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The Diffraction Matrix for a Discontinuity in Curvature

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Abstract

To determine how the early time behavior of a pulse radiated by an EMP antenna depends on the geometry of that antenna, it is sufficient to concentrate on the high frequency components of the spectrum and examine the manner in which each of the high frequency waves emitted from the source is scattered by the geometrical features of the antenna. In principal at least, this can be accomplished using the geometrical theory of diffraction. A key requirement of this theory is a knowledge of the diffraction matrix associated with any surface singularity present in the problem, and in order that we may explore a variety of antenna configurations, the matrix is required for singularities other than the wedge-type for which it is presently available.

A valid expression for the diffraction matrix for a discontinuity in curvature is here derived. Using a model consisting of two parabolic cylinders of different latus recta joined at the front, an asymptotic development of the surface field in the vicinity of the join is first obtained, from which the elements of the complete diffraction matrix are then obtained by integration. The results differ significantly from the physical optics estimates, and some of the consequences of this are examined. The diffraction matrix is cast in a form directly analogous to that for a wedge-type singularity, thereby facilitating its incorporation within existing programs for analyzing the EMP problem, but no explicit consideration of EMP antennas is here included.

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1. Introduction

A problem of some interest in EMP studies is the effect of the antenna geometry on the temporal behavior of the field at times shortly after the onset of the pulse. To achieve the rapid rise that is desirable, the feeding section of the antenna is usually taken to be biconical, but since it is not practical that this geometry persist to infinity, the most elementary model of an EMP antenna is a circular cylinder with a biconical feed. The junction between each conical and cylindrical section can now be the source of a diffracted field which, at any point in space, will change the character of the radiated pulse at those times when both the direct and the diffracted contributions are present.

There are, of course, other possible transitions between sections which are conical at the feed, but have finite diameter far away. It is not even necessary that the generators of either portion be straight lines, and since the nature of the diffracted field is a function of the geometry, the choice of transition geometry will affect the manner in which the pulse is perturbed. A geometry which is now of interest to explore is one in which the antenna surface has no discontinuity in slope, but has at most a discontinuity in curvature. It would appear that the removal of any wedge-like surface discontinuity will reduce the diffracted field throughout those portions of space which are directly illuminated by the feed, but no quantitative estimate of this reduction is available at the moment.

The early time behavior of the radiated pulse is produced by the high frequency components in the spectrum, and can be computed from a knowledge of the high frequency CW solution. This method has been used by Sancer and Varvatsis (1971) to analyze an antenna consisting of a bicone mated directly to the cylindrical portions, and is clearly feasible only to the extent that it is possible to obtain a high frequency solution of the required accuracy for the geometry in question.

When a metallic object is illuminated by an electromagnetic wave, a powerful method for estimating its high frequency scattering behavior is the geometrical theory of diffraction, originated by Keller (1962). The theory is basically an extension of ray techniques to include the concept of diffracted rays which arise from surface singularities of the body. The strength of each such ray contribution to the scattering is proportional to a diffraction coefficient which is determined, to the first order at least, by the local surface geometry at the point of diffraction. In those cases where the diffraction coefficients are known, their expressions have been obtained from exact solutions of selected canonical problems displaying the geometry in question, and thus it is that the coefficients for an edge or wedge-like singularity (slope discontinuity) are deduced from the solution of the two-dimensional problem of scattering of a plane wave by a half-plane or wedge. In the problem treated by Sancer and Varvatsis, the surface singularity was indeed wedge-like, and the higher order contributions to the high frequency diffracted field could be (and were) determined by GTD.

The diffraction coefficients are the key to the GTD method and one particular but important case where they are not yet known is when the surface slope (first derivative) is continuous, but the curvature (involving the second derivative) is discontinuous. Their derivation for this geometrical feature is vital to an analysis of the more general antenna configurations discussed above. It is also necessary for an adequate treatment of scattering by bodies such as a cone-sphere or hemispherically-capped cylinder, and in the absence of any exact canonical solution from which to deduce the coefficients, it has been necessary to rely (see, for example, Senior, 1965) on the crude estimates offered by physical optics. Automatically, therefore, the polarization dependence has been suppressed (Knott and Senior, 1971).

Although an exact canonical solution would be desirable, it is not, in fact, essential to the determination of a diffraction coefficient, and an adequate description of the surface field in a vicinity of the geometric feature can suffice. For a discontinuity in curvature, we can obtain such a surface field description using the model that was employed by Weston (1962, 1965) in studying the creeping waves launched by the discontinuity. Weston considered only the case of a plane wave incident with its magnetic vector parallel to the (line) discontinuity (H polarization). This is treated in Section 3 and the initial part of the analysis follows closely that which was given by Weston (1962). The analogous case of a plane wave incident with its electric vector parallel to the discontinuity (E polarization) is discussed in Section 4. The corresponding diffraction coefficients for H and E polarized waves are derived in Sections 5 and 6 respectively, and the general diffraction matrix is constructed in Section 7. The results differ from the physical optics estimates for almost all angles of incidence and diffraction, and some of the consequences of these new and rigorous formulae are explored.

The form in which the diffraction matrix is expressed is directly analogous to that for a discontinuity in slope, thereby facilitating the incorporation of our results within a program such as that developed by Sancer and Varvatsis (1971) for the bicone cylinder. However, the main focus of this present report is on the determination of the diffraction matrix alone, and the application to the calculation of the early time pulse behaviors for a variety of antenna geometries will be treated in a subsequent report.

2. Preliminary Considerations

We consider a two-dimensional perfectly conducting surface consisting of two half parabolic cylinders of different latus recta joined at the front. In terms of the Cartesian coordinates (x, y, z) with the z axis coincident with the join, the surface is defined as:

$$x = -\frac{1}{2} a_2 y^2, \qquad y > 0 \text{, all } z$$
$$x = -\frac{1}{2} a_1 y^2, \qquad y < 0 \text{, all } z \tag{1}$$

so that the positive x axis is in the direction of the outward normal to the surface at the join. For convenience we shall henceforth write (1) as

$$x = -\frac{1}{2} a y^2, \qquad \text{all } z \tag{2}$$

where $a = a_2 (y > 0)$, $a = a_1 (y < 0)$. It is easily verified that the slope (first derivative) of the surface is continuous at the join (it is infinite there), but since the radius of curvature is

$$\rho_c(y) = -\frac{1}{a}(1 + a^2 y^2)^{3/2}, \tag{3}$$

the curvature is discontinuous at $y = 0$ unless $a_2 = a_1$.

A plane electromagnetic wave is incident with its propagation vector lying in the xy plane and making an angle α with the negative x axis, where* $|\alpha| \leq \frac{\pi}{2}$ (see Fig. 1) If the wave has its magnetic vector in the z direction (H polarization), we can write

$$\underline{H}^i = \hat{z}\, e^{ik(-x\cos\alpha + y\sin\alpha)},$$
$$\underline{E}^i = -Z(\hat{x}\sin\alpha + \hat{y}\cos\alpha)\, e^{ik(-x\cos\alpha + y\sin\alpha)}, \tag{4}$$

*In the derivation of the surface field about the join it is, in fact, necessary to assume that $\pi/2 - |\alpha|$ is bounded away from zero to ensure that the shadow boundary is sufficiently far removed from the z axis, but since the diffraction coefficients (see Secs. 5 and 6) remain finite as $\alpha \to \pm\pi/2$, our final results are valid even for grazing incidence. This is vital for a treatment of the antenna problem.

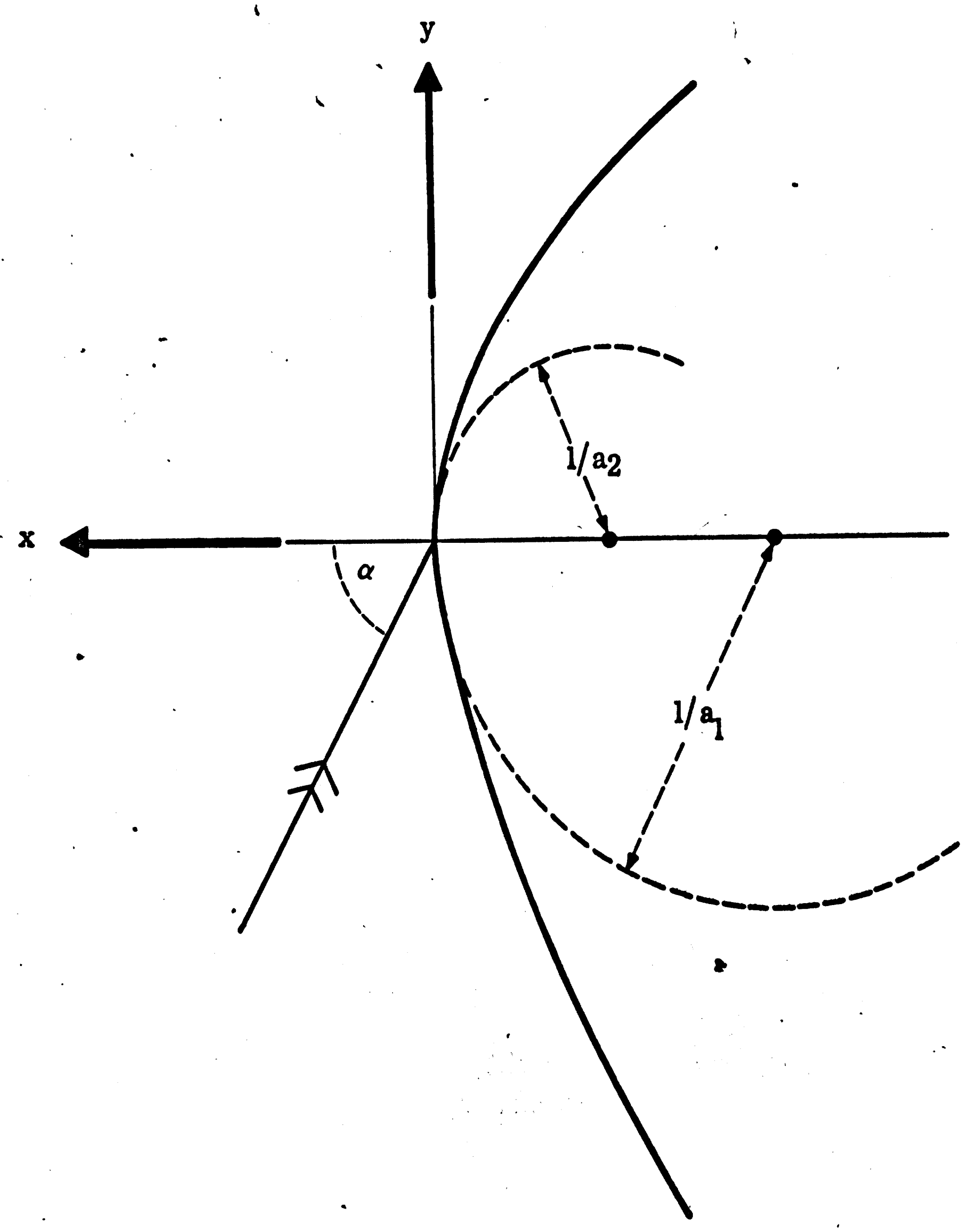

FIG. 1: Diffraction Coefficient Model

where $Z = 1/Y$ is the intrinsic impedance of free space and a time factor $e^{-i\omega t}$ has been assumed and suppressed. Due to the presence of the perfectly conducting surface, a scattered field $(\underline{E}^s, \underline{H}^s)$ will be generated satisfying the boundary condition

$$\hat{n}\wedge(\underline{E}^i + \underline{E}^s) = 0$$

at the surface, where $\hat{n}$ is a unit vector normal in the outwards direction. Our task is to find the total (incident plus scattered) magnetic field at the surface, with particular reference to a region in the immediate vicinity of the join.

Since the problem is two-dimensional (being independent of the coordinate z), it can be expressed as a scalar problem for the total magnetic field component, $H_z = u$, which is required to satisfy the Neumann boundary condition $(\partial u/\partial n) = 0$ at the surface, with $u-u_o$ obeying the radiation condition, where

$$u_o = H_z^i = e^{ik(-x\cos\alpha + y\sin\alpha)} . \tag{5}$$

We now have a hard-body problem, and this is treated in Section 3.

If, on the other hand, the incident plane wave has its electric vector in the z direction, then

$$\underline{E}^i = \hat{z}\, e^{ik(-x\cos\alpha + y\sin\alpha)} ,$$
$$\underline{H}^i = Y\,(\hat{x}\sin\alpha + \hat{y}\cos\alpha)\, e^{ik(-x\cos\alpha + y\sin\alpha)} . \tag{6}$$

Our task is again to find the total (incident plus scattered) magnetic field at the surface, and since the problem is two-dimensional, it can be expressed as a scalar one for the total electric field component $E_z = u$. This is required to satisfy the Dirichlet boundary condition $u=0$ at the surface, with $u-u_o$ obeying the radiation condition, where

$$u_o = E_z^i = e^{ik(-x\cos\alpha + y\sin\alpha)} . \tag{7}$$

The resulting soft-body problem is treated in Section 4.

3. H Polarization

3.1 The Integral Equation

Maue's integral equation for the field on an acoustically hard surface is

$$u(P) = 2u_o(P) + \frac{1}{2\pi} \int \frac{\partial G}{\partial n_Q} u(Q)\, dS_Q \tag{8}$$

where

$$G = \frac{e^{ikR}}{R} \tag{9}$$

with

$$R = \left\{ (x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2 \right\}^{1/2} , \tag{10}$$

and because of the two-dimensional nature of the present problem, the z integration can be carried out immediately using

$$\int_{-\infty}^{\infty} \frac{e^{ikR}}{R}\, dz_Q = i\pi H_o^{(1)}(kr) \tag{11}$$

where

$$r = \left\{ (x_P - x_Q)^2 + (y_P - y_Q)^2 \right\}^{1/2} . \tag{12}$$

Thus, Eq. (8) reduces to

$$u(P) = 2u_o(P) + \frac{i}{2} \int \frac{\partial}{\partial n_Q} \left(H_o^{(1)}(kr) \right) u(Q)\, ds_Q . \tag{13}$$

If we regard y as the variable of integration and denote it by t, then

$$ds_Q = (1+\bar{a}^2 t^2)^{1/2}\, dt \tag{14}$$

with t running from $-\infty$ to ∞, where $\bar{a} = a_2$ $(t > 0)$, $\bar{a} = a_1 (t < 0)$. Also

$$\hat{n}_Q = (\hat{x} + \bar{a}\, t\, \hat{y})\,(1+\bar{a}^2 t^2)^{-1/2} \tag{15}$$

so that

$$\frac{\partial r}{\partial n_Q} = \frac{1}{r}\left\{(x_Q - x_P) + \bar{a}\,t\,(y_Q - y_P)\right\}(1+\bar{a}^2 t^2)^{-1/2}$$

$$= \frac{1}{r}\left\{-\frac{1}{2}\bar{a}\,t^2 - x_P + \bar{a}\,t\,(t - y_P)\right\}(1+\bar{a}^2 t^2)^{-1/2} \quad .$$

Hence

$$\frac{\partial}{\partial n_Q} H_o^{(1)}(kr) = -\frac{k}{r}\left\{\bar{a}\,t\,(t-y) - \frac{1}{2}(\bar{a}\,t^2 - ay^2)\right\}\frac{H_1^{(1)}(kr)}{(1+\bar{a}^2 t^2)^{1/2}} \quad ,$$

and the integral equation now takes the form

$$u(y) = 2u_o(y) - \frac{ik}{2}\int_{-\infty}^{\infty} u(t)\, H_1^{(1)}(kr)\left\{\bar{a}t(t-y) - \frac{1}{2}(\bar{a}t^2 - ay^2)\right\}\frac{dt}{r} \tag{16}$$

with

$$r = \left\{(y-t)^2 + \frac{1}{4}(ay^2 - \bar{a}t^2)^2\right\}^{1/2} \quad . \tag{17}$$

This is in agreement with the result obtained by Weston (1962; Eq. 9), and our task is to find an asymptotic development of u(y) for large k/a, with particular reference to a vicinity of the join (y = 0) .

