{\frac{1}{2}} \cos \phi}{F \Delta}, \quad \frac{\partial \xi}{\partial y} = \frac{\xi \Sigma^{\frac{1}{2}} T^{\frac{1}{2}} \sin \phi}{F \Delta}, \quad \frac{\partial \xi}{\partial z} = \frac{\eta T}{F \Delta}, \qquad \text{(II-2)}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{F \Sigma^{\frac{1}{2}} T^{\frac{1}{2}}}, \quad \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{F \Sigma^{\frac{1}{2}} T^{\frac{1}{2}}}, \quad \frac{\partial \phi}{\partial z} = 0,$$

where

$$\Delta(\xi, \eta) = \xi^2 - \eta^2, \Sigma(\eta) = 1 - \eta^2, \ T(\xi) = \xi^2 - 1. \qquad \text{(II-3)}$$

(These three abbreviations will be used throughout Sections II and III.) The cartesian unit vectors are given in terms of unit vectors of the spheroidal system by

$$\underline{e}_x = -\frac{T^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}} \eta \cos \phi \, \underline{e}_\eta + \frac{\Sigma^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}} \xi \cos \phi \, \underline{e}_\xi - \sin \phi \, \underline{e}_\phi,$$

$$\underline{e}_y = -\frac{T^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}} \eta \sin \phi \, \underline{e}_\eta + \frac{\Sigma^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}} \xi \sin \phi \, \underline{e}_\xi + \cos \phi \, \underline{e}_\phi, \qquad \text{(II-4)}$$

$$\underline{e}_z = \frac{\xi \Sigma^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}} \underline{e}_\eta + \frac{\eta T^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}} \underline{e}_\xi .$$

The scalar wave equation (I-4) has solutions of the form

$$f_{mn} = U_{mn}(\xi)\, V_{mn}(\eta)\, (a_{mn} \cos m\phi + b_{mn} \sin m\phi).$$

The prolate spheroidal functions U_{mn} and V_{mn} must satisfy the differential equations

$$\frac{d}{d\xi}(\xi^2 - 1)\frac{dU_{mn}}{d\xi} - \left[\frac{m^2}{\xi^2 - 1} - A_{mn} - c^2\xi^2\right]U_{mn} = 0, \qquad \text{(II-5)}$$

$$\frac{d}{d\eta}(1 - \eta^2)\frac{dV_{mn}}{d\eta} - \left[\frac{m^2}{1 - \eta^2} + A_{mn} + c^2\eta^2\right]V_{mn} = 0, \qquad \text{(II-6)}$$

respectively, where $c = kF$. The separation constants A_{mn} are fixed by the requirement that the angular functions of the first kind $V_{mn} = S^{(1)}_{mn}(c,\eta)$ be finite for $-1 \le \eta \le 1$. The radial functions of the first kind $U_{mn} = R^{(1)}_{mn}(c,\xi)$ are solutions of the equation (II-5) for the same values of A_{mn}; they are finite in the range $1 \le \xi < \infty$, and are normalized to behave asymptotically like the spherical Bessel functions:

$$R^{(1)}_{mn}(c,\xi) \sim \frac{1}{c\xi}\cos\left(c\xi - \frac{m+n+1}{2}\pi\right) \qquad \text{(II-7)}$$

as $c \to \infty$, or more precisely, if $(c\xi)^2(\xi^2 - 1) \gg m^2$ and $(c\xi)^2 \gg A_{mn}$. The radial functions of the fourth kind are of interest for present purposes, as indicated in Section I; these are determined by the radiation condition (I-5) as the solutions of (II-5) with the asymptotic behavior

$$R^{(4)}_{mn}(c,\xi) \sim \frac{1}{c\xi}\exp\left[-i\left(c\xi - \frac{m+n+1}{2}\pi\right)\right] \qquad \text{(II-8)}$$

as $c\xi \to \infty$. The function $R^{(3)}_{mn}$ has similar behavior with $-i$ taking the place of i. The radial functions of the second kind, appearing as the imaginary part of those of the third kind [cf. (I-7)], thus behave asymptotically as follows:

$$R^{(2)}_{mn}(c,\xi) \sim \frac{1}{c\xi}\sin\left(c\xi - \frac{m+n+1}{2}\pi\right). \qquad \text{(II-9)}$$

The asymptotic behavior of the derivatives of these functions may be obtained by differentiation:[4]

$$R_{mn}^{(1)'}(c,\xi) \sim -\frac{1}{\xi}\sin\left(c\xi - \frac{m+n+1}{2}\pi\right), \qquad \text{(II-10)}$$

$$R_{mn}^{(2)'}(c,\xi) \sim \frac{1}{\xi}\cos\left(c\xi - \frac{m+n+1}{2}\pi\right). \qquad \text{(II-11)}$$

The angular functions of the second kind will not be needed in the present work. Both the radial and the angular functions of the second kind have singularities, at $\xi = 1$ and at $\eta = \pm 1$, respectively. Since the argument c will remain fixed except in the tabulated results of machine computations, it will be omitted in writing the spheroidal functions.

The radial functions $W_{mn} = R_{mn}$ and the angular functions $W_{mn} = S_{mn}$ are solutions of the same differential equation,

$$\frac{d}{d\zeta}\left[(1-\zeta^2)\frac{dW_{mn}}{d\zeta}\right] + \left[\frac{m^2}{1-\zeta^2} - c^2\zeta^2 - A_{mn}\right]W_{mn} = 0, \qquad \text{(II-12)}$$

but refer to different portions of the real ζ-axis as range of the independent variable. This equation is similar to the Lamé or the Mathieu equation, with regular singular points at $\zeta = \pm 1$ and with an irregular singular point at $\zeta = \infty$. The older literature about the theory of this equation is listed by H. Bateman (Ref. 13) and M. J. O. Strutt (Ref. 14), the more recent literature by J. A. Stratton et al. (Ref. 15) and by C. Flammer (Ref. 16). In the present report, the notation of Reference 15 has been adopted, except that the subscript ℓ has been replaced by n for typographical reasons.

By multiplying equation (II-12) for $R_{mn}^{(1)}$ by $R_{mn}^{(2)}$, multiplying equation (II-12) for $R_{mn}^{(2)}$ by $R_{mn}^{(1)}$, and subtracting, one obtains a linear first-order differential equation for the Wronskian of these functions, and the value of the arbitrary constant is easily deduced from the asymptotic value of the Wronskian by use of (II-7), (II-10), and (II-11)

4. A proof of this fact appears in J. F. Ritt (Ref. 12).

[cf. Reference 16, Formula (67)]:

$$R_{mn}^{(1)} R_{mn}^{(2)'} - R_{mn}^{(1)'} R_{mn}^{(2)} = [c\, T(\zeta)]^{-1}. \qquad \text{(II-13)}$$

The Wronskian of the angular functions is similarly found to be

$$S_{mn}^{(1)} S_{mn}^{(2)'} - S_{mn}^{(1)'} S_{mn}^{(2)} = [h\, \Sigma(\zeta)]^{-1}, \qquad \text{(II-14)}$$

and the constant h will now be determined. Since $R_{mn}^{(1)}(\zeta)$ and $S_{mn}^{(1)}(\zeta)$ are both solutions of (II-12) regular at $\zeta = 1$, their ratio must be a constant, and the functions of the second kind are also so selected that their ratio is constant:

$$S_{mn}^{(1)}(\zeta) = K_{mn}^{(1)} R_{mn}^{(1)}(\zeta), \quad S_{mn}^{(2)}(\zeta) = K_{mn}^{(2)} R_{mn}^{(2)}(\zeta) \qquad \text{(II-15)}$$

(in the notation of Reference 16, Formulas (68) and (69); but note that Flammer's index n coincides with n + m in the notation of Reference 15 and of the present report). If the resulting identity $K_{mn}^{(2)} R_{mn}^{(2)}/R_{mn}^{(1)} = K_{mn}^{(1)} S_{mn}^{(2)}/S_{mn}^{(1)}$ is differentiated and compared to (II-13) and (II-14), one finds that $K_{mn}^{(2)}/c[R_{mn}^{(1)}]^2 = - K_{mn}^{(1)}/h[S_{mn}^{(1)}]^2$, since $\Sigma(\zeta) = - T(\zeta)$. Finally, the first equation of (II-15) gives the desired evaluation:

$$h = - c/K_{mn}^{(1)} K_{mn}^{(2)}. \qquad \text{(II-16)}$$

In the literature about spheroidal functions and their applications one sometimes finds the statement that great simplifications could be achieved if recurrence relations for the functions S_{mn} and R_{mn} were known. By a method employed for Mathieu functions by E. T. Whittaker (Ref. 17) the existence of such relations for the radial functions of the first kind was established by I. Marx (Ref. 18), and integral representations for the coefficients were obtained. Further methods, detailed in References 19 and 20, give alternative representations that permit deeper insight into the structure of these relations, and incidentally enable one to extend the recurrence relations to radial functions of the second kind and to both kinds of angular functions. The coefficients, in either representation, are at least as difficult to evaluate as the functions whose computation they are supposed to simplify. What is really needed is a representation of the solutions of such non-hypergeometric equations as (II-12), with the simplicity of the hypergeometric series--enabling one to identify the recurrence coefficients as simpler functions of ζ, n, and c -- and the possibility of such a representation is still an open question. The methods used to obtain the existence of the recurrence relations and to evaluate their coefficients are sketched below.

The spheroidal functions are factors of the spheroidal wave functions. The wave functions may conveniently be separated into a set even in ϕ and a set odd in ϕ, and here only the former set

$$\psi_{mn}^{(1)}(x,y,z) = \psi_{mn}^{(1)}(\eta,\xi,\phi) = S_{mn}^{(1)}(\eta)\, R_{mn}^{(1)}(\xi)\cos m\phi \qquad \text{(II-17)}$$

will be needed. Since the scalar wave operator $\nabla^2 + k^2$ commutes with all differentiations, the functions $\partial\psi_{mn}/\partial z$ and $\partial\psi_{mn}/\partial x$ are likewise solutions of the scalar wave equation (I-4), and they are regular solutions which are even functions of ϕ, as is easily verified by means of formulas (II-2) and the series representations given in Reference 16. Consequently the theorem related in the preceding section ensures that they may be expanded in double series (I-6) of the wave functions $\psi_{mn}^{(1)}$, convergent in any closed, bounded subregion of space. Since both the function sets cos m ϕ, for m = 0, 1, 2, . . ., and $S_{mn}^{(1)}(\eta)$, for fixed m and for n = 0, 1, 2, . . ., are orthogonal as in (I-8) over the ranges of their respective arguments, the coefficients in the expansions may be isolated by double integration. The method of Whittaker consists in comparing the series representations of $\partial\psi_{mn}^{(1)}/\partial z$ and $\partial\psi_{mn}^{(1)}/\partial x$ with the expressions obtained from the differentiation formulas (II-2):

$$\frac{\partial\psi_{mn}^{(1)}}{\partial z} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} f_{pq}^{mn}\, \psi_{pq}^{(1)}(\eta,\xi,\phi)$$

$$= \frac{\cos m\phi}{F\Delta}\left[\xi\, \Sigma\, S_{mn}^{(1)\prime} R_{mn}^{(1)} + \eta T\, S_{mn}^{(1)} R_{mn}^{(1)\prime}\right],$$

$$\frac{\partial\psi_{mn}^{(1)}}{\partial x} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} g_{pq}^{mn}\,\psi_{pq}^{(1)}(\eta,\xi,\phi)$$

$$= \frac{\cos\phi\cos m\phi}{F\Delta}\Sigma^{1/2}T^{1/2}\left[-\eta S_{mn}^{(1)'}R_{mn}^{(1)} + \xi S_{mn}^{(1)}R_{mn}^{(1)'}\right]$$

$$+ \frac{m\sin\phi\sin m\phi}{F\Sigma^{1/2}T^{1/2}} S_{mn}^{(1)} R_{mn}^{(1)} \qquad \text{(II-18)}$$

$$= \frac{\cos(m-1)\phi}{2F}\left\{\frac{\Sigma^{1/2}T^{1/2}}{\Delta}\left[-\eta S_{mn}^{(1)'}R_{mn}^{(1)} + \xi S_{mn}^{(1)}R_{mn}^{(1)'}\right] + \frac{m}{\Sigma^{1/2}T^{1/2}} S_{mn}^{(1)}R_{mn}^{(1)}\right\}$$

$$+ \frac{\cos(m+1)\phi}{2F}\left\{\frac{\Sigma^{1/2}T^{1/2}}{\Delta}\left[-\eta S_{mn}^{(1)'}R_{mn}^{(1)} + \xi S_{mn}^{(1)}R_{mn}^{(1)'}\right] - \frac{m}{\Sigma^{1/2}T^{1/2}} S_{mn}^{(1)}R_{mn}^{(1)}\right\},$$

the primes denoting differentiation with respect to the appropriate variable. The orthogonality of the wave functions implies that only coefficients f_{pq}^{mn} with $p = m$ and coefficients g_{pq}^{mn} with $p = m - 1$ or $p = m + 1$ are different from zero, and that these coefficients are equal to integrals over the right hand members. These integrals, moreover, are zero whenever the integrand is an odd function of η. Since $S_{mn}^{(1)}(\eta)$ is an even or an odd function of η according as n is even or odd, it follows further that only coefficients f_{pq}^{mn} with n and q of opposite parity, and coefficients g_{pq}^{mn} with n and q of equal parity, are different from zero.

