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# STUDIES IN RADAR CROSS SECTIONS XL — SURFACE ROUGHNESS AND IMPEDANCE BOUNDARY CONDITIONS

by

R. E. Hiatt, T. B. A. Senior and V. H. Weston

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## Preface

This is the fortieth in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers; and (d) low and high density ionization phenomena.

K. M. Siegel

## Foreword

This report is concerned with the effect of minor surface roughnesses on the scattering cross sections of targets, and like Gaul it is divided into three parts. In discussing these it is most convenient to consider them in reverse order.

In Part III it is shown that when an electromagnetic field is incident on a perfectly conducting surface having a random (but statistically uniform) distribution of small geometrical irregularities, the boundary condition can be replaced by a generalized impedance condition applied at a neighboring mean surface. The surface impedance is, in general, a tensor function of the direction at which the field is incident as well as of the statistical properties of the irregularities, but in certain instances the tensor nature either disappears or can be suppressed. In these cases, the impedance is a scalar and the boundary condition then reduces to the standard form of the Leontovich or impedance boundary condition. A condition of this type is frequently employed at the (smooth) surface of a material of large but finite refractive index, and is discussed in detail in Part II; its appearance in the present work implies that to some extent a rough but perfectly conducting surface behaves in the same manner as a smooth but imperfectly conducting one, and enables us to trade roughness for conductivity.

Part I is primarily concerned with minor surface roughnesses as they affect model scattering experiments. The boundary condition derived in

Part III is here used to determine the scattering cross section of a large rough sphere, and the results are then compared with experiment. It is found that even for a sphere whose 'depth' of roughness is as large as $10^{-2}\lambda$, the measured change in cross section is no more than about 0.1 db. This is in good agreement with the theoretical prediction and goes some way to confirm the usefulness of an approach whereby the roughness is incorporated in the boundary condition.

As is true in so many fields of basic research, once results are found they are applicable to many problems. In any investigation of modeling, it is necessary to know the required tolerance on the surface finish of the target, and it is desirable to study this first in the case of linear modeling before proceeding to the more complex subject of non-linear modeling. Likewise, the results have application to radar camouflage problems, since the surface roughness of either the camouflage material or of the body to which it is applied will affect the performance of the material. Roughness considerations are also important in the field of target discrimination, as they are in any detailed study of the radar scattering properties of large (natural) bodies such as the earth or moon. And finally, it almost goes without saying that the techniques developed in this report can be used to investigate the behavior of rough surfaces in acoustics.

In consequence, it is felt that the results obtained here are significant to a number of contracts held by the Radiation Laboratory, and we are happy to acknowledge that the work was supported jointly by the Air Force Cambridge

Research Center under Contracts AF 19(604)-5470 and AF 19(604)-4993, by the Rome Air Development Center under Contracts AF 30(602)-1808 and AF 30(602)-2099, and by the Autometric Corporation under Subcontract 33-S-101.

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# PART I

# THE EFFECT OF SURFACE ROUGHNESS ON SCATTERING CROSS SECTIONS

## §1. Introduction

During the last two years it has become apparent that a difference of opinion exists as to the influence of surface imperfections in model scattering experiments. On the one hand there are those who believe that an RMS surface finish good to $10^{-5}\lambda$ (approx.) is required if the effects of surface roughness are to be discounted, and that an increase in the roughness to $10^{-4}\lambda$ could produce a detectable change (of order 1 db) in the scattering cross section. In comparison with this, a tolerance of about $10^{-3}\lambda$ on the absolute dimensions of the body is regarded as sufficient.

The above viewpoint is held by several experimentalists of considerable reputation, and if the restrictions are indeed necessary it is questionable whether the scattering cross section of any practical shape can be predicted satisfactorily by means of model experiments. For example, individual rivets would then become important.

On the other hand, there are many who do not accept the necessity for these restrictions, and who feel that surface imperfections of as much as $10^{-2}\lambda$ will seldom (if ever) affect the scattering cross section in any detectable manner. The only exceptions are those cases where the return from the smooth (unperturbed) body is either small in magnitude (as in backscattering from an infinite cone nose-on), or the result of a particular phase relationship which is seriously disturbed by the presence of the roughness. This opinion is shared by the authors, and as a contribution towards a better understanding of roughness effects in general, some results obtained from a study of a particular type of roughness are presented here.

The type of roughness considered here is one in which the surface irregularities are distributed at random, but in a statistically uniform and isotropic manner. The surface slopes are assumed small, and the minimum (effective) radius of curvature (or dimension) of the mean (unperturbed) surface is assumed large compared with the wavelength. The effects of the surface roughness can then be discussed in terms of an impedance boundary condition applied at the mean surface, and this approach is described briefly in §2. As an illustration the method has been used to determine the back scattering cross section of a rough sphere, and the results obtained appear in §3. The details of the analysis are given in the Appendix to this Part.

In order to test the theoretical predictions a series of experiments has been carried out in which the scattering cross section of a suitably chosen rough sphere has been measured relative to the cross section of a smooth sphere of about the same diameter. Three different frequencies were employed thereby simulating three different scales of roughness. The results are presented in §5 and confirm that even for a sphere whose roughness depth is as large as $10^{-2}\,\lambda$ the change in cross section is no more than about 0.1 db. This is in reasonable agreement with the theory.

§2. Approximate Boundary Conditions

Let us consider first an infinite perfectly conducting plane which is perturbed in some manner so as to yield a type of rough surface. Let $z = \zeta(x,y)$ be the amplitude of the perturbation measured from a mean surface which, for convenience, is taken to be the plane $z = 0$. Then if the perturbed surface is defined in a statistical manner so that $\zeta$ is effectively a random variable, and if the statistical properties are uniform and isotropic, the boundary conditions at the actual surface $z = \zeta$ can be written as relations connecting the tangential components of the electric and magnetic fields at the mean surface $z = 0$. The only characteristics of the surface which enter into these equations are the correlation function F (and its derivatives) and the standard deviation $\zeta_0$ of the amplitudes. It is assumed that F is a function of the distance $\rho$ between neighboring points on the surface, and falls rapidly to zero for $\rho >> \ell$, where $\ell$ can be interpreted as the scale of the irregularities (or the size of a typical "hump"). The details of the analysis are given in Part III, and it is there shown that the above results are valid providing $\zeta$ and its first derivatives are continuous and the slope of the surface is everywhere small. In the practical case to be investigated here we shall only be concerned with values of $\ell$ for which $k\ell < 1$, where $k = 2\pi/\lambda$, and a sufficient condition upon the slope is then $\zeta_0 << \ell$.

The boundary conditions on the mean surface are functions of the angle at which the field in incident, and in any general application of the conditions this variation is a severe handicap. For the present purposes, however, only the approximate magnitude of the perturbation effect is required, and it seems reasonable to expect that the accuracy of the boundary condition will not be seriously impaired if an average is taken over all directions of the incident field. The boundary conditions which result from the averaging process are

$$E_x = -\eta Z H_y \tag{1}$$

$$E_y = \eta Z H_x \tag{2}$$

where $Z = 1/Y$ is the intrinsic impedance of free space and $\eta$ is a parameter defined in terms of the surface characteristics by the equation

$$\eta = -\frac{ik\zeta_0^2}{4}\left[ik + \int_0^\infty \left\{\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + k^2\right)\left(F J_0\left(\frac{k\rho}{\sqrt{2}}\right)\right) - \frac{k}{\sqrt{2}}\frac{\partial F}{\partial\rho} J_1\left(\frac{k\rho}{\sqrt{2}}\right)\right\} e^{ik\rho}\, d\rho\right] \tag{3}$$

Equations (1) and (2) can be written as

$$\underline{E} - (\hat{n} \cdot \underline{E})\hat{n} = \eta Z \hat{n} \wedge \underline{H} \quad , \tag{4}$$

where $\hat{n}$ is a unit vector normal in the outwards direction, and this will be recognized as the usual impedance boundary condition for a material whose effective surface impedance is $\eta$.

A boundary condition of the form (4) is frequently applied at the surface of a medium whose refractive index N is large compared with unity; $\eta$ is then interpreted as a function of the electrical properties of the material and is proportional to 1/N. A rigorous derivation of the boundary condition as applied to such materials is given in Part II, where it is shown that (4) is also valid for surfaces of varying curvature providing

$$\left|\text{Im } N\right| kd \gg 1 \quad ,$$

where d is the smallest radius of curvature (or dimension) of the surface. If the permeability $\mu$ is not large compared with $\mu_0$ , a sufficient restriction upon d is

$$kd \gg \left| \operatorname{Im} \eta \right| . \tag{5}$$

Returning now to the boundary condition at a rough surface, this can be generalized so as to apply to a mean surface which is curved by an analysis similar in all respects to that given in Part II, providing the minimum radius of curvature (or dimension) of the mean surface satisfies the inequality (5). In addition, it is noted that the roughness parameter $\eta$ enters into the problem only via the boundary condition (4), and accordingly a rough (but perfectly conducting) surface is equivalent to an imperfectly conducting (but smooth) surface so far as its scattering properties are concerned. This enables us to associate an effective conductivity s with the rough surface. In the particular case $k\ell \ll 1$, equation (3) gives

$$\eta \sim i \frac{\sqrt{\pi}}{2} \frac{k\zeta_0^2}{\ell} \tag{6}$$

and hence for small scale roughnesses,

$$s \sim i \frac{4}{\pi} Y \frac{\ell^2}{k\zeta_0^4} \text{ mhos/m} . \tag{7}$$

As an example, if $k\ell = 1/5$ and $k\zeta_0 = 1/100$,

$$s \sim \frac{10^5}{\lambda} \text{ mhos/m}$$

and at X band this is comparable to the conductivity of ordinary metals. In this instance at least it would not appear that the roughness can have any appreciable effect.

## § 3. Scattering by a Rough Sphere

The impedance condition (4) is an approximation to the exact boundary conditions, and apart from any errors introduced by the averaging over the incident field directions, (4) is correct to the first order in $\eta$. Accordingly, in any solution obtained using this condition there is no physical justification for retaining terms which are of a higher order in $\eta$; and in consequence, if the fields are capable of expansion in series of ascending (positive) powers of $\eta$, the perfectly smooth approximation (corresponding to $\eta = 0$) can be inserted into the right-hand side of (4). In general, such expansions will be valid and lead to solutions which are essentially "perturbations" about the solutions for the surface without roughness.

We shall now use this fact to determine the backscattered field when a plane wave is incident on a uniformly rough sphere for which $ka \gg 1$, where a is the mean radius. If the incident field is polarized with its electric vector in the x direction, the scattered electric field at a distance R from the center of the sphere is

$$E_x = -\frac{a}{2R-a}\, e^{ik(R-2a)} \left\{ A_o + \frac{A_1}{ka} + O(\overline{ka}^{-2}) \right\} \tag{8}$$

where

$$A_o = 1 - 2\eta \tag{9}$$

$$A_1 = -\frac{2i(R-a)^2}{(2R-a)^2}(1-2\eta) + \frac{2R}{2R-a}(1-i)\eta \quad , \tag{10}$$

and this result is valid if $|\eta| \ll (ka)^{-1/3}$. The detailed analysis is given in the Appendix to this Part.