It is easily verified (as Weston does) that u and $\partial u/\partial y$ are continuous at y = 0. $\partial^2 u/\partial y^2$ is, however, discontinuous, and the limits from left and right are infinite. These facts are evident from the asymptotic expansion.

3.2 Asymptotic Solution

If the incident field (5) were to impinge on the complete and uniform parabolic surface

$$x = -\frac{1}{2} a y^2 , \qquad \text{all } y , z$$

an asymptotic expansion of the surface field could be obtained by the Luneberg-Kline method, and as shown in Appendix A, Eq. (A.22), we would then have

$$u(y) = U(y, a)\, e^{ikf(y, a)} \tag{18}$$

with

$$U(y, a) = 2 - \frac{ia}{k} (\cos \alpha - a\, y \sin \alpha)^{-3} + O(k^{-2}) , \tag{19}$$

$$f(y, a) = y \sin \alpha + \frac{1}{2} a y^2 \cos \alpha . \tag{20}$$

Though it is a relatively straightforward task to derive the actual term of order k^{-2} in the expression for U(y, a), this proves to be unnecessary for our purposes.

Following Weston (1962), we now write the field on the conjoint surface of Eq. (2) as the sum of two parts: that which would exist on that particular section were the whole surface a continuation of it, plus a perturbation created by the join. Thus,

$$u(y) = U(y, a) e^{ikf(y, a)} + \frac{1}{k} I(y, a)\, e^{iks(y, a)} \tag{21}$$

where

$$s(y,a) = \int_0^{|y|} (1 + a^2 \tau^2)^{1/2} \, d\tau \tag{22}$$

with U (y, a) and f (y, a) as given in Eqs. (19) and (20) respectively. The only unknown quantity in (21) is I (y, a), and since the discontinuity in U(y, a) is of order k^{-1} at y=0, it is clear that I(y, a) must be $O(k^0)$ for small y.

If we now substitute the expression (21) for u(y) into the integral equation (16) and, for convenience, write

$$\frac{H_1^{(1)}(kr)}{r} \left\{ \bar{a}\, t\, (t-y) - \frac{1}{2} (\bar{a} t^2 - a y^2) \right\} = K\, (t, y, a, \bar{a}) \, , \tag{23}$$

we have

$$U\,(y, a)\, e^{ikf(y,a)} + \frac{1}{k} I\,(y, a) e^{iks(y,a)} = 2\, u_o(y)$$

$$- \frac{ik}{2} \int_{-\infty}^{\infty} U\,(t, \bar{a})\, e^{ikf(t, \bar{a})}\, K(t, y, a, \bar{a})\, dt$$

$$- \frac{i}{2} \int_{-\infty}^{\infty} I(t, \bar{a})\, e^{iks(t, \bar{a})}\, K(t, y, a, \bar{a})\, dt$$

which can be recast in the form

$$U(y,a)e^{ikf(y,a)} + \frac{1}{k} I(y,a)e^{iks(y,a)} = 2\,u_o(y)$$

$$-\frac{ik}{2}\int_{-\infty}^{\infty}\left\{U(t,\bar{a})e^{ikf(t,\bar{a})}K(t,y,a,\bar{a}) - U(t,a)e^{ikf(t,a)}K(t,y,a,a)\right\}dt$$

$$-\frac{ik}{2}\int_{-\infty}^{\infty} U(t,a)e^{ikf(t,a)}K(t,y,a,a)dt - \frac{i}{2}\int_{-\infty}^{\infty} I(t,\bar{a})e^{iks(t,\bar{a})}K(t,y,a,\bar{a})\,dt \quad . \tag{24}$$

But $U(y,a)e^{ikf(y,a)}$ is, by definition, the field on a single parabolic cylinder formed by continuing that portion on which the observation point lies, and hence

$$U(y,a)e^{ikf(y,a)} = 2\,u_o(y) - \frac{ik}{2}\int_{-\infty}^{\infty} U(t,a)e^{ikf(t,a)}K(t,y,a,a)\,dt\,, \tag{25}$$

implying

$$I(y,a)e^{iks(y,a)} = -\frac{ik}{2}\int_{-\infty}^{\infty} I(t,\bar{a})e^{iks(t,\bar{a})}K(t,y,a,\bar{a})dt - \frac{ik^2}{2}\,Q \tag{26}$$

where

$$Q = \int_{\infty}^{\infty}\left\{U(t,\bar{a})e^{ikf(t,\bar{a})}K(t,y,a,\bar{a}) - U(t,a)e^{ikf(t,a)}K(t,y,a,a)\right\}dt \quad . \tag{27}$$

Let us now examine the quantity Q. Since the integrand is a known function we could, in principle at least, evaluate the integral precisely, but for our purposes an asymptotic evaluation will suffice. We take first the case $y < 0$. Then $a = a_1$ and since $\bar{a} = a_2(a_1)$ for $t > 0$ (< 0), as always, the integrand in (27) is identically zero for $t < 0$, so that

$$Q = \int_0^{\infty} \left\{ U(t, a_2) e^{ikf(t, a_2)} K(t, y, a_1, a_2) - U(t, a_1) e^{ikf(t, a_1)} K(t, y, a_1, a_1) \right\} dt \quad . \tag{28}$$

From Eq. (23), using the expression (17) for r,

$$K(t, y, a_1, a_2) = \frac{1}{2} \frac{H_1^{(1)}\left(k\,|y-t| \sqrt{1 + \left\{ \frac{a_2 t^2 - a_1 y^2}{2(y-t)} \right\}^2} \right)}{|y-t| \sqrt{1 + \left\{ \frac{a_2 t^2 - a_1 y^2}{2(y-t)} \right\}^2}} \left\{ a_2 (y-t)^2 + (a_1 - a_2) y^2 \right\} .$$

Write

$$kt = \zeta \ , \quad -ky = \zeta_0 \quad .$$

Clearly, $\zeta, \zeta_0 \geq 0$, and

$$K(t,y,a_1,a_2) = \frac{1}{2k} \frac{H_1^{(1)}\left((\zeta+\zeta_o) \sqrt{1+\frac{1}{4k^2}\left\{\frac{a_2\zeta^2-a_1\zeta_o^2}{\zeta+\zeta_o}\right\}^2}\right)}{(\zeta+\zeta_o)\sqrt{1+\frac{1}{4k^2}\left\{\frac{a_2\zeta^2-a_1\zeta_o^2}{\zeta+\zeta_o}\right\}^2}} \left\{a_2(\zeta+\zeta_o)^2+(a_1-a_2)\zeta_o^2\right\}$$

$$=\frac{1}{2k} H_1^{(1)}(\zeta+\zeta_o)\left\{a_2(\zeta+\zeta_o)+(a_1-a_2)\frac{\zeta_o^2}{\zeta+\zeta_o}\right\}\left\{1+O(k^{-2})\right\} , \quad (29)$$

which also implies

$$K(t,y,a_1,a_1)= \frac{1}{2k} H_1^{(1)}(\zeta+\zeta_o)a_1(\zeta+\zeta_o)\left\{1+O(k^{-2})\right\} . \quad (30)$$

Moreover,

$$U(t,a_2)= 2 + O(k^{-1})= U(t, a_1) \quad (31)$$

and

$$e^{ikf(t,a_2)} = e^{i\zeta\sin\alpha} \exp\left(i \frac{a_2}{2k}\zeta^2\cos\alpha\right)$$

$$= e^{i\zeta\sin\alpha}\left\{1+O(k^{-1})\right\}$$

$$= e^{ikf(t,a_1)} . \quad (32)$$

Hence, from Eqs. (28) through (32),

$$Q = \frac{a_2-a_1}{k^2}\int_0^{\infty} e^{i\zeta\sin\alpha} H_1^{(1)}(\zeta+\zeta_o)\left(\zeta+\zeta_o-\frac{\zeta_o^2}{\zeta+\zeta_o}\right)\left\{1+O(k^{-1})\right\} d\zeta$$

for $y < 0$, and by a simple shift of the variable of integration, this becomes

$$Q = \frac{a_2 - a_1}{k^2} \int_{\zeta_o}^{\infty} e^{i(\zeta - \zeta_o)\sin\alpha} H_1^{(1)}(\zeta) \left(\zeta - \frac{\zeta_o^2}{\zeta}\right) \left\{1 + O(k^{-1})\right\} d\zeta \qquad (33)$$

where

$$\zeta_o = -ky > 0 .$$

For $y > 0$ we return to Eq. (27), and noting that now $a = a_2$, we have

$$Q = \int_{-\infty}^{0} \left\{ U(t, a_1) e^{ikf(t, a_1)} K(t, y, a_2, a_1) - U(t, a_2) e^{ikf(t, a_2)} K(t, y, a_2, a_2) \right\} dt$$

$$= \int_{0}^{\infty} \left\{ U(-t', a_1) e^{ikf(-t', a_1)} K(-t', y, a_2, a_1) - U(-t', a_2) e^{ikf(-t', a_2)} K(-t', y, a_2, a_2) \right\} dt' . \qquad (34)$$

Fortunately it is not necessary to repeat the entire derivation that was carried out for $y < 0$. If we now define

$$kt' = -kt = \zeta , \qquad ky = \zeta_o$$

so that $\zeta, \zeta_o > 0$, the expression for $K(-t', y, a_2, a_1)$ follows immediately from Eq. (29) on interchanging a_2 and a_1. Eq. (30) similarly yields $K(-t', y, a_2, a_2)$, and whereas (31) applies directly, Eq. (32) shows

$$e^{ikf(-t', a_2)} = e^{-i\zeta\sin\alpha} \left\{1 + O(k^{-1})\right\} = e^{ikf(-t', a_1)} .$$

Hence, for $y > 0$,

$$Q = -\frac{a_2-a_1}{k^2} \int_0^{\infty} e^{-i\zeta \sin\alpha} H_1^{(1)}(\zeta+\zeta_o)\left(\zeta+\zeta_o - \frac{\zeta_o^2}{\zeta+\zeta_o}\right)\left\{1+O(k^{-1})\right\} d\zeta$$

and by a simple shift in the variable of integration, this becomes

$$Q = -\frac{a_2-a_1}{k^2} \int_{\zeta_o}^{\infty} e^{-i(\zeta-\zeta_o)\sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{\zeta_o^2}{\zeta}\right)\left\{1+O(k^{-1})\right\} d\zeta \qquad (35)$$

where now

$$\zeta_o = ky > 0 .$$

Eqs. (33) and (34) can be combined into the following single expression:

$$Q = \mp \frac{a_2-a_1}{k^2} \int_{\zeta_o}^{\infty} e^{\mp i(\zeta-\zeta_o)\sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{\zeta_o^2}{\zeta}\right)\left\{1+O(k^{-1})\right\} d\zeta \qquad (36)$$

and with the upper (lower) sign for $y > 0$ (< 0), and $\zeta_o = k|y|$.

Examination of (36) shows that we still have to treat the integral involving I. This is a trivial matter to the order in k that we require. Using (29), we have

$$\int_0^{\infty} I(t,\bar{a}) e^{iks(t,\bar{a})} K(t,y,a,\bar{a})dt$$

$$= \frac{1}{2k^2} \int_0^{\infty} I\left(\frac{\zeta}{k}, a_2\right) e^{iks\left(\frac{\zeta}{k}, a_2\right)} H_1^{(1)}(\zeta+\zeta_o)\left\{a_2(\zeta+\zeta_o) + (a-a_2)\frac{\zeta_o^2}{\zeta+\zeta_o}\right\}\left\{1+O(k^{-2})\right\} d\zeta$$

$$= O(k^{-2}) \qquad (37)$$

since I is $O(k^0)$. Similarly for the integral over the negative range of t, and hence

$$I(y) = \pm \frac{i}{2}(a_2-a_1)e^{-ik\{s(y,a)-y\sin\alpha\}} \int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\xi)\left(\zeta-\frac{\zeta_o^2}{\zeta}\right) d\zeta$$

$$+ O(k^{-1}) \qquad (38)$$

with the upper (lower) sign as before and $\zeta_o = k|y|$. This is in agreement with Eq. (46) of Weston (1962).

The perturbation field I(y) is all that we require to specify the total surface field u(y), and from Eq. (21), we have

$$u(y) = e^{iky\sin\alpha}\left\{U(y,a)\, e^{ia\frac{\zeta_o^2}{2k}\cos\alpha} \pm \frac{i}{2k}(a_2-a_1)\int_{\zeta_o}^{\infty} e^{\mp i\zeta\sin\alpha}\right.$$

$$\left. \cdot\, H_1^{(1)}(\zeta)\left(\zeta-\frac{\zeta_o^2}{\zeta}\right)d\zeta + O(k^{-2})\right\} . \qquad (39)$$

This proves adequate for the determination of the diffraction coefficient to the leading order in k .

3.3 Surface Field Behavior

Although Eq. (39) constitutes our main result for H polarization, it is of interest to examine the behavior of the total surface field u(y) in the immediate vicinity of the join, i.e. for $\zeta_o = k|y| << 1$.

Consider

$$\int_{\zeta_o}^{\infty} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{\zeta_o^2}{\zeta}\right) d\zeta = L_1(\zeta_o) - \zeta_o^2 L_{-1}(\zeta_o) \tag{40}$$

where

$$L_1(\zeta_o) = \int_{\zeta_o}^{\infty} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta)\, \zeta\, d\zeta \tag{41}$$

$$L_{-1}(\zeta_o) = \int_{\zeta_o}^{\infty} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta) \frac{d\zeta}{\zeta} \quad . \tag{42}$$

Since the integral expression for $L_1(\zeta_o)$ is convergent even for $\zeta_o=0$, we can write

$$L_1(\zeta_o) = L_1(0) - \int_0^{\zeta_o} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta)\, \zeta\, d\zeta \quad .$$

But

$$\begin{aligned} L_1(0) &= \int_0^{\infty} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta)\, \zeta\, d\zeta \\ &= \sec^3\alpha \left\{ 1 \pm \frac{2}{\pi} (\sin\alpha\cos\alpha + \alpha) \right\} \end{aligned} \tag{43}$$

as shown in Appendix B (Eq. B.3). Moreover

$$\begin{aligned} H_1^{(1)}(\zeta) &= -\frac{2i}{\pi\zeta} + \left(1 + \frac{2i}{\pi}\log\frac{\zeta}{2}\right) J_1(\zeta) \\ &\quad - \frac{i\zeta}{2\pi} \sum_{m=o}^{\infty} \frac{(-1)^m \left(\frac{\zeta}{2}\right)^{2m}}{m!\,(m+1)!} \left\{ \psi(m) + \psi(m+1) \right\} \end{aligned} \tag{44}$$

(see Watson, 1948, p. 62), so that

$$H_1^{(1)}(\zeta) = -\frac{2i}{\pi}\left(\frac{1}{\zeta} - \frac{\zeta}{2}\log\zeta\right) + O(\zeta, \zeta^3\log\zeta) . \tag{45}$$

Hence

$$\int_0^{\zeta_0} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\zeta\, d\zeta = -\frac{2i}{\pi}\zeta_0 + O(\zeta_0^2, \zeta_0^3\log\zeta_0) ,$$

giving

$$L_1(\zeta_0) = \sec^3\alpha\left\{1 \pm \frac{2}{\pi}(\sin\alpha\cos\alpha + \alpha)\right\} + \frac{2i}{\pi}\zeta_0 + O(\zeta_0^2, \zeta_0^3\log\zeta_0) . \tag{46}$$

The treatment of the integral expression for $L_{-1}(\zeta_0)$ is a little more complex due to the failure of the integral to converge when $\zeta_0=0$. Let us therefore write

$$L_{-1}(\zeta_0) = \int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha}\left\{H_1^{(1)}(\zeta) + \frac{2i}{\pi\zeta}\right\}\frac{d\zeta}{\zeta} - \frac{2i}{\pi}\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha}\frac{d\zeta}{\zeta^2} .$$