A differential formula for raising the index n, a formula for lowering the index n, a formula for raising m, and one for lowering m are sufficient, since all other recurrence relations may be derived from them. Consequently relations among $R_{mn}^{(1)'}$, $R_{mn}^{(1)}$ and $R_{m,n-1}^{(1)}$, among $R_{m,n-1}^{(1)'}$, $R_{m,n-1}^{(1)}$, and $R_{mn}^{(1)}$, among $R_{mn}^{(1)'}$, $R_{mn}^{(1)}$, and $R_{m-1,n}^{(1)}$, and among $R_{m-1,n}^{(1)'}$, $R_{m-1,n}^{(1)}$, and $R_{mn}^{(1)}$ are taken as the standard forms. Multiplying both sides of (II-18) by $S_{pq}^{(1)}(\eta)\cos p\phi$ with a suitable choice of the indices p and q, and integrating both sides with respect to η and ϕ, one obtains the formulas

$$F\, f^{mn}_{m,n-1}\, N_{m,n-1}\, R^{(1)}_{m,n-1} = T\, R^{(1)'}_{mn} \int_{-1}^{1} \frac{\eta}{\Delta} S^{(1)}_{mn}\, S^{(1)}_{m,n-1}\, d\eta$$

$$+ R^{(1)}_{mn}\, \xi \int_{-1}^{1} \frac{\Sigma}{\Delta} S^{(1)'}_{mn}\, S^{(1)}_{m,n-1}\, d\eta, \qquad \text{(II-19)}$$

$$F\, f^{m,n-1}_{mn}\, N_{mn}\, R^{(1)}_{mn} = T\, R^{(1)'}_{m,n-1} \int_{-1}^{1} \frac{\eta}{\Delta} S^{(1)}_{m,n-1}\, S^{(1)}_{m,n}\, d\eta$$

$$+ R^{(1)}_{m,n-1}\, \xi \int_{-1}^{1} \frac{\Sigma}{\Delta} S^{(1)'}_{m,n-1}\, S^{(1)}_{mn}\, d\eta, \qquad \text{(II-20)}$$

and

$$2\, F\, g^{mn}_{m-1,n}\, N_{m-1,n}\, R^{(1)}_{m-1,n} = T^{1/2}\, R^{(1)'}_{mn}\, \xi \int_{-1}^{1} \frac{\Sigma^{1/2}}{\Delta} S^{(1)}_{mn}\, S^{(1)}_{m-1,n}\, d\eta$$

$$- R^{(1)}_{mn} \int_{-1}^{1} S^{(1)}_{m-1,n} \left[\frac{\eta \Sigma^{1/2} T^{1/2}}{\Delta} S^{(1)'}_{mn} - \frac{m}{\Sigma^{1/2} T^{1/2}} S^{(1)}_{mn} \right] d\eta, \qquad \text{(II-21)}$$

$$2\, F\, g^{m-1,n}_{mn}\, N_{mn}\, R^{(1)}_{mn} = T^{1/2}\, R^{(1)'}_{m-1,n}\, \xi \int_{-1}^{1} \frac{\Sigma^{1/2}}{\Delta} S^{(1)}_{m-1,n}\, S^{(1)}_{mn}\, d\eta$$

$$- R^{(1)}_{m-1,n} \int_{-1}^{1} S^{(1)}_{mn} \left[\frac{\eta \Sigma^{1/2} T^{1/2}}{\Delta} S^{(1)'}_{m-1,n} + \frac{m-1}{\Sigma^{1/2} T^{1/2}} S^{(1)}_{m-1,n} \right] d\eta, \qquad \text{(II-22)}$$

where

$$N_{mn} = \int_{-1}^{1} \left[S^{(1)}_{mn}(\eta) \right]^2 d\eta. \qquad \text{(II-23)}$$

The constants in the left hand members of (II-19)--(II-22) may be evaluated by an examination of the asymptotic forms of these relations. If (II-7) and (II-10), with suitable choices of m and n, are substituted into the formulas, one finds from the dominant terms that

$$F\, f^{mn}_{m,n-1}\, N_{m,n-1} = -F\, f^{m,n-1}_{mn}\, N_{mn} = c\int_{-1}^{1} \eta\, S^{(1)}_{mn}\, S^{(1)}_{m,n-1}\, d\eta \qquad \text{(II-24)}$$

and

$$2F\, g^{mn}_{m-1,n}\, N_{m-1,n} = -2F\, g^{m-1,n}_{mn}\, N_{mn} = c\int_{-1}^{1} \Sigma^{1/2}\, S^{(1)}_{mn}\, S^{(1)}_{m-1,n}\, d\eta. \qquad \text{(II-25)}$$

The recurrence relations are now seen to have the form

$$T\,\alpha_{mn}(\xi)\, R^{(1)\prime}_{mn} + \beta_{mn}(\xi)\, R^{(1)}_{mn} = R^{(1)}_{m,n-1}, \qquad \text{(II-26)}$$

$$T\,\alpha_{mn}(\xi)\, R^{(1)\prime}_{m,n-1} + \gamma_{mn}(\xi)\, R^{(1)}_{m,n-1} = -R^{(1)}_{mn}, \qquad \text{(II-27)}$$

and

$$T\,\pi_{mn}(\xi)\, R^{(1)\prime}_{mn} + \rho_{mn}(\xi)\, R^{(1)}_{mn} = R^{(1)}_{m-1,n}, \qquad \text{(II-28)}$$

$$T\,\pi_{mn}(\xi)\, R^{(1)\prime}_{m-1,n} + \sigma_{mn}(\xi)\, R^{(1)}_{m-1,n} = -R^{(1)}_{mn}. \qquad \text{(II-29)}$$

Integral representations of the coefficients in (II-26)--(II-29) are given in formulas (II-19)--(II-22), with constants as in (II-23)--(II-25).

For a further study of the coefficients it is adequate to concentrate on the recurrence relations (II-26) and (II-27) which change the index n. Differentiation under the integral sign in the coefficient α_{mn}, easily proved to be valid, leads to a verification that the equation

$$T\, d\alpha_{mn}/d\xi + \beta_{mn} + \gamma_{mn} = 0 \qquad \text{(II-30)}$$

connects the three coefficients. Eliminating β_{mn} and γ_{mn} from (II-26) and (II-27) by means of (II-30), one obtains the identity

$$d/d\xi\,[\alpha_{mn}/R^{(1)}_{mn}\,R^{(1)}_{m,n-1}] = T^{-1}\left\{[R^{(1)}_{m,n-1}]^{-2} - [R^{(1)}_{mn}]^{-2}\right\}. \qquad \text{(II-31)}$$

With (II-31) and the Wronskian identity (II-13) it is possible to prove that the radial functions of the second kind $R^{(2)}_{mn}$ and $R^{(2)}_{m,n-1}$ satisfy recurrence relations of the form (II-26) and (II-27) with the same coefficients α_{mn}, β_{mn}, and γ_{mn}.

Consider the functions

$$F_{mn}(\xi) = [T\,\alpha_{mn}\,R^{(2)\prime}_{mn} + \beta_{mn}\,R^{(2)}_{mn} - R^{(2)}_{m,n-1}]/R^{(1)}_{m,n-1},$$

$$G_{mn}(\xi) = [T\,\alpha_{mn}\,R^{(2)\prime}_{m,n-1} + \gamma_{mn}\,R^{(2)}_{m,n-1} + R^{(2)}_{m,n}]/R^{(1)}_{mn}.$$

In view of (II-13), $R^{(2)\prime}_{mn}$ may be eliminated from F_{mn}, and the function rewritten as

$$F_{mn} = [R^{(2)}_{mn}/R^{(1)}_{mn}\,R^{(1)}_{m,n-1}]\,[T\,\alpha_{mn}\,R^{(1)\prime}_{mn} + \beta_{mn}\,R^{(1)}_{mn} - R^{(1)}_{m,n-1}]$$
$$+ \alpha_{mn}/c\,R^{(1)}_{mn}\,R^{(1)}_{m,n-1} + R^{(2)}_{mn}/R^{(1)}_{mn} - R^{(2)}_{m,n-1}/R^{(1)}_{m,n-1};$$

(II-26) shows that the quantity in brackets is zero. Differentiation of the rewritten form of F_{mn} and substitution of (II-31), and of (II-13) for both the values n and n - 1 of the second subscript, gives the result $F'_{mn} \equiv 0$. Similarly G'_{mn} is seen to be zero. It follows that there are constants a, b such that

$$\alpha_{mn}\,R^{(2)\prime}_{mn} + \beta_{mn}\,R^{(2)}_{mn} - R^{(2)}_{m,n-1} = a\,R^{(1)}_{m,n-1},$$

$$\alpha_{mn}\,R^{(2)\prime}_{m,n-1} + \gamma_{mn}\,R^{(2)}_{m,n-1} + R^{(2)}_{mn} = b\,R^{(1)}_{mn}.$$

Examination of the asymptotic form of these relations, in the light of (II-7) and (II-9)--(II-11), shows finally that a and b are both zero. Equations (II-26) and (II-27) remain valid if functions of the second kind are substituted for those of the first kind. A similar proof may be carried out for formulas (II-28) and (II-29).

In Reference 19 it is proved that recurrence relations having the property just established--validity for two linearly independent pairs of functions--have uniquely determined coefficients, which in the present case assume the form

$$\alpha_{mn} = c[R^{(1)}_{m,n-1} R^{(2)}_{mn} - R^{(1)}_{mn} R^{(2)}_{m,n-1}], \tag{II-32}$$

$$\beta_{mn} = c\,T\,[R^{(1)}_{m,n-1} R^{(2)\prime}_{mn} - R^{(1)\prime}_{mn} R^{(2)}_{m,n-1}]$$

$$= [R^{(1)}_{m,n-1} R^{(2)\prime}_{mn} - R^{(1)\prime}_{mn} R^{(2)}_{m,n-1}]/[R^{(1)}_{mn} R^{(2)\prime}_{mn} - R^{(1)\prime}_{mn} R^{(2)}_{mn}], \tag{II-33}$$

$$\gamma_{mn} = -\,c\,T\,[R^{(1)}_{mn} R^{(2)\prime}_{m,n-1} - R^{(1)\prime}_{m,n-1} R^{(2)}_{mn}]$$

$$= -\,[R^{(1)}_{mn} R^{(2)\prime}_{m,n-1} - R^{(1)\prime}_{m,n-1} R^{(2)}_{mn}]/[R^{(1)}_{m,n-1} R^{(2)\prime}_{m,n-1} - R^{(1)\prime}_{m,n-1} R^{(2)}_{m,n-1}]. \tag{II-34}$$

The coefficients π_{mn}, ρ_{mn}, and σ_{mn} in (II-28) and (II-29) have representations identical with (II-32), (II-33), and (II-34), respectively, except that the subscript pair m, n - 1 is replaced by the pair m - 1, n wherever it appears. To obtain formulas similar to (II-26)--(II-29) for the angular functions $S^{(1)}_{mn}(\eta)$ and $S^{(2)}_{mn}(\eta)$, one needs to make only the modifications evident from an examination of equations (II-13), (II-14), and (II-16): besides replacing, in (II-32)--(II-34), $T(\xi)$ by $\Sigma(\eta)$ and each radial function $R(\xi)$ by the angular function $S(\eta)$ having the same subscripts and superscripts, one must change the constant c to $h = -c/K^{(1)}_{mn} K^{(2)}_{mn}$. Equations (II-15) may also be used, together with previous results, to put into integral form the coefficients in the recurrence relations for the angular functions.

In an obvious manner, recurrence relations among three functions with contiguous subscript indices m or n can be derived from the differential formulas just established. Certain new identities among the spheroidal functions are obtained from a comparison of the formulas (II-32)--(II-34) with the integral forms of the coefficients given in (II-19)--(II-25). A listing of formulas for the spheroidal functions obtained from the results of this section is given in Appendix A.

III

EXACT SOLUTION

The application of the theory related in Section I to the problem of scattering from a prolate spheroid was carried out by F. V. Schultz (Ref. 21). A plane wave moving in the negative z-direction along the z-axis is scattered by the prolate spheroid $\xi = \xi_0$. The electric vector $\underline{E}'$ and the magnetic vector $\underline{H}'$ constituting this wave are assumed (without loss of generality) to be parallel to the positive y-axis and the positive x-axis, respectively, so that

$$\underline{E}' = E' \, e^{ikz} \, \underline{e}_y, \; \underline{H}' = H' \, e^{ikz} \, \underline{e}_x, \tag{III-1}$$

where $H' = \sqrt{\epsilon/\mu}\, E'$. The expansion of a plane scalar wave of unit amplitude in prolate spheroidal functions was given by P. M. Morse (Ref. 22). Specialized to the present case, the expansion becomes

$$e^{ikz} = \sum_{n=0}^{\infty} A_n S_{on}^{(1)}(\eta)\, R_{on}^{(1)}(\xi), \tag{III-2}$$

where

$$A_n = 2\, i^n \, S_{on}^{(1)}(1)/N_{on}, \tag{III-3}$$

with N_{on} as in (II-23). Substitution of (III-2) and (III-3) in the first of Maxwell's equations (I-12) gives for $\underline{E}'$ the expansion

$$\underline{E}' = -\frac{i}{k}\sqrt{\frac{\mu}{\epsilon}}\, H' \sum_{n=0}^{\infty} A_n \, \nabla \times S_{on}^{(1)}(\eta)\, R_{on}^{(1)}(\xi)\, \underline{e}_x. \tag{III-4}$$

As in Section I, the spheroidal scalar wave functions (II-17) are used to define spheroidal vector wave functions

$$ {}^{x}\underline{M}_{mn} = \nabla \times \underline{e}_x \psi_{mn} (\eta, \xi, \phi) = \nabla\psi_{mn} \times \underline{e}_x, \; {}^{y}\underline{M}_{mn}, \; {}^{z}\underline{M}_{mn} \qquad \text{(III-5)} $$

In terms of these functions and of the amplitude factor E' one obtains from (III-4) the expression

$$ \underline{E}' = -\frac{i}{k} E' \sum_{n=0}^{\infty} A_n \, {}^{x}\underline{M}_{0n}^{(1)} \qquad \text{(III-6)} $$

for the incident wave. Using formulas (II-2)--(II-4), one may write (III-6) in the extended form

$$ \underline{E}' = \frac{iE'}{kF} \sum_{n=0}^{\infty} A_n \Bigg[\underline{e}_\eta \left(\frac{T}{\Delta}\right)^{\frac{1}{2}} S_{on}^{(1)} \, dR_{on}^{(1)}/d\xi \, \sin\phi $$

$$ - \underline{e}_\xi \left(\frac{\Sigma}{\Delta}\right)^{\frac{1}{2}} dS_{on}^{(1)}/d\eta \, R_{on}^{(1)} \sin\phi \qquad \text{(III-7)} $$

$$ - \underline{e}_\phi \left(\frac{\eta T}{\Delta} S_{on}^{(1)} \, dR_{on}^{(1)}/d\xi + \frac{\xi \Sigma}{\Delta} dS_{on}^{(1)}/d\eta \, R_{on}^{(1)} \right) \cos\phi \Bigg]. $$

According to (I-2), the electric vector $\underline{E}$ of the scattered wave must satisfy the boundary conditions

$$ E_\eta (\xi_0, \eta, \phi) = -\frac{iE'}{kF} \sum_{n=0}^{\infty} A_n \left(\frac{T_0}{\Delta_0}\right)^{\frac{1}{2}} S_{on}^{(1)}(\eta) \, dR_{on}^{(1)}(\xi_0)/d\xi \, \sin\phi \qquad \text{(III-8)} $$

and

$$ E_\phi (\xi_0, \eta, \phi) = \frac{iE'}{kF} \sum_{n=0}^{\infty} A_n \left(\frac{\eta T_0}{\Delta_0} S_{on}^{(1)}(\eta) \, dR_{on}^{(1)}(\xi_0)/d\xi \right. $$

$$ \left. + \frac{\xi_0 \Sigma}{\Delta_0} dS_{on}^{(1)}(\eta)/d\eta \, R_{on}^{(1)}(\xi_0) \right) \cos\phi, \qquad \text{(III-9)} $$

where

$$ \Delta_0 = \xi_0^2 - \eta^2, \quad \text{and } T_0 = \xi_0^2 - 1. \qquad \text{(III-10)} $$

These are identities in η and ϕ.