Equation (8) is of particular interest in showing the variation of the roughness effect as a function of the distance R. The dominant contribution to the overall effect is provided by the term $A_0$ and this is independent of R. The first contribution which is a function of R comes from the term $A_1$ and is reduced in magnitude by the (large) factor ka in the denominator. As $R \rightarrow \infty$, $A_1 \rightarrow -\frac{i}{2}(1 + 2i\eta)$ but as the receiver approaches the sphere $A_1 \rightarrow 2(1 - i)\eta$. Since $|\eta|$ is small compared with unity, the ratio of these two terms is approximately $4(1 + i)\eta$, which implies a decrease in the effect of surface roughness as the receiver moves into the near field. In practice, however, it is unlikely that such a change would be detected in view of the factor ka by which the term $A_1$ is divided, and to a first approximation $A_1$ and all subsequent terms can be neglected. The magnitude of the scattered field is then

$$|E_x| \quad \frac{a}{2R - a} \; |1 - 2\eta| \tag{11}$$

which only differs from the "smooth" result by at most a few percent for the type of roughness being considered here. Moreover, for $k\ell \ll 1$, $\eta$ is purely imaginary and equation (11) shows that the cross section is increased by the presence of the small roughness. As $k\ell$ increases, however, the approximate formula (6) ultimately ceases to apply, and the impedance assumes a resistive part as indicated by equation (3); the cross section of the sphere may then be either increased or decreased depending on the relative magnitudes of the real and imaginary parts of $\eta$. This is discussed at more length in Part II.

## § 4. An Experiment

To test the above conclusions and, at the same time, to obtain some direct measurements of the effect of roughness, an experiment was carried out in which the back scattering cross section of a rough sphere was measured against the cross section of a smooth (standard sphere at a variety of different distances ranging from 16 feet down to (about) 6 inches. Each sphere was case in aluminum, a hemisphere at a time, and it was found that the casting process could be modified so as to provide a suitable degree of roughness. The first sphere was left in its rough initial state, while the second was machined to give a smooth sphere of radius approximately equal to the mean radius of the other. The dimensions (in cm) were found to be as follows:

| | Rough Sphere | Smooth Sphere |
|---|---|---|
| a | $12.857 \pm 0.013$ | $12.697 \pm 0.010$ |
| $\zeta_0$ | 0.037 | ------ |
| $\ell$ | 0.101 | ------ |

where a is the mean radius (the variation is a consequence of slight asymmetries), $\zeta_0$ is the RMS amplitude of the roughness, and $\ell$ is the scale. The measurement of $\zeta_0$ and $\ell$ was made using a vernier caliper and although there was some variation from point to point on the sphere, the above values are typical of those obtained.

The two spheres are shown in Figure 1 and the close up photograph of the rough sphere in Figure 2 gives some idea of both the type of surface and the degree of roughness.

In order to simulate three different degrees of roughness, the cross sections of the two spheres were measured at the frequencies 2.87, 9.7 and 23 KMc, corresponding to the wavelengths 10.5, 3.1 and 1.3 cm respectively. The measurements were made in an indoor anechoic room 30 feet wide by 60 feet long using conventional

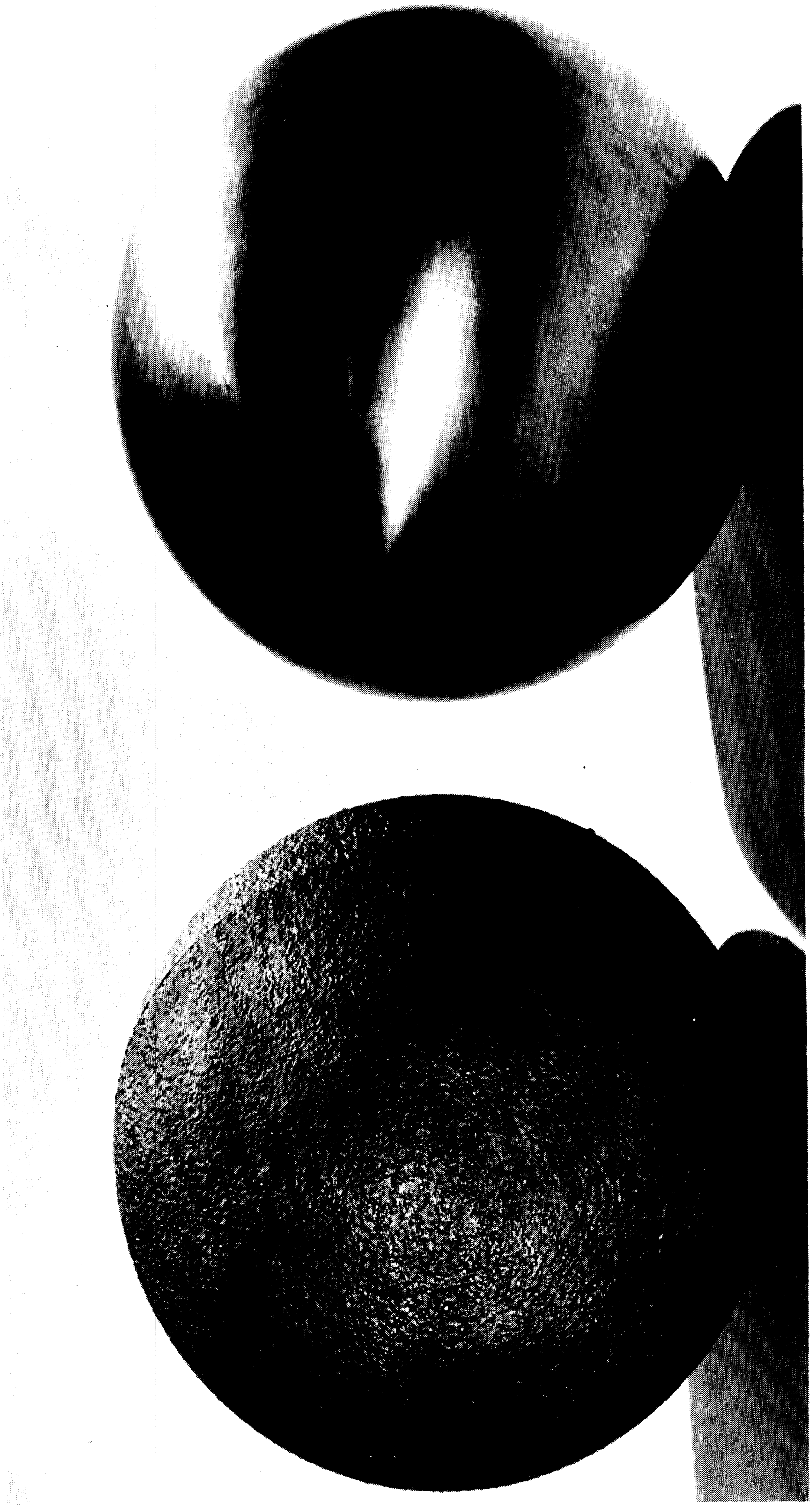

Figure 1. 10-inch Metal Spheres, Rough and Smooth Surface

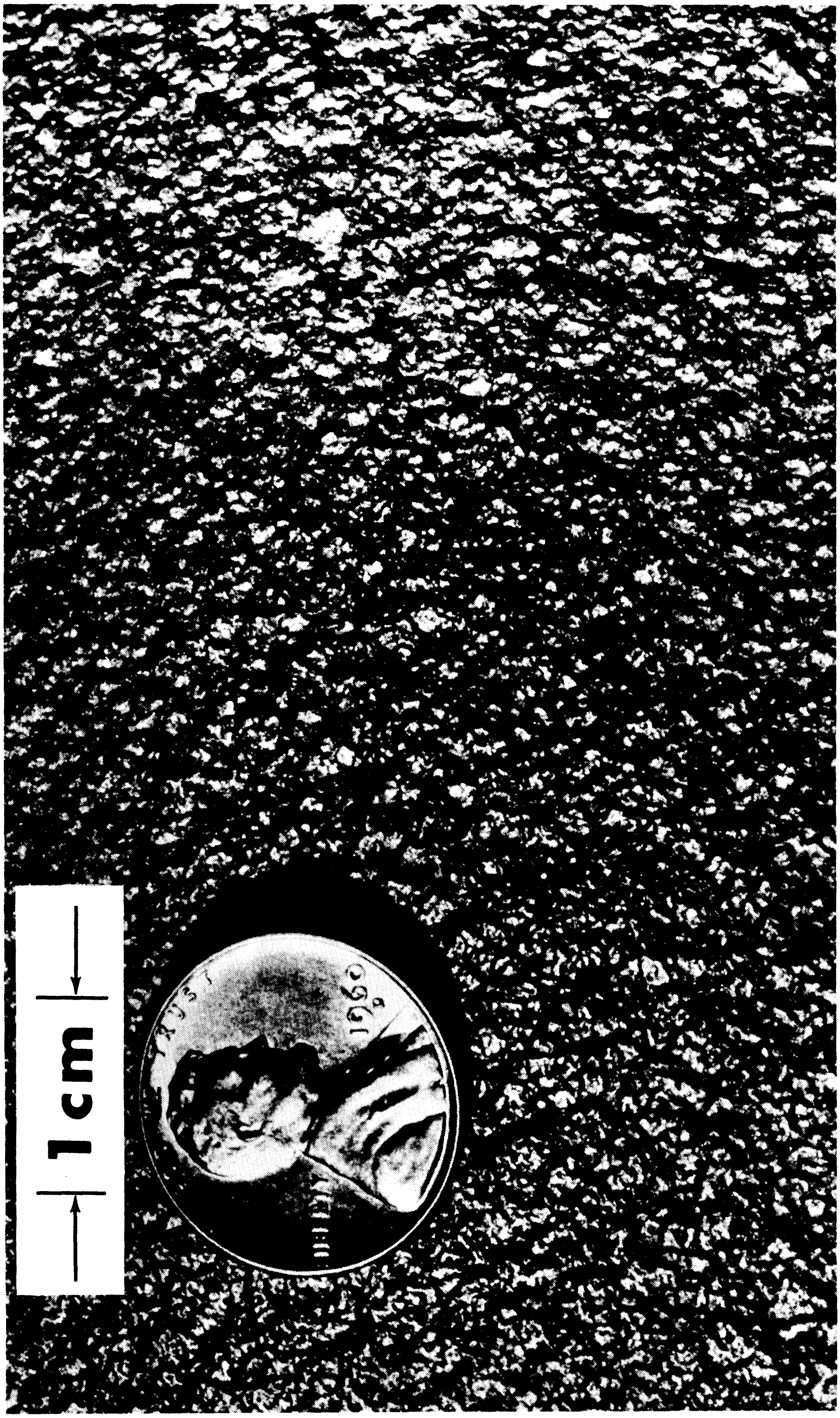


Figure 2. Surface Condition of Rough 10-inch Sphere

equipment and technique. Particular care was taken to achieve the greatest possible accuracy, and it is believed that the results for the relative cross sections are good to about 0.2 db.

A block diagram of the equipment is shown in Figure 3. At X band a cavity stabilized oscillator was employed, and at the S and K band frequencies the stability was obtained from a crystal oscillator. The receiver was of the microwave superheterodyne type using a separate mixer for each frequency band. The models were supported on a styrofoam column resting on a pedestal which could be rotated about its axis, and this in turn was mounted on a trolley to facilitate measurements as a function of range. A photograph of the room and part of the equipment is given in Figure 4.

The comparison between the cross sections of the spheres was carried out in two different ways. In the first, the cross sections of the spheres were individually recorded as each was rotated through $360^{o}$. This procedure proved adequate at the lowest frequency where the roughness produced a negligible effect. At the higher frequencies point by point data was taken in addition to the $360^{o}$ plots. In obtaining this further data eight points on the rough sphere were selected, four on each hemisphere, in such a way that the plane of the junction between the two hemispheres was never parallel or perpendicular to either the direction of the illuminating beam or the electric vector. The antenna beam was then "directed" successively at each of these points, and the average signal recorded as the range was varied through $\pm\lambda/4$. The contribution due to the background was thereby minimized. The change in this contribution as a function of range could generally be kept to less than one or two decibels, and much of the time it was no more than one.