As evident from Eq. (45), the first integral now converges even for $\zeta_0 = 0$, implying

$$L_{-1}(\zeta_0) = O(1) - \int_0^{\zeta_0} e^{\mp i\zeta\sin\alpha}\left\{H_1^{(1)}(\zeta) + \frac{2i}{\pi\zeta}\right\}\frac{d\zeta}{\zeta} - \frac{2i}{\pi}\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha}\frac{d\zeta}{\zeta^2} .$$

But

$$\int_0^{\zeta_0} e^{\mp i\zeta\sin\alpha}\left\{H_1^{(1)}(\zeta) + \frac{2i}{\pi\zeta}\right\}\frac{d\zeta}{\zeta} = 0(\zeta_0, \zeta_0\log\zeta_0)$$

and from integration by parts

$$\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha}\frac{d\zeta}{\zeta^2} = \left(\frac{1}{\zeta_0} \pm i\sin\alpha\log\zeta_0\right) e^{\mp i\zeta_0\sin\alpha}$$

$$+\sin^2\alpha\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha}\log\zeta\, d\zeta \quad .$$

Since this last integral converges even for $\zeta_0=0$, it is $O(1)$, and we now have

$$L_{-1}(\zeta_0) = -\frac{2i}{\pi}\left(\frac{1}{\zeta_0} \pm i\ \sin\alpha\log\zeta_0\right) e^{\mp i\zeta_0\sin\alpha} + O(1,\zeta_0\log\zeta_0)$$

$$= -\frac{2i}{\pi}\left(\frac{1}{\zeta_0} \pm i\sin\alpha\log\zeta_0\right) + O\,(1,\zeta_0\log\zeta_0) \quad . \tag{47}$$

Hence, on substituting the expressions for $L_1(\zeta_0)$ and $L_{-1}(\zeta_0)$ into Eq. (40), we obtain

$$\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{\zeta_0^2}{\zeta}\right) d\zeta = \sec^3\alpha\left\{1 \pm \frac{2}{\pi}(\sin\alpha\cos\alpha+\alpha)\right\}$$

$$+\frac{4i}{\pi}\zeta_0 \mp \frac{2}{\pi}\sin\alpha\,\zeta_0^2\log\zeta_0 + O\,(\zeta_0^2,\ \zeta_0^3\log\zeta_0) \quad . \tag{48}$$

It is evident that the integral and its first derivative with respect to ζ_0 are finite at the join ($\zeta_0 = 0$), but the second derivative is infinite there. The same is therefore true of $I(y)$.

To provide the analogous results for the total surface field $u(y)$, it is necessary to examine the leading term on the right hand side of Eq. (39) for small ζ_0.

From Eq. (19),

$$U(y,a)e^{ia\frac{\zeta_o^2}{2k}\cos\alpha} = 2-\frac{ia}{k}\,(\sec^3\alpha-\zeta_o^2\cos\alpha)+O(k^{-2})$$

$$=2-\frac{i}{2k}\left\{(a_2+a_1)\pm(a_2-a_1)\right\}(\sec^3\alpha-\zeta_o^2\cos\alpha)+O(k^{-2}) \qquad (49)$$

where the upper (lower) sign again refers to $y>0$ (<0). Hence, to the first two orders in k,

$$u(y)=e^{iky\sin\alpha}\left\{2-\frac{i}{2k}(a_2+a_1)\sec^3\alpha+\frac{i}{\pi k}(a_2-a_1)\sec^3\alpha\,(\sin\alpha\cos\alpha+\alpha)\right.$$

$$\left.\mp\frac{2}{\pi k}(a_2-a_1)\zeta_o-\frac{i}{\pi k}(a_2-a_1)\sin\alpha\,\zeta_o^2\log\zeta_o+O(\zeta_o^2,\zeta_o^3\log\zeta_o)\right\} \qquad (50)$$

and since

$$\mp\frac{\zeta_o}{k} = -y$$

it is apparent that $u(y)$ is continuous at $y=0$, i.e.

$$u(0+)=u(0-) \qquad (51)$$

as is $u'(y)$, i.e.

$$u'(0+)=u'(0-)\,, \qquad (52)$$

but $u''(y)$ is infinite at $y=0$, unless $\alpha=0$. Indeed

$$u''(0)=-\frac{2ik}{\pi}(a_2-a_1)\sin\alpha\lim_{y\to 0}\log|y|\,,$$

and thus a discontinuity in curvature is, for this polarization, characterized by a surface field 'singularity' of the form $y^2\log|y|$.

4. E Polarization

We now turn to the case in which the incident field is the E polarized plane wave of Eq. (6). As previously remarked, the problem of determining the surface field can be expressed as a soft body one.

4.1 The Integral Equation

Maue's integral equation for an acoustically soft surface is

$$\frac{\partial}{\partial n_P} u(P) = 2 \frac{\partial}{\partial n_P} u_o(P) - \frac{1}{2\pi} \int \frac{\partial}{\partial n_Q} u(Q) \frac{\partial G}{\partial n_P} dS_Q \tag{53}$$

where G is as defined in Eq. (9), and since the two-dimensional nature of the problem again enables us to carry out the z integration directly, Eq. (53) can be reduced to

$$\frac{\partial}{\partial n_P} u(P) = 2 \frac{\partial}{\partial n_P} u_o(P) - \frac{i}{2} \int \frac{\partial}{\partial n_Q} u(Q) \frac{\partial}{\partial n_P} H_o^{(1)}(kr)\, ds_Q \tag{54}$$

where r is as given in Eq. (12).

With the variable of integration t defined as before, Eqs. (14) and (15) again apply. Also,

$$\hat{n}_P = (\hat{x} + a\, y\, \hat{y})(1 + a^2 t^2)^{-1/2}$$

and hence

$$\frac{\partial r}{\partial n_P} = \frac{1}{r} \left\{ (x_P - x_Q) + ay\,(y_P - y_Q) \right\} (1 + a^2 y^2)^{-1/2} ,$$

giving

$$\frac{\partial}{\partial n_P} H_o^{(1)}(kr) = -\frac{k}{r}\left\{ay(y-t) + \frac{1}{2}(\bar{a}t^2 - ay^2)\right\}\frac{H_1^{(1)}(kr)}{(1+a^2y^2)^{1/2}} \quad .$$

The integral equation is now

$$\frac{\partial}{\partial n_P}u(P) = 2\frac{\partial}{\partial n_P}u_o(P) + \frac{ik}{2}\int_{-\infty}^{\infty}\frac{\partial}{\partial n_Q}u(Q)\left\{ay(y-t) + \frac{1}{2}(\bar{a}t^2 - ay^2)\right\}H_1^{(1)}(kr) \cdot \left\{\frac{1+\bar{a}^2t^2}{1+a^2y^2}\right\}^{1/2}\frac{dt}{r} \tag{55}$$

and if we define

$$(1+a^2y^2)^{1/2}\frac{\partial}{\partial n_P}u(P) = v(y) \ , \tag{56}$$

$$(1+a^2y^2)^{1/2}\frac{\partial}{\partial n_P}u_o(P) = v_o(y) \ , \tag{57}$$

so that

$$(1+\bar{a}^2t^2)^{1/2}\frac{\partial}{\partial n_Q}u(Q) = v(t) \ , \tag{58}$$

then

$$v(y) = 2\,v_o(y) - \frac{ik}{2}\int_{-\infty}^{\infty} v(t)H_1^{(1)}(kr)\left\{ay(t-y) - \frac{1}{2}(\bar{a}t^2 - ay^2)\right\}\frac{dt}{r} \ , \tag{59}$$

cf Eq. (16), where r is as given in Eq. (17). For future reference we note that Eqs. (7) and (57) imply

$$v_o(y) = -ik(\cos\alpha - ay\sin\alpha)e^{iky(\sin\alpha + \frac{a}{2}y\cos\alpha)} \quad . \tag{60}$$

4.2 Asymptotic Solution

We again postulate a representation of the field v(y) as the sum of that which would exist on a complete parabolic cylinder and a perturbation term originating at the discontinuity, viz.

$$v(y) = -ik\left\{V(y,a)e^{ikf(y,a)} + \frac{1}{k} J(y,a)e^{iks(y,a)}\right\} . \tag{61}$$

The first term on the right hand side is that associated with a complete cylinder, and as shown in Appendix A, Eq. (A.15),

$$V(y,a) = (\cos\alpha - ay\sin\alpha)\left\{2 + \frac{ia}{k}(\cos\alpha - ay\sin\alpha)^{-3} + O(k^{-2})\right\} , \tag{62}$$

$$f(y,a) = y\sin\alpha + \frac{1}{2}ay^2\cos\alpha . \tag{63}$$

The second term in (61) is that due to the join. Since

$$s(y,a) = \int_0^{|y|} (1+a^2\tau^2)^{1/2}\, d\tau , \tag{64}$$

the only unknown quantity is J(y, a), and this is evidently $O(k^0)$ for small y.

If we now substitute the expression (61) for v(y) into the integral equation (59) and, for convenience, write

$$\frac{H_1^{(1)}(kr)}{r}\left\{ay(t-y) - \frac{1}{2}(\bar{a}t^2 - ay^2)\right\} = \tilde{K}(t,y,a,\bar{a}) , \tag{65}$$

we obtain

$$-ikV(y,a)e^{ikf(y,a)} - iJ(y,a)e^{iks(y,a)} = 2v_o(y)$$

$$-\frac{k^2}{2}\int_{-\infty}^{\infty} V(t,\bar{a})e^{ikf(t,\bar{a})}\tilde{K}(t,y,a,\bar{a})\,dt$$

$$-\frac{k}{2}\int_{-\infty}^{\infty} J(t,\bar{a})e^{iks(t,\bar{a})}\tilde{K}(t,y,a,\bar{a})\,dt$$

which can be recast in the form

$$-ikV(y,a)e^{ikf(y,a)} - iJ(y,a)e^{iks(y,a)} = 2v_o(y)$$

$$-\frac{k^2}{2}\int_{-\infty}^{\infty}\left\{V(t,\bar{a})e^{ikf(t,\bar{a})}\tilde{K}(t,y,a,\bar{a}) - V(t,a)e^{ikf(t,a)}\tilde{K}(t,y,a,a)\right\}dt$$

$$-\frac{k^2}{2}\int_{-\infty}^{\infty} V(t,a)e^{ikf(t,a)}\tilde{K}(t,y,a,\bar{a})\,dt - \frac{k}{2}\int_{-\infty}^{\infty} J(t,\bar{a})e^{iks(t,\bar{a})}\tilde{K}(t,y,a,\bar{a})\,dt \;. \tag{66}$$

But $-ikV(y,a)e^{ikf(y,a)}$ is, by definition, the solution for a single parabolic cylinder formed by continuing the portion on which the observation point lies. Hence

$$-ikV(y,a)e^{ikf(y,a)} = 2v_o(y) - \frac{k^2}{2}\int_{-\infty}^{\infty} V(t,a)e^{ikf(t,a)}\tilde{K}(t,y,a,\bar{a})\,dt \tag{67}$$

implying

$$J(y, a)e^{iks(y, a)} = -\frac{ik}{2}\int_{-\infty}^{\infty} J(t, \bar{a})\, e^{iks(t, \bar{a})}\, \tilde{K}(t, y, a, \bar{a})dt - \frac{ik^2}{2}\tilde{Q} \tag{68}$$

where

$$\tilde{Q} = \int_{-\infty}^{\infty}\left\{V(t, \bar{a})e^{ikf(t, \bar{a})}\tilde{K}(t, y, a, \bar{a}) - V(t, a)e^{ikf(t, a)}\tilde{K}(t, y, a, a)\right\}dt\,. \tag{69}$$

The asymptotic evaluation of $\tilde{Q}$ is directly analogous to that for the quantity Q given in Section 3.2. If $y < 0$, then

$$\tilde{Q} = \int_{0}^{\infty}\left\{V(t, a_2)e^{ikf(t, a_2)}\tilde{K}(t, y, a_1, a_2) - V(t, a_1)e^{ikf(t, a_1)}\tilde{K}(t, y, a_1, a_1)\right\}dt \tag{70}$$

and writing

$$kt = \zeta > 0\,, \qquad -ky = \zeta_0 > 0\,,$$

we have

$$\tilde{K}(t, y, a_1, a_2) = -\frac{1}{2k}\, H_1^{(1)}(\zeta+\zeta_0)\left\{a_1(\zeta+\zeta_0) + (a_2 - a_1)\frac{\zeta^2}{\zeta+\zeta_0}\right\}\left\{1 + O(k^{-2})\right\}\,,$$

$$\tilde{K}(t, y, a_1, a_1) = -\frac{1}{2k}H_1^{(1)}(\zeta+\zeta_0)a_1(\zeta+\zeta_0)\left\{1 + O(k^{-2})\right\}\,.$$

Also

$$V(t, a_2) = 2\cos\alpha + O(k^{-1})$$

$$= V(t, a_1)$$

and since

$$e^{ikf(t,a_2)} = e^{i\zeta\sin\alpha}\left\{1+O(k^{-1})\right\} = e^{ikf(t,a_1)} ,$$

it follows that, for $y < 0$,

$$\tilde{Q} = -\frac{a_2-a_1}{k^2}\cos\alpha\int_0^\infty e^{i\zeta\sin\alpha} H_1^{(1)}(\zeta+\zeta_o)\frac{\zeta^2}{\zeta+\zeta_o}\left\{1+O(k^{-1})\right\}d\zeta$$

which, by a change of variable of integration, becomes

$$\tilde{Q} = -\frac{a_2-a_1}{k^2}\cos\alpha\int_{\zeta_o}^\infty e^{i(\zeta-\zeta_o)\sin\alpha} H_1^{(1)}(\zeta)\frac{(\zeta-\zeta_o)^2}{\zeta}\left\{1+O(k^{-1})\right\}d\zeta \quad (71)$$

where $\zeta_o = -ky > 0$.

The procedure for $y > 0$ is rather similar. The integration in the expression for Q now extends from $t = -\infty$ to $t = 0$, and writing

$$kt = -\zeta < 0, \qquad ky = \zeta_o > 0 ,$$

the expression can be reduced to

$$\tilde{Q} = \frac{a_2-a_1}{k^2}\cos\alpha\int_{\zeta_o}^\infty e^{-i(\zeta-\zeta_o)\sin\alpha} H_1^{(1)}(\zeta)\frac{(\zeta-\zeta_o)^2}{\zeta}\left\{1+O(k^{-1})\right\}d\zeta \ . \quad (72)$$

Equations (71) and (72) can be combined into the single result

$$\tilde{Q} = \pm\frac{a_2-a_1}{k^2}\cos\alpha\int_{\zeta_o}^\infty e^{\mp i(\zeta-\zeta_o)\sin\alpha} H_1^{(1)}(\zeta)\frac{(\zeta-\zeta_o)^2}{\zeta}\left\{1+O(k^{-1})\right\}d\zeta \quad (73)$$

with the upper (lower) sign for $y > 0$ (< 0), and $\zeta_o = k|y|$.

Examination of Eq. (68) shows that we still have to estimate the integral involving $J(t, \bar{a})$. However, it is trivial to prove that

$$\int_{-\infty}^{\infty} J(t, \bar{a}) e^{iks(t, \bar{a})} \tilde{K}(t, y, a, \bar{a})\, dt = O(k^{-2})$$

using the fact that $J(t, \bar{a}) = O(k^0)$, and hence

$$J(y) = \mp \frac{i}{2} (a_2 - a_1) \cos\alpha\, e^{-ik\left\{s(y, a) - y \sin\alpha\right\}} \int_{\zeta_o}^{\infty} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta) \frac{(\zeta - \zeta_o)^2}{\zeta} d\zeta + O(k^{-1}). \tag{74}$$

The perturbation field $J(y)$ is all that is required to specify the total field quantity $v(y)$, and from Eq. (61) we now have

$$v(y) = -ike^{iky\sin\alpha}\left\{V(y, a) e^{ia \frac{\zeta_o^2}{2k} \cos\alpha} \mp \frac{i}{2k} (a_2 - a_1) \cos\alpha \cdot \right.$$
$$\left. \cdot \int_{\zeta_o}^{\infty} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta) \frac{(\zeta - \zeta_o)^2}{\zeta} d\zeta + O(k^{-2}) \right\}. \tag{75}$$

This is our main result for E polarization, and the form is strikingly similar to that given in Eq. (39).