The electric vector $\underline{E}$ of the wave scattered from the spheroid has an expansion (I-16) in vector wave functions of the fourth kind. It is clear from the boundary conditions that the coefficients in this expansion are zero for all terms except those whose η and ϕ components depend on the variable ϕ in the same way as the right members of (III-8) and (III-9), respectively. Accordingly one may write

$$\underline{E} = \sum_{n=0}^{\infty} \left[\alpha_n \, {}^{x}\underline{M}_{on}^{(4)} + \beta_n \, {}^{z}\underline{M}_{1n}^{(4)} \right]. \qquad \text{(III-11)}$$

In extended form, the expansion (III-11) becomes

$$\begin{aligned} \underline{E} = -\frac{1}{F} \sum_{n=0}^{\infty} \Bigg\{ \alpha_n \Bigg[& \left(\underline{e}_\eta \left(\frac{T}{\Delta}\right)^{\frac{1}{2}} S_{on}^{(1)} \, dR_{on}^{(4)}/d\xi - \underline{e}_\xi \left(\frac{\Sigma}{\Delta}\right)^{\frac{1}{2}} dS_{on}^{(1)}/d\eta \, R_{on}^{(4)} \right) \sin\phi \\ & - \underline{e}_\phi \left(\frac{\eta T}{\Delta} S_{on}^{(1)} \, dR_{on}^{(4)}/d\xi + \frac{\xi \Sigma}{\Delta} dS_{on}^{(1)}/d\eta \, R_{on}^{(4)} \right) \cos\phi \Bigg] \\ & - \beta_n \Bigg[\left(\underline{e}_\eta \frac{\eta}{(\Sigma\,\Delta)^{\frac{1}{2}}} - \underline{e}_\xi \frac{\xi}{(T\,\Delta)^{\frac{1}{2}}} \right) S_{1n}^{(1)} \, R_{1n}^{(4)} \sin\phi \\ & + \underline{e}_\phi \frac{(\Sigma\,T)^{\frac{1}{2}}}{\Delta} \left(\eta \, dS_{1n}^{(1)}/d\eta \, R_{1n}^{(4)} - \xi S_{1n}^{(1)} dR_{1n}^{(4)}/d\xi \right) \cos\phi \Bigg] \Bigg\}. \end{aligned} \qquad \text{(III-12)}$$

The electric vector $\underline{E}$ in (III-12) together with the magnetic vector $\underline{H}$ obtained by the same method gives an exact solution to the problem of scattering from the prolate spheroid $\xi = \xi_0$, if the coefficients α_n, β_n are so determined that the following two equations are satisfied:

$$\begin{aligned} & \sum_{n=0}^{\infty} \left[\alpha_n T_0^{\frac{1}{2}} S_{on}^{(1)}(\eta) \, dR_{on}^{(4)}(\xi_0)/d\xi - \beta_n \, \eta \Sigma^{-\frac{1}{2}} S_{1n}^{(1)}(\eta) \, R_{1n}^{(4)}(\xi_0) \right] \\ & = \frac{i}{k} E' \sum_{n=0}^{\infty} A_n \, T_0^{\frac{1}{2}} S_{on}^{(1)}(\eta) \, dR_{on}^{(1)}(\xi_0)/d\xi, \end{aligned} \qquad \text{(III-13)}$$

$$\sum_{n=0}^{\infty}\Big\{\alpha_n\Big[\eta T_0\ S_{on}^{(1)}(\eta)\, dR_{on}^{(4)}(\xi_0)/d\xi + \xi_0\Sigma\ dS_{on}^{(1)}(\eta)/d\eta\, R_{on}^{(4)}(\xi_0)\Big]$$

$$+\beta_n(\Sigma T_0)^{\frac{1}{2}}\Big[\eta dS_{1n}^{(1)}(\eta)/d\eta\, R_{1n}^{(4)}(\xi_0) - \xi_0\, S_{1n}^{(1)}(\eta)\, dR_{1n}^{(4)}(\xi_0)/d\xi\Big]\Big\}$$

$$= \frac{i}{k} E' \sum_{n=0}^{\infty} A_n\Big[\eta T_0\ S_{on}^{(1)}(\eta)\, dR_{on}^{(1)}(\xi_0)/d\xi \tag{III-14}$$

$$+ \xi_0\Sigma\ dS_{on}^{(1)}(\eta)/d\eta\, R_{on}^{(1)}(\xi_0)\Big].$$

Once a solution of the system (III-13), (III-14) for the coefficients α_n and β_n has been obtained, equation (III-11) with these values of the coefficients substituted gives an exact representation of the electric vector $\underline{E}$ of the scattered field. The electric vector $\underline{E}'$ of the incident field is given by equation (III-5). To compute the radar cross-section from formula (I-19), one must examine the asymptotic behavior of $\underline{E}$, and therefore of the vector functions ${}^{x}\underline{M}_{on}^{(4)}$ and ${}^{z}\underline{M}_{on}^{(4)}$, as the distance r from the origin becomes arbitrarily large. Since r is asymptotically equal to $F\xi = c\xi/k$, the expression (II-8) gives the desired information. Combining this formula and the differentiation formulas with the definition (II-17), (III-5) of the spheroidal vector wave functions, one obtains the asymptotic expressions

$${}^{x}\underline{M}_{on}^{(4)} \sim \frac{S_{on}^{(1)}(\eta)}{F\xi}\left[-\underline{e}_\eta \sin\phi + \underline{e}_\phi\ \eta\cos\phi\right]\exp\left[-i\left(c\xi - \frac{n}{2}\pi\right)\right]$$

$$= \frac{S_{on}^{(1)}(\eta)}{r}\left[-\underline{e}_\eta \sin\phi + \underline{e}_\phi\ \eta\cos\phi\right]\exp\left[-i\left(kr - \frac{n}{2}\pi\right)\right],$$

and

$${}^{z}\underline{M}_{1n}^{(4)} \sim \frac{S_{1n}^{(1)}(\eta)}{F\ \xi}\left[-\underline{e}_\phi\ (1-\eta^2)^{1/2}\cos\phi\right]\exp\left[-i\left(c\xi - \frac{n+1}{2}\pi\right)\right]$$

$$= \frac{S_{1n}^{(1)}(\eta)}{r}\left[-\underline{e}_\phi\ (1-\eta^2)^{1/2}\cos\phi\right]\exp\left[-i\left(kr - \frac{n+1}{2}\pi\right)\right],$$

which hold for fixed k as $c\xi \to \infty$ or $r \to \infty$, provided that $(c\xi)^2 \gg A_{mn}$ and $(c\xi)^2 (\xi^2 - 1) \gg m^2$. Using these formulas, one obtains from (III-11) the asymptotic expression for the scattered wave[5]

$$\underline{E} \sim \frac{e^{-ikr}}{r} \sum_{n=0}^{\infty} i^n \Big\{ -\underline{e}_\eta \alpha_n S_{on}^{(1)}(\eta) \sin\phi + \underline{e}_\phi \left[\alpha_n \eta S_{on}^{(1)}(\eta) - i \beta_n (1-\eta^2)^{1/2} S_{1n}^{(1)}(\eta)\right] \cos\phi \Big\}. \tag{III-15}$$

It is no loss of generality to assume that the incident wave has zero phase, so that $E' = (\underline{E}' \cdot \underline{E}'^*)^{\frac{1}{2}} = E'_0$, a positive real constant. The scattered wave will then in general have non-zero phase.

In order to facilitate the division indicated in (I-19) and to display the radar cross-section as a function of a significant dimension of the scattering body, namely the semi-major axis $a = F\xi_0$ of the prolate spheroid $\xi = \xi_0$, one may introduce a new set of expansion coefficients

$$\alpha'_n = \alpha_n / E' a, \quad \beta'_n = \beta_n / E' a \tag{III-16}$$

into the asymptotic expression (III-15). The complex conjugates of α'_n and β'_n may be denoted by α'^*_n and β'^*_n, respectively. Substitution of (III-15) and (III-16) into the formula (I-19) gives for the radar cross-section of a prolate spheroid the expression

$$\sigma(\eta,\phi) = 4\pi a^2 \Big\{ \sin^2\phi \left[\sum_{n=0}^{\infty} i^n \alpha'_n S_{on}^{(1)}(\eta)\right] \left[\sum_{n=0}^{\infty} (-i)^n \alpha'^*_n S_{on}^{(1)}(\eta)\right] + \cos^2\phi \left[\sum_{n=0}^{\infty} i^n \Big(\alpha'_n \eta S_{on}^{(1)}(\eta) - \right.$$

5. Since A_{mn} increases indefinitely with n (cf. Ref. 16), the validity of this formula is not immediately clear because of the requirement that $(c\xi)^2 \gg A_{mn}$. For analytical proof that the expression is actually valid, see Reference 29.

$$- i\ \beta_n' (1-\eta^2)^{1/2} S_{1n}^{(1)}(\eta)\Big)\Big] \cdot \Big[\sum_{n=0}^{\infty} (-i)^n \Big(\alpha_n'^{*} \eta S_{0n}^{(1)}(\eta)$$

$$+ i\ \beta_n'^{*} (1-\eta^2)^{1/2} S_{1n}^{(1)}(\eta)\Big)\Big]\Big\}. \qquad \text{(III-17)}$$

The formula (III-17) is an exact formula if the expansion coefficients are obtained from an exact solution of the system (III-13), (III-14).

For the case of back-scattering, that is, the scattering in the direction opposite to the direction of propagation of the incident wave, the value $\eta = 1$ is substituted into (III-17) and the cross-section becomes independent of ϕ as well. Since $S_{mn}^{(1)}(1) = 0$ for $m > 0$, the back-scattering cross-section is given by the simpler formula

$$\sigma = \sigma(1, \phi) = 4\pi a^2 \left| \sum_{n=0}^{\infty} i^n \alpha_n' S_{0n}^{(1)}(1) \right|^2 . \qquad \text{(III-18)}$$

It is this quantity which was computed in the work reported in later sections.

The problem of obtaining numerical answers in the "bistatic" case, where the transmitter is located on the axis of symmetry and the receiver at an arbitrary point, has also been investigated. The results are discussed in Appendix C.

IV

SCALAR SCATTERING

For a wave derived from a velocity potential that satisfies the scalar wave equation (I-4) - such as an acoustic wave - there is an analogous scattering problem, which will be briefly described. Again a plane wave with direction of propagation parallel to the major axis of a prolate spheroid is incident on the spheroid. The body is assumed to be smooth and rigid, so that the entire incident energy is scattered. For each ray from the origin, the cross-section of an isotropic scatterer yielding, asymptotically, the same mean power density is defined as the scalar scattering cross-section σ' of the spheroid.

Because of the difficulty of computing the radar cross-section σ, the scalar scattering cross-section σ' is sometimes used as an approximation to σ. The order of magnitude of error in making such an approximation can be better understood if one analyzes Figure 3 in Section VIII of this report, where the scattering body is a prolate spheroid, or Figure VI-1 in Reference 23, where the scattering body is a cone. Similar comparisons are known for the conducting wedge, sphere, and paraboloid.

All the numerical quantities needed to find the scalar scattering cross-section are computed in the process of finding the radar cross-section. Consequently the former has been obtained for the same parameter values as the latter, as reported in Sections V, VII, and VIII.

A monochromatic scalar wave is derived from a velocity potential of the form $\Psi = \psi e^{i\omega t}$, where ψ is a scalar function of position which satisfies equation (I-4) for $f = \psi$. For simplicity, the function ψ alone will be called the velocity potential. Let ψ' denote the velocity potential of the incident wave, let ψ denote the velocity potential of the scattered wave, and let $\psi'' = \psi' + \psi$ denote the total velocity potential. On the surface of the scattering body, the gradient of the total velocity potential must be a tangential vector. In terms of spheroidal coordinates and other previously introduced notation, this condition may be written

$$\nabla \psi'' (\eta, \xi_0, \phi) \cdot \underline{e}_\xi (\eta, \xi_0, \phi) = 0$$

or

$$\partial \psi (\eta, \xi_0, \phi)/\partial \xi + \partial \psi' (\eta, \xi_0, \phi)/\partial \xi = 0. \qquad \text{(IV-1)}$$

At infinity, the velocity potential of the scattered wave must behave like that of a diverging spherical wave, i.e., ψ must satisfy the scalar radiation condition (I-5). The incident wave has a velocity potential described by the function

$$\psi' (\eta, \xi, \phi) = \psi_0' \, e^{ikz} = \psi_0' \sum_{n=0}^{\infty} A_n \, S_{0n}^{(1)} (\eta) \, R_{0n}^{(1)} (\xi). \qquad \text{(IV-2)}$$

In formula (IV-2), the expansion (III-2) for e^{ikz} has been employed, with coefficients A_n as described in (III-3). If zero phase is assumed for the incident wave, the constant ψ_0' is the maximum amplitude, a positive real number. The velocity potential of the scattered wave must be independent of ϕ if (IV-1) is to be satisfied: it may have mathematical singularities situated inside the scattering body; and it must satisfy the radiation condition (I-5) at infinity. Consequently, an expansion of the form

$$\psi (\eta, \xi, \phi) = \sum_{n=0}^{\infty} \gamma_n \psi_{0n}^{(4)} (\eta, \xi, \phi) = \sum_{n=0}^{\infty} \gamma_n \, S_{0n}^{(1)} (\eta) \, R_{0n}^{(4)} (\xi) \qquad \text{(IV-3)}$$

is indicated, where $\psi_{0n}^{(4)}$ are the spheroidal scalar wave functions introduced in Section III. The boundary condition (IV-1) implies that

$$\psi_0' \sum_{n=0}^{\infty} A_n \, S_{0n}^{(1)} (\eta) \frac{dR_{0n}^{(1)} (\xi_0)}{d\xi} + \sum_{n=0}^{\infty} \gamma_n \, S_{0n}^{(1)} (\eta) \frac{dR_{0n}^{(4)} (\xi_0)}{d\xi} = 0, \qquad \text{(IV-4)}$$

and the orthogonality of the angular spheroidal functions of the first kind permits one to obtain the constants γ_n directly from (IV-4). For any non-negative integer N, multiplication of both members of (IV-4) by

$S_{0N}^{(1)}(\eta)$ and integration with respect to η from -1 to 1 gives

$$\psi_0' A_N \, dR_{0N}^{(1)}(\xi_0)/d\xi + \gamma_N \, dR_{0N}^{(4)}(\xi_0)/d\xi = 0. \qquad \text{(IV-5)}$$