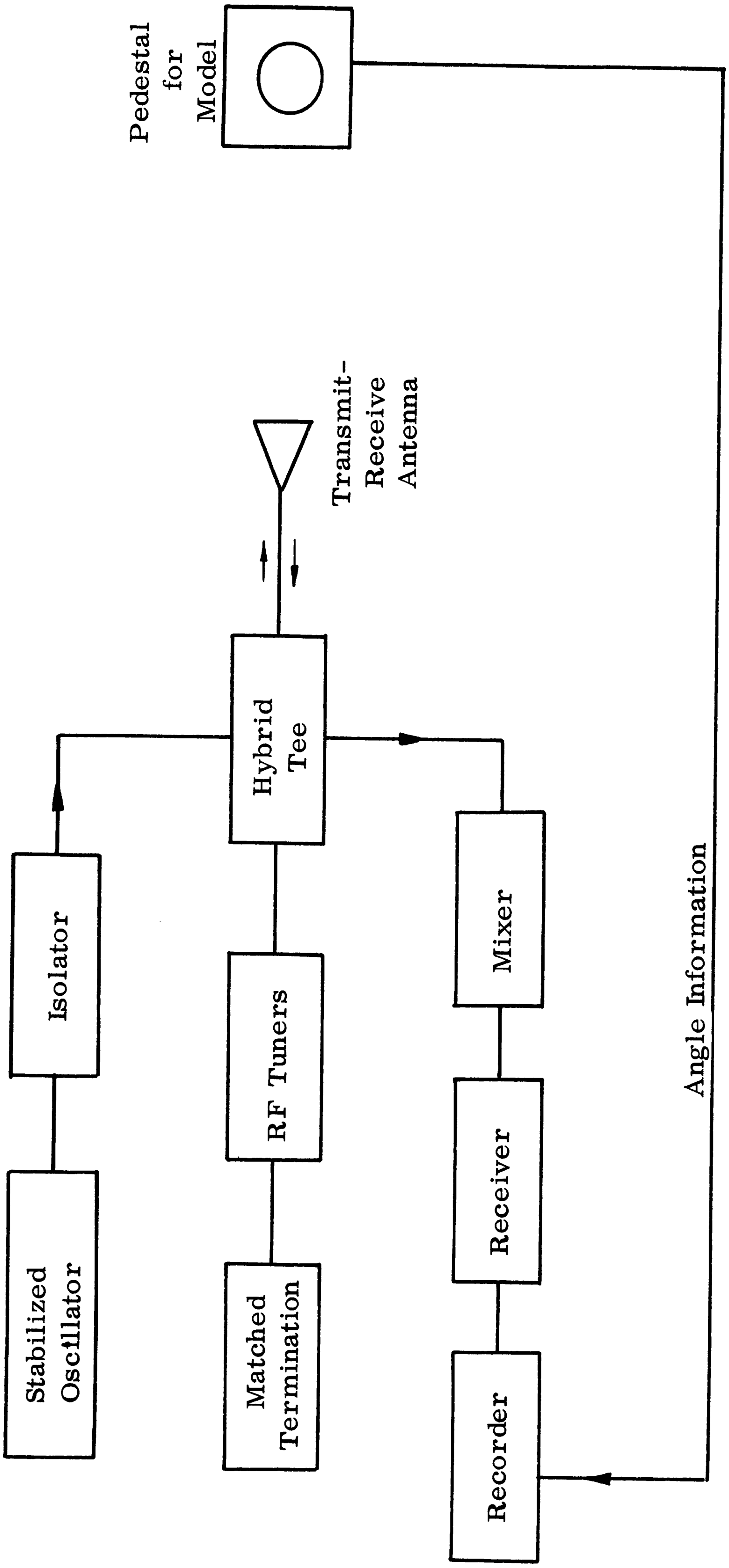


Figure 3. Block Diagram of Equipment

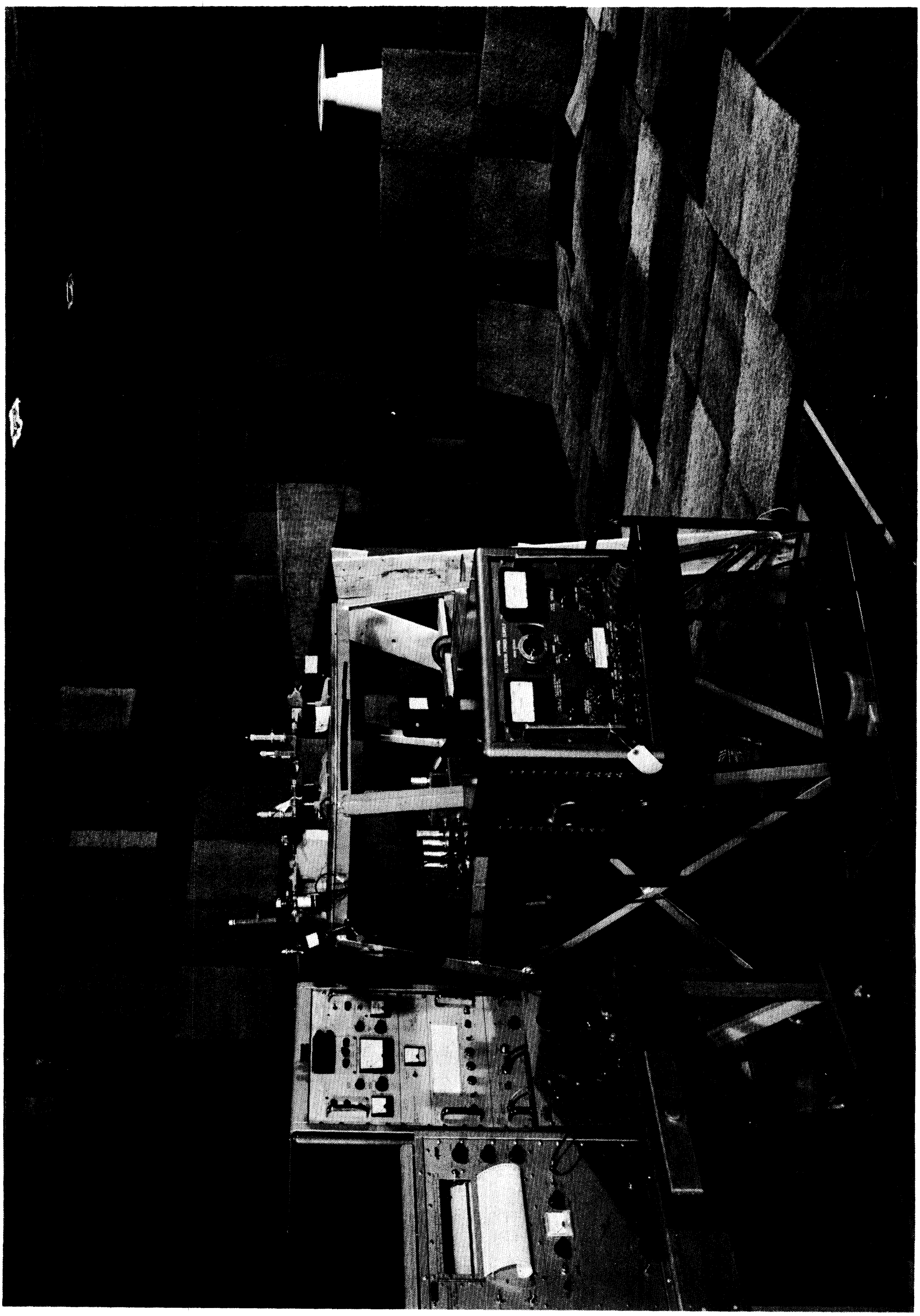

Figure 4. Anechoic Room and K-Band Equipment

Interspersed between these eight readings, three readings were obtained with the smooth sphere, and the difference in averages was then recorded or plotted as one point (see, for example, Figure 7). Point by point data of this type was obtained at both X and K bands, although at 23 KMc the number of readings was further increased to 24 by rotating the sphere through $\pm 5^{\circ}$ at each of the above-mentioned eight points.

## §5. Results

At all the frequencies at which the experimental work was carried out the values of $k\ell$ are small compared with unity and since $\zeta_0 << \ell$ equation (6) can be used to calculate the effective surface impedance consequent upon the presence of the roughness. Thus we have

$$\lambda = 10.5 \text{ cm} , \qquad \eta = 0.009\, i ,$$

$$\lambda = 3.1 \text{ cm} , \qquad \eta = 0.03\, i ,$$

$$\lambda = 1.3 \text{ cm} , \qquad \eta = 0.07\, i ,$$

Using now equation (8), the roughness is found to increase the back scattering cross section of the sphere by an amount which increases from $2 \times 10^{-3}$ db at $\lambda = 10.5$ cm , through $2 \times 10^{-2}$ db at $\lambda = 3.1$ cm to $10^{-1}$ db at $\lambda = 1.3$ cm . In addition, however, there is the change in the cross section of the rough sphere over the smooth (standard) sphere produced by its larger mean radius. At S band where the sphere is near the upper end of the resonant region the change is - 0.1 db; for the X and K band frequencies the change is 0.1 db. These changes plus the theory outlined in § 2 and § 3 then predict that the cross section of the rough sphere will exceed the cross section of the smooth (standard) sphere by the following amounts:

$\lambda$ = 10.5 cm , -0.10 db ,

$\lambda$ = 3.1 cm , 0.12 db ,

$\lambda$ = 1.3 cm , 0.20 db .

Before going on to compare these with the values found experimentally, it may be of interest to list the degrees of roughness appropriate to the frequencies employed. If d denotes the total depth of the roughness (approximately 2 $\zeta_0$) and w denotes the total width of a typical bump (rather than the width between 3 db points used in the specification of the scale $\ell$ ), the various parameters are:

| $\lambda$(cm) | ka | $d/\lambda$ | $w/\lambda$ |
|---|---|---|---|
| 10.5 | 7.6 | $7 \times 10^{-3}$ | $3 \times 10^{-2}$ |
| 3.1 | 25.4 | $2 \times 10^{-2}$ | $10^{-1}$ |
| 1.3 | ~~77.5~~ 621.~~8~~ | $5 \times 10^{-2}$ | $2 \times 10^{-1}$ |

In view of the large values of ka it is not to be expected that any change in the relative cross sections as a function of distance will be detectable.

In Figure 5 the experimental results at 2.87 KMc are shown in the form of $360^{\circ}$ plots for three different ranges. Each plot contains four traces -- one for the smooth sphere and one for each of the three orientations of the rough sphere -- and in general the traces are more or less coincident with one another. By inspection of these traces (and other similar traces not presented here), it is concluded that there is no measurable effect due to the roughness at this frequency. In passing it should be noted that the thickness of the