4.3 Surface Field Behavior

It is convenient to proceed as we did in the case of H polarization, and to examine the behavior of the surface field in the immediate vicinity of the join, i.e. for $\zeta_o << 1$.

Consider

$$\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\frac{(\zeta-\zeta_0)^2}{\zeta}\,d\zeta = L_1(\zeta_0)-2\zeta_0 L_0(\zeta_0)+\zeta_0^2 L_{-1}(\zeta_0) \tag{76}$$

where

$$L_1(\zeta_0)=\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\zeta\,d\zeta\ , \tag{77}$$

$$L_0(\zeta_0)=\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,d\zeta\ , \tag{78}$$

$$L_{-1}(\zeta_0)=\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,\frac{d\zeta}{\zeta}\ . \tag{79}$$

Since the expansions of $L_1(\zeta_0)$ and $L_{-1}(\zeta_0)$ have already been determined (see Eqs. (43) and (47)), it is only necessary to examine $L_0(\zeta)$, and though the integral does not converge when $\zeta_0=0$, we have

$$L_0(\zeta_0)= -\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_0^{(1)'}(\zeta)\,d\zeta$$

$$= -\left[e^{\mp i\zeta\sin\alpha} H_0^{(1)}(\zeta)\right]_{\zeta_0}^{\infty} \mp i\sin\alpha\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_0^{(1)}(\zeta)\,d\zeta$$

on integrating by parts. This new integral is convergent when $\zeta_o=0$ and writing

$$\int_{\zeta_o}^{\infty} e^{\mp i\zeta\sin\alpha} H_o^{(1)}(\zeta)d\zeta = \left(\int_0^{\infty} - \int_0^{\zeta_o}\right) e^{\mp i\zeta\sin\alpha} H_o^{(1)}(\zeta)\, d\zeta$$

we can use Eq. (B.2) of Appendix B to show

$$\int_0^{\infty} e^{\mp i\zeta\sin\alpha} H_o^{(1)}(\zeta)\, d\zeta = \sec\alpha\left(1 \pm \frac{2\alpha}{\pi}\right).$$

Moreover,

$$H_o^{(1)}(\zeta) = \left\{1 + \frac{2i}{\pi}\log\frac{\zeta}{2}\right\} J_o(\zeta) - \frac{2i}{\pi}\sum_{m=o}^{\infty} \frac{(-1)^m\left(\frac{\zeta}{2}\right)^{2m}}{(m!)^2}\,\psi(m+1)$$

$$= 1 + \frac{2i}{\pi}\left(\gamma + \log\frac{\zeta}{2}\right) + O(\zeta^2, \zeta^2\log\zeta)$$

and therefore

$$\int_0^{\zeta_o} e^{\mp i\zeta\sin\alpha} H_o^{(1)}(\zeta)\, d\zeta = O(\zeta_o, \zeta_o\log\zeta_o).$$

Hence

$$L_o(\zeta_o) = 1 + \frac{2i}{\pi}\left(\gamma + \frac{\zeta_o}{2}\right) \pm i\left(1 \pm \frac{2\alpha}{\pi}\right)\tan\alpha + O(\zeta_o, \zeta_o\log\zeta_o)$$

and on inserting the results of Eqs. (43), (47) and (80) into Eq. (76), we find

$$\int_{\zeta_0}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,\frac{(\zeta-\zeta_0)^2}{\zeta}\,d\zeta = \sec^3\alpha\left\{1 \pm \frac{2}{\pi}(\sin\alpha\cos\alpha + \alpha)\right\}$$

$$-2\zeta_0\left\{1 + \frac{2i\gamma}{\pi} + i\left(\pm 1 + \frac{2\alpha}{\pi}\right)\tan\alpha + \frac{2i}{\pi}\log\zeta_0\right\} + O(\zeta_0^2,\ \zeta_0^2\log\zeta_0)\ . \qquad (81)$$

It is evident that the integral is finite at the join ($\zeta_0 = 0$), but that its first derivative with respect to ζ_0 is infinite there. The same is therefore true of $J(y)$.

To provide the analogous results for the total field $v(y)$, it is necessary to examine the leading term on the right hand side of Eq. (75) for small ζ_0. From Eq. (62),

$$V(y,a)e^{ia\frac{\zeta_0^2}{2k}\cos\alpha} = 2\cos\alpha + \frac{ia}{k}(\sec^2\alpha + 2i\zeta_0\sin\alpha + \zeta_0^2\cos^2\alpha) + O(k^{-2})$$

$$= 2\cos\alpha + \frac{i}{2k}\left\{(a_2+a_1) \pm (a_2-a_1)\right\}(\sec^2\alpha + 2i\zeta_0\sin\alpha$$

$$+ \zeta_0^2\cos^2\alpha) + O(k^{-2}) \qquad (82)$$

where the upper (lower) sign again refers to $y > 0$ (< 0). Hence, to the first two orders in k,

$$v(y) = -ike^{iky\sin\alpha}\left\{2\cos\alpha + \frac{i}{2k}(a_2-a_1)\sec^2\alpha - \frac{i}{\pi k}(a_2-a_1)\sec^2\alpha\right.$$

$$\left.\cdot(\sin\alpha\cos\alpha + \alpha) \mp \frac{2}{\pi k}(a_2-a_1)\cos\alpha\ \zeta_0\log\zeta_0 + O(\zeta_0,\ \zeta_0^2\log\zeta_0)\right\}. \qquad (83)$$

It is obvious that v(y) is continuous at y = 0, i.e.

$$v(0+) = v(0-)\,, \tag{84}$$

but v'(y) is infinite at y = 0 unless $|\alpha| = \pi/2$, which values have been excluded. In fact, since

$$\mp \frac{\zeta_o}{k} = -y\,,$$

we have

$$v'(0) = \frac{2ik}{\pi}(a_2 - a_1)\cos\alpha \lim_{y \to 0} \log|y|\,,$$

so that a discontinuity in curvature is, for this polarization, characterized by a field 'singularity' of the form $y\log|y|$. This should be compared with the behavior $y^2\log|y|$ found for H polarization.

5. Diffraction Coefficient for H Polarization

5.1 General Expression

For a two dimensional geometry such as that shown in Fig. 1, the scattered magnetic field is

$$H_z^s(\underline{r}') = \frac{i}{4}\int_S H_z(\underline{r})\frac{\partial}{\partial n}H_o^{(1)}(k|\underline{r}-\underline{r}'|)dS \tag{85}$$

where

$$\underline{r}' = x'\hat{x} + y'\hat{y}$$

denotes the point of observation, and

$$\underline{r} = x\hat{x} + y\hat{y}$$

represents a variable point of integration.

If $R = |\underline{r}-\underline{r}'|$, then

$$\frac{\partial}{\partial n} = \left(n_x \frac{x-x'}{R} + n_y \frac{y-y'}{R}\right)\frac{\partial}{\partial R} \sim -(n_x \cos\theta + n_y \sin)\frac{\partial}{\partial R}$$

at large distances, where $x' = r'\cos\theta$ and $y' = r'\sin\theta$. Hence

$$\frac{\partial}{\partial n} H_o^{(1)}(kR) \sim -ik(n_x\cos\theta + n_y\sin\theta)\sqrt{\frac{2}{\pi kR}}\; e^{ikR - i\frac{\pi}{4}}$$

$$\sim -ik(n_x\cos\theta + n_y\sin\theta)\sqrt{\frac{2}{\pi kr'}}\; e^{ikr' - i\frac{\pi}{4} - ik\hat{r}'\cdot\underline{r}}.$$

implying that in the far zone

$$H_z^s(\underline{r}') \sim \sqrt{\frac{2}{\pi kr'}}\; e^{ikr' - i\frac{\pi}{4}}\; \frac{k}{4}\int_S (n_x\cos\theta + n_y\sin\theta) H_z(\underline{r}) e^{-ik\hat{r}'\cdot\underline{r}}\, dS\,.$$

In the notation of Keller (1962), the diffraction coefficient D_H for H polarization is

$$\sqrt{r'}\, e^{-ikr'}\, H_z^s(\underline{r}')\,,$$

so that

$$D_H = \sqrt{\frac{2}{\pi k}}\; e^{-i\frac{\pi}{4}}\; \frac{k}{4}\int_S (n_x\cos\theta + n_y\sin\theta) H_z(\underline{r}) e^{-ik\hat{r}'\cdot\underline{r}}\, dS\,,$$

but it proves more convenient to work with the far field amplitude P as defined by Bowman et al (1970), namely

$$P_H = \frac{k}{4}\int_S (n_x\cos\theta + n_y\sin\theta) H_z(\underline{r})\, e^{-ik\hat{r}'\cdot\underline{r}}\, dS\,, \qquad (86)$$

in terms of which

$$D_H = \sqrt{\frac{2}{\pi k}}\, e^{-i\frac{\pi}{4}}\, P_H \, . \tag{87}$$

For the specific geometry of Fig. 1,

$$\hat{n} = (\hat{x} + a y \hat{y}) \left\{1 + (ay)^2\right\}^{-1/2}$$

and

$$dS = \left\{1 + (ay)^2\right\}^{1/2} dy \, ,$$

so that

$$P_H = \frac{k}{4} \int_{-\ell}^{\ell} (\cos\theta + ay\sin\theta)\; u(y) e^{-iky\sin\theta + ika\frac{y^2}{2}\cos\theta}\, dy, \tag{88}$$

where we have introduced the symbol u(y) of Section 3 to denote the surface field component $H_z(\underline{r})$, and where the integration has been limited to some region about the join because our concern is only with the scattering that originates there. If we now introduce a new variable of integration, t = ky, the range of integration extends from $t = -k\ell$ to $t = k\ell$, and since k is, by assumption, large, the limits can be replaced by $\pm\infty$. Hence

$$P_H = \frac{1}{4} \int_{-\infty}^{\infty} (\cos\theta + \frac{1}{k}\, at\sin\theta)\; u(\frac{t}{k})\; e^{-it\sin\theta + ia\frac{t^2}{2k}\cos\theta}\, dt \, , \tag{89}$$

and this is the expression that must be evaluated.

5.2 Physical Optics

According to the physical optics approximation, the surface field throughout the illuminated region is

$$u(y) = 2\, e^{iky(\sin\alpha + \frac{a}{2} y \cos\alpha)} \, ,$$

and the restriction (later relaxed) that $\frac{\pi}{2} - |\alpha|$ is bounded away from zero is sufficient to ensure that a region about the join is directly illuminated. The corresponding value of P_H is therefore

$$P_H^{p.o.} = \frac{1}{2} \int_{-\infty}^{\infty} e^{it(\sin\alpha - \sin\theta)} (\cos\theta + \frac{1}{k}\, at \sin\theta) \cdot \exp\left\{ ia \frac{t^2}{2k} (\cos\alpha + \cos\theta) \right\} dt \;, \tag{90}$$

which is equivalent to retaining only the leading term in U(y, a) in the general expression (39) for u(y).

If we are to identify the join contribution, it is necessary that the specular point be away from the join, i.e. $|\alpha - \theta|$ bounded away from zero, and this restriction applies throughout the subsequent analysis. With

$$p = \sin\alpha - \sin\theta \;, \tag{91}$$

Eq. (90) becomes

$$P_H^{p.o.} = \frac{1}{2} \cos\theta \int_{-\infty}^{\infty} e^{ipt} \left[1 + \frac{a}{k} \left\{ t \tan\theta + \frac{it^2}{2} (\cos\alpha + \cos\theta) \right\} + O(k^{-2}) \right] dt \;.$$

But

$$\int_0^{\infty} t^n e^{ipt} dt = (\frac{i}{p})^{n+1}\, n\,!$$

and (92)

$$\int_{-\infty}^{0} t^n e^{ipt} dt = -\,(\frac{i}{p})^{n+1}\, n!$$

for $n \geq 0$. Hence

$$\int_{-\infty}^{\infty} e^{ipt}\, dt = 0 ,$$

and since $a = a_2$ $(t > 0)$, $a = a_1$ $(t < 0)$,

$$P_H^{p.o.} = \frac{a_2 - a_1}{2k} \cos\theta \left[\left(\frac{i}{p}\right)^2 \tan\theta + \frac{i}{2} \left(\frac{i}{p}\right)^3 2(\cos\alpha + \cos\theta) + O(k^{-1}) \right]$$

$$= \frac{a_2 - a_1}{2k} \frac{1 + \cos(\alpha+\theta)}{p^3} + O(k^{-2}) . \tag{93}$$

As we shall later see, this approximation to P_H is in error even in the leading term and is inadequate for many purposes.

5.3 Precise Evaluation

An expression for the total surface field u(y) that is correct through two orders in k was given in Eq. (39), and on expanding the leading term on the right hand side of this equation, we obtain

$$u\left(\frac{t}{k}\right) = 2e^{it\sin\alpha} \left\{ 1 - \frac{ia}{2k} (\sec^3\alpha - t^2 \cos\alpha) \pm \frac{i}{4k} (a_2 - a_1) L(t) + O(k^{-2}) \right\} , \tag{94}$$

where

$$L(t) = \int_{|t|}^{\infty} e^{\mp i \zeta \sin\alpha} H_1^{(1)}(\zeta) \left(\zeta - \frac{t^2}{\zeta}\right) d\zeta \tag{95}$$

with the upper (lower) sign according as $t > 0$ (< 0). The first term on the right hand side of Eq. (94) is, of course, the physical optics approximation.