One may solve (IV-5) for γ_N and substitute into (IV-3) to obtain an exact expression for the velocity potential of the scattered wave:

$$\psi = -\psi_0' \sum_{n=0}^{\infty} A_n H_n(\xi_0) S_{0n}^{(1)}(\eta) R_{0n}^{(4)}(\xi), \qquad \text{(IV-6)}$$

where the abbreviation H_n is defined by

$$H_n(\xi) = dR_{0n}^{(1)}(\xi)/d\xi \, [dR_{0n}^{(4)}(\xi)/d\xi]^{-1}. \qquad \text{(IV-7)}$$

Analogously to the radar cross-section, the scalar scattering cross-section σ' of a scalar wave is defined by

$$\sigma = \lim_{r \to \infty} 4\pi r^2 |\psi|^2 / |\psi_0'|^2, \qquad \text{(IV-8)}$$

and under the assumption of zero phase for the incident wave one may omit the absolute value bars in the denominator. To compute (IV-8) one needs an asymptotic expression for (IV-7) as $r \to \infty$. Besides the asymptotic formula (II-8) for the radial spheroidal functions of the fourth kind, one may conveniently use the fact that $1/c\xi = a/c\,\xi_0\, r$, where a is the semi-major axis of the spheroid $\xi = \xi_0$. Substituting into (IV-7) one obtains[6]

$$\psi \sim \frac{1}{c\xi_0} \sum_{n=0}^{\infty} A_n H_n(\xi_0) S_{0n}^{(1)} \exp\left[-i\left(kr - \frac{n+1}{2}\pi\right)\right] \qquad \text{(IV-9)}$$

as $c\xi \to \infty$ or $r \to \infty$, with H_n as in (IV-7). Finally, equation (IV-9) is substituted into (IV-8) to give the formula

6. See footnote 5, p. 32.

$$\sigma' = \frac{4\pi a^2}{c^2 \xi_0^2} \sum_{n=0}^{\infty} i^{n+1} A_n H_n(\xi_0) S_{0n}^{(1)}(\eta) \sum_{n=0}^{\infty} (-i)^{n+1} A_n^* H_n(\xi_0) S_{0n}^{(1)}(\eta) \qquad \text{(IV-10)}$$

for the scalar scattering cross-section σ' of a plane scalar wave scattered by a prolate spheroid. Here A_n^* denotes the complex conjugate of A_n. It may be noted that the scalar cross-section is axially symmetric, i.e., depends on η alone. For the back-scattering case, the formula (IV-10) is evaluated for $\eta = 1$.

V

APPROXIMATIONS

The equations (III-13) and (III-14) represent an exact solution to the vector scattering problem, in the sense that they define, at least in theory, the exact values of all the expansion coefficients in the expression for the scattered field. However, as they stand they are obviously of no value in computing a finite number of the coefficients. Schultz (Ref. 21) suggests the possibility of obtaining simpler expressions for the quantities α_n and β_n by expanding the equations in powers of η and equating the coefficients of like powers, but also points out that the amount of algebra involved in this process is prohibitive. The method developed by Schultz, which has been adopted for the computations to be reported here, is to multiply both members of each equation by $S_{0N}^{(1)}(\eta)$ and integrate with respect to η from -1 to 1, for each integer N in the range of the summation index n. The result is an infinite set of linear equations in the infinite set of unknowns α_n and β_n, with constant coefficients. In the notation of Reference 21, which will be followed with minor modifications in this section and in the computations, these equations are

$$\sum_{n=0}^{\infty} (\alpha_n C_{Nn} + \beta_n D_{Nn}) = E' a \sum_{n=0}^{\infty} B_{Nn}, \tag{V-1}$$

$$\sum_{n=0}^{\infty} (\alpha_n V_{Nn} + \beta_n W_{Nn}) = E' a \sum_{n=0}^{\infty} U_{Nn} \tag{V-2}$$

(N = 0, 1, 2, ...), where

$$B_{Nn} = \frac{iA_n}{c\xi_0} (\xi_0^2 - 1)^{1/2} \frac{dR_{0n}^{(1)}(\xi_0)}{d\xi} \int_{-1}^{1} S_{0n}^{(1)}(\eta)\, S_{0N}^{(1)}(\eta)\, d\eta, \tag{V-3}$$

$$C_{Nn} = (\xi_0^2 - 1)^{1/2} \frac{dR_{0n}^{(4)}(\xi_0)}{d\xi} \int_{-1}^{1} S_{0n}^{(1)}(\eta)\, S_{0N}^{(1)}(\eta)\, d\eta, \tag{V-4}$$

$$D_{Nn} = -R_{nN}^{(4)}(\xi_0) \int_{-1}^{1} (1-\eta^2)^{1/2} S_{1n}^{(1)}(\eta)\, S_{0N}^{(1)}(\eta)\, d\eta, \tag{V-5}$$

$$U_{Nn} = \frac{iA_n}{c\xi_0}\Bigg[(\xi_0^2 - 1) \frac{dR_{0n}^{(1)}(\xi_0)}{d\xi} \int_{-1}^{1} S_{0n}^{(1)}(\eta)\, S_{0N}^{(1)}(\eta)\, d\eta$$
$$+ \xi_0 R_{0n}^{(1)}(\xi_0) \int_{-1}^{1} (1-\eta^2) \frac{dS_{0n}^{(1)}(\eta)}{d\eta} S_{0N}^{(1)}(\eta)\, d\eta\Bigg], \tag{V-6}$$

$$V_{Nn} = (\xi_0^2 - 1) \frac{dR_{0n}^{(4)}(\xi_0)}{d\xi} \int_{-1}^{1} S_{0n}^{(1)}(\eta)\, S_{0N}^{(1)}(\eta)\, d\eta$$
$$+ \xi_0 R_{0n}^{(4)}(\xi_0) \int_{-1}^{1} (1-\eta^2) \frac{dS_{0n}^{(1)}(\eta)}{d\eta} S_{0N}^{(1)}(\eta)\, d\eta, \tag{V-7}$$

$$W_{Nn} = -\xi_0 (\xi_0^2 - 1)^{1/2} \frac{dR_{1n}^{(4)}(\xi_0)}{d\xi} \int_{-1}^{1} (1-\eta^2)^{1/2} S_{1n}^{(1)}(\eta)\, S_{0N}^{(1)}(\eta)\, d\eta$$
$$+ (\xi_0^2 - 1)^{1/2} R_{1n}^{(4)}(\xi_0) \int_{-1}^{1} (1-\eta^2)^{1/2} \frac{dS_{1n}^{(1)}(\eta)}{d\eta} S_{0N}^{(1)}(\eta)\, d\eta. \tag{V-8}$$

The problem of finding the exact solution to the infinite system (V-1), (V-2) is of course beyond the capacity even of a machine. It is physically plausible that the exact solution is considerably more than is needed for practical purposes, and that an adequate approximation is obtained by truncation of the series. For any n, the quantities α_n and β_n are measures of the radiated energy contained in the n'th-mode oscillation, and this is known to die out fairly rapidly with increasing n as long as n exceeds the characteristic dimension times the wave number. Therefore by taking only the first few terms of the series, and the same range of values of N, one may derive a finite system of linear equations whose solution should approximate the exact values. The degree of approximation will of course depend on the order of the system. A fourth order system was actually used for the computations, the choice being dictated primarily by limitations on machine capacity, time, and money. It was hoped that the degree of approximation obtained in this way would be at least comparable to that afforded by the other approximations inherent in the general method. Also of course the problem was simplified in that computations were made only for the case of back-scattering, so that the coefficients β_n drop out of the picture.

An estimate of the exact degree of error resulting from the truncation of the infinite determinants, such as might be carried out by an extension of the method of Wintner (Ref. 24), was not made. It was originally planned to program the computation for both third-order and fourth-order determinants, and for the three fineness ratios (ratios of major to minor axis) 10:1, 5:1, and $(1 + \epsilon)$:1. At that time it was felt that an adequate criterion of accuracy would be furnished by comparison of the results for third-order and fourth-order determinants for all three fineness ratios, particularly since it was believed that an exact analysis could be carried out for the $(1 + \epsilon)$:1 ratio. Unfortunately economic limitations made it impossible to obtain the desired values at the 5:1 ratio, and when theoretical results for the ratio $(1 + \epsilon)$:1 were obtained and showed no unexpected behavior, further study of this case was likewise abandoned. Improved accuracy might have been obtained with less effort if, prior to programming, the minimization procedure used by E. Schmidt (Ref. 27) to determine the solution in Hilbert space of an infinite number of linear equations in an infinite number of unknowns had been adapted to the case of n equations in m unknowns for $m > n$.

Among the approximations made in the computations, the principal one concerns the representations of the spheroidal functions $S_{mn}(\eta)$ and $R_{mn}(\xi)$. Because of the lack of knowledge of the theory of these functions, and because of the limited range of existing tables, it was necessary to represent the functions S_{mn} and R_{mn} by series of associated Legendre functions and of spherical Bessel functions, respectively. The particular spheroidal functions required here are represented as follows:

$$S^{(1)}_{mn}(\eta) = \sum_{k=0}^{\infty} \mu_{n,k}\, d^{mn}_{k}\, P^{m}_{k+m}(\eta), \qquad \text{(V-9)}$$

$$R^{(1)}_{mn}(\xi) = (\xi^2-1)^{m/2}\, \frac{\sum_{k=0}^{\infty} \mu_{n,k}\, i^{n-k}\, d^{mn}_{k}(k+2m)!\; j_{m+k}(c\xi)/k!}{\xi^m \sum_{k=0}^{\infty} \mu_{n,k}\, d^{mn}_{k}(k+2m)!/k!} \qquad \text{(V-10)}$$

$$R^{(2)}_{mn}(\xi) = K(c, m, n) \sum_{k=-\infty}^{\infty} \mu_{n,k}\, d^{mn}_{k}\, Q^{m}_{k+m}(\xi), \qquad \text{(V-11)}$$

$$R^{(4)}_{mn}(\xi) = R^{(1)}_{mn}(\xi) - i\, R^{(2)}_{mn}(\xi), \qquad \text{(V-12)}$$

where $P^{m}_{k+m}(\eta)$ and $Q^{m}_{k+m}(\xi)$ are associated Legendre functions, $j_{k+m}(c\xi)$ are spherical Bessel functions of the first kind, and the constants d^{mn}_{k} are the spheroidal coefficients, essentially defined by equation (V-9) and discussed in detail in References 15 and 16. The symbol $\mu_{i,j}$ as used here and throughout the rest of this report is defined as

$$\mu_{i,j} \equiv \frac{1}{2}[1 + (-1)^{i+j}]; \quad \text{i.e.,} \quad \mu_{i,j} = \begin{cases} 0 & \text{if } i+j \text{ is odd} \\ 1 & \text{if } i+j \text{ is even} \end{cases},$$

and K(c, m, n) is defined below:

$$K(c,m,n) = \frac{2\, c^{m-1}\, \Gamma(\frac{n+2m+1}{2})}{\Gamma(\frac{n+2}{2})\, \Gamma(m-\frac{1}{2})\, d^{mn}_{-2m} \sum_{k=0}^{\infty} \mu_{n,k}\, d^{mn}_{k}(k+2m)!/k!}, \text{ (n even)}, \qquad \text{(V-13)}$$

$$K(c,m,n) = \frac{-8c^{m-1}\Gamma\left(\frac{n+2m+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(m-\frac{3}{2}\right) d_{1-2m}^{mn} \sum_{k=0}^{\infty} \mu_{n,k}\, d_k^{mn}\, (k+2m)!/k!}, \text{ (n odd).} \qquad \text{(V-14)}$$

The expansion (V-11), (V-13), (V-14) for $R_{mn}^{(2)}(\xi)$ is preferred over an expansion in spherical Neumann functions analogous to (V-10) because of its superior convergence in the region $\xi \cong 1$.

The spheroidal coefficients d_k^{mn} are tabulated for certain ranges of the indices in References 15 and 16. However, the ranges covered are inadequate for the purposes of the present investigation. For this reason and because of the possibilities of errors in existing tables, it seemed advisable from the outset to compute all the coefficients used. The computation is carried out in the manner described in Reference 16. First the series (V-9) is substituted into the differential equation for the angular spheroidal functions. By means of the differential equation and the recurrence formulas for the associated Legendre functions, all differentiations and all variable coefficients are eliminated. In the simplified series, each associated Legendre function appears with a constant coefficient whose value must be zero. The result is a set of recurrence formulas for the spheroidal coefficients:

$$E_{k+2}^{m}\, d_{k+2}^{mn} + F_k^{mn}\, d_k^{mn} + G_{k-2}^{m}\, d_{k-2}^{mn} = 0, \qquad \text{(V-15)}$$

where

$$E_{k+2}^{m} = \frac{(2m+k+2)(2m+k+1)}{(2m+2k+3)(2m+2k+5)}, \qquad \text{(V-16)}$$

$$F_k^{mn} = \frac{(m+k)(m+k+1) + A_{mn}}{c^2} + \frac{2(m+k)(m+k+1) - 2m - 1}{(2m+2k-1)(2m+2k+3)} \qquad \text{(V-17)}$$

$$G_{k-2}^{m} = \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}. \qquad \text{(V-18)}$$

The indices n, m, and k range over the following values:

$$n = 0, 1, 2, \ldots, \quad m = 0, 1, 2, \ldots,$$

$$k = -24, -23, \ldots, -1, 0, 1, \ldots, 15, 16.$$

The condition that the function represented by the series (V-9) be finite over the whole range $-1 \leq \eta \leq 1$ implies that it is an entire function, since -1 and 1 are the only finite singular points of the differential equation satisfied by the spheroidal functions. The ratio test for infinite series of functions therefore shows that d_{k+2}^{mn} / d_k^{mn} must approach zero as $k \to \infty$. The convergent series (V-9) is truncated to give an approximate expression for the angular functions; to fix the number of terms used for the approximation, experience is the only guide available.