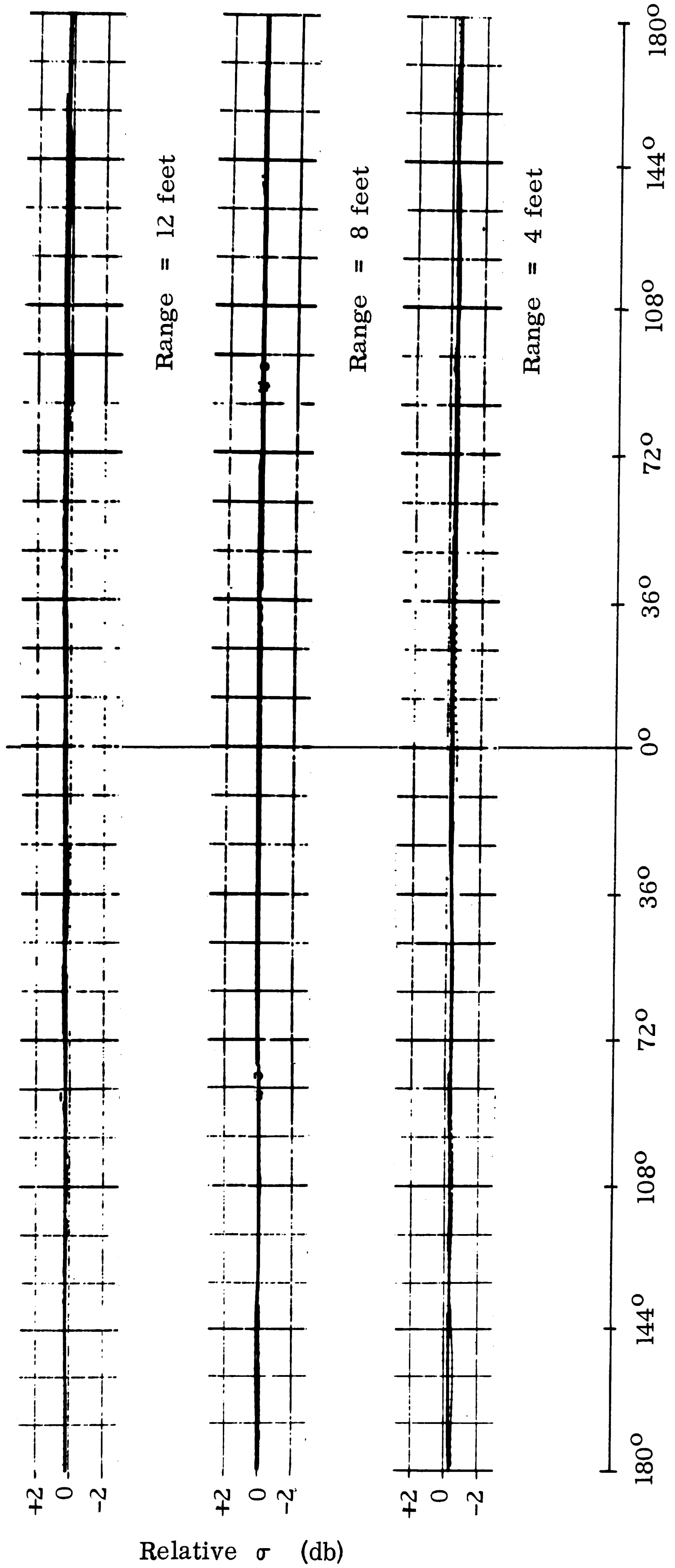


Figure 5. Relative Radar Cross Section, $\sigma$, of Smooth and Rough Sphere at $\lambda = 10$ cm. Each of the Three Multiple Traces Shows $\sigma$ for the Smooth Sphere and for Three Different Orientations of the Rough Sphere.

traces in Figure 5 is of order 0.1 db, and accordingly a more detailed analysis would be necessary if the predicted change due to the different sphere radii is to be detected. Since this change is not truly a roughness effect, no such analysis was performed.

The 360$^{o}$ plots at 9.7 KMc are shown in Figure 6. The effect of the surface roughness is quite apparent here, but the peak deviations from the smooth sphere return are limited to about 1 db, and the average difference in the returns is even less. This is brought out more clearly in Figure 7 in which the point by point measurements are shown as a function of the range R. For comparison with the Rayleigh distance, the maximum range (R = 16 feet) is equivalent to $R = 9.4\,(a^2/\lambda)$, where a is the sphere radius, and at the minimum range (R = 6 inches) $R = 0.29\,(a^2/\lambda)$.

The results in Figure 7 show no statistically significant range dependence, though there appears to be a tendency for the relative cross section to decrease with decreasing range. This is in accordance with the theory. When all the points in Figure 7 are averaged regardless of range, the cross section of the rough sphere is found to be 0.12 db above that for the smooth sphere, and whilst the standard deviation of the experimental values is somewhat large (0.40 db), the average is in truly remarkable agreement with the theory. The extent of the agreement is, perhaps, a little fortuitous, but does provide confirmation of the theoretical approach.

The final set of 360$^{o}$ plots are given in Figure 8 and are for 23 KMc. The surface roughness now has a marked effect, and the peak deviations from the cross section of the smooth sphere are as much as 4 db. The multiple traces shown result from changing the range by $\pm\,\lambda/4$ and serve to indicate the effect of the

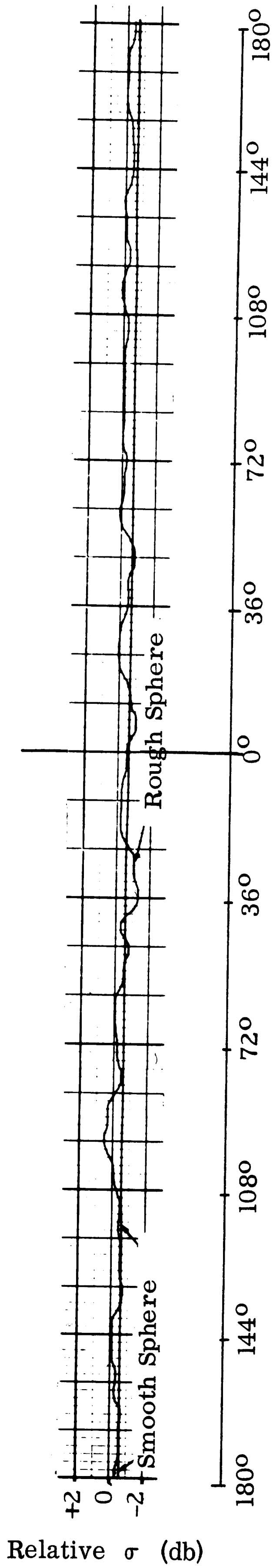


Figure 6. Relative Radar Cross Section, $\sigma$, of Smooth and Rough Sphere at $\lambda = 3.1$ cm

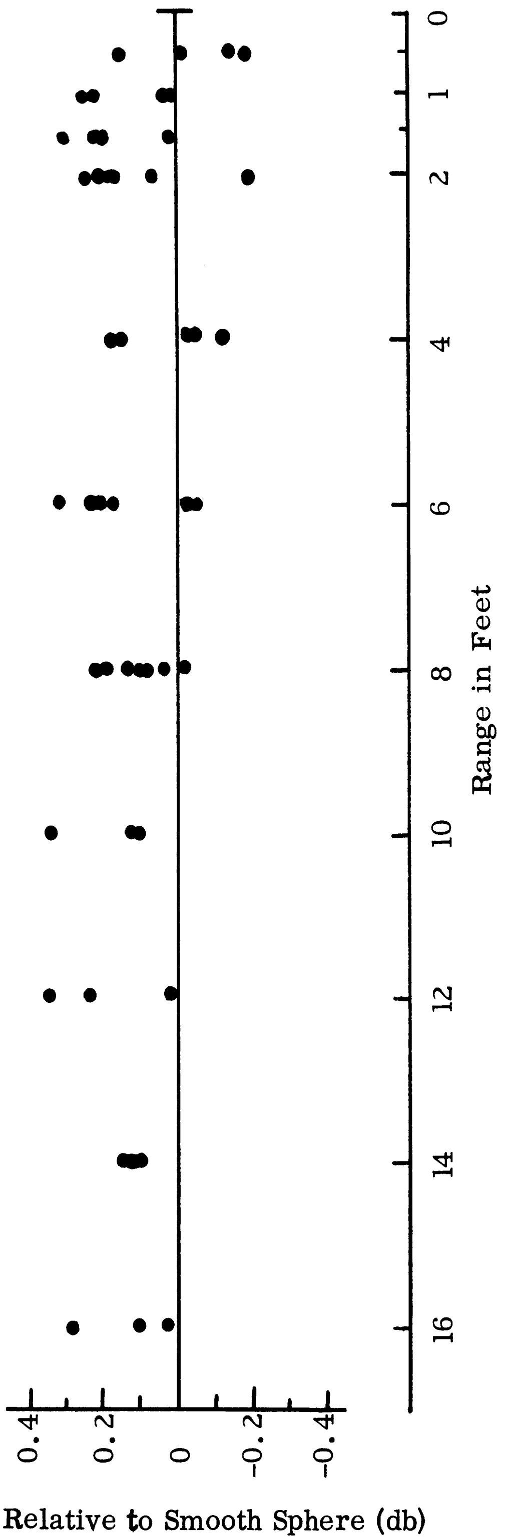


Figure 7. Effect of Range on the Ratio of $\sigma$ of Rough to Smooth Sphere $\lambda$ = 3.1 cm. Points Plotted are the Average of 16 or More Readings.

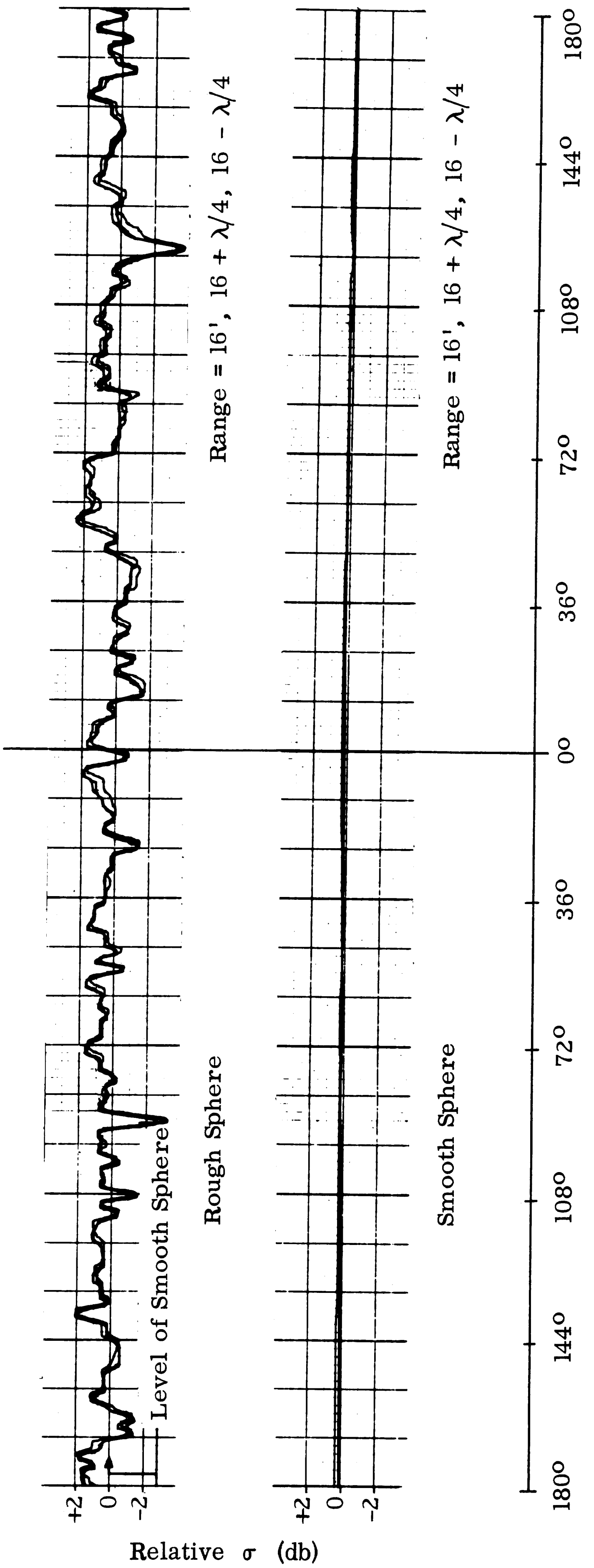


Figure 8. Radar Cross Section, $\sigma$, of Rough and Smooth Sphere at $\lambda = 1.3$ cm

background signal. Such traces as these were entirely reproducible, and providing sufficient care was taken the sphere could be removed from its pedestal and then replaced, with the same traces obtained once again.

At this frequency the bumps on the sphere are about $\lambda/5$ wide by $\lambda/20$ deep and it seems probable that the bumps are here acting singly or in combination of 2 or 3 at a time to produce the individual features in the traces. The fine structure in the traces is no more than $2^{o}$ in width and corresponds to a displacement of the sphere's surface of approximately 0.4 cm . It can be seen from Figure 2 that this is comparable to the width of the bumps, and under these circumstances a theory based on the random addition of the returns from many small irregularities is clearly inappropriate. It therefore comes as no surprise that the predicted change in cross section differs from that observed.