On substituting the expression for $u(\frac{t}{k})$ into Eq. (89) and again expanding the quadratic factor in powers of k, we have

$$P_H = \frac{\cos\theta}{2}\int_{-\infty}^{\infty} e^{ipt}\left\{1-\frac{ia}{2k}\left(\sec^3\alpha+2it\tan\theta-t^2(\cos\alpha+\cos\theta)\right)\right.$$
$$\left.\pm\frac{i}{4k}(a_2-a_1)L(t)+O(k^{-2})\right\}dt$$

and using now the results (92), the expression for P_H becomes

$$P_H=\frac{\cos\theta}{2}\left(-\frac{i}{2k}\right)(a_2-a_1)\left\{\frac{i}{p}\sec^3\alpha+\left(\frac{i}{p}\right)^2 2i\tan\theta-2\left(\frac{i}{p}\right)^3(\cos\alpha+\cos\theta)\right\}$$
$$+\frac{i\cos\theta}{8k}(a_2-a_1)\left(\int_0^{\infty}-\int_{-\infty}^{0}\right)e^{ipt}L(t)dt+O(k^{-2})$$
$$=\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha+\theta)}{p^3}+\frac{a_2-a_1}{4k}\,\frac{\cos\theta\sec^3\alpha}{p}$$
$$+\frac{i\cos\theta}{8k}(a_2-a_1)\left(\int_0^{\infty}-\int_{-\infty}^{0}\right)e^{ipt}L(t)\,dt+O(k^{-2})\,.$$

Since the first term on the right hand side is simply $P_H^{p.o.}$, it follows that

$$P_H=P_H^{p.o.}+i\,\frac{a_2-a_1}{8k}\cos\theta\left\{-\frac{2i}{p}\sec^3\alpha+\left(\int_0^{\infty}-\int_{-\infty}^{0}\right)e^{ipt}L(t)dt\right\}+O(k^{-2}). \tag{96}$$

The only remaining task is to evaluate (precisely) the integrals containing L(t). We have

$$\left(\int_0^\infty - \int_{-\infty}^0\right) e^{ipt} L(t)dt = \int_0^\infty e^{ipt} \int_t^\infty e^{-i\zeta \sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{t^2}{\zeta}\right) d\zeta\, dt$$

$$- \int_{-\infty}^0 e^{ipt} \int_{-t}^\infty e^{i\zeta \sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{t^2}{\zeta}\right) d\zeta\, dt$$

$$= \int_0^\infty \left\{ e^{ipt} \int_t^\infty e^{-i\zeta \sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{t^2}{\zeta}\right) d\zeta - e^{-ipt} \int_t^\infty e^{i\zeta \sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{t^2}{\zeta}\right) d\zeta \right\} dt$$

$$= \int_0^\infty \left\{ e^{ipt} L^{(-)}(t) - e^{-ipt} L^{(+)}(t) \right\} dt$$

where

$$L^{(-)}(t) = \int_t^\infty e^{-i\zeta \sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{t^2}{\zeta}\right) d\zeta \quad ,$$

$$L^{(+)}(t) = \int_t^\infty e^{i\zeta \sin\alpha} H_1^{(1)}(\zeta)\left(\zeta - \frac{t^2}{\zeta}\right) d\zeta \ .$$

But

$$L^{(\mp)}(0) = \int_0^\infty e^{\mp i\zeta \sin\alpha} H_1^{(1)}(\zeta)\, \zeta\, d\zeta = \sec^3\alpha \left\{ 1 \mp \frac{2}{\pi}(\sin\alpha\cos\alpha + \alpha) \right\}$$

as shown in Appendix B, and

$$L^{(\mp)\prime}(t) = -2t \int_t^\infty e^{\mp i\zeta \sin\alpha} \frac{H_1^{(1)}(\zeta)}{\zeta} d\zeta$$

where the prime denotes differentiation with respect to t. Hence, on integration by parts,

$$\left(\int_0^\infty - \int_{-\infty}^0\right) e^{ipt} L(t)dt = \frac{1}{ip}\left[e^{ipt}L^{(-)}(t) + e^{-ipt}L^{(+)}(t)\right]_0^\infty$$

$$- \frac{1}{ip}\int_0^\infty \left\{e^{ipt}L^{(-)'}(t) + e^{-ipt}L^{(+)'}(t)\right\} dt$$

$$= \frac{i}{p}\left\{L^{(-)}(0)+L^{(+)}(0)\right\} + \frac{i}{p}\int_0^\infty \left\{e^{ipt}L^{(-)'}(t) + e^{-ipt}L^{(+)'}(t)\right\} dt$$

$$= \frac{2i}{p}\sec^3\alpha - \frac{2i}{p}\int_0^\infty \left\{ e^{ipt}\int_t^\infty e^{-i\zeta\sin\alpha}\,\frac{H_1^{(1)}(\zeta)}{\zeta}\,d\zeta\right.$$

$$\left. + e^{-ipt}\int_t^\infty e^{i\zeta\sin\alpha}\,\frac{H_1^{(1)}(\zeta)}{\zeta}\,d\zeta\right\} t\ dt$$

and when this is substituted into (96), the expression for P_H becomes

$$P_H = P_H^{p.o.} + \frac{a_2-a_1}{4kp}\cos\theta\int_0^\infty \left\{ e^{ipt}\int_t^\infty e^{-i\zeta\sin\alpha}\,\frac{H_1^{(1)}(\zeta)}{\zeta}\,d\zeta\right.$$

$$\left. + e^{-ipt}\int_t^\infty e^{i\zeta\sin\alpha}\,\frac{H_1^{(1)}(\zeta)}{\zeta}\,d\zeta\right\} t\,dt + O(k^{-2})\,. \qquad (97)$$

We now define $\gamma_+'(t)$ such that

$$\gamma_+'(t) = t\,e^{ipt}$$

with $\gamma_+(0)=0$. Thus,

$$\gamma_+(t)=\int_0^t \tau e^{ip\tau}d\tau = \frac{1}{p^2}\left\{(1-ipt)e^{ipt}-1\right\} , \tag{98}$$

and we note, that for small t, $\gamma_+(t)=\frac{1}{2}t^2+O(t^3)$. We likewise define

$$\gamma_-(t) = \frac{1}{p^2}\left\{(1+ipt)e^{-ipt}-1\right\} , \tag{99}$$

so that

$$\gamma_-'(t) = t\, e^{-ipt} .$$

Hence

$$\int_0^\infty e^{ipt}\int_t^\infty e^{-i\zeta\sin\alpha}\frac{H_1^{(1)}(\zeta)}{\zeta}d\zeta\, t\, dt=\int_0^\infty \gamma_+'(t)\int_t^\infty e^{-i\zeta\sin\alpha}\frac{H_1^{(1)}(\zeta)}{\zeta}d\zeta\, dt$$

$$= \left[\gamma_+(t)\int_t^\infty e^{-i\zeta\sin\alpha}\frac{H_1^{(1)}(\zeta)}{\zeta}d\zeta\right]_0^\infty + \int_0^\infty e^{-it\sin\alpha}\gamma_+(t)\frac{H_1^{(1)}(t)}{t}dt , \tag{100}$$

using integration by parts again. But

$$\int_t^\infty e^{-i\zeta\sin\alpha}\frac{H_1^{(1)}(\zeta)}{\zeta}d\zeta = O(t^{-1})$$

for small t, implying

$$\lim_{t\to 0}\ \gamma_+(t)\int_t^\infty e^{-i\zeta\sin\alpha}\frac{H_1^{(1)}(\zeta)}{\zeta}d\zeta = 0 .$$

Thus, the lower limit, t=0, provides no contribution to the leading term on the right hand side of (100), and since the upper limit also contributes zero,

$$\int_0^{\infty} e^{ipt} \int_t^{\infty} e^{-i\zeta \sin\alpha} \frac{H_1^{(1)}(\zeta)}{\zeta} d\zeta\, t\, dt = \int_0^{\infty} e^{-it \sin\alpha} \gamma_+(t) \frac{H_1^{(1)}(t)}{t} dt \,, \quad (101)$$

$$\int_0^{\infty} e^{-ipt} \int_t^{\infty} e^{i\zeta \sin\alpha} \frac{H_1^{(1)}(\zeta)}{\zeta} d\zeta\; t\, dt = \int_0^{\infty} e^{it \sin\alpha} \gamma_-(t) \frac{H_1^{(1)}(t)}{t} dt, \quad (102)$$

giving

$$P_H = P_H^{p.o.} + \frac{a_2 - a_1}{4kp} \cos\theta \int_0^{\infty} \left\{ e^{-it\sin\alpha} \gamma_+(t) + e^{it\sin\alpha} \gamma_-(t) \right\} \frac{H_1^{(1)}(t)}{t} dt + O(k^{-2}) \,. \quad (103)$$

We have now reached the last step. From the expressions for $\gamma_+(t)$ and p given in Eqs. (98) and (91) respectively,

$$e^{-it \sin\alpha} \gamma_+(t) = \frac{1}{p^2} \left\{ (1 - ipt) e^{-it \sin\theta} - e^{-it \sin\alpha} \right\} .$$

A similar result holds for $e^{it \sin\alpha} \gamma_-(t)$, and hence

$$\int_0^{\infty} \left\{ e^{-it\sin\alpha} \gamma_+(t) + e^{it\sin\alpha} \gamma_-(t) \right\} \frac{H_1^{(1)}(t)}{t} dt = \frac{i}{p} \int_0^{\infty} (e^{it\sin\theta} - e^{-it\sin\theta}) H_1^{(1)}(t) dt$$

$$+ \frac{1}{p^2} \int_0^{\infty} \left\{ (e^{it\sin\theta} + e^{-it\sin\theta}) - (e^{it\sin\alpha} + e^{-it\sin\alpha}) \right\} \frac{H_1^{(1)}(t)}{t} dt$$

$$= -\frac{2}{p} \tan\theta + \frac{2}{p^2} (\cos\theta - \cos\alpha) = \frac{2}{p^2} \frac{1 - \cos(\alpha - \theta)}{\cos\theta} \,, \quad (104)$$

where we have used the fact that

$$\int_0^{\infty} \sin (t \sin \gamma) H_1^{(1)}(t) dt = \tan \gamma \ , \qquad |\gamma| < \frac{\pi}{2} \tag{105}$$

(see Eq. (B. 1) in the limit $\nu \rightarrow 1$) and

$$\int_0^{\infty} \left\{ \sin(t \sin \gamma_1) - \sin(t \sin \gamma_2) \right\} \frac{H_1^{(1)}(t)}{t} \, dt = i \, (\cos \gamma_2 - \cos \gamma_1) \tag{106}$$

(which follows, formally at least, on multiplying Eq. (105) through by i cos γ and integrating with respect to γ).

The final expression for P_H is therefore

$$P_H = P_H^{p.o.} + \frac{a_2 - a_1}{2k} \, \frac{1 - \cos(\alpha - \theta)}{p^3} + O(k^{-2}) \ . \tag{107}$$

The difference from the physical optics result is quite apparent, but we postpone exploration of this solution until after consideration of the analogous problem of E polarization.

6. Diffraction Coefficient for E Polarization

6.1 General Expression

The derivation of the general expression for the diffraction coefficient so closely parallels that in Section 5.1 that only the briefest of comments are required.

In place of Eq. (85) we now have

$$E_z^s(\underline{r}') = -\frac{i}{4} \int_S \frac{\partial}{\partial n} E_z(\underline{r}) \, H_0^{(1)}(k |\underline{r} - \underline{r}'|) \, dS \ . \tag{108}$$

On proceeding to the far field and defining the diffraction coefficient D_E for E polarization as

$$\sqrt{r'}\, e^{-ikr'} E_z^s(\underline{r}') ,$$

it is found that

$$D_E = -\sqrt{\frac{2}{\pi k}}\, e^{-i\frac{\pi}{4}} \frac{i}{4} \int_S \frac{\partial E_z}{\partial n} e^{-ik\hat{r}' \cdot \underline{r}} dS ,$$

but for convenience we shall again work with the far field amplitude

$$P_E = -\frac{i}{4} \int_S \frac{\partial E_z}{\partial n} e^{-ik\hat{r}' \cdot \underline{r}} dS , \tag{109}$$

in terms of which

$$D_E = \sqrt{\frac{2}{\pi k}}\, e^{-i\frac{\pi}{4}} P_E . \tag{110}$$

For the specific geometry of Fig. 1,

$$P_E = -\frac{i}{4} \int_{-\ell}^{\ell} \frac{\partial E_z}{\partial n} e^{-iky\sin\theta + ika\frac{y^2}{2}\cos\theta} \left\{1+(ay)^2\right\}^{1/2} dy$$

and on introducing a new variable of integration t = ky, together with the quantity v(y) defined in Section 4, the above equation becomes

$$P_E = -\frac{i}{4k} \int_{-\infty}^{\infty} v\left(\frac{t}{k}\right) e^{-it\sin\theta + ia\frac{t^2}{2k}\cos\theta} dt . \tag{111}$$

This is the expression that must be evaluated.

6.2 Physical Optics

The physical optics approximation postulates that over the illuminated portion of the surface

$$\frac{\partial E_z}{\partial n} = 2\,\frac{\partial E_z^i}{\partial n}$$

and hence, for the incident field of Eq. (7),

$$v(y) = -2ik(\cos\alpha - ay\sin\alpha)\,e^{iky(\sin\alpha + \frac{a}{2} y\cos\alpha)} ,$$

which is equivalent to retaining only the leading term in V(y, a) in the general expression (61) for v(y). It follows that

$$P_E^{p.o.} = -\frac{1}{2}\int_{-\infty}^{\infty} e^{it(\sin\alpha - \sin\theta)}(\cos\alpha - a\frac{t}{k}\sin\alpha)\exp\left\{ia\frac{t^2}{2k}(\cos\alpha+\cos\theta)\right\}dt \tag{112}$$

and with p defined as before, Eq. (112) can be written as

$$\begin{aligned} P_E^{p.o.} &= -\frac{1}{2}\cos\alpha\int_{-\infty}^{\infty} e^{ipt}\left[1 - \frac{a}{k}\left\{t\tan\alpha - \frac{it^2}{2}(\cos\alpha+\cos\theta)\right\} + O(k^{-2})\right] dt \\ &= \frac{a_2 - a_1}{2k}\cos\alpha\left[\left(\frac{i}{p}\right)^2\tan\alpha - \frac{i}{2}\left(\frac{i}{p}\right)^3 2(\cos\alpha+\cos\theta) + O(k^{-1})\right] \\ &= -\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha+\theta)}{p^3} + O(k^{-2}) , \end{aligned} \tag{113}$$

so that

$$P_E^{p.o.} = -\,P_H^{p.o.}$$

as expected. This is consistent with the known fact (Knott and Senior, 1971) that the physical optics approximation is intrinsically polarization independent.

6.3 Precise Evaluation

An expression for v(y) that is correct through two orders in k was given in Eq. (61) and if we expand the leading term on the right hand side of this equation in powers of k, we obtain

$$v(\frac{t}{k}) = -2ik\cos\alpha\, e^{it\sin\alpha}\left\{1+\frac{ia}{2k}(\sec^3\alpha+t^2\cos\alpha+2it\tan\alpha) \mp \frac{i}{4k}(a_2-a_1)M(t)+O(k^{-2})\right\}, \qquad (114)$$

where

$$M(t)=\int_{|t|}^{\infty} e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,\frac{(\zeta-t)^2}{\zeta}\,d\zeta \qquad (115)$$

with the upper (lower) sign according as $t = ky > 0\ (< 0)$.

When this is substituted into Eq. (111) and the quadratic factor also expanded in powers of k,

$$P_E = -\frac{\cos\alpha}{2}\int_{-\infty}^{\infty} e^{ipt}\left\{1+\frac{ia}{2k}\left(\sec^3\alpha+2it\tan\alpha+t^2(\cos\alpha+\cos\theta)\right) \mp \frac{i}{4k}(a_2-a_1)M(t)+O(k^{-2})\right\}dt$$

$$= -\frac{\cos\alpha}{2}\left(\frac{i}{2k}\right)(a_2-a_1)\left\{\frac{i}{p}\sec^3\alpha+\left(\frac{i}{p}\right)^2 2i\tan\alpha+2\left(\frac{i}{p}\right)^3(\cos\alpha+\cos\theta)\right\}$$

$$+\frac{i\cos\alpha}{8k}(a_2-a_1)\left(\int_0^{\infty}-\int_{-\infty}^{0}\right)e^{ipt}M(t)dt+O(k^{-2})$$

(continued)

$$= -\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha+\theta)}{p^3} + \frac{a_2-a_1}{4k}\,\frac{\sec^2\alpha}{p}$$

$$+\frac{i\cos\alpha}{8k}(a_2-a_1)\left(\int_0^\infty - \int_{-\infty}^0\right)e^{ipt}M(t)dt + O(k^{-2}),$$

and since the first term on the right hand side is simply $P_E^{p.o.}$, it follows that

$$P_E = P_E^{p.o.} + i\,\frac{a_2-a_1}{8k}\cos\alpha\left\{-\frac{2i}{p}\sec^3\alpha + \left(\int_0^\infty - \int_{-\infty}^0\right)e^{ipt}M(t)dt\right\} + O(k^{-2})\;. \tag{116}$$