Since E_k^m in (V-15) vanishes for k = -2m, the coefficients d_k^{mn} are all zero for $k < -2m$. For this range of values of k, however, the functions $Q_{k+m}^m(\xi)$ are undefined. A detailed study of the spheroidal equation is carried out in Reference 15 for all values of the separation constant, not merely those that lead to the functions of the first kind. The coefficients d_k^{mn} as functions of the index k are defined for non-integral values of k, and it is found that the products $d_k^{mn} Q_{k+m}^m(\xi)$ for non-integral k have finite limit values as k approaches an integral value, even for $k < -2m$. The products are put into the form

$$d_{-|k|}^{mn} \; Q_{-|k|+m}^{mn} = \frac{1}{\rho} d_{-|k|}^{m} \; P_{|k|-m-1}^{m} \qquad \text{(V-19)}$$

for $|k| > 2m$, where ρ is a quantity that tends to zero as k approaches integral values. The coefficients d_k^{mn}/ρ are then obtained by means of the same recurrence formulas used for the coefficients d_k^{mn} when $k > -2m$. The expression (V-19) may be substituted into (V-11) to give

$$R_{mn}^{(2)}(\xi) = K(c,m,n)\left[\sum_{k=-\infty}^{-2m-1} \mu_{n,k}\, d_k^{mn}\, P_{-n-1}(\xi)/\rho + \sum_{k=-2m}^{\infty} \mu_{n,k}\, d_k^{mn}\, Q_n(\xi)\right]. \qquad \text{(V-20)}$$

The constants d_k^{mn}/ρ also form a convergent series, and the first series in (V-20) is truncated at some finite negative k to give an approximation.

The quantity A_{mn} which appears in the formula (V-11) for F_k^{mn} is the separation constant of equation (II-12). This separation constant is a root of the transcendental equation formed by iteration of the recurrence formula (V-15) and application of the conditions $E_{-2m}^{m} = 0$ and $d_{k+2}^{mn}/d_k^{mn} \rightarrow 0$ as $k \rightarrow \infty$, as outlined in Section VII. The actual values of A_{mn} may be approximated by series in positive or negative powers of $c = k\,F$, according as this quantity is small or large; these series are given in Reference 16. In practice it is usually necessary to refine the approximation by an iteration scheme such as the one described in Section VII.

The expressions for the angular spheroidal functions and for the derivatives of the radial spheroidal functions which are needed to obtain an approximate solution of the system (V-1), (V-2), serve at the same time to compute the scalar scattering cross-section σ' from the formula (IV-10). The results of this computation are compared to the results for the radar cross-section σ, for corresponding values of the parameters, in Section VIII.

Even after the infinite system (V-1), (V-2) for the radar cross-section coefficients is replaced by a finite approximation, the computations needed to solve the system are far beyond the practical capacity of manual computing machines. The principal difficulty is that a large number of previously untabulated functional values must be obtained and stored. At this point, therefore, the services of an electronic digital computer are indispensable. The next section is devoted to a description of the machine that was used.

VI

MARK III COMPUTER

The machine employed for the present computations was the Mark III electronic digital computer. This computing machine was constructed for the Bureau of Ordnance, U.S. Navy, by the Harvard Computation Laboratory, and is now located at the Naval Proving Ground at Dahlgren, Virginia. The Mark III utilizes magnetic drum storage units, and at the time of its selection for this work had the largest capacity of any digital machine available. Previously, one value of a radar cross-section had been obtained at the cost of several man months of manual computation. To compute radar cross-sections with the necessary accuracy for the desired number of values of the parameters, it was necessary to use a machine with large storage capacity and large word length; this was the principal consideration leading to the choice of the Mark III. The difference in operating costs between this machine and faster machines with lower capacities was outweighed by the reduction in computing time made possible by the storage facilities of the Mark III.

These facilities include channels capable of storing 4350 sixteen decimal digit numbers, of which 150 are permanently stored constants. In addition, the machine has a capacity of 4000 three-address instructions, and carries permanently stored routines for elementary operations and computational procedures. In this problem floating arithmetic sub-routines were employed using fifteen decimal digit coefficients, four digit exponents and signs. The machine operates with a coded decimal system, in which each decimal digit is represented by four binary digits. Input is supplied on magnetic tapes, and results are printed in decimal numbers by five electric typewriters. Because of the use of moving drums and tapes and of mechanical relays, the Mark III does not operate as rapidly as machines of some other types.

Even with the storage capacity available, the various components of the problem could not be coded for simultaneous computation. It was found necessary to split the problem into five individually coded runs, described in some detail in the following section.

VII

MACHINE COMPUTATIONS

The wavelength λ of the radiation is considered to be in the "resonance region" if it is nearly equal to the "characteristic dimension" of the spheroid. In Section VIII it will be indicated why the quantity $2\pi a$ should be considered the characteristic dimension, a being the semi-major axis of the spheroid. The range of the parameter $2\pi a/\lambda$ from which the computed values were selected was determined by a numerical analysis likewise described in Section VIII. The objective was to investigate how the radar cross-section changes from the region $2\pi a/\lambda < 1$ but not too far from 1, where the values of the cross-section are close to those predicted by the Rayleigh approximation, to the resonance region where $2\pi a/\lambda$ nearly equals 1 (and actually somewhat exceeds 1). In the latter region, it was first required to make a "guess" as to the location of the first maximum and the width of the arc containing it. In the range indicated by the guess, it was necessary to obtain enough values to determine the location and height of the first maximum and the width of the arc on which it occurs, and outside this range it was necessary to obtain enough values to permit rectification of a slightly incorrect initial guess. If the guess had been wrong in the direction of smaller wavelength by as much as a factor of five, a completely new computation would have been needed. Fortunately the original guess was adequate, and the choice of parameter values listed below permitted determination of the desired quantities.

Once it was clear that the peak region had been correctly guessed, the values for which computations were to be made could be selected with a certain amount of freedom. This fact was exploited to side-step values for which it was felt that certain difficulties inherent in the method of computation might arise (e.g. the divergence of two quantities whose quotient is finite).

The values of $2\pi a/\lambda$ for which the radar cross-section was computed are listed on the following page:

.094, .105, .157, .236, .262, .314, .377, .524, .754, .785, .942, 1.26 , 1.57 , 1.89 , 2.10 , 2.51 , 3.14 , 3.77 , 4.71 , 5.89 , 6.28 , 10.5. In the computations a system of four equations in four unknowns replaced the infinite system (V-1), (V-2). For purposes of comparison, the system of three equations in three unknowns was computed for the value $\frac{2\pi a}{\lambda} = 1.89$.

Figure 2 gives a schematic diagram of the subdivision of the problem, indicating the interdependence of the various intermediate steps. The distribution of these steps among the five runs of the calculator is shown by listing in a separate column the quantities computed in each run. The process runs from left to right, with the first column showing known information. The order in which items appear in their column is not significant.

A table of Bessel and associated Legendre functions and their derivatives was first computed, since existing tables were inadequate in the ranges required for this work. The values of the Bessel functions were obtained from an expansion of $J_{n+1/2}(c\xi)$ in powers of $(c\xi)^{-2}$, and further values and values of the derivatives were obtained from the usual recurrence formulas. The associated Legendre functions of the first kind and their derivatives were expanded in powers of ξ^{-2} and those of the second kind obtained from the functions of the first kind. The actual formulas used are listed in Appendix B.

The approximate values of the separation constants A_{mn} were first obtained from the infinite series given in Reference 16, which after truncation may be written

$$A_{mn} = \sum_{k=0}^{4} g_k^{mn} \; c^{2k} \qquad (c<5), \qquad \text{(VII-1)}$$

$$A_{mn} = \sum_{k=0}^{4} h_k^{mn} \; c^{-k} \qquad (c>5), \qquad \text{(VII-2)}$$

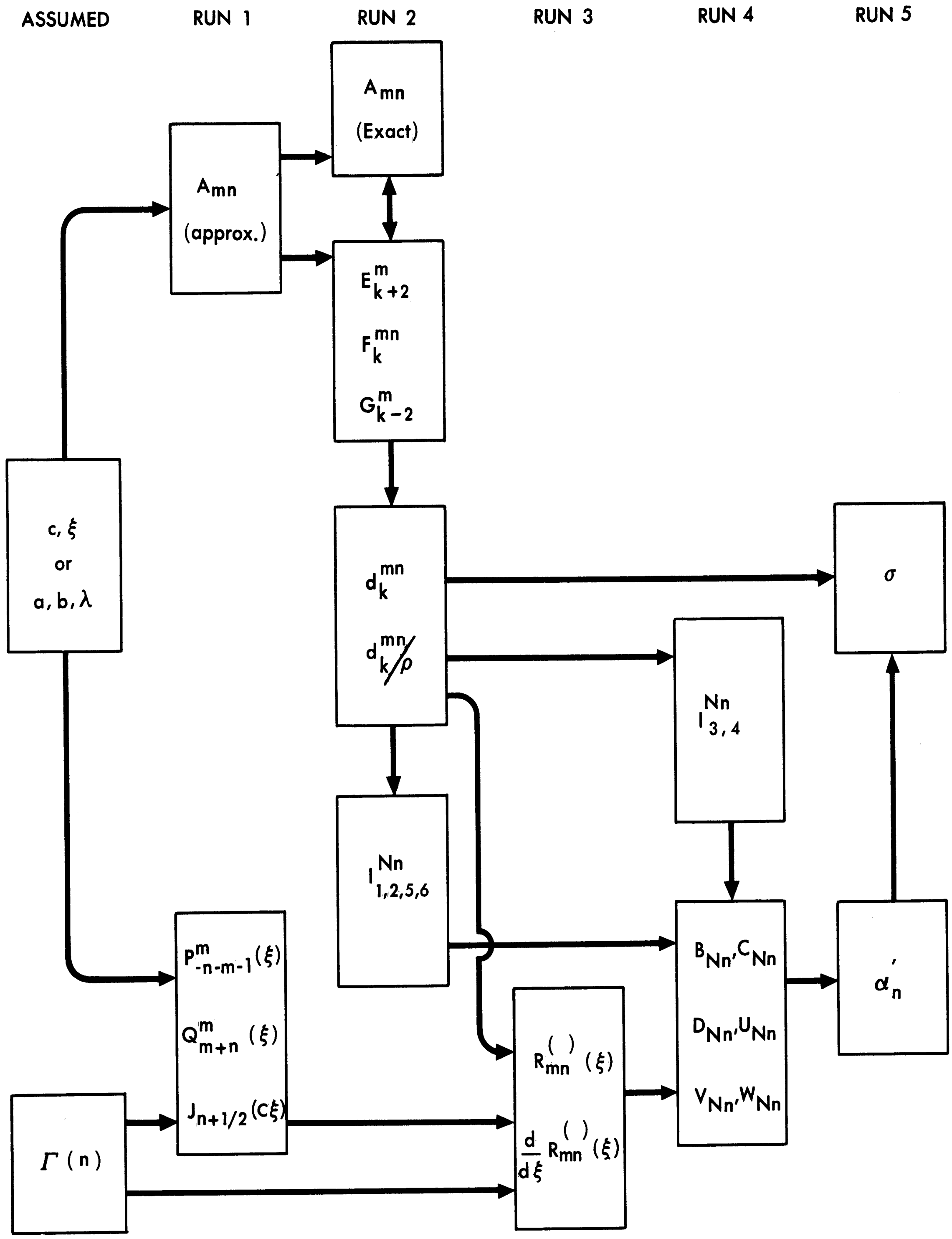

FIG. 2 LOGICAL AND SEQUENTIAL STRUCTURE OF COMPUTATIONS

with coefficients g_k^{mn} and h_k^{mn} given explicitly in Appendix B. In the course of manual computations it was discovered that for most values of c used, and particularly for those near c = 5, the number of coefficients given in (VII-1) and (VII-2) was not enough to determine the separation constants to the required degree of approximation, while the formulas for additional coefficients were prohibitively complicated. A small error in the value of A_{mn} led to errors in the recurrence formulas for the spheroidal coefficients d_k^{mn} of the order of magnitude of the quantities computed. For this reason an iterative procedure was devised that served to refine the values of A_{mn} sufficiently in all cases computed. The procedure is outlined as applied for even values of k; for odd k there are minor modifications:

1) The quantities E_{k+2}^{m}, F_k^{mn}, and G_{k-2}^{m} are computed as in (V-16), (V-17), and (V-18), with the value $A_{mn} = A_{mn}^{(1)}$ determined by the series (VIII-1) or (VII-2).

2) The value of $K_k^{mn} = d_{k+2}^{mn} / d_k^{mn}$ is found for k = 14 from the formula $K_{14}^{mn} = - G_{14}^{m} / F_{16}^{mn}$, obtained from (V-15) with k = 16 if $d_{18}^{mn} / d_{14}^{mn}$ is replaced by its limit zero.

3) The values of K_k^{mn} for k = 12, 10, ..., 0 are found successively from the recurrence formula (V-15), used in the form

$$1 / K_{k-2}^{mn} = - (F_k^{mn} + E_{k+2}^{m} K_k^{mn}) / G_{k-2}^{m} .$$

4) The quantity $F_0^{mn} + E_2^{m} K_0^{mn} = - G_{-2}^{m} / K_{-2}^{mn}$ is computed. As follows from (V-18), $G_{-2}^{m} = 0$ is the exact value that should be obtained, and so $\left| G_{-2}^{m} / K_{-2}^{mn} \right|$ is a measure of the error in the first approximation $A_{mn}^{(1)}$ to the separation constant. If the error is not considered excessive, the values K_k^{mn} obtained in step 3) may be used to compute the spheroidal coefficients d_k^{mn}.

5) If the error exceeds the bounds of required accuracy, a second approximation $A_{mn} = A_{mn}^{(2)}$ to the separation constant is derived from the formula (V-15) for k = 0, which may be written

$$K_0^{mn} = - [c^{-2} (m^2 + m + A_{mn}) + 1/(2m + 3)] / E_2^m;$$

the value K_0^{mn} used is the one computed in 4). An average of the two approximations $A_{mn}^{(1)}$ and $A_{mn}^{(2)}$ is then used to begin the process again at step 1).

If the procedure is stable, the error in 4) will decrease to within the desired range of accuracy after a sufficient number of iterations. If the error does not decrease, the method is inapplicable; in the present computations, this difficulty did not arise.

The procedure described above is similar to a method developed by C. J. Bouwkamp (Ref. 26). The method of Bouwkamp is theoretically superior in improving the approximations to A_{mn}, converging more rapidly and for a wider range of parameter values. However, it is also much more difficult to program for a digital machine like the Mark III, and this fact led to its rejection in favor of the procedure outlined above.

In the second and fourth runs of the machine, the quantities I_k^{Nn} were computed. These are integrals involving the associated Legendre functions, and arise from the integration of equations (III-13) and (III-14) to derive the system (V-1) and (V-2). The integrals I_k^{Nn} have been reduced by Schultz to series in the spheroidal coefficients d_k^{mn}, as shown in the appendix. If enough values of the d_k^{mn} are known, computation of the integrals offers no difficulties.

The computation of the remaining quantities listed in the table presented no special problems. The specific formulas used are given in Appendix B. The scalar cross-sections were computed manually from the tables listing the results of the first three machine runs.