Information on the average change in measured cross section was obtained by the point by point method. Almost 800 readings were averaged regardless of range and showed that the cross section of the rough sphere exceeded that of the standard by 0.51 db . The standard deviation of the points was, however, 1.04 db .

In order to facilitate comparison of the returns at the three frequencies, sample recordings of the rough sphere data are given in Figure 9. The way in which the roughness effect increases with increasing frequency is clearly visible.

## 6. Theoretical Discussion

The theory outlined in §2 and §3 is based on an impedance boundary condition derived in Part III. This condition is accurate to the first order

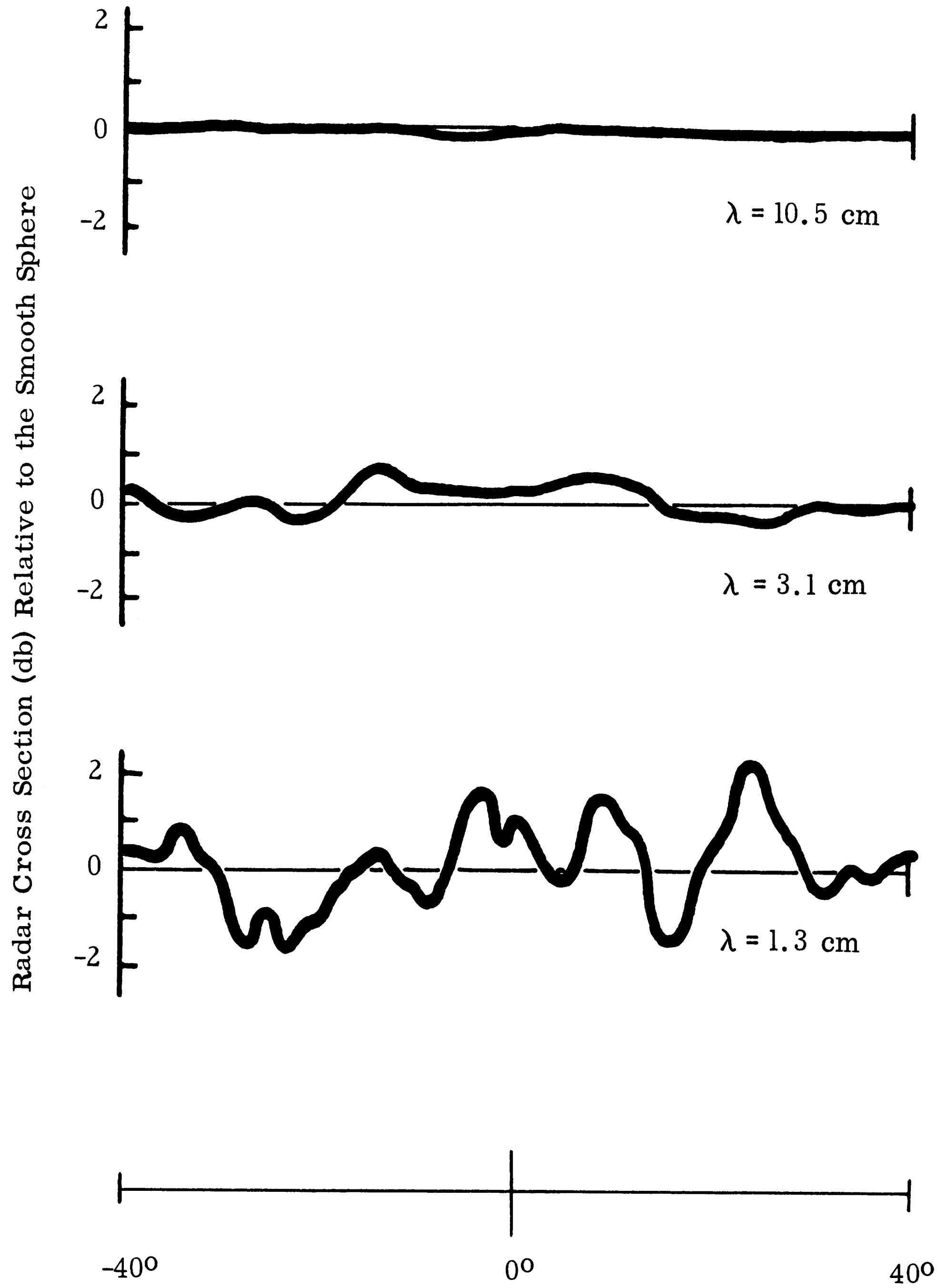


Figure 9. Effect of Surface Roughness on Backscatter Pattern of 25 cm Metal Sphere. Average Bump Size About 0.7 mm Deep by 3 mm Wide.

in the roughness effect providing the inclination of the actual surface to a mean surface is everywhere small, and providing the irregularities are distributed at random but in a statistically uniform and isotropic manner. If these restrictions are fulfilled, the main effect of the surface roughness is to slightly modify in phase and amplitude the field scattered by each surface element of the smooth body.

If any of the above restrictions are relaxed, the boundary condition may cease to hold. Thus, if the slopes of the irregularities are large there is the possibility of scattering taking place from the sides of the individual humps, so producing a field in a direction other than that for the smooth body and of a magnitude which is no longer negligible. If the surface of an infinite cone were roughened in this manner, a contribution could be expected which was not from the tip. Similarly, if the irregularities are not distributed at random, then in certain directions the fields produced by the individual element may add up in phase, and here again the boundary condition is not applicable. As an example of this, if small concentric grooves are cut in the sides of an infinite cone, a first order modification to the field may result, particularly for backscattering in a direction normal to the rings.

If the surface irregularities do not satisfy the above restrictions, alternative methods must be employed for assessing the effect of the surface roughness, and only for certain special types of irregularity are appropriate methods available. Thus, for one or more isolated bumps whose dimensions are small compared with the wavelength, the total scattered field can be obtained by using the Rayleigh scattering formula for each individual bump and neglecting the interaction with the field of the smooth body. Since the cross section in Rayleigh scattering is proportional to the sixth power of a linear dimension, the percentage change in the total scattering cross section will be small unless the field of the unperturbed body is itself small, or the number of bumps is large. If, on the other hand, the surface perturbations are of a very regular kind and can be approximated by a series of corrugations, the effect may be estimated by using the known solutions for scattering by a corrugated sheet (Senior[1]) or by a corrugated cylinder (Clemmow and Weston[2]). In this case, the perturbation field may be comparable to the field of the smooth body.

## §7. Conclusions

In embarking on a study of surface roughness and its effect on radar scattering cross sections, one of the objectives was to consider the degree of surface finish which is necessary in model scattering experiments. As part of the experimental program the back scattering cross section of a suitably chosen rough sphere has been measured at S, X and K band frequencies and compared with the cross section of a smooth sphere of approximately the same diameter. It was found that even with a surface roughness which would normally be regarded as completely unacceptable for model work the change in cross section due to roughness was relatively small. Thus, at X band the sphere used had a roughness whose depth was $0.02\,\lambda$, but still the average change in cross section which could be attributed to the roughness was less than 0.1 db, and at S band no change could be detected. As is to be expected, the effect increases with increasing frequency, and at K band where the bumps were $0.05\,\lambda$ in depth the scattering patterns were quite irregular.

On the theoretical side an analysis of the general problem of roughness has shown that geometrical irregularities characterized by small surface gradients and random but statistically uniform properties can be handled by the usual type of impedance boundary condition. This implies that the roughness produces a comparable effect to a change in the conductivity of the unperturbed surface. Results obtained with this approach are in good agreement with the experimental data.

The authors wish to express their thanks to Theodore Hon for his assistance with the experimental work.

# APPENDIX

## Backscattering from a Large Rough Sphere

Following the theory outlined in Section 2 it is assumed that the surface roughness produces an effective surface impedance $\eta$. The boundary condition which is applied at the surface $R = a$ is then

$$\underline{E} - (\hat{n} \cdot \underline{E})\,\hat{n} = \eta\, Z\, \hat{n} \wedge \underline{H} \quad . \tag{A1}$$

If the incident field is a plane wave travelling in the positive z direction with an electric vector

$$\begin{aligned} \underline{E}^{i} &= \hat{i}_x\, e^{ikz - i\omega t} \\ &= e^{-i\omega t} \sum_{n=1} i^n \frac{2n+1}{n(n+1)} \left\{ \underline{m}^{(1)}_{o1n} - i\, \underline{n}^{(1)}_{e1n} \right\} \quad , \end{aligned}$$

where $\underline{m}_{o1n}$ and $\underline{n}_{e1n}$ are the spherical vector wave functions defined by Stratton[3] the scattered field can be written as

$$\underline{E}^{s} = e^{-i\omega t} \sum_{n=1} i^n \frac{2n+1}{n(n+1)} \left\{ a_n\, \underline{m}^{(3)}_{o1n} - i b_n\, \underline{n}^{(3)}_{e1n} \right\}$$

and by application of the boundary condition (A1) the coefficients $a_n$ and $b_n$ are found to be

$$a_n = - \frac{\rho j_n(\rho) - i\eta \left[\rho j_n(\rho)\right]'}{\rho h_n(\rho) - i\eta \left[\rho h_n(\rho)\right]'} \tag{A2}$$

$$b_n = - \frac{\left[\rho j_n(\rho)\right]' + i\eta\, \rho j_n(\rho)}{\left[\rho h_n(\rho)\right]' + i\eta\, \rho h_n(\rho)} \tag{A3}$$

where $\rho$ = ka and the primes denote differentiation with respect to the entire argument. The back scattered field is then

$$\underline{E}^s = \hat{i}_x \sum_{n=1}^{\infty} (-i)^n (n+\tfrac{1}{2}) \left\{ a_n h_n(kR) + i b_n \frac{1}{kR} \left[ kR\, h_n(kR) \right]' \right\} . \tag{A4}$$

To evaluate this expression for large ka it is convenient to separate out the portion appropriate to a smooth sphere ($\eta = 0$). If E(0) is the x component of the electric vector in the back scattered field for this case, and if $\underline{E}^s = \hat{i}_x E(\eta)$, equation (A4) can be written in the form

$$E(\eta) = E(0) + \sum_{n=1}^{\infty} (-i)^n (n+\tfrac{1}{2}) \left\{ h_n(kR) \left[ a_n(\eta) - a_n(0) \right] \right.$$

$$\left. + i \frac{1}{kR} \left[ kR\, h_n(kR) \right]' \left[ b_n(\eta) - b_n(0) \right] \right\} , \tag{A5}$$

where the new coefficients are defined by the equations

$$a_n(\eta) - a_n(0) = \frac{\eta}{\rho\, h_n(\rho)} \left\{ \rho\, h_n(\rho) - i\eta \left[ \rho\, h_n(\rho) \right]' \right\}^{-1}$$

$$b_n(\eta) - b_n(0) = \frac{\eta}{\left[ \rho\, h_n(\rho) \right]'} \left\{ \left[ \rho\, h_n(\rho) \right]' + i\, \eta\, \rho\, h_n(\rho) \right\}^{-1} .$$

If $|\eta|$ is sufficiently small,

$$a_n(\eta) - a_n(0) \sim \eta \left\{ \rho\, h_n(\rho) \right\}^{-2} , \qquad |\eta| << 1 \tag{A6}$$

$$b_n(\eta) - b_n(0) \sim \eta \left\{ \left[ \rho\, h_n(\rho) \right]' \right\}^{-2} , \qquad |\eta| << \rho^{-1/3} \tag{A7}$$

and the coefficients will be replaced by these asymptotic values. This has the effect of neglecting the residues produced by the first order poles of $a_n(\eta) - a_n(0)$ and

$b_n(\eta) - b_n(0)$ when the series in (A5) is transformed by means of a Watson transformation into a contour integral plus a residue series. Since these residues correspond to the diffracted field, the approximation represented by equations (A6) and (A7) is sufficient whenever the reflected field is dominant.