Equation (114) is very similar to the analogous equation (96) for H polarization and the steps that are taken to evaluate (precisely) the integrals containing M(t) are likewise similar. Thus

$$\left(\int_0^\infty - \int_{-\infty}^0\right)e^{ipt}M(t)dt = \int_0^\infty e^{ipt}\int_t^\infty e^{-i\zeta\sin\alpha}H_1^{(1)}(\zeta)\,\frac{(\zeta-t)^2}{\zeta}\,d\zeta\,dt$$

$$-\int_{-\infty}^0 e^{ipt}\int_{-t}^\infty e^{i\zeta\sin\alpha}H_1^{(1)}(\zeta)\,\frac{(\zeta+t)^2}{\zeta}\,d\zeta\,dt$$

$$= \int_0^\infty \left\{e^{ipt}\int_t^\infty e^{-i\zeta\sin\alpha}H_1^{(1)}(\zeta)\,\frac{(\zeta-t)^2}{\zeta}d\zeta - e^{-ipt}\int_t^\infty e^{i\zeta\sin\alpha}H_1^{(1)}(\zeta)\,\frac{(\zeta-t)^2}{\zeta}d\zeta\right\}dt$$

$$= \int_0^\infty\left\{e^{ipt}M^{(-)}(t) - e^{-ipt}M^{(+)}(t)\right\}dt$$

where

$$M^{(-)}(t) = \int_t^\infty e^{-i\zeta\sin\alpha}H_1^{(1)}(\zeta)\,\frac{(\zeta-t)^2}{\zeta}\,d\zeta\;,$$

$$M^{(+)}(t)=\int_t^\infty e^{i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,\frac{(\zeta-t)^2}{\zeta}\,d\zeta\,.$$

But

$$M^{(\mp)}(0)=\int_0^\infty e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\zeta\,d\zeta=\sec^3\alpha\left\{1\pm\frac{2}{\pi}(\sin\alpha\cos\alpha+\alpha)\right\}$$

as shown in Appendix B, and

$$M^{(\mp)'}(t)=-2\int_t^\infty e^{\mp i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,\frac{\zeta-t}{\zeta}\,d\zeta$$

where the prime denotes differentiation with respect to t. Hence, on integration by parts,

$$\left(\int_0^\infty-\int_{-\infty}^0\right)e^{ipt}M(t)dt=\frac{1}{ip}\left[e^{ipt}M^{(-)}(t)+e^{-ipt}M^{(+)}(t)\right]_0^\infty$$

$$-\frac{1}{ip}\int_0^\infty\left\{e^{ipt}M^{(-)'}(t)+e^{-ipt}M^{(+)'}(t)\right\}dt$$

$$=\frac{i}{p}\left\{M^{(-)}(0)+M^{(+)}(0)\right\}+\frac{i}{p}\int_0^\infty\left\{e^{ipt}M^{(-)'}(t)+e^{-ipt}M^{(+)'}(t)\right\}dt$$

and since the first term is simply $\frac{2i}{p}\sec^3\alpha$, we now have

$$P_E=P_E^{p.o.}+\frac{a_2-a_1}{4kp}\cos\alpha\int_0^\infty\left\{e^{ipt}\int_t^\infty e^{-i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,\frac{\zeta-t}{\zeta}\,d\zeta\right.$$

$$\left.+e^{-ipt}\int_t^\infty e^{i\zeta\sin\alpha} H_1^{(1)}(\zeta)\,\frac{\zeta-t}{\zeta}d\zeta\right\}dt+O(k^{-2})\,. \qquad (117)$$

As shown in Section 5.3,

$$\int_0^\infty \left\{ e^{ipt} \int_t^\infty e^{-i\zeta \sin\alpha} \frac{H_1^{(1)}(\zeta)}{\zeta} d\zeta + e^{-ipt} \int_t^\infty e^{i\zeta \sin\alpha} \frac{H_1^{(1)}(\zeta)}{\zeta} d\zeta \right\} tdt$$

$$= \frac{2}{p^2} \frac{1-\cos(\alpha-\theta)}{\cos\theta} .$$

Also

$$\int_0^\infty \left\{ e^{ipt} \int_t^\infty e^{-i\zeta \sin\alpha} H_1^{(1)}(\zeta) d\zeta + e^{-ipt} \int_t^\infty e^{i\zeta \sin\alpha} H_1^{(1)}(\zeta) d\zeta \right\} dt$$

$$= 2 \int_0^\infty \left\{ \cos pt \int_t^\infty \cos(\zeta \sin\alpha) H_1^{(1)}(\zeta) d\zeta + \sin pt \int_t^\infty \sin(\zeta \sin\alpha) H_1^{(1)}(\zeta) d\zeta \right\} dt$$

$$= \frac{2}{p} \left[\sin pt \int_t^\infty \cos(\zeta \sin\alpha) H_1^{(1)}(\zeta) d\zeta - \cos pt \int_t^\infty \sin(\zeta \sin\alpha) H_1^{(1)}(\zeta) d\zeta \right]_0^\infty$$

$$+ \frac{2}{p} \int_0^\infty \left\{ \sin pt \cos(t \sin\alpha) - \cos pt \sin(t \sin\alpha) \right\} H_1^{(1)}(t) dt$$

$$= \frac{2}{p} \int_0^\infty \sin(\zeta \sin\alpha) H_1^{(1)}(\zeta) d\zeta - \frac{2}{p} \int_0^\infty \sin(t \sin\theta) H_1^{(1)}(t) dt$$

$$= \frac{2}{p} (\tan\alpha - \tan\theta)$$

as follows on using Eq. (105) and the expression (91) for p. Hence

$$P_E = P_E^{p.o.} - \frac{a_2 - a_1}{2k} \cos\alpha \left\{ \frac{1-\cos(\alpha-\theta)}{p^3 \cos\theta} + \frac{\tan\theta - \tan\alpha}{p^2} \right\} + O(k^{-2})$$

which reduces to

$$P_E = P_E^{p.o.} + \frac{a_2 - a_1}{2k} \frac{1-\cos(\alpha-\theta)}{p^3} + O(k^{-2}) \; . \qquad (118)$$

As in the case of H polarization, the difference from the physical optics result is quite apparent.

7. General Considerations

7.1 The Nature of the Diffraction Coefficients

It is convenient to start by summarizing the results obtained for the diffraction coefficients. Using a specific model consisting of two parabolic cylinders of different latus recta joined so as to create a two-dimensional (line) discontinuity in curvature at the front (see Fig. 1), it has been shown that if

$$\underline{E}^i = \hat{z}\, e^{ik(-x\cos\alpha + y\sin\alpha)} \qquad (119)$$

then

$$\underline{E}^s \sim \hat{z} \sqrt{\frac{2}{\pi k r}}\; e^{i(kr - \frac{\pi}{4})}\, P_E \qquad (120)$$

with

$$P_E = F - G \; ; \qquad (121)$$

whereas if

$$\underline{H}^i = \hat{z}\; e^{ik(-x\cos\alpha + y\sin\alpha)} \; , \qquad (122)$$

implying

$$\underline{E}^i = -Z\, (\hat{x}\sin\alpha + \hat{y}\cos\alpha)\, e^{ik(-x\cos\alpha + y\sin\alpha)} \; ,$$

then

$$\underline{H}^s \sim \hat{z} \sqrt{\frac{2}{\pi k r}}\; e^{i(kr - \frac{\pi}{4})}\, P_H$$

implying

$$\underline{E}^s = -Z\,(\hat{x}\sin\theta - \hat{y}\cos\theta)\sqrt{\frac{2}{\pi kr}}\; e^{i(kr-\frac{\pi}{4})}\, P_H \tag{123}$$

with

$$P_H = F + G\,, \tag{124}$$

where

$$F = \frac{a_2-a_1}{2k}\;\frac{1-\cos(\alpha-\theta)}{(\sin\alpha-\sin\theta)^3} + O(k^{-2})\,, \tag{125}$$

$$G = \frac{a_2-a_1}{2k}\;\frac{1+\cos(\alpha+\theta)}{(\sin\alpha-\sin\theta)^3} + O(k^{-2})\,. \tag{126}$$

As demanded by the reciprocity condition concerning the interchange of receiver and transmitter, the expressions for F and G are unaffected if θ is replaced by $-\alpha$ and α by $-\theta$.

The terms shown on the right hand sides of Eqs. (125) and (126) are the leading terms in a high frequency asymptotic development of F and G, and are valid provided that $|\theta-\alpha|$ is bounded away from zero (to separate the contribution of the specular point from that of the discontinuity). In the expansion of the surface field it was found necessary to assume that $\frac{\pi}{2} - |\alpha|$ is bounded away from zero (to ensure that the discontinuity is fully illuminated), and on physical grounds it would also appear necessary that $\frac{\pi}{2} - |\theta|$ be bounded from zero so that the discontinuity will be directly visible to the field point. However, the expressions for F and G are finite and continuous in the limit $\alpha, \theta \to \pm\frac{\pi}{2}$, $|\sin\alpha - \sin\theta| \neq 0$, allowing us to replace the conditions on α and θ individually by the less restrictive ones $|\alpha|$, $|\theta| \leq \frac{\pi}{2}$. This extension is vital to the application of our results to the problem of an EMP generator.

Equations (121) and (124) can be written more compactly as

$$P_{E,H} = F \mp G \tag{127}$$

and since the physical optics approximation is simply

$$P^{p.o.}_{E,H} = \mp G , \tag{128}$$

the differences from the physical optics estimates are obvious. Under most circumstances, F is small compared with G: for example, in the particular case of backscattering ($\theta = -\alpha$),

$$F = \frac{a_2-a_1}{8k} \operatorname{cosec} \alpha + O(k^{-2})$$

$$G = \frac{a_2-a_1}{8k} \operatorname{cosec}^3 \alpha + O(k^{-2})$$

and F is less than G by a factor 10 or more if α is less than 18.4 degrees. Nevertheless, F is the sole source of the polarization dependence of the scattering. Its inclusion is therefore important in any analysis which seeks to reproduce the polarization characteristics, and is vital in any study aimed at the cross polarized component of the backscattered field.

7.2 Complete Diffraction Matrix

A derivation of the diffraction matrix associated with a discontinuity in curvature is essential for the incorporation of our results within the general framework of the geometrical theory of diffraction (see, for example, Keller, 1962). In order to obtain this, it is helpful to proceed in two steps, drawing first the analogy between a discontinuity in curvature (involving the second derivative) and a discontinuity in slope (or first derivative), i.e., a wedge-like singularity.

Consider a perfectly conducting wedge of half angle Ω on which a plane electromagnetic wave is incident as shown in Fig. 2. Following Keller (1957), we introduce a set of base vectors, $\hat{T}$, $\hat{N}$, $\hat{B}$, where $\hat{T}$ is a unit vector parallel to the edge, $\hat{N}$ is a unit vector 'normal' to the edge, and $\hat{B}$ is the unit vector

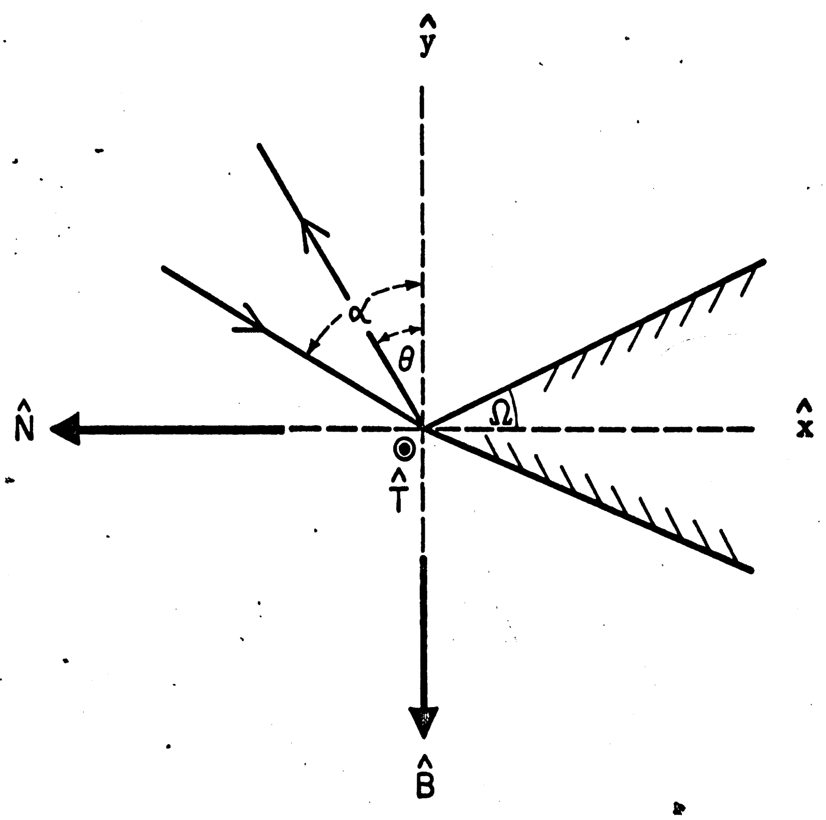

FIG. 2: Geometry for First Derivative (wedge) Discontinuity.

'binormal' to the edge and pointing into the shadowed half space. The direction of $\hat{T}$ is chosen to make $\hat{T}$, $\hat{N}$, $\hat{B}$ a right handed system, i.e. $\hat{T} = \hat{N} \wedge \hat{B}$.

With the coordinate system of Fig. 2, it is known that an incident electric field

$$\underline{E}^i = \hat{e}^i e^{ik\hat{i}\cdot\underline{r}} \tag{129}$$

having

$$\hat{i} = -\hat{N}\sin\alpha + \hat{B}\cos\alpha \qquad \left(-\frac{\pi}{2} + \Omega \leq \alpha \leq \frac{\pi}{2}\right) \tag{130}$$

produces a diffracted electric field whose expression at points far from the edge and away from all geometrical optics boundaries is

$$\underline{E}^d = \hat{e}^d \left(-\frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi kr}}\right) e^{ik\hat{s}\cdot\underline{r}} \tag{131}$$

with

$$\hat{s} = \hat{N}\sin\theta - \hat{B}\cos\theta \qquad \left(-\frac{\pi}{2} + \Omega \leq \theta \leq \frac{3\pi}{2} - \Omega\right). \tag{132}$$

In particular (see, for example, Senior and Uslenghi, 1971), if

$$\hat{e}^i = \hat{T} \tag{133}$$

then

$$\hat{e}^d = -\hat{T}\,(X - Y) \tag{134}$$

and if

$$\hat{e}^i = \hat{N}\cos\alpha + \hat{B}\sin\alpha \tag{135}$$

then

$$\hat{e}^d = (\hat{N}\cos\theta + \hat{B}\sin\theta)\,(X + Y), \tag{136}$$

where

$$X = \frac{1}{n}\sin\frac{\pi}{n}\left\{\cos\frac{\pi}{n} - \cos\frac{1}{n}(\alpha-\theta)\right\}^{-1}, \tag{137}$$

$$Y = \frac{1}{n}\sin\frac{\pi}{n}\left\{\cos\frac{\pi}{n} + \cos\frac{1}{n}(\pi-\alpha-\theta)\right\}^{-1} \tag{138}$$

with

$$n = 2\,(1 - \frac{\Omega}{\pi}) \quad . \tag{139}$$

Let us now adopt a similar coordinate system for a discontinuity in curvature. This is shown in Fig. 3 and necessitates the following coordinate transformations of our earlier results in order that we may employ them:

$$\hat{x} \rightarrow \hat{N}, \;\; \hat{y} \rightarrow -\hat{B}, \;\; \hat{z} \rightarrow -\hat{T}, \;\; \alpha \rightarrow \alpha - \frac{\pi}{2}, \quad \theta \rightarrow \frac{\pi}{2} - \theta \; . \tag{140}$$

Equations (129) through (132) now apply as for a wedge-like discontinuity but under the conditions that

$$|\alpha| \; , |\theta| \leq \frac{\pi}{2}$$

with*

$|\sin\alpha - \sin\theta|$ bounded away from zero .

Moreover, from Eqs. (119) through (126) we have that if

$$\hat{e}^{\,i} = \hat{T} \tag{141}$$

then

$$\hat{e}^{\,d} = \hat{T}\; 2i(F-G) \; , \tag{142}$$

whereas if

$$\hat{e}^{\,i} = \hat{N}\cos\alpha + \hat{B}\sin\alpha \tag{143}$$

then

$$\hat{e}^{\,d} = -(\hat{N}\cos\theta + \hat{B}\sin\theta)2i(F+G) \; , \tag{144}$$

where the expressions for F and G in terms of the new angles α and θ are

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*Observe that this last condition excludes backscattering at grazing incidence as well as the specular direction.