An examination of the formulas in Appendix B, particularly of those referring to runs 4 and 5 of the calculator, gives some indication of the increased labor that would have to be expended to compute the next approximation by means of a system of five equations in five unknowns. Besides the additional values of the Bessel and associated Legendre functions, of the quantities A_{mn}, d_k^{mn}, and I_k^{Nn}, and of the elements B_{Nn}, C_{Nn}, D_{Nn}, U_{Nn}, V_{Nn}, and W_{Nn} of the determinants α_n, it would be necessary to compute an additional determinant with complex elements, and all determinants computed would be of the fifth instead of the fourth order. Conside·ation of the estimated costs, using the computing facilities and techniques at present available, ruled out any effort to improve the results by such an extension.

In Appendix C it is demonstrated that the quantities already computed for the back-scattering cross-section can also be used to compute the bistatic cross-section $\sigma(\theta, \phi)$, where the transmitter is on the major axis of the spheroid and the receiver is on the ray with spherical coordinates θ, ϕ.

VIII
RESULTS

The problem at hand was to determine the nose-on radar cross-section at the Rayleigh side of the resonance region, and to determine the magnitude of the cross-section at the first maximum, for a prolate spheroid with fineness ratio of 10:1. Since it was known a priori that the computations would be limited by economic considerations to only a few points, it became necessary to predict the location of the resonance region and to see that most of the points chosen for computation were in the Rayleigh region near the first maximum. It was decided, however, that as a safety measure a few points should be computed at fairly wide intervals in the region of larger characteristic-dimension-to-wavelength ratio, to be used if the predicted location of the resonance region was inaccurate. The margin of safety was cut sharply after the second run of the machine computations, when it became necessary to reduce the number of points being computed.

The reasoning behind the prediction of location of the resonance region is outlined here for the benefit of anyone who may have to make similar predictions in the future.

It was expected that, for wavelengths small compared to $2\pi a$, the vector cross-section would oscillate about the approximate solution given by geometrical optics,

$$\sigma_{G.O.} = \frac{\pi b^4}{a^2}.$$

It was expected also that, for wavelengths large compared to $2\pi a$, the Rayleigh approximation would yield an asymptote to the graph of the vector cross-section. The Rayleigh solution for the vector case is

$$\sigma_{Ray.} = \frac{64\,\pi^3\,T^2\,k^4}{N^2\,(4\pi - N)^2}$$

where

$$k = \frac{2\pi}{\lambda}, \quad T = \frac{4}{3}\pi a b^2,$$

$$\text{and } N = \frac{2\pi a^2}{a^2 - b^2}\left\{1 - \frac{b^2}{2a\sqrt{a^2 - b^2}} \log \frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}\right\}.$$

For $\frac{a}{b} = 10$, this yields

$$\frac{\sigma_{Ray.}}{\sigma_{G.O.}} = 7.117\left(\frac{2\pi a}{\lambda}\right)^4.$$

It was also believed that an upper bound on the abscissa of the first maximum could be obtained from the approximate solution given by physical optics,

$$\sigma_{P.O.} = \frac{\pi b^4}{a^2}\left\{1 - 2 \cos ka \left(\frac{\sin ka}{ka}\right) + \left(\frac{\sin ka}{ka}\right)^2\right\},$$

and that a lower bound on the abscissas of most of the succeeding maxima could be deduced from the thin-wire theory discussed in Reference 25, which gives the following formulas for the locations of the maxima:

$$\frac{\pi}{4}\left\{2 \log \frac{\lambda}{\pi b} + \frac{1}{2} \log \frac{2\pi a}{\lambda} - 1.87\right\}^{-1} = \begin{cases} \cot \frac{2\pi a}{\lambda} & \text{for the odd maxima} \\ -\tan \frac{2\pi a}{\lambda} & \text{for the even maxima.} \end{cases}$$

For the spheroid considered, with a/b = 10, the left hand side of this expression reduces to

$$\frac{\pi}{4}\left\{4.12 - \frac{3}{2} \log \frac{2\pi a}{\lambda}\right\}^{-1}.$$

All the relationships listed seem clearly to point to the choice of $2\pi a$ as the characteristic dimension of the spheroid.

Use of this theory in predicting the locations of the maxima is vindicated by the results of the computations. The predictions seem to be very accurate indeed.

In the interest of completeness the scalar solutions have been plotted for comparison with the vector solutions. The Rayleigh solution for the scalar case is

$$\sigma' = \frac{4\pi^3\, T^2}{\lambda^4}\left(\frac{2 - L}{1 - L}\right)^2 ,$$

where $T = \frac{4}{3}\pi a\, b^2$,

and $$L = \frac{b^2}{a^2 - b^2}\left\{\frac{a}{2\sqrt{a^2 - b^2}}\log\frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}} - 1\right\}.$$

For a/b = 10, this yields

$$\frac{\sigma'}{\sigma_{G.O.}} = 1.815\left(\frac{2\pi a}{\lambda}\right)^4 .$$

Some particular values of the scalar cross-sections of prolate spheroids were previously published by Spence and Granger (Ref. 26). Those values which correspond to the present problem are compared with the Mark III results.

Figure 3 is a graph of all the results mentioned above. The ordinate in each case is the nose-on cross-section σ divided by the geometrical-optics result $\frac{\pi b^4}{a^2}$, while the abscissa is 2π times the ratio of semi-major axis a to wavelength λ. The exact vector and scalar curves were extended to the points where the question of reliability made further plotting undesirable (partly because of the limited number of points obtained and partly because of the convergence question). The abscissas of the maxima as predicted by the thin-wire theory of Reference 25 are shown at the top of the graph. These serve to emphasize further that the number of points obtained was inadequate to define the curves in the range beyond the first two maxima. However, this range was not of particular interest in the problem under consideration. Several results based on third-order systems have been included for purposes of comparison and to give some indication of the convergence situation.

Numerical results for the nose-on radar cross-sections of prolate spheroids with fineness ratios other than 10:1 can be obtained in exactly the same way as those presented here. The algebraic results have been generalized in Appendix C to cover the case of transmitter on the axis of symmetry and receiver at an arbitrary point. The additional computations needed for this case are also noted in Appendix C. The algebraic expressions for the oblate spheroid problem are similar to those for the prolate spheroid (see Reference 28).

The tabulated results obtained in the present investigation may be of use in solving the above-mentioned problems. In Appendix D the printed results available from the Mark III computations are listed. Possibilities are being investigated for the publication of these tables elsewhere; in any case, numerical values of any quantity tabulated will be furnished upon request.

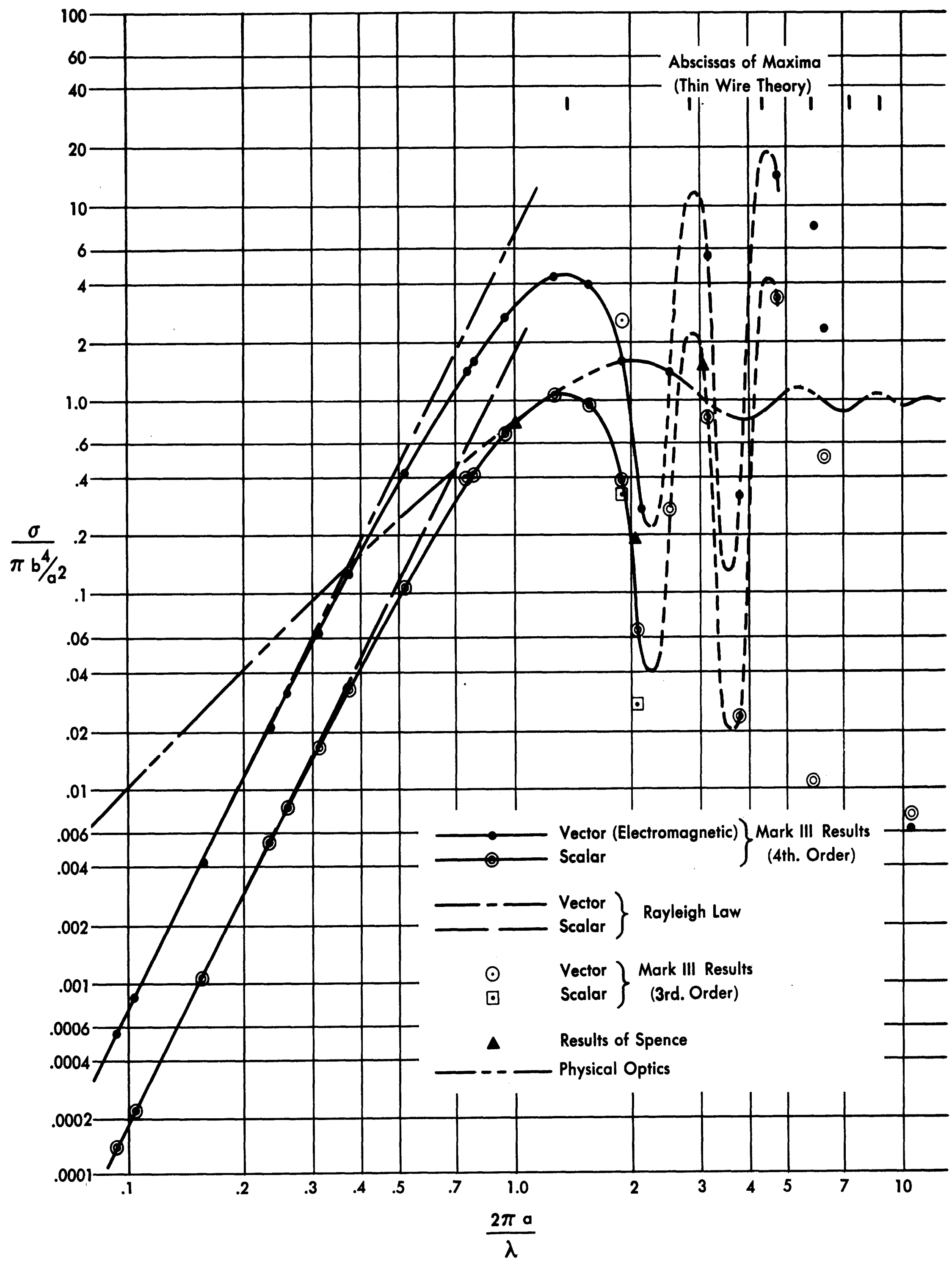

FIG. 3 BACK-SCATTERING FROM A PROLATE SPHEROID, NOSE-ON

APPENDIX A

RECURRENCE RELATIONS AND OTHER FORMULAS FOR PROLATE SPHEROIDAL FUNCTIONS AND WAVE FUNCTIONS

Differential recurrence relations for radial prolate spheroidal functions (j = 1 or 2):

$$(\xi^2 - 1)\alpha_{mn}(\xi)\, dR^{(j)}_{mn}/d\xi + \beta_{mn}(\xi)\, R^{(j)}_{mn} = R^{(j)}_{m,n-1},$$

$$(\xi^2 - 1)\alpha_{mn}(\xi)\, dR^{(j)}_{m,n-1}/d\xi + \gamma_{mn}(\xi)\, R^{(j)}_{m,n-1} = -R^{(j)}_{mn};$$

$$(\xi^2 - 1)\pi_{mn}(\xi)\, dR^{(j)}_{mn}/d\xi + \rho_{mn}(\xi)\, R^{(j)}_{mn} = R^{(j)}_{m-1,n},$$

$$(\xi^2 - 1)\pi_{mn}(\xi)\, dR^{(j)}_{m-1,n}/d\xi + \sigma_{mn}(\xi)\, R^{(j)}_{m-1,n} = -R^{(j)}_{mn};$$

$$\alpha_{mn} = \int_{-1}^{1} \frac{\tau}{\xi^2 - \tau^2} S^{(1)}_{mn}(\tau)\, S^{(1)}_{m,n-1}(\tau)\, d\tau \Big/ c \int_{-1}^{1} \tau S^{(1)}_{mn}(\tau)\, S^{(1)}_{m,n-1}(\tau)\, d\tau$$

$$= c[R^{(1)}_{m,n-1} R^{(2)}_{mn} - R^{(1)}_{mn} R^{(2)}_{m,n-1}];$$

$$\beta_{mn} = \xi \int_{-1}^{1} \frac{1 - \tau^2}{\xi^2 - \tau^2} dS^{(1)}_{mn}(\tau)/d\tau\, S^{(1)}_{m,n-1}(\tau)\, d\tau \Big/ c \int_{-1}^{1} \tau S^{(1)}_{mn}(\tau)\, S^{(1)}_{m,n-1}(\tau)\, d\tau$$

$$= c(\xi^2 - 1)\, [R^{(1)}_{m,n-1} R^{(2)'}_{mn} - R^{(1)'}_{mn} R^{(2)}_{m,n-1}]$$

$$= [R^{(1)}_{m,n-1} R^{(2)'}_{mn} - R^{(1)'}_{mn} R^{(2)}_{m,n-1}] \Big/ [R^{(1)}_{mn} R^{(2)'}_{mn} - R^{(1)'}_{mn} R^{(2)}_{mn}]$$

$$\gamma_{mn} = \xi \int_{-1}^{1} \frac{1 - \tau^2}{\xi^2 - \tau^2} S^{(1)}_{mn}(\tau)\, dS^{(1)}_{m,n-1}(\tau)/d\tau\, d\tau \Big/ c \int_{-1}^{1} \tau S^{(1)}_{mn}(\tau)\, S^{(1)}_{m,n-1}(\tau)\, d\tau$$

$$= -c(\xi^2 - 1)\,[R^{(1)}_{mn} R^{(2)'}_{m,n-1} - R^{(1)'}_{m,n-1} R^{(2)}_{mn}]$$

$$= -[R^{(1)}_{mn} R^{(2)'}_{m,n-1} - R^{(1)'}_{m,n-1} R^{(2)}_{mn}] \Big/ [R^{(1)}_{m,n-1} R^{(2)'}_{m,n-1} - R^{(1)'}_{m,n-1} R^{(2)}_{m,n-1}];$$

$$\pi_{mn} = \frac{\dfrac{\xi}{(\xi^2-1)^{1/2}} \displaystyle\int_{-1}^{1} \frac{(1-\tau^2)^{1/2}}{\xi^2 - \tau^2} S^{(1)}_{mn}(\tau)\, S^{(1)}_{m-1,n}(\tau)\, d\tau}{c \displaystyle\int_{-1}^{1} (1-\tau^2)^{1/2} S^{(1)}_{mn}(\tau)\, S^{(1)}_{m-1,n}(\tau)\, d\tau}$$

$$= c\,[R^{(1)}_{m-1,n} R^{(2)}_{mn} - R^{(1)}_{mn} R^{(2)}_{m,n-1}];$$

$$\rho_{mn} = \Bigg[\frac{m}{(\xi^2-1)^{1/2}} \int_{-1}^{1} \frac{1}{(1-\tau^2)^{1/2}} S^{(1)}_{mn}(\tau)\, S^{(1)}_{m-1,n}(\tau)\, d\tau$$