By using the relations

$$j_n(\rho)h_n'(\rho) - j_n'(\rho)h_n(\rho) = \frac{i}{\rho^2}$$

$$\left[\rho j_n(\rho)\right]'' + \left\{1 - \frac{n(n+1)}{\rho^2}\right\}\rho j_n(\rho) = 0$$

equation (A5) can now be simplified to give

$$E(\eta) = \left(1 - i\eta\frac{\partial}{\partial\rho}\right)E(0) + \eta\, S \tag{A8}$$

where

$$S = \sum_{n=1}^{\infty} (-i)^n \left(n + \frac{1}{2}\right) \frac{n(n+1)\left[kR\, h_n(kR)\right]'}{kR\left\{\rho\left[\rho h_n(\rho)\right]'\right\}^2} \tag{A9}$$

Taking first the portion corresponding to a smooth sphere, the analysis in Weston[4] shows that

$$E(0) = -\frac{a}{2R-a}\, e^{ik(R-2a)}\left[1 + \frac{A_1(0)}{ka} + \frac{A_2(0)}{(ka)^2} + \dots\right] \tag{A10}$$

for large ka , where

$$A_1(0) = -\frac{2i(R-a)^2}{(2R-a)^2}, \tag{A11}$$

$$A_2(0) = \frac{a(R-a)(2R^2 - 4Ra + 3a^2)}{(2R-a)^4} \quad . \tag{A12}$$

The remaining task is to sum the series S , and for this purpose the Watson transform technique is used. If a contour C is drawn surrounding the poles of cos $\nu\pi$ on the positive real axis of the $\nu$ plane, S can be written as

$$S = \sqrt{\frac{2}{\pi}} \frac{e^{i\pi/4}}{2kR\rho^2} \int_C \frac{e^{i\nu\pi/2}\nu(\nu^2 - 1/4)\left[\sqrt{kR}\; H_\nu(kR)\right]'}{\cos\nu\pi\left\{\left[\sqrt{\rho}\; H_\nu(\rho)\right]'\right\}^2} d\nu \; .$$

The contour C may now be deformed into a straight line path from $-\infty$ to $+\infty$ and passing through the origin at an angle $\beta$ to the positive real $\nu$ axis, where $-\pi/2 < \beta < 0$ . The odd portion of the integrand then integrates to zero, so that

$$S = \sqrt{\frac{2}{\pi}} \frac{e^{3i\pi/4}}{kR\rho^2} \int_0^{\infty\exp(-i\beta)} \frac{\nu\tan\nu\pi(\nu^2 - 1/4)\left[\sqrt{kR}\; H_\nu(kR)\right]'}{e^{i\nu\pi/2}\left\{\left[\sqrt{\rho} H_\nu(\rho)\right]'\right\}^2} d\nu \; .$$

To evaluate this new integral it is convenient to break it into two parts by setting

$$\tan\nu\pi = \frac{ie^{-i\nu\pi}}{\cos\nu\pi} - i \quad .$$

Taking first the integral $S_1$ corresponding to the first of these two terms, it is permissable to put $\beta = \pi/2$ . Writing $\nu = -ip$ we then have

$$S_I = -\sqrt{\frac{2}{\pi}} \frac{e^{i\pi/4}}{kR\rho^2} \int_0^{\infty} p(p^2 + 1/4)\frac{e^{-\pi p}}{\cosh\pi p} \cdot \frac{\left[\sqrt{kR}\; H_{ip}(kR)\right]' dp}{e^{-\pi p/2}\left\{\left[\sqrt{\rho}\; H_{ip}(\rho)\right]'\right\}^2}$$

Since the dominant behavior of the integrand is provided by the factor $e^{-\pi p}/\cosh \pi p$ , $S_1$ may be approximated as

$$S_1 \sim -\frac{1}{kR\,\rho^2}\, e^{i(kR-2a)} \int_0^{\infty} p(p^2+1/4)\, \frac{e^{-\pi p}}{\cosh \pi p}\, dp$$

and although this integral can be easily evaluated, for present purposes it is only necessary to note that

$$S_1 \sim \frac{1}{kR(ka)^2}\, e^{i(kR-2a)} \quad \times \quad \text{constant} \; .$$

The other integral $S_2$ is given by

$$S_2 = \sqrt{\frac{2}{\pi}}\, \frac{e^{i\,\pi/4}}{kR\,\rho^2} \int_0^{\infty \exp(-i\beta)} \nu(\nu^2-1/4)\, \frac{\left[\sqrt{kR}\; H_\nu(kr)\right]' \, d\nu}{e^{i\nu\pi/2}\left\{\left[\sqrt{\rho}\; H_\nu(\rho)\right]'\right\}^2}$$

and is evaluated by replacing the Hankel functions by their asymptotic expansions for $|\nu| < \sqrt{\rho}$ (see Scott[5]). We then have

$$\left[\sqrt{\rho}\; H_\nu(\rho)\right]' \sim \sqrt{\frac{2}{\pi}} \exp\left\{ i\rho - (\nu-1/2)\, i\, \frac{\pi}{2} + \frac{i\nu^2}{2\rho} - \frac{i}{8\rho}\right\}\left\{1 - (\nu^2-1/4)\,\frac{1}{4\rho^2} - \frac{i\nu^4}{24\rho^3} + \dots\right\}$$

and the integral can now be approximated as

$$S_2 \sim \frac{e^{i(kR-2a)}}{kR\,\rho^2} \int_0^{\infty \exp(-i\beta)} \nu^3 \exp\left\{\frac{i\nu^2}{2}\left(\frac{1}{kR} - \frac{2}{ka}\right)\right\} d\nu$$

$$= -\frac{2}{kR(2-\frac{a}{R})^2}\, e^{ik(R-2a)} \left\{1 + 0\left(\frac{1}{ka}\right)\right\}$$

Hence,

$$S = -\frac{a}{2R-a}\, e^{ik(R-2a)} \left\{ \frac{2R}{ka(2R-a)} + 0\left(\frac{1}{\overline{ka}^2}\right) \right\} \tag{A13}$$

and the back scattered field for the rough sphere is therefore

$$E(\eta) = -\frac{a}{2R-a}\, e^{ik(R-2a)} \left\{ A_0(\eta) + \frac{A_1(\eta)}{ka} + \frac{A_2(\eta)}{(ka)^2} + \ldots \right\} \tag{A14}$$

where

$$A_0(\eta) = 1 - 2\eta$$

$$A_1(\eta) = -\frac{2i(R-a)^2}{(2R-a)^2}(1-2\eta) + \frac{2R}{2R-a}\,\eta(1-i) \ .$$

This result holds for sufficiently large ka (such that the diffracted field is negligible) and for sufficiently small $|\eta|$ (i.e. $|\eta| << (ka)^{-1/3}$).

# PART II

## IMPEDANCE BOUNDARY CONDITIONS FOR IMPERFECTLY CONDUCTING SURFACES

Summary

It is shown how the exact electromagnetic boundary conditions at the surface of a material of large refractive index can be approximated to yield the usual impedance or Leontovich boundary conditions. These conditions relate the tangential components of the electric and magnetic fields (or the normal components and their normal derivatives) via a surface impedance which is a function only of the electromagnetic properties of the material. They are valid for surfaces whose radii of curvature are large compared with the penetration depth, and also for materials which are not homogeneous but whose properties vary slowly from point to point. As the refractive index (or conductivity) increases to infinity, the conditions go over uniformly to the conditions for perfect conductivity.

### §1. Introduction

In its most straightforward form an impedance boundary condition is one which relates the tangential components of the electric and magnetic fields via an impedance factor which is a function of the properties of the surface and, possibly, of the field which is incident upon it. The concept of a surface impedance is, of course, not new, and has long been used in a variety of engineering calculations. On the other hand, the idea of incorporating this impedance into the initial formulation of a boundary value problem appears to date only from the beginning of the last war.

During the early 1940's a considerable number of Russian papers were published dealing with various aspects of propagation over the earth, and in these an attempt was made to take into account the properties of actual ground materials by specifying an impedance boundary condition at the surface. This represented a departure from the (then accepted practice of studying in complete detail certain problems of a very idealized nature, and paved the way for a discussion of propagation over an inhomogeneous, as well as a rough, earth. It was shown that the impedance boundary condition is a valid approximation to the exact condition when the refractive index of the ground is large compared with unity, and the surface impedance can be expressed directly in terms of the electromagnetic properties of the material. These boundary conditions are usually attributed to Leontovich (see, for example, Fock[6]) and were described by Leontovich[7] himself in 1948. They were first applied to a physical problem by Alpert[8] in 1940, and were used extensively in Russian work throughout the war. A short summary of their application to propagation problems has been given by Feinberg[9].

Unfortunately, the proofs associated with these conditions are not readily accessible. Although the conditions are frequently employed in modern electromagnetic theory, it would often appear that either their degree of generality or the restrictions which they require are not fully appreciated. It is the purpose of the present paper to collect in one place some of the proofs associated with these conditions as they apply to the surface of a material of large but finite refractive index. This also serves to provide the necessary background for a subsequent paper in which impedance boundary conditions are developed for a surface which is perfectly conducting but geometrically rough.

In § 2 the exact electromagnetic boundary conditions are briefly discussed. The approximate conditions for a flat interface between a homogeneous isotropic medium and free space are derived in § 3, and the flat interface is generalized to a surface of large radius of curvature in § 4. The necessary modifications when the properties of the medium vary from point to point are given in § 5.

## § 2. Exact Boundary Conditions

At the interface between two homogeneous isotropic media neither of which is perfectly conducting, an electromagnetic field satisfies the boundary conditions

$$\left[\hat{n} \wedge \underline{E}\right] = 0 \tag{1}$$

$$\left[\hat{n} \cdot \underline{D}\right] = 0 \tag{2}$$

$$\left[\hat{n} \wedge \underline{H}\right] = 0 \tag{3}$$

$$\left[\hat{n} \cdot \underline{B}\right] = 0 \tag{4}$$

where $\hat{n}$ is a unit vector normal and the square brackets denote the discontinuities in the corresponding field components on crossing the boundary. In these equations $\underline{E}$ and $\underline{H}$ are the electric and magnetic field vectors in terms of which

$$\underline{B} = \mu \underline{H}$$

where $\mu$ is the permeability, and

$$\underline{D} = \epsilon \underline{E}$$

where $\epsilon$ is the complex permittivity*

$$\epsilon = \epsilon' + i \frac{\sigma}{\omega} .$$

A consequence of using a complex (rather than a real) permittivity is that no surface charge distribution appears on the right-hand side of (2).

Equations (1) through (4) are not all independent and therefore constitute a set of boundary conditions at the interface which are more than sufficient. If, for example, the first two are selected, the use of Maxwell's equations shows that (3) and (4) are satisfied automatically. Similarly if the conditions (3) and (4) upon the magnetic field are selected; and indeed, a specification of all the tangential components ($\underline{E}$ and $\underline{H}$), or both normal components will suffice. On the other hand, (1) and (4) or (2) and (3) do not constitute sufficient sets since, for example, (1) is not independent of (4).

It should be emphasized that in spite of the so-called "proofs" presented in many textbooks the boundary conditions (1) through (4) cannot be verified by experiments carried out in a homogeneous medium, nor is the author aware of any method by which they can be deduced from Maxwell's equations. In consequence, it appears necessary to regard them as an essential postulate of electromagnetic

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* A time variation $e^{i\omega t}$ is assumed in this Part alone.

theory, and the consequent agreement between theory and experiment then provides the evidence in favor of their validity.