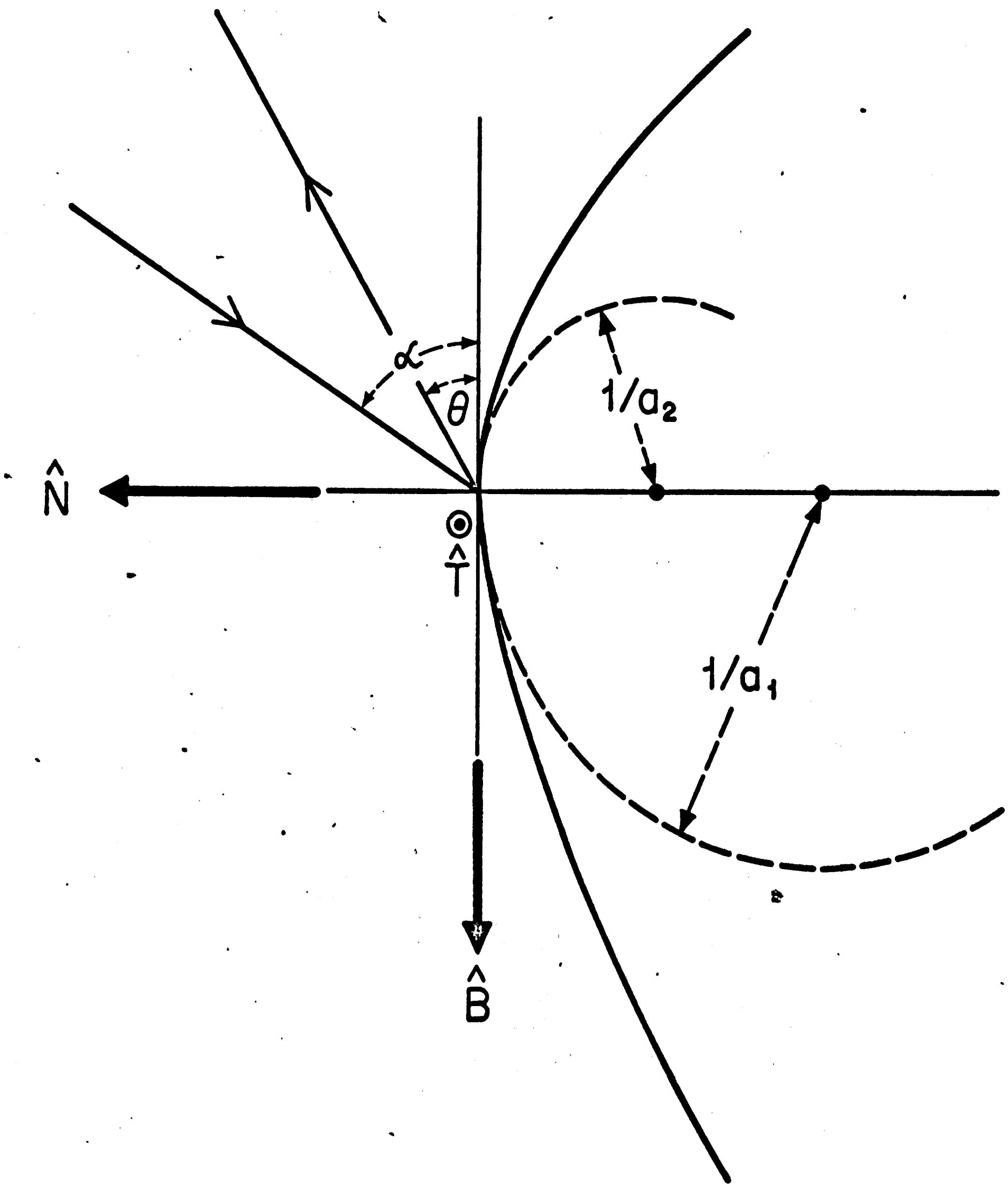

FIG. 3: Modified Coordinate System for a Second Derivative Discontinuity.

$$F = -\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha+\theta)}{(\cos\alpha+\cos\theta)^3} + O(k^{-2}), \tag{145}$$

$$G = -\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha-\theta)}{(\cos\alpha+\cos\theta)^3} + O(k^{-2})\ . \tag{146}$$

On comparing Eqs. (141) through (144) with (133) through (136), it is seen that the diffracted field produced by a discontinuity in the second derivative is obtainable from that created by a discontinuity in the first derivative by replacing

$$X \quad \text{by} \quad -2iF = -\frac{a_2-a_1}{2ik}\,\frac{\sec^2\frac{\alpha-\theta}{2}}{\cos\alpha+\cos\theta} \tag{147}$$

and

$$Y \quad \text{by} \quad -2iG = -\frac{a_2-a_1}{2ik}\,\frac{\sec^2\frac{\alpha+\theta}{2}}{\cos\alpha+\cos\theta}\ . \tag{148}$$

This analogy between the diffracted fields produced by the two types of surface discontinuity is rather interesting. In a physical sense, a discontinuity in the second derivative is like a very 'subdued' version of a first derivative discontinuity, and the latter may occur (in the form of a surface kink) if insufficient care is taken in the fabrication of a model. From an examination of the surface in the immediate vicinity of the discontinuity, it could be hard to tell whether the discontinuity was in the second derivative (as, perhaps, intended) or was instead in the first derivative. Consider, therefore, the expressions for X and Y for small ϵ (> 0) where

$$n = 1+\epsilon\ ,$$

implying

$$\Omega = \frac{\pi}{2}(1-\epsilon)\ .$$

We now have

$$X = -\frac{\epsilon}{2} \sec^2 \frac{\alpha-\theta}{2} + O(\epsilon^2) ,$$

$$Y = -\frac{\epsilon}{2} \sec^2 \frac{\alpha+\theta}{2} + O(\epsilon^2) ,$$

and from Eqs. (147) and (148) it then follows that a second derivative discontinuity is equivalent (as regards the diffracted field) to a kink having

$$\epsilon = \frac{a_2-a_1}{ik} (\cos\alpha + \cos\theta)^{-1} ,$$

that is , to a wedge of half angle

$$\Omega = \frac{\pi}{2} \left\{ 1 - \frac{a_2-a_1}{ik} (\cos\alpha + \cos\theta)^{-1} \right\} . \tag{149}$$

In addition to the expected dependence on k, we observe that Ω is also a function of the angles of incidence and diffraction.

A further consequence of the analogy between the two types of surface discontinuity is that we can write down immediately the complete diffraction matrix for a discontinuity in curvature by making the substitutions (147) and (148) in the known matrix for a wedge-like discontinuity. With the base vectors $\hat{T}$, $\hat{N}$, $\hat{B}$ and the angles α and θ as shown in Fig. 3, we consider an incident plane wave having an electric field

$$\underline{E}^i = \hat{e}^i e^{ik\hat{i}\cdot\underline{r}} \tag{150}$$

where

$$\hat{i} = \hat{T}\cos\beta - \hat{N}\sin\beta\sin\alpha + \hat{B}\sin\beta\cos\alpha \tag{151}$$

with $0 < \beta < \pi$. At points far from the discontinuity and in directions satisfying the previously-mentioned restrictions on α and θ, the diffracted electric field can be written as

$$\underline{E}^d = \hat{e}^d \left(- \frac{e^{i\frac{\pi}{4}}}{\sin\beta\sqrt{2\pi kr}} \right) e^{ik\hat{s}\cdot\underline{r}} \tag{152}$$

(see Keller, 1957), where

$$\hat{s} = \hat{T}\cos\beta + \hat{N}\sin\beta\sin\theta - \hat{B}\sin\beta\cos\theta \tag{153}$$

and

$$\hat{e}^d = \Delta\hat{e}^i \tag{154}$$

in which Δ is a 3 x 3 matrix (or dyadic) provided $\hat{e}^i$ is treated as a column vector in the base $\hat{T}$, $\hat{N}$, $\hat{B}$. An adequate expression for Δ now follows by making the substitutions (147) and (148) in Eq. (A.13) of Senior and Uslenghi (1971), and is

$$\Delta = -\begin{pmatrix} -2i(F-G) & 0 & 0 \\ 2i(F-G)\cot\beta\sin\theta & 2i(F+G)\cos\theta\cos\alpha & 2i(F+G)\cos\theta\sin\alpha \\ -2i(F-G)\cot\beta\cos\theta & 2i(F+G)\sin\theta\cos\alpha & 2i(F+G)\sin\theta\sin\alpha \end{pmatrix} \tag{155}$$

where F and G are as given in Eqs. (145) and (146) respectively.

8. Conclusions

The rigorous derivation of the diffraction matrix for a discontinuity in curvature significantly enlarges the scope of the geometrical theory of diffraction, and permits application of this technique to surface singularities other than the wedge-type to which it has been restricted heretofore. Since the form of the matrix is so similar to that for a discontinuity in surface slope, it would be relatively simple to extend existing programs for calculating the diffracted fields produced by discontinuities in slope to include this new and more suppressed form of discontinuity. In the context of EMP generators, for example, it is now possible to investigate the short time radiated field associated with near-source geometries which are more general than the bicone-cylinder configurations that have been considered so far (Sancer and Varvatsis, 1971), and this application of the matrix will be examined in a subsequent report.

9. References

Bowman, J. J., T. B. A. Senior and P. L. E. Uslenghi (1969), Electromagnetic and Acoustic Scattering by Simple Shapes, North-Holland Publishing Company, Amsterdam.

Keller, J.B. (1957), "Diffraction by an Aperture," J. Appl. Phys., 28, 426-444.

Keller, J.B. (1962), "Geometrical Theory of Diffraction," J. Opt. Soc. Amer., 52, 116-130.

Keller, J.B., R. M. Lewis and B. D. Seckler (1956), "Asymptotic Solution of Some Diffraction Problems," Comm. Pure Appl. Math., 9, 207-265.

Kline, M. (1951), "An Asymptotic Solution of Maxwell's Equations," Comm. Pure Appl. Math., 4, 225-263

Knott, E. F., and T. B. A. Senior (1971), "The Polarization Characteristics of Scattered Fields," Electronics Letters, 7, 183-184.

Luneburg, R. K. (1944), The Mathematical Theory of Optics, Brown University Press, Providence, R. I.

Sancer, M. I. and A. D. Varvatsis (1971), "Geometrical Diffraction Solution for the High Frequency Early Time Behavior of the Field Radiated by an Infinite Cylindrical Antenna with a Biconical Feed," Sensor and Simulation Note No.

Senior, T. B. A. (1965), "The Backscattering Cross Section of a Cone-Sphere," IEEE Trans. Antennas and Propagation, AP-13, 271-277.

Senior, T. B. A. and P. L. E. Uslenghi (1971), "High-Frequency Backscattering from a Finite Cone," Radio Science, 6, 393-406.

Watson, G. N. (1945), Bessel Functions, Cambridge University Press.

Weston, V. H. (1962), "The Effect of a Discontinuity in Curvature in High-Frequency Scattering," IRE Trans. Antennas and Propagation, AP-10, 775-780.

Weston, V. H. (1965), "Extension of Fock Theory for Currents in the Penumbra Region," Radio Science, 69D, 1257-1270.

Appendix A — Luneburg-Kline Expansion for a Parabolic Cylinder

When an electromagnetic wave is incident on a smooth convex body, the reflected field at high frequencies can be expanded in a series of negative powers of the wave number k. The procedures for exploiting this fact were originally developed by Luneburg (1944), and because of their subsequent extension by Kline (1951), the resulting expansion is now known as the Luneburg-Kline expansion. A variety of specific applications to both two- and three-dimensional problems has been given by Keller et al (1956).

The particular problem of concern to us here is the determination of the first two terms in the asymptotic expansion of the surface field for a plane electromagnetic wave at oblique incidence on a parabolic cylinder. The equation of the cylinder is taken as

$$y^2 = -\frac{2x}{a}, \qquad -\infty < z < \infty \tag{A.1}$$

(see Fig. A.1), where a is the reciprocal of the latus rectum and the origin of coordinates is at the front of the cylinder. The propagation vector of the incident field is assumed to be

$$\hat{k} = -\hat{x}\cos\alpha + \hat{y}\sin\alpha ,$$

and by treating in succession the cases in which the field has its electric or magnetic vector in the z direction, we are led to two scalar problems satisfying Dirichlet and Neumann boundary conditions respectively at the surface. We therefore write

$$u^i = e^{ik(-x\cos\alpha + y\sin\alpha)}, \tag{A.2}$$

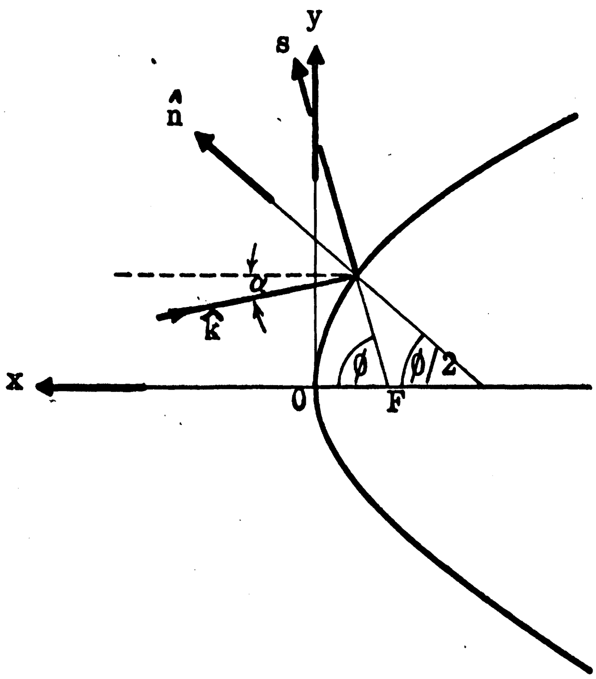

Fig. A.1: Geometry

where $u^i = E_z^i$ for an E-polarized wave and $u^i = H_z^i$ for H-polarization, and seek the scattered (or reflected) field u^s satisfying the boundary condition

$$u^i + u^s = 0 \qquad \text{(E-polarization)} \qquad (A.3)$$

or

$$\frac{\partial u^i}{\partial n} + \frac{\partial u^s}{\partial n} = 0 \qquad \text{(H-polarization)} \qquad (A.4)$$

at the surface, where n is in the direction of the outward normal to the cylinder. In both cases it is assumed that u^s is capable of asymptotic expansion in the form

$$u^s(x,y) \sim e^{ik\psi(x,y)} \sum_{m=o}^{\infty} \frac{V_m(x,y)}{(ik)^m} , \qquad (A.5)$$

and we seek the first two terms in the expansion of the surface field.

It is convenient to reverse the order adopted in the main body of this Report by treating the case of E polarization first. This is followed by H polarization, and certain of the purely mathematical operations involved are presented in the third and final section.

In addition to the Cartesian coordinates (x, y, z), we also employ cylindrical polar coordinates (ρ, ϕ, z) referred to an origin at the focus, in terms of which

$$x = \rho \cos\phi - \frac{1}{2a}, \qquad y = \rho \sin\phi .$$

On the surface,

$$\rho = \rho_o = \frac{1}{a(1+\cos\phi)},$$

implying

$$x = -\frac{1}{2a}\,\frac{1-\cos\phi}{1+\cos\phi}, \qquad y = \frac{1}{a}\,\frac{\sin\phi}{1+\cos\phi},$$

so that

$$x = \frac{1}{2a} - \rho_o .$$

The radius of curvature of the cylinder is $2a(1-{}^{x}/_{a})^{3/2}$.