$$- (\xi^2-1)^{1/2} \int_{-1}^{1} \frac{\tau(1-\tau^2)^{1/2}}{\xi^2-\tau^2}\, dS^{(1)}_{mn}(\tau)/d\tau\, S^{(1)}_{m-1,n}(\tau)\, d\tau\Bigg]$$

$$\cdot \left[c \int_{-1}^{1} (1-\tau^2)^{1/2} S^{(1)}_{mn}(\tau)\, S^{(1)}_{m-1,n}(\tau)\, d\tau\right]^{-1}$$

$$= c\,(\xi^2-1)\,[R^{(1)}_{m-1,n} R^{(2)'}_{mn} - R^{(1)'}_{mn} R^{(2)}_{m-1,n}]$$

$$= [R^{(1)}_{m-1,n} R^{(2)'}_{mn} - R^{(1)'}_{mn} R^{(2)}_{m-1,n}] \Big/ [R^{(1)}_{mn} R^{(2)'}_{mn} - R^{(1)'}_{mn} R^{(2)}_{mn}];$$

$$\sigma_{mn} = \Bigg[\frac{-m+1}{(\xi^2-1)^{1/2}} \int_{-1}^{1} \frac{1}{(1-\tau^2)^{1/2}} S^{(1)}_{mn}(\tau)\, S^{(1)}_{m-1,n}(\tau)\, d\tau$$

$$- (\xi^2-1)^{1/2} \int_{-1}^{1} \frac{\tau(1-\tau^2)^{1/2}}{\xi^2-\tau^2} S^{(1)}_{mn}(\tau)\, dS^{(1)}_{m-1,n}(\tau)/d\tau\, d\tau\Bigg]$$

$$\cdot \left[c \int_{-1}^{1} (1-\tau^2)^{1/2} S^{(1)}_{mn}(\tau)\, S^{(1)}_{m-1,n}(\tau)\, d\tau\right]^{-1}$$

$$= - c(\xi^2-1)\,[R^{(1)}_{mn}\,R^{(2)'}_{m-1,n} - R^{(1)'}_{m-1,n}\,R^{(2)}_{mn}]$$
$$= -[R^{(1)}_{mn}R^{(2)'}_{m-1,n} - R^{(1)'}_{m-1,n}R^{(2)}_{mn}]\Big/[R^{(1)}_{m-1,n}R^{(2)'}_{m-1,n} - R^{(1)'}_{m-1,n}R^{(2)}_{m-1,n}].$$

Differential recurrence relations for angular prolate spheroidal functions ($j = 1$ or 2):

$$(1-\eta^2)\,a_{mn}(\eta)\,dS^{(j)}_{mn}/d\eta + b_{mn}(\eta)\,S^{(j)}_{mn} = S^{(j)}_{m,n-1},$$
$$(1-\eta^2)\,a_{mn}(\eta)\,dS^{(j)}_{m,n-1}/d\eta + c_{mn}(\eta)\,S^{(j)}_{m,n-1} = -S^{(j)}_{mn};$$

$$(1-\eta^2)\,p_{mn}(\eta)\,dS^{(j)}_{mn}/d\eta + r_{mn}(\eta)\,S^{(j)}_{mn} = S^{(j)}_{m-1,n},$$
$$(1-\eta^2)\,p_{mn}(\eta)\,dS^{(j)}_{m-1,n}/d\eta + s_{mn}(\eta)\,S^{(j)}_{m-1,n} = -S^{(j)}_{mn};$$

$$a_{mn}(\zeta) = -\alpha_{mn}(\zeta),\quad b_{mn}(\zeta) = -\beta_{mn}(\zeta),\quad c_{mn}(\zeta) = -\gamma_{mn}(\zeta);$$

$$p_{mn}(\zeta) = -\pi_{mn}(\zeta),\quad r_{mn}(\zeta) = -\rho_{mn}(\zeta),\quad s_{mn}(\zeta) = -\sigma_{mn}(\zeta).$$

Recurrence relations for three contiguous functions:

$$\alpha_{m,n+1}\,R_{m,n-1} + (\gamma_{m,n+1}\,\alpha_{mn} - \beta_{mn}\alpha_{m,n+1})\,R_{mn} + \alpha_{mn}\,R_{m,n+1} = 0;$$

$$\pi_{m,n+1}\,R_{m-1,n} + (\sigma_{m+1,n}\,\pi_{mn} - \rho_{mn}\,\pi_{m+1,n})\,R_{mn} + \pi_{mn}\,R_{m+1,n} = 0;$$

and similarly for angular functions.

Series for derivatives of spheroidal scalar wave functions:

$X^{(1)}_{mn}(x,y,z) = \chi^{(1)}_{mn}(\eta,\xi,\phi) = S^{(1)}_{mn}(\eta)\,R^{(1)}_{mn}(\xi)\,e^{im\phi}$; the symbol $\mu_{n,j}$ is defined on p. 41;

$$\partial X^{(1)}_{mn}/\partial z = \sum_{j=0}^{\infty}\mu_{n+1,j}\;f^{mn}_j\,\chi^{(1)}_{mj};\quad \partial X^{(1)}_{mn}/\partial x + i\,\partial X^{(1)}_{mn}/\partial y$$
$$= \sum_{j=0}^{\infty}\mu_{n,j}\;g^{mn}_j\,\chi^{(1)}_{m+1,j};\quad \partial X^{(1)}_{mn}/\partial x - i\,\partial X^{(1)}_{mn}/\partial y = \sum_{j=0}^{\infty}\mu_{n,j}\;h^{mn}_j\,\chi^{(1)}_{m-1,j};$$

$$f_j^{mn} = k \sin\tfrac{1}{2}(j-n)\pi \int_{-1}^{1} \eta \; S_{mn}^{(1)}(\tau)\, S_{mj}^{(1)}(\tau)\, d\tau \Big/\!\!\Big/ \int_{-1}^{1} [S_{mj}^{(1)}(\tau)]^2\, d\tau;$$

$$g_j^{mn} = -k \cos\tfrac{1}{2}(j-n)\pi \int_{-1}^{1} (1-\tau^2)^{\frac{1}{2}} S_{mn}^{(1)}(\tau)\, S_{m+1,j}^{(1)}(\tau)\, d\tau \Big/\!\!\Big/ \int_{-1}^{1} [S_{m+1,j}^{(1)}(\tau)]^2\, d\tau;$$

$$h_j^{mn} = k \cos\tfrac{1}{2}(j-n)\pi \int_{-1}^{1} (1-\tau^2)^{\frac{1}{2}} S_{mn}^{(1)}(\tau)\, S_{m-1,j}^{(1)}(\tau)\, d\tau \Big/\!\!\Big/ \int_{-1}^{1} [S_{m-1,j}^{(1)}(\tau)]^2\, d\tau.$$

Series for components of spheroidal vector wave functions:

$${}^{x}\underline{M}_{mn}^{(1)} = \nabla X_{mn}^{(1)} \times \underline{e}_x, \quad {}^{y}\underline{M}_{mn}^{(1)} = \nabla X_{mn}^{(1)} \times \underline{e}_y, \quad {}^{z}\underline{M}_{mn}^{(1)} = \nabla X_{mn}^{(1)} \times \underline{e}_z;$$

$${}^{x}\underline{M}_{mn}^{(1)} \pm i\,{}^{y}\underline{M}_{mn}^{(1)} = \mp\, i \left[\frac{\partial X_{mn}^{(1)}}{\partial z}(\underline{e}_x \pm i\,\underline{e}_y) - \left(\frac{\partial X_{mn}^{(1)}}{\partial x} \pm i \frac{\partial X_{mn}^{(1)}}{\partial y}\,\underline{e}_z\right)\right];$$

$${}^{z}\underline{M}_{mn}^{(1)} = \frac{1}{2i}\left[\left(\frac{\partial X_{mn}^{(1)}}{\partial x} + i\,\frac{\partial X_{mn}^{(1)}}{\partial y}\right)(\underline{e}_x - i\,\underline{e}_y) - \left(\frac{\partial X_{mn}^{(1)}}{\partial x} - i\,\frac{\partial X_{mn}^{(1)}}{\partial y}\right)(\underline{e}_x + i\,\underline{e}_y)\right];$$ substitute series from

preceding set of formulas.

APPENDIX B

FORMULAS USED IN COMPUTATIONS

The essential formulas used in the computations and not given elsewhere in this report are listed here:

1) Legendre and Bessel Functions: these were computed from the following polynomials and recurrence relations. The symbol [x] denotes the largest integer not exceeding x:

$$J_{n+1/2}(c\xi) = \sqrt{\frac{2}{\pi c\xi}}\left[\sin(c\xi - \tfrac{1}{2}n\pi)\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!}(2c\xi)^{-2k}\right.$$

$$\left. + \cos(c\xi - \tfrac{1}{2}n\pi)\sum_{k=0}^{[\frac{1}{2}(n-1)]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!}(2c\xi)^{-2k-1}\right],$$

(n = 2, 3, . . ., 17).

$$P_{-n-1}(\xi) = 2^{n+1}\sum_{k=0}^{[-\frac{1}{2}(n+1)]} \frac{(-1)^k(-2n-2k-2)!\,\xi^{-n-2k-1}}{k!(-n-k-1)!(-n-2k-1)!},$$

(n = -1, -2, . . ., -16).

$$P^1_{-n-2}(\xi) = 2^{n+2}(\xi^2-1)^{1/2}\sum_{k=0}^{[-\frac{1}{2}(n+3)]} \frac{(-1)^k(-2n-2k-4)!\,\xi^{-n-2k-3}}{k!(-n-k-2)!(-n-2k-3)!},$$

(n = -3, -4, . . ., -16).

$$\frac{d}{d\xi}P_{-n-1}(\xi) = \left(\frac{2}{\xi}\right)^n \frac{n}{\xi^2-1}\left[2\sum_{k=0}^{[-\frac{1}{2}(n+1)]} \frac{(-1)^k(-2n-2k-2)!\,\xi^{-2k}}{k!(-n-k-1)!(-n-2k-1)!}\right.$$

$$\left. - \sum_{k=0}^{[-\frac{1}{2}n]} \frac{(-1)^k(-2n-2k)!\,\xi^{-2k}}{k!(-n-k)!(-n-2k)!}\right],$$

(n = -1, -2, . . ., -16).

$$\frac{d}{d\xi} P^1_{-n-2}(\xi) = \left(\frac{2}{\xi}\right)^{n+2} (\xi^2 - 1)^{-1/2}$$

$$\times \left[-\frac{1}{2}(n+2) \sum_{k=0}^{[-\frac{1}{2}(n+2)]} \frac{(-1)^k (-2n-2k-2)!\,\xi^{-2k}}{k!(-n-k-1)!(-n-2k-2)!} \right.$$

$$\left. + (n+1) \sum_{k=0}^{[-\frac{1}{2}(n+3)]} \frac{(-1)^k (-2n-2k-4)!\,\xi^{-2k}}{k!(-n-k-2)!(-n-2k-3)!} \right]$$

$(n = -3, -4, \ldots, -16)$.

$$Q_n(\xi) = \frac{1}{2} P_n(\xi) \log_e \frac{\xi+1}{\xi-1} - \sum_{k=0}^{[\frac{1}{2}(n-1)]} \frac{2n-4k-1}{(1+2k)(n-k)} P_{n-1-2k}(\xi),$$

$(n = 1, 2, \ldots, 15)$,

$$Q_0(\xi) = \frac{1}{2} \log_e \frac{\xi+1}{\xi-1}.$$

$$Q^1_{n+1}(\xi) = \frac{1}{2} P^1_{n+1}(\xi) \log_e \frac{\xi+1}{\xi-1} - (\xi^2-1)^{-1/2} P_{n+1}(\xi)$$

$$- \sum_{k=0}^{[\frac{1}{2}n]} \frac{2n-4k+1}{(1+2k)(n-k+1)} P^1_{n-2k}(\xi),$$

$(n = 0, 1, \ldots, 15)$,

$$Q^1_0(\xi) = -(\xi^2-1)^{-1/2}, \quad Q^1_{-1}(\xi) = -\xi(\xi^2-1)^{-1/2}.$$

$$\frac{d}{d\xi} Q_n(\xi) = \frac{n+1}{\xi^2-1} \left\{ \frac{1}{2} [P_{n+1}(\xi) - \xi P_n(\xi)] \log_e \frac{\xi+1}{\xi-1} \right.$$

$$\left. - \sum_{k=0}^{[\frac{1}{2}n]} \frac{(2n-4k+1)}{(1+2k)(n-k+1)} P_{n-2k}(\xi) + \sum_{k=0}^{[\frac{1}{2}(n-1)]} \frac{(2n-4k+1)}{(1+2k)(n-k)} P_{n-2k-1}(\xi) \right\},$$

$(n = 1, 2, \ldots, 15)$,

$$\frac{d}{d\xi} Q_0(\xi) = -(\xi^2-1)^{-3/2}$$

$$\frac{d}{d\xi} Q^1_{n+1}(\xi) = (\xi^2-1)^{-1} \Big\{ (n+1)\Big[\frac{1}{2} P_{n+2}(\xi) \log_e \frac{\xi+1}{\xi-1} - \frac{1}{(\xi^2-1)^{1/2}} P_{n+2}(\xi)$$

$$- \sum_{k=0}^{[\frac{1}{2}(n+1)]} \frac{(2n-4k+3)}{(1+2k)(n-k+2)} P^1_{n-2k+1}(\xi)\Big]$$

$$-(n+2)\,\xi \Big[\frac{1}{2} P^1_{n+1}(\xi) \log_e \frac{\xi+1}{\xi-1} - \frac{1}{(\xi^2-1)^{1/2}} P_{n+1}(\xi)$$

$$- \sum_{k=0}^{[\frac{1}{2}n]} \frac{(2n-4k+1)}{(1+2k)(n-k+1)} P^1_{n-2k}(\xi)\Big]\Big\},$$

$(n = 0, 1, \ldots, 15)$,

$$\frac{d}{d\xi} Q^1_0(\xi) = \xi(\xi^2-1)^{-3/2}, \quad \frac{d}{d\xi} Q^1_{-1}(\xi) = (\xi^2-1)^{-3/2}.$$