It will be observed that the boundary conditions relate the field in the first medium (which we shall henceforth regard as free space) to that in the second medium, and in practice are not always easy to apply in the solution of problems. When the second medium is perfectly conducting, however, the fields therein are identically zero and the only fields to be considered are those in free space. In this case equations (1) and (4) reduce to

$$\hat{n} \wedge \underline{E} = 0 \tag{5}$$

$$\hat{n} \cdot \underline{B} = 0 \quad , \tag{6}$$

but equations (2) and (3) are replaced by

$$\hat{n} \cdot \underline{D} = \delta \tag{7}$$

$$\hat{n} \wedge \underline{H} = \underline{K} \quad , \tag{8}$$

where $\delta$ and $\underline{K}$ are surface distributions of charge and current respectively. Since these are known only when the fields $\underline{E}$ and $\underline{H}$ have been determined, (7) and (8) do not represent boundary conditions in the usual sense, and we are therefore left with equations (5) and (6) from which to determine the fields in free space. On the other hand, a further degeneracy now appears and whereas two conditions were required when the medium was not perfectly conducting, a single equation now suffices. Thus, for example, equation (5) alone* specifies the fields at all points, and equations (6), (7) and (8) can all be deduced therefrom.

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* Although a radiation condition (or its equivalent) must also be imposed if the region is infinite in extent, and an edge condition if this is appropriate.

When the refractive index N of the second medium relative to free space is large compared with unity, boundary conditions can be derived which are analogous to (5) and (6) in that the only fields which appear are those in free space (medium 1). This permits a considerable simplification in the analysis of any scattering or diffraction problem involving bodies which are not perfectly conducting, since it avoids the need to calculate the fields within the body. These new conditions are an approximation to (1) through (4), and their derivation is based on the neglect of terms $O(1/N^2)$ in comparison with unity. We shall first obtain the conditions for an infinite flat interface and later generalize the results so as to apply to a more practical set of circumstances.

## §3. Approximate Boundary Conditions for a Flat Interface

Consider a homogeneous isotropic medium whose permittivity, permeability and conductivity are $\epsilon'$, $\mu$ and $\sigma$ respectively. It is assumed that this medium occupies the region $z < 0$ of a Cartesian coordinate system $(x, y, z)$. The half-space $z > 0$ is free space, the permittivity and permeability of which are $\epsilon_0$ and $\mu_0$.

Relative to free space the complex refractive index of the medium is

$$N = \sqrt{\frac{\mu}{\mu_0}\left(\frac{\epsilon'}{\epsilon_0} + i\,\frac{\sigma}{\omega\epsilon_0}\right)}\;,$$

and boundary conditions at the interface $z = 0$ will now be derived under the assumption that $|N|$ is large compared with unity. It will be appreciated that this requirement is satisfied by a material whose dielectric constant $\frac{\epsilon'}{\epsilon_0}$ is large, as well as by a material of high conductivity. For the purposes of the analysis it is convenient to introduce the parameter $\eta$ defined by $\eta = \frac{\mu}{\mu_o N}$ ; thus

$$\eta = \frac{1}{\sqrt{\frac{\mu_o}{\mu}\left(\frac{\epsilon'}{\epsilon_o} + i\frac{\sigma}{\omega\epsilon_o}\right)}} \tag{9}$$

and is zero for perfect conductivity.

Let us denote by $(\underline{E}, \underline{H})$ the electromagnetic field in $z > 0$, and by $(\underline{E}', \underline{H}')$ the field in $z < 0$. From the divergence condition we have

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{10}$$

and similarly

$$\frac{\partial E'_x}{\partial x} + \frac{\partial E'_y}{\partial y} + \frac{\partial E'_z}{\partial z} = 0 \ . \tag{11}$$

At the interface $z = 0$ the tangential components of the electric field are continuous, so that

$$E_x = E'_x$$

$$E_y = E'_y$$

and hence, by tangential differentiation,

$$\frac{\partial E_x}{\partial x} = \frac{\partial E'_x}{\partial x}$$

$$\frac{\partial E_y}{\partial y} = \frac{\partial E'_y}{\partial y} \ .$$

Equations (10) and (11) then give

$$\frac{\partial E_z}{\partial z} = \frac{\partial E'_z}{\partial z} \ . \tag{12}$$

In the medium, however,

$$\frac{\partial^2 E_z'}{\partial x^2} + \frac{\partial^2 E_z'}{\partial y^2} + \frac{\partial^2 E_z'}{\partial z^2} + k^2 N^2 E_z' = 0 \tag{13}$$

where k is the propagation constant in free space. If $|N| >> 1$, the field is rapidly varying in the z direction, leading to a large value of $\frac{\partial^2 E_z'}{\partial z^2}$, and by comparison with this the x and y derivatives are small. This fact is, perhaps, most clearly seen by considering a plane wave incident on the boundary from the direction of free space. Because of the large value of $|N|$, application of Snell's law shows that the transmitted field is deflected toward the normal. For a fixed direction of incidence, the angle between the direction of the transmitted field and the normal is $O(1/|N|)$, which implies that $\frac{\partial^2 E_z'}{\partial x^2}$ and $\frac{\partial^2 E_z'}{\partial y^2}$ are smaller than $\frac{\partial^2 E_z'}{\partial z^2}$ by a factor of order $|N|^2$. Accordingly, in equation (13) the first two derivatives can be neglected in comparison with the third, and the equation then becomes

$$\frac{\partial^2 E_z'}{\partial z^2} + k^2 N^2 E_z' = 0 \ . \tag{14}$$

The solution of this is

$$E_z' = A e^{ikNz} + B e^{-ikNz} , \tag{15}$$

where A and B are constants as yet undetermined. If N is defined to have positive imaginary part, the fact that the medium is infinite in extent implies that A must be zero, since the field $E_z'$ must correspond to propagation in the negative z direction. Hence

$$E_z' = B e^{-ikNz} \tag{16}$$

from which we obtain

$$\frac{\partial E'_z}{\partial z} = -ikN\, E'_z \quad . \tag{17}$$

But from equation (2)

$$E'_z = \frac{\epsilon_o}{\epsilon} E_z \tag{18}$$

at the interface, and this can be combined with equation (17) to give

$$\begin{aligned} \frac{\partial E'_z}{\partial z} &= -ikN \frac{\epsilon_o}{\epsilon} E_z \\ &= -ik\eta E_z \end{aligned} \tag{19}$$

at $z = 0$. Using equation (12) we now have

$$\frac{\partial E_z}{\partial z} = -ik\eta E_z \tag{20}$$

and this is one of the required boundary conditions at the interface. Equation (20) is accurate to the first order in $\eta$.

A similar analysis can be developed for the normal component of the magnetic field. From the divergence condition we obtain

$$\frac{\partial H_z}{\partial z} = \frac{\partial H'_z}{\partial z}$$

(cf equation 12), and since we also have

$$\frac{\partial H'_z}{\partial z} = -ikN\, H'_z$$

(cf equation 17), it follows that

$$\frac{\partial H_z}{\partial z} = -ikN\, H'_z \quad .$$

But at the boundary $z = 0$, $H'_z = \frac{\mu_o}{\mu} H_z$

and hence

$$\frac{\partial H_z}{\partial z} = -\frac{ik}{\eta} H_z \quad . \tag{21}$$

This is the second of two boundary conditions at the interface, and is accurate to order $\eta$.

It will be observed that (21) differs from (20) in having $\eta$ replaced by $1/\eta$, and this is in accordance with the interpretation of $\eta$ as an impedance associated with the surface. The point will be elaborated upon in a moment, but for the time being it is sufficient to note that equations (20) and (21) specify the behavior of the normal components of both $\underline{E}$ and $\underline{H}$ at the interface, and therefore represent a sufficient set of boundary conditions.

For some applications an alternative (but entirely equivalent) representation of these boundary conditions proves more convenient. Taking first equation (20), since

$$\underline{E} = -\frac{Z}{ik} \nabla \wedge \underline{H}$$

where $Z = \frac{1}{Y} = \sqrt{\frac{\mu_o}{\epsilon_o}}$ is the intrinsic impedance of free space, and since $\nabla \cdot \underline{E} = 0$, the boundary condition can be written as

$$\frac{\partial}{\partial x}(E_x + \eta Z H_y) = -\frac{\partial}{\partial y}(E_y - \eta Z H_x) \; . \qquad (22)$$

Similarly, the boundary condition (21) gives

$$\frac{\partial}{\partial y}(E_x + \eta Z H_y) = \frac{\partial}{\partial x}(E_y - \eta Z H_x) \qquad (23)$$

and by eliminating $E_x + \eta Z H_y$ and $E_y - \eta Z H_x$ successively between these equations, we have

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad (24)$$

where $\Phi = E_x + \eta Z H_y$ or $E_y - \eta Z H_x$. This equation can be solved by assuming a separable form for $\Phi$. If

$$\Phi = \Phi_1(x)\ \Phi_2(y) \quad ,$$

then

$$\frac{\partial^2 \Phi_1}{\partial x^2} + a^2 \Phi_1 = 0$$

$$\frac{\partial^2 \Phi_2}{\partial y^2} - a^2 \Phi_2 = 0$$

where $a^2$ is some separation constant, and the solutions are

$$\Phi_1 = A_1 e^{iax} + B_1 e^{-iax}$$

$$\Phi_2 = A_2 e^{ay} + B_2 e^{-ay}$$

where $A_1$, $B_1$, $A_2$ and $B_2$ are constants as yet undefined. If a is not purely real, both $A_1$ and $B_1$ must be identically zero since otherwise $\Phi_1$ would become exponentially large for large x (either positive or negative). In this case $\Phi_1$, and hence $\Phi$, is zero. If a is purely real, the same argument applied to the variable y shows that $\Phi_2$ is zero, leading to the same conclusion as regards $\Phi$. Since $\Phi$ is therefore zero,

$$E_x = -\eta Z H_y \tag{25}$$

$$E_y = \eta Z H_x \quad , \tag{26}$$

and this is the alternative statement of the boundary conditions at the interface $z = 0$. In this form the conditions simply state that $\eta Z$ is the effective impedance of the surface as seen by a field in free space. For comparison with this, the impedance of a perfectly conducting surface is zero.

## §4. Extension to a Curved Interface

In order to generalize these conditions for application to surfaces which are not flat, it is first necessary to express equations (20) and (21), (25) and (26) in forms which do not explicitly involve the coordinate system. If $E_n$ and $H_n$ are the field components normal to the boundary, and if n is a coordinate whose positive direction is outwards as regards the medium, equations (20) and (21) can be written as

$$\frac{\partial E_n}{\partial n} = - ik\eta E_n \tag{27}$$

$$\frac{\partial H_n}{\partial n} = - \frac{ik}{\eta} H_n \quad . \tag{28}$$

For the second pair of conditions a vector form is more convenient, and following Leontovich[7], equations (25) and (26) are combined to give

$$\underline{E} - (\hat{n} \cdot \underline{E})\, \hat{n} = \eta Z\, \hat{n} \wedge \underline{H} \quad . \tag{29}$$

Of the three scalar equations contained herein, only two are independent.

We now turn to a consideration of the boundary conditions at a curved interface between the medium and free space. As in the case of the flat interface the object is to determine approximate boundary conditions in which only the fields in free space appear. It is clear, however, that unless restrictions are placed upon the shape of the boundary, these conditions will involve the geometrical properties of the surface as well as the electrical parameters of the medium, and in consequence may vary from point to point on the surface. Such conditions would be of little practical value. On the other hand, by restricting the type of surface to be allowed, the curvature effects can be made negligible, and the boundary conditions then reduce to those obtained for an infinite flat surface.

A rigorous derivation of the restrictions which must be placed on the type of surface in order that equations (27) through (29) be valid is beyond the scope of this paper, and for details of the analysis reference is made to Rytov[10] and Leontovich[7]. The actual limitations, however, can be arrived at by a semi-intuitive argument.