E-Polarization

The scalar quantity u now represents the component E_z, and on application of the boundary condition (A.3), we obtain

$$e^{-ik\rho_o\cos(\phi+\alpha)} + e^{ik\psi(\rho_o,\phi)} \sum_{m=o}^{\infty} \frac{V_m(\rho_o,\phi)}{(ik)^m} \sim 0 .$$

Hence

$$\psi(\rho_0, \phi) = -\rho_0 \cos(\phi + \alpha), \tag{A.6}$$

$$V_0(\rho_0, \phi) = -1 \tag{A.7}$$

and

$$V_m(\rho_0, \phi) = 0 , \quad m \geq 1 . \tag{A.8}$$

At any point $P = P(\rho, \phi)$ off the surface, the field is attributable to a ray which is reflected at the appropriate point P_0 of the surface according to the laws of geometrical optics. Following Keller et al (1956, pp. 249 et seq.), we introduce new (ray) coordinates s, β where s is the distance along the reflected ray at P measured from the caustic, and β is the angle which this ray makes with the positive x axis. Thus

$$\beta = \phi + \alpha$$

and if s_0 is the value of s at P_0, then

$$s_0 = \frac{1}{2} \rho_0 \cos \frac{\beta + \alpha}{2} \quad .$$

Hence, at P,

$$\psi(s, \beta) = \psi(s_0, \beta) + s - s_0 \tag{A.9}$$

with

$$s = s_0 + \left[P_0 P \right] \quad .$$

The expression for $V_0(s, \beta)$ is

$$V_0(s, \beta) = V_0(s_0, \beta) \left(\frac{s_0}{s}\right)^{1/2} \quad ,$$

as follows from the conservation of power. Hence, using (A.7),

$$V_0(s, \beta) = -\left(\frac{s_0}{s}\right)^{1/2}$$

implying

$$u^s \sim -\left(\frac{s_o}{s}\right)^{1/2} e^{ik\left\{\psi(s_o,\beta)+s-s_o\right\}}\left\{1+O(k^{-1})\right\} . \tag{A.10}$$

Since

$$\hat{n} = \hat{x}\cos\frac{\phi}{2} + \hat{y}\sin\frac{\phi}{2} , \tag{A.11}$$

$$\left.\frac{\partial u^i}{\partial n}\right|_{\rho=\rho_o} = -ik\cos\left(\frac{\phi}{2}+\alpha\right) e^{-ik\rho_o\cos(\phi+\alpha)} . \tag{A.12}$$

Also

$$\frac{\partial u^s}{\partial n} = \left\{ik\frac{\partial}{\partial n}\psi(s,\beta)\sum_{m=o}^{\infty}\frac{V_m(s,\beta)}{(ik)^m} + \sum_{m=o}^{\infty}\frac{1}{(ik)^m}\frac{\partial}{\partial n}V_m(s,\beta)\right\} e^{ik\psi(s,\beta)}$$

and because $V_m=0$, $m \geq 1$, on the surface, with $V_o = -1$,

$$\left.\frac{\partial u^s}{\partial n}\right|_{\rho=\rho_o} = \left\{-ik\frac{\partial\psi}{\partial n} + \frac{\partial V_o}{\partial n} + O(k^{-1})\right\} \left.e^{ik\psi(s,\beta)}\right|_{\rho=\rho_o} .$$

On inserting the results of Eqs. (A.29) and (A.30), this becomes

$$\left.\frac{\partial u^s}{\partial n}\right|_{\rho=\rho_o} = \left\{-ik\cos\left(\alpha+\frac{\phi}{2}\right) + \frac{a\cos^3\frac{\phi}{2}}{\cos^2\left(\alpha+\frac{\phi}{2}\right)} + O(k^{-1})\right\} e^{-ik\rho_o\cos(\phi+\alpha)} \tag{A.13}$$

which can be combined with Eq. (A. 12) to give

$$\frac{\partial}{\partial n}(u^i+u^s)\Big|_{\rho=\rho_o} = -2ik\cos\left(\alpha+\frac{\phi}{2}\right)\left\{1+\frac{ia}{2k}\,\frac{\cos^3\frac{\phi}{2}}{\cos^3\left(\alpha+\frac{\phi}{2}\right)}+O(k^{-2})\right\}e^{-ik\rho_o\cos(\phi+\alpha)} \quad \text{(A. 14)}$$

In terms of Cartesian coordinates,

$$\frac{\partial}{\partial n}(u^i+u^s)\Big|_{\rho=\rho_o} = -\frac{2ik}{\sqrt{1-2ax}}(\cos\alpha-ay\sin\alpha)\left\{1+\frac{ia}{2k}(\cos\alpha-ay\sin\alpha)^{-3}\right.$$

$$\left.+O(k^{-2})\right\}e^{ik(-x\cos\alpha+y\sin\alpha)}, \quad \text{(A. 15)}$$

which is the result required. For $\alpha = 0$ it is equivalent to the solution obtained by Keller et al (1956) for the special case of axial incidence.

H-Polarization

The procedure is rather similar to the above. The scalar u now represents the field component H_z, and on application of the boundary condition (A.4), we obtain

$$-ik\cos\left(\alpha-\frac{\phi}{2}\right)e^{-ik\rho_o\cos(\phi+\alpha)}+\left\{ik\,\frac{\partial\psi}{\partial n}\sum_{m=o}^{\infty}\frac{1}{(ik)^m}\,\frac{\partial V_m}{\partial n}\right\}e^{ik\psi}\sim 0$$

for $\rho=\rho_o$. Hence

$$\psi(\rho_o,\phi) = -\rho_o \cos(\phi+\alpha)$$

as before, and

$$\cos(\alpha-\frac{\phi}{2}) - V_o(\rho_o,\phi)\frac{\partial\psi}{\partial n} = 0\,, \qquad \text{(A. 16)}$$

$$\frac{\partial V_{m-1}}{\partial n} + V_m(\rho_o,\phi)\frac{\partial\psi}{\partial n} = 0\,, \quad m \geq 1\,. \qquad \text{(A. 17)}$$

Introducing the ray coordinates (s,β) once again, $\psi(s,\beta)$ assumes the form (A. 9). Hence

$$V_o(s_o,\beta) = (\frac{\partial\psi}{\partial n})^{-1}\cos(\alpha+\frac{\phi}{2})$$

and on inserting the expression for $\partial\psi/\partial n$ given in Eq. (A. 30), this becomes

$$V_o(s_o,\beta) = 1\,, \qquad \text{(A. 18)}$$

implying

$$V_o(s,\beta) = (\frac{s_o}{s})^{1/2}\quad . \qquad \text{(A. 19)}$$

Equation (A. 17) serves to specify V_1 on the surface as

$$V_1(s_o,\beta) = -(\frac{\partial\psi}{\partial n})^{-1}\left.\frac{\partial V_o}{\partial n}\right|_{s=s_o}$$

and using (A. 29) in conjunction with (A. 30), we have

$$V_1(s_o,\beta) = -\frac{a\cos^3\frac{\phi}{2}}{\cos^3(\alpha+\frac{\phi}{2})}\quad . \qquad \text{(A. 20)}$$

Combining (A. 2), (A. 6), (A. 18) and (A. 20), the surface field is found to be

$$(u^i+u^s)\Big|_{\rho=\rho_0} = 2\left\{1 - \frac{ia}{2k}\,\frac{\cos^3\frac{\phi}{2}}{\cos^3(\alpha+\frac{\phi}{2})} + O(k^{-2})\right\} e^{-ik\rho_0\cos(\phi+\alpha)} \tag{A.21}$$

in agreement with the result given by Keller et al (1956) for the special case of axial incidence ($\alpha=0$). In terms of Cartesian coordinates,

$$(u^i+u^s)\Big|_{\rho=\rho_0} = 2\left\{1 - \frac{ia}{2k}(\cos\alpha - ay\sin\alpha)^{-3} + O(k^{-2})\right\} e^{ik(-x\cos\alpha + y\sin\alpha)}, \tag{A.22}$$

which is the result required.

<u>Mathematical Relations</u>

We present here a derivation of the two formulae that have been employed in the preceding sections.

From Eq. (A.11), we have

$$\frac{\partial}{\partial n} = \cos\frac{\beta-\alpha}{2}\,\frac{\partial}{\partial x} + \sin\frac{\beta-\alpha}{2}\,\frac{\partial}{\partial y}$$

and since $s=s(x,y)$, $\beta=\beta(x,y)$, it follows that

$$\frac{\partial}{\partial n} = \left(\cos\frac{\beta-\alpha}{2}\,\frac{\partial s}{\partial x} + \sin\frac{\beta-\alpha}{2}\,\frac{\partial s}{\partial y}\right)\frac{\partial}{\partial s} + \left(\cos\frac{\beta-\alpha}{2}\,\frac{\partial \beta}{\partial x} + \sin\frac{\beta-\alpha}{2}\,\frac{\partial \beta}{\partial y}\right)\frac{\partial}{\partial \beta}\,. \tag{A.23}$$

To determine $\partial s/\partial x$, $\partial s/\partial y$, $\partial\beta/\partial x$ and $\partial\beta/\partial y$, we first observe that

$$x = -\frac{1}{2a} + (s-s_0)\cos\beta + \rho_0\cos(\beta-\alpha) \tag{A.24}$$

$$y = (s-s_0)\sin\beta + \rho_0\sin(\beta-\alpha) \tag{A.25}$$

where

$$\rho_0 = \frac{1}{2a}\sec^2\frac{\beta-\alpha}{2} \quad \text{and} \quad s_0 = \rho_0 \sec^3\frac{\beta-\alpha}{2}\cos\frac{\beta+\alpha}{2} .$$

On eliminating s from (A.24) and (A.25), we obtain

$$(x+\frac{1}{2a})\sin\beta - y\cos\beta = \frac{1}{2a}\sin\alpha\sec^2\frac{\beta-\alpha}{2} \tag{A.26}$$

and differentiation with respect to x keeping y constant then shows that

$$\frac{\partial\beta}{\partial x} = -\frac{\sin\beta}{s} . \tag{A.27}$$

Likewise, on differentiating (A.26) with respect to y keeping x constant, we have

$$\frac{\partial\beta}{\partial y} = \frac{\cos\beta}{s} .$$

To find $\partial s/\partial x$, it is convenient to express x in the form

$$x = s\cos\beta + \frac{1}{2a}\left(\frac{\sin\frac{3\beta-\alpha}{2}\sin\alpha}{\cos^3\frac{\beta-\alpha}{2}} - 1\right)$$

as follows by eliminating s_0 and ρ_0 from Eq. (A.24). If we now differentiate with respect to x allowing s and β to vary, and then use Eq. (A.27), we find

$$\frac{\partial s}{\partial x} = \cos\beta + \frac{3\sin\beta\sin\alpha}{\cos\beta+\cos\alpha} .$$

Similarly, from Eq. (A.25),

$$y = s\sin\beta - \frac{1}{2a}\,\frac{\sin\frac{3\beta-\alpha}{2}\sin\alpha}{\cos^3\frac{\beta-\alpha}{2}}$$

and differentiation with respect to y gives

$$\frac{\partial s}{\partial y} = \sin\beta - \frac{3\cos\beta\sin\alpha}{\cos\beta+\cos\alpha} .$$

On substituting the above results into Eq. (A.23), the normal derivative becomes

$$\frac{\partial}{\partial n} = \left(\cos\frac{\beta+\alpha}{2} + 3\,\frac{\sin\alpha\sin\frac{\beta+\alpha}{2}}{\cos\beta+\cos\alpha}\right)\frac{\partial}{\partial s} - 2a\tan\frac{\beta-\alpha}{2}\cos^3\frac{\beta-\alpha}{2}\,\frac{\partial}{\partial\beta} . \qquad \text{(A.28)}$$

We first apply this operation to $(s_o/s)^{1/2}$. Since

$$\frac{\partial s_o}{\partial\beta} = -\frac{1}{2a}\,\frac{\sin\alpha - \frac{1}{2}\sin\beta}{\cos^4\frac{\beta-\alpha}{2}} ,$$

it follows that

$$\frac{\partial}{\partial n}\left(\frac{s_o}{s}\right)^{1/2} = \frac{1}{2}\left(\frac{s_o}{s}\right)^{1/2}\left\{\frac{1}{s_o}\,\frac{\tan\frac{\beta+\alpha}{2}}{\cos\frac{\beta-\alpha}{2}}\left(\sin\alpha - \frac{1}{2}\sin\beta\right) - \frac{1}{s}\left(\cos\frac{\beta+\alpha}{2} + 3\,\frac{\sin\alpha\sin\frac{\beta+\alpha}{2}}{\cos\beta+\cos\alpha}\right)\right\}$$

and when $s=s_o$, this reduces to

$$\left.\frac{\partial}{\partial n}\left(\frac{s_o}{s}\right)^{1/2}\right|_{s=s_o} = -\frac{1}{2s_o\cos\frac{\beta+\alpha}{2}} .$$

Hence

$$\left.\frac{\partial}{\partial n}\left(\frac{s_o}{s}\right)^{1/2}\right|_{s=s_o} = -a\,\frac{\cos^3\frac{\phi}{2}}{\cos\left(\alpha+\frac{\phi}{2}\right)} . \qquad \text{(A.29)}$$

Our final task is to apply the operation (A.28) to $\psi(s,\beta)$. Since

$$\psi(s,\beta) = \psi(s_0,\beta)+s-s_0$$

it is obvious that

$$\frac{\partial\psi}{\partial s} = 1 .$$

Using the known expressions for $\psi(s_0,\beta)$ and s_0, it can also be shown that

$$\frac{\partial\psi}{\partial\beta} = \frac{3}{4a}\sin\alpha\sec^4\frac{\beta-\alpha}{2} ,$$

and on substituting these into Eq. (A.28), we find

$$\frac{\partial\psi}{\partial n} = \cos\frac{\beta+\alpha}{2} .$$

In terms of the oridinal coordinate ϕ,

$$\frac{\partial\psi}{\partial n} = \cos\left(\alpha+\frac{\phi}{2}\right) . \qquad \text{(A.30)}$$

Appendix B — Some Integral Expressions

The results that we require can be deduced from the discontinuous integral of Weber and Schafheitlin (Watson, 1948, pp. 398 et seq.). From formulae (4) and (5) on p. 405,

$$\int_0^\infty J_\nu(t)\, e^{\pm it\sin\gamma}\, dt = \sec\gamma\, e^{\pm i\nu\gamma}, \quad \mathrm{Re}\,\nu > -1 ,$$

and since

$$H_\nu^{(1)}(t) = \left\{ ie^{-i\nu\pi} J_\nu(t) - i J_{-\nu}(t) \right\} \operatorname{cosec} \nu\pi ,$$

we have

$$\int_0^\infty H_\nu^{(1)}(t)\, e^{\pm it\sin\gamma}\, dt = 2\sec\gamma\, e^{-i\nu\pi/2}\, \frac{\sin\nu\left(\frac{\pi}{2} \mp \gamma\right)}{\sin\nu\pi} \qquad \text{(B. 1)}$$

valid for $|\mathrm{Re}.\nu| < 1$. In particular, for $\nu = 0$,

$$\int_0^\infty H_0^{(1)}(t) e^{\pm it\sin\gamma}\, dt = \sec\gamma\left(1 \mp \frac{2\gamma}{\pi}\right) . \qquad \text{(B.2)}$$

Differentiation of Eq. (B. 1) with respect to γ yields

$$\int_0^\infty H_\nu^{(1)}(t)\, t e^{\pm it\sin\gamma}\, dt = 2i\sec^3\gamma \operatorname{cosec}\nu\pi\, e^{-i\nu\pi/2} \left\{ \nu\cos\gamma\cos\nu\left(\frac{\pi}{2} \mp \gamma\right) \mp \sin\gamma\sin\nu\left(\frac{\pi}{2} \mp \gamma\right) \right\} .$$

The integral converges when $\nu = 1$ and hence, on taking the limit as $\nu \rightarrow 1$,

$$\int_0^{\infty} H_1^{(1)}(t) t e^{\pm it \sin \gamma} dt = \sec^3 \gamma \left\{ 1 \mp \frac{2}{\pi} (\sin \gamma \cos \gamma + \gamma) \right\}, \qquad \text{(B.3)}$$

which is the other result that is needed.