2) <u>Separation Constants</u>: The separation constant A_{mn} was obtained from a) or b) below, depending on the value of c:

a) $c \geq 5.0$: $$A_{mn} \approx \sum_{k=-1}^{4} h_k^{mn} c^{-k},$$

where

$$h_{-1}^{mn} = -(2n+1), \quad h_0^{mn} = (2n^2+2n+3-4m^2)\cdot 2^{-2},$$

$$h_1^{mn} = (2n+1)(n^2+n+3-8m^2)\cdot 2^{-4},$$

$$h_2^{mn} = [5(n^4+2n^3+8n^2+7n+3) - 48m^2(2n^2+2n+1)]\cdot 2^{-8},$$

$$h_3^{mn} = [66n^5+165n^4+962n^3+1278n^2+1321n+453 - m^2(2368n^3+3552n^2+4448n+1632) + m^4(256n+128)]\cdot 2^{-10},$$

$$h_4^{mn} = [252n^6 + 756n^5 + 5885n^4 + 10510n^3 + 18478n^2 + 13349n + 4425$$
$$- m^2(14720n^4 + 29440n^3 + 64000n^2 + 49280n + 17280)$$
$$+ m^4(6144\, n^2 + 6144n + 3072)] \cdot 2^{-12}.$$

b) $c < 5$: $\quad A_{mn} \approx \sum_{k=0}^{4} g_k^{mn} \; c^{2k}$,

where

$$g_0^{mn} = -(n+m)(n+m+1), \; g_1^{mn} = 1/2 \left[\frac{(2m-1)(2m+1)}{(2n+2m-1)(2n+2m+3)} - 1\right],$$

$$g_2^{mn} = 1/2 \left[\frac{(n+1)(n+2)(n+2m+1)(n+2m+2)}{(2n+2m+1)(2n+2m+3)^3(2n+2m+5)} - \frac{(n+2m)(n-1)\,n\,(n+2m-1)}{(2n+2m-3)(2n+2m-1)^3(2n+2m+1)}\right],$$

$$g_3^{mn} = (4m^2-1)\left[\frac{(n-1)\,n\,(n+2m-1)(n+2m)}{(2n+2m-5)(2n+2m-3)(2n+2m-1)^5(2n+2m+1)(2n+2m+3)} - \frac{(n+1)(n+2)(n+2m+1)(n+2m+2)}{(2n+2m-1)(2n+2m+1)(2n+2m+3)^5(2n+2m+5)(2n+2m+7)}\right],$$

$$g_4^{mn} = 2(4m^2-1)^2\left[\frac{(n+1)(n+2)(n+2m+1)(n+2m+2)}{(2n+2m-1)^2(2n+2m+1)(2n+2m+3)^7(2n+2m+5)(2n+2m+7)^2} - \frac{(n-1)\,n\,(n+2m-1)(n+2m)}{(2n+2m-5)^2(2n+2m-3)(2n+2m-1)^7(2n+2m+1)(2n+2m+3)^2}\right]$$
$$+ 2^{-4}\left[\frac{(n+1)(n+2)(n+3)(n+4)(n+2m+1)(n+2m+2)(n+2m+3)(n+2m+4)}{(2n+2m+1)(2n+2m+3)^4(2n+2m+5)^3(2n+2m+7)^2(2n+2m+9)} - \frac{(n-3)(n-2)(n-1)\,n\,(n+2m-3)(n+2m-2)(n+2m-1)(n+2m)}{(2n+2m-7)(2n+2m+5)^2(2n+2m-3)^3(2n+2m-1)^4(2n+2m+1)}\right]$$
$$+ 2^{-3}\left[\frac{(n-1)^2n^2(n+2m-1)^2(n+2m)^2}{(2n+2m-3)^2(2n+2m-1)^7(2n+2m+1)^2} - \frac{(n+1)^2(n+2)^2(n+2m+1)^2(n+2m+2)^2}{(2n+2m+1)^2(2n+2m+3)^7(2n+2m+5)^2}\right]$$
$$- 1/2\left[\frac{(n-1)\,n\,(n+1)(n+2)(n+2m-1)(n+2m)(n+2m+1)(n+2m+2)}{(2n+2m-3)(2n+2m-1)^4(2n+2m+1)^2(2n+2m+3)^4(2n+2m+5)}\right].$$

3) Integrals: the integrals involving the angular spheroidal functions, which arise in the process of determining the expansion coefficients for the scattered wave, have been reduced in Reference 21 to expressions equivalent to the following formulas. Here, $\delta_{j,k}$ is the Kronecker symbol, which is equal to 0 if $j \neq k$ and equal to 1 if $j = k$. The symbol $\mu_{i,j}$ is defined on p. 41.

$$I_1^{Nn} \equiv \int_{-1}^{1} S_{on}^{(1)}(\eta)\, S_{on}^{(1)}(\eta)\, d\eta = 2\delta_{N,n} \sum_{k=0}^{\infty} \mu_{N,k} (d_k^{on})^2/(2k+1).$$

$$I_2^{Nn} \equiv \int_{-1}^{1} \eta (1-\eta^2)^{-1/2}\, S_{1n}^{(1)}(\eta)\, S_{on}^{(1)}(\eta)\, d\eta$$

$$= -2\mu_{N,n+1} \left[\sum_{k=0}^{\infty} \mu_{n,k} \frac{k+1}{2k+3} d_k^{1n} d_{k+1}^{oN} + \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \mu_{N,k}\, \mu_{n,j}\, d_k^{oN} d_j^{1n} \right].$$

$$I_3^{Nn} \equiv \int_{-1}^{1} \eta\, S_{on}^{(1)}(\eta)\, S_{oN}^{(1)}(\eta)\, d\eta = \mu_{N,n+1} \sum_{k=0}^{\infty} \frac{2k+2}{(2k+1)(2k+3)} (d_k^{on} d_{k+1}^{oN} + d_{k+1}^{on} d_k^{oN}).$$

$$I_4^{Nn} \equiv \int_{-1}^{1} (1-\eta^2)\, S_{oN}^{(1)}(\eta) \frac{d}{d\eta} S_{on}^{(1)}(\eta)\, d\eta$$

$$= 2\mu_{N,n+1} \sum_{k=0}^{\infty} \frac{k+1}{(2k+1)(2k+2)} \left[(k+1)\, d_{k+1}^{on}\, d_k^{oN} - k\, d_k^{on}\, d_{k+1}^{oN} \right].$$

$$I_5^{Nn} \equiv \int_{-1}^{1} (1-\eta^2)^{1/2}\, S_{1n}^{(1)}(\eta)\, S_{oN}^{(1)}(\eta)\, d\eta$$

$$= 2\mu_{N,n} \sum_{k=0}^{\infty} \frac{k+1}{(2k+1)(2k+3)} \left[k\, d_{k-1}^{1n}\, d_{k+1}^{oN} - (k+2)\, d_k^{1n}\, d_k^{oN} \right].$$

$$I_6^{Nn} \equiv \int_{-1}^{1} \eta(1-\eta^2)^{1/2} S_{0N}^{(1)}(\eta) \frac{d}{d\eta} S_{1n}^{(1)}(\eta)\, d\eta$$

$$= 2\mu_{N,n}\left\{\sum_{k=0}^{\infty} \frac{(k+1)^2}{(2k+1)(2k+3)}\left[(k+2)\, d_k^{1n}\, d_k^{0N} + k\, d_{k-1}^{1n}\, d_{k+1}^{0N}\right]\right.$$

$$- \sum_{k=0}^{\infty} \frac{\mu_{n,k}}{2k+1} d_k^{0N}\left[\frac{k(k-1)}{2k-1} d_{k-2}^{1n} + \frac{(k+1)^2}{2k+3} d_k^{1n} + k\, d_k^{1n}\right]$$

$$\left. - \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu_{n,k}\,\mu_{n,j}\, d_k^{0N}\, d_j^{1n}\right\}.$$

4) Elements of Determinants: these are given in terms of the above integrals as follows, with A_n as defined in (III-3):

$$B_{Nn} = \frac{i}{c\xi_0} A_n (\xi_0^2-1)^{1/2} \frac{d}{d\xi} R_{on}^{(1)}(\xi_0)\, I_1^{Nn},$$

$$C_{Nn} = \delta_{N,n} (\xi_0^2-1)^{1/2} \frac{d}{d\xi} R_{on}^{(4)}(\xi_0)\, I_1^{Nn},$$

$$D_{Nn} = - R_{1n}^{(4)}(\xi_0)\, I_2^{Nn},$$

$$U_{Nn} = \frac{i}{c\xi_0} A_n \left[(\xi_0^2-1)\frac{d}{d\xi} R_{on}^{(1)}(\xi_0)\, I_3^{Nn} + \xi_0 R_{on}^{(1)}(\xi_0)\, I_4^{Nn}\right],$$

$$V_{Nn} = (\xi_0^2-1)\frac{d}{d\xi} R_{1n}^{(4)}(\xi_0)\, I_3^{Nn} + \xi_0 R_{on}^{(4)}(\xi_0)\, I_4^{Nn},$$

$$W_{Nn} = -\xi_0(\xi_0^2-1)\frac{d}{d\xi} R_{1n}^{(4)}(\xi_0)\, I_5^{Nn} + (\xi_0^2-1)^{1/2} R_{1n}^{(4)}(\xi_0)\, I_6^{Nn}.$$

5) <u>Cross-Section Coefficients</u>: The coefficients in the expansion for the back-scattered wave ($\eta = 1$) are quotients of the following determinants, whose elements are described in 4):

$$\alpha_0' = \frac{1}{G}\begin{vmatrix} B_{00} & 0 & D_{01} & D_{03} \\ B_{22} & C_{22} & D_{21} & D_{23} \\ U_{10} + U_{12} & V_{12} & W_{11} & W_{13} \\ U_{30} + U_{32} & V_{32} & W_{31} & W_{33} \end{vmatrix}$$

$$\alpha_1' = \frac{1}{H}\begin{vmatrix} B_{11} & 0 & D_{10} & D_{12} \\ B_{33} & C_{33} & D_{30} & D_{32} \\ U_{01} + U_{03} & V_{03} & W_{00} & W_{02} \\ U_{21} + U_{23} & V_{23} & W_{20} & W_{22} \end{vmatrix}$$

$$\alpha_2' = \frac{1}{G}\begin{vmatrix} C_{00} & B_{00} & D_{01} & D_{03} \\ 0 & B_{22} & D_{21} & D_{23} \\ V_{10} & U_{10} + U_{12} & W_{11} & W_{13} \\ V_{30} & U_{30} + U_{32} & W_{31} & W_{33} \end{vmatrix}$$

$$\alpha_3' = \frac{1}{H}\begin{vmatrix} C_{11} & B_{11} & D_{10} & D_{12} \\ 0 & B_{33} & D_{30} & D_{32} \\ V_{01} & U_{01} + U_{03} & W_{00} & W_{02} \\ V_{21} & U_{21} + U_{23} & W_{20} & W_{22} \end{vmatrix}$$

where

$$G = \begin{vmatrix} C_{00} & 0 & D_{01} & D_{03} \\ 0 & C_{22} & D_{21} & D_{23} \\ V_{10} & V_{12} & W_{11} & W_{13} \\ V_{30} & V_{32} & W_{31} & W_{33} \end{vmatrix} \quad \text{and } H = \begin{vmatrix} C_{11} & 0 & D_{10} & D_{12} \\ 0 & C_{33} & D_{30} & D_{32} \\ V_{01} & V_{03} & W_{00} & W_{02} \\ V_{21} & V_{23} & W_{20} & W_{22} \end{vmatrix}$$

6) Radar Back-Scattering Cross-Section: This is computed from the formula

$$\sigma = 4\pi a^2 \left| \sum_{n=0}^{\infty} i^n \alpha_n' \sum_{k=0}^{\infty} d_k^{on} \right|^2 .$$

APPENDIX C

COMPUTATION OF BISTATIC CROSS-SECTION

Formula (III-18) indicates that of the two sets of expansion coefficients α_n and β_n, only the α_n are used in computing the back-scattering cross-section. On the other hand, to obtain the bistatic cross-section $\sigma(\theta,\phi)$--where the transmitter is on the major axis of the spheroid and the receiver is on the ray with angles θ, ϕ--one requires both the α_n and the β_n, as follows from formula (III-17). The only additional quantities required which are not used in obtaining the back-scattering cross-section are the angular functions $S_{mn}^{(1)}(\cos\theta)$. These are given by the expression

$$S_{mn}^{(1)}(\cos\theta) = \sum_{k=0}^{\infty} \mu_{n,k}\, d_k^{mn}\, P_{n+k}^{m}(\cos\theta)$$

where the $P_{n+k}^{m}(\cos\theta)$ are associated Legendre functions. The spheroidal coefficients d_k^{mn} have been computed as described, and the associated Legendre functions are tabulated. It might be observed that in solving the systems of equations (III-13) and (III-14) for the α_n on a large scale machine, very little additional expense is involved in obtaining the β_n also.

In the case where $\theta = \pi/2$, i.e., the receiver is in the plane of the minor axes, the computation of $\sigma(\theta,\phi)$ becomes particularly simple, since $\cos\theta = 0$ and

$$S_{0,2n}^{(1)}(0) = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$$

$$S_{0,2n+1}^{(1)}(0) = \frac{(-1)^n(2n+1)!}{2^{2n}(n!)^2}\ .$$

APPENDIX D

QUANTITIES TABULATED IN MARK III OUTPUT

A list of the parameter values computed is given on page 47 of the text. For each value of $\frac{2\pi a}{\lambda}$ listed, the following quantities are recorded:

1. Associated Legendre functions $P^{m}_{-n-m-1}(\xi)$ and $Q^{m}_{m+n}(\xi)$ and their derivatives with respect to ξ,
   for m = 0, 1,
   n = -1, -2, . . . -16.

2. Bessel functions $J_{n+\frac{1}{2}}(c\xi)$, for n = 1, 2, . . . 17.

3. Separation constants A_{mn}, for m = 0, 1; n = 0, 1, 2, 3.

4. Spheroidal coefficients d_k^{mn} (or d_k^{mn}/ρ in range of $k < 0$, where $d_k^{mn} = 0$),
   for m = 0, 1,
   n = 0, 1, 2, 3,
   k = all necessary values in range -16 to +16 (see Appendix B, Part 3).

5. Radial spheroidal functions $R^{(1)}_{mn}(\xi)$ and $R^{(2)}_{mn}(\xi)$ and derivatives of these with respect to ξ,
   for $\xi = 1.005$
   m = 0, 1,
   n = 0, 1, 2, 3.

6. Boundary integrals I_k^{Nn}
   for k = 1, 2, 3, 4, 5, 6,
   all combinations of N and n in range
   N = 0, 1, 2, 3,
   n = 0, 1, 2, 3.

7. Determinantal elements B_{Nn}, C_{Nn}, D_{Nn}, U_{Nn}, V_{Nn}, W_{Nn}, for all combinations of N and n in range
   $N = 0, 1, 2, 3,$
   $n = 0, 1, 2, 3.$

8. Radar cross-section σ.

With the exception of σ, all quantities are given to 15 significant figures. The values of σ are rounded off to 5 significant figures.

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