It will be recalled that in the analysis of the flat boundary the assumption was made that

$$|N| >> 1 \quad , \tag{30}$$

and this is sufficient to ensure that within the medium the field is slowly varying along the surface and behaves essentially as a plane wave propagating in the direction of the inward normal. Let us now seek to apply equation (29) or (27) and (28) to each point on a curved surface. In order that the field shall vary little within a wavelength along the surface, a restriction must be placed upon the radii of curvature, and a trivial analysis shows that the requirement is

$$|N| \quad k\rho >> 1 \tag{31}$$

where $\rho$ is the smallest radius of curvature at the point in question. If (31) is satisfied, any correction to the boundary condition (29) consequent upon the curvature is negligible (see Leontovich[7]).

For a surface which is open (implying that the medium is infinite in extent) and which has no inward normal intersecting the surface in a second point, the restrictions (30) and (31) are sufficient to justify the application of the flat surface conditions. For a closed surface, however, a difficulty arises when the conduction current in the medium is negligible compared with the displacement current. The inward travelling field then suffers little or no attenuation, and accordingly may appear as an outward travelling field on the farther side of the surface. This is contrary to the assumption made in the derivation of the flat surface condition. For this reason it is necessary for the field within the medium to be attenuated at a rate such that the penetration depth $\delta$ is small compared with $\rho$, giving rise to the additional restriction

$$\delta << \rho \quad . \tag{32}$$

If $\sigma >> \omega \epsilon'$, equation (32) can be written as

$$\sqrt{\frac{\mu}{\mu_0} \frac{\sigma}{2\omega \epsilon_0}} \quad k\rho >> 1 \quad ,$$

which in turn reduces to the inequality (31) if the conduction current dominates. On the other hand, if the displacement current dominates, the inequality (32) represents an additional restriction which is stronger than (31).

The difficulty which arises with a dielectric medium has been noted by Leontovich[7], who also points out that for a body made of this material the boundary condition (29) can be justified only under very restricted circumstances. For a body of general shape the boundary conditions are only applicable if the medium is conducting and satisfies the inequality (32). The importance of this restriction, rather than (31), can be seen from a study of the few exact solutions which are known for bodies which are not perfectly conducting. For example, if a plane wave is incident on a sphere of radius $\rho$, the exact solution can be found as a sum of vector wave functions whose coefficients are functions of N. If it is now assumed that $|N| k\rho \gg 1$, these coefficients reduce to the forms which would have been obtained by using the condition (29) apart from additional terms involving $\tan Nk\rho$. Such terms only disappear if $\tan Nk\rho$ can be replaced by -i to the leading order in N, i.e. if $|\text{Im } N| k\rho \gg 1$. Similarly, if a field is incident upon an infinite slab of (uniform) thickness d, the exact solution contains an exponential factor $e^{2iNkd}$ corresponding to internal reflection from the lower surface, and the approximate boundary conditions would then be valid only if the terms containing this factor can be taken zero. This in turn requires an attenuation of the inward travelling field subject to a restriction of the form (32) with $\rho$ replaced by d. It is of interest to note that (32) is here required even though the surface is flat.

In summary, we now have that for a homogeneous isotropic body whose refractive index N and smallest radius of curvature or dimension $\rho$ are such that

$$|N| \gg 1 \tag{30}$$

$$|\text{Im } N| \, k\rho \gg 1 \ , \tag{33}$$

the boundary conditions at its surface can be written as

$$\frac{\partial E_n}{\partial n} = - ik\eta \, E_n \tag{27}$$

$$\frac{\partial H_n}{\partial n} = -\frac{ik}{\eta} H_n \quad , \tag{28}$$

where $\eta = \dfrac{\mu}{\mu_o N}$ . These are equivalent to the single vector condition

$$\underline{E} - (\hat{n} \cdot \underline{E})\hat{n} = \eta Z \hat{n} \wedge \underline{H} \quad . \tag{29}$$

In some circumstances it may be possible to replace (33) by the weaker restriction

$$|N| \, k\rho >> 1 \quad , \tag{31}$$

but such cases must be regarded as exceptional. In this connection it is of interest to note that Fock[6] in his discussion of these boundary conditions ignores the distinction between dielectric and conducting media, and gives only the restrictions (30) and (31).

Equations (27) through (29) are approximations to the exact boundary conditions correct to the first order in $\eta$, and accordingly in any solution obtained using these conditions there is no (physical) justification for retaining terms which are of a higher order in $\eta$. A consequence of this is that if the fields are capable of expansion in series of ascending (positive) powers of $\eta$, the perfectly conducting approximation (corresponding to $\eta = 0$) can be inserted into the right-hand sides of (27) and (29) and into the left-hand side of (28). In general, such expansions will be vlaid, though a problem in which this is not true is the incidence of an H-polarized plane wave on an imperfectly conducting half-plane (Senior[11, 12]). In this case, however, the failure may well be due to the additional assumption of a "thin" body.* implicit in the problem.

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*The mathematical requirement here is $d << \lambda$, where d is the thickness of the half-plane, and by assuming that the half-plane is tipped with a semi-circular cylinder it can be shown that the boundary conditions are applicable if $\delta << d << \lambda$.

## §5. An Inhomogeneous Medium

Let us now go on to consider the problem in which the medium is not homogeneous, so that the refractive index varies from point to point. This variation will be attributed to $\epsilon$ alone and $\mu$ will be regarded as spatially invariant. We shall again begin by assuming an infinite flat interface between the medium and free space.

At any point (x, y, z) within the medium

$$\nabla \cdot (\epsilon \underline{E}') = 0 \tag{34}$$

and since

$$\nabla \cdot (\epsilon \underline{E}') = \epsilon \nabla \cdot \underline{E}' + \underline{E}' \cdot \nabla \epsilon$$

equation (34) can be written as

$$\frac{\partial E_x'}{\partial x} + \frac{\partial E_y'}{\partial y} = -\frac{\partial E_z'}{\partial z} - \frac{1}{\epsilon} \underline{E}' \cdot \nabla \epsilon \ . \tag{35}$$

In free space, however,

$$\nabla \cdot \underline{E} = 0 \tag{36}$$

and using the continuity of the tangential components across the interface, we now have

$$\begin{aligned} \frac{\partial E_z}{\partial z} &= -\frac{\partial E_x'}{\partial x} - \frac{\partial E_y'}{\partial y} \\ &= \frac{\partial E_z'}{\partial z} + \frac{1}{\epsilon} \underline{E}' \cdot \nabla \epsilon \end{aligned} \tag{37}$$

at z = 0. In terms of $\eta$, however,

$$\epsilon = \frac{\mu}{\mu_o} \frac{\epsilon_o}{\eta^2}$$

so that

$$\frac{1}{\epsilon}\nabla\epsilon = -\frac{2}{\eta}\nabla\eta ,$$

and this can be inserted into equation (37) to give

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z'}{\partial z} - \frac{2}{\eta}\left(E_x' \frac{\partial\eta}{\partial x} + E_y' \frac{\partial\eta}{\partial y} + E_z' \frac{\partial\eta}{\partial z}\right) \tag{38}$$

at $z = 0$. But at the interface

$$E_x' = E_x$$

$$E_y' = E_y$$

$$E_z' = \frac{\mu_o}{\mu}\eta^2 E_z$$

and using these relations equation (38) becomes

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z'}{\partial z} - \frac{2}{\eta}\left(E_x \frac{\partial\eta}{\partial x} + E_y \frac{\partial\eta}{\partial y} + \frac{\mu_o}{\mu}\eta^2 E_z \frac{\partial\eta}{\partial z}\right). \tag{39}$$

In arriving at this equation no approximations have been made, and the second term on the right-hand side can be interpreted as a correction to the boundary condition resulting from the variation of $\eta$ throughout the medium. If

$$\left|\frac{1}{k\eta}\nabla\eta\right| << 1 \tag{40}$$

which implies that the relative variation is small, $E_x$, $E_y$ and $E_z$ will not differ substantially from the values appropriate to a homogeneous medium. For such a medium it was shown in § 3 that

$$E_x , E_y = O(\eta) , \quad E_z = O(1)$$

and in equation (39) it is now seen that the lateral variation of $\eta$ is more important than the normal variation. Indeed, if $\frac{\partial\eta}{\partial x}$, $\frac{\partial\eta}{\partial y}$ and $\frac{\partial\eta}{\partial z}$ are all comparable with one another, the effect produced by the z variation of $\eta$ is smaller by an order of magnitude. The z variation can therefore be neglected and henceforth $\eta$ will be assumed to be a function of x and y only.

The next step is to obtain an expression for $\frac{\partial E_z'}{\partial z}$ in terms of the free space field. From the field equations

$$\underline{E}' = -\frac{\sqrt{\mu_o \epsilon_o}}{ik\epsilon} \nabla_\wedge \underline{H}' \tag{41}$$

$$\underline{H}' = \frac{\sqrt{\mu_o \epsilon_o}}{ik\mu} \nabla_\wedge \underline{E}' \tag{42}$$

we have

$$\underline{E}' = \frac{1}{k^2 N^2} \nabla_\wedge \nabla_\wedge \underline{E}' \quad , \tag{43}$$

and since

$$\nabla_\wedge \nabla_\wedge \underline{E}' = \nabla (\nabla \cdot \underline{E}') - \nabla^2 \underline{E}'$$

$$= -\nabla\left(\frac{1}{\epsilon} \underline{E}' \cdot \nabla \epsilon\right) - \nabla^2 \underline{E}' \ ,$$

the equation for the field within the medium can be written

$$\nabla^2 \underline{E}' + k^2 N^2 \underline{E}' + \nabla \left(\frac{1}{\epsilon} \underline{E}' \cdot \nabla \epsilon\right) = 0 \ . \tag{44}$$

If $\epsilon$ is now expressed in terms of the refractive index N using

$$\epsilon = \frac{\mu_o}{\mu} \epsilon_o N^2 \ ,$$

equation (44) becomes

$$\nabla^2 \underline{E}' + k^2 N^2 \underline{E}' + 2\nabla \left(\frac{1}{N} \underline{E}' \cdot \nabla N\right) = 0 \ . \tag{45}$$

and since the tangential derivatives of $\underline{E}'$ are again negligible in comparison with the normal derivative, (45) reduces to

$$\frac{\partial^2 \underline{E}'}{\partial z^2} + k^2 N^2 \underline{E}' + 2\nabla\left(\frac{1}{N}\underline{E}' \cdot \nabla N\right) = 0 \quad . \tag{46}$$

In particular,

$$\frac{\partial^2 E_z'}{\partial z^2} + k^2 N^2 E_z' + 2\frac{\partial}{\partial z}\left\{\frac{1}{N}\left(E_x'\frac{\partial N}{\partial x} + E_y'\frac{\partial N}{\partial y}\right)\right\} = 0 \quad . \tag{47}$$

In order to determine $E_z'$ from (47) it is necessary to know the variation of $E_x'$ and $E_y'$ in the z direction, and for this purpose the x and y components of equation (46) are employed. To the first order the variation of N can be neglected, and we then have

$$\frac{\partial^2 E_x'}{\partial z^2} + k^2 N^2 E_x' = 0$$

the solution of which is

$$E_x' = (E_x')_{z=0}\, e^{-ikNz} \tag{48}$$

since the field in the medium must behave as a wave travelling in the negative z direction. Moreover, at $z = 0$

$$E_x' = E_x$$

and hence

$$E_x' = E_x\, e^{-ikNz} \quad . \tag{49}$$

Similarly,

$$E_y' = E_y\, e^{-ikNz} \tag{50}$$

and equation (47) can now be written as

$$\frac{\partial^2 E_z'}{\partial z^2} + k^2 N^2 E_z' + \alpha\, e^{-ikNz} = 0 \tag{51}$$

where

$$\alpha = -2ik\left(E_x \frac{\partial N}{\partial x} + E_y \frac{\partial N}{\partial y}\right) . \tag{52}$$

$\alpha$ is, of course, independent of z.

The complete solution of equation (51) is obtained by adding a particular integral to the general solution $A\,e^{-ikNz}$, where A is some constant. The former can be taken as

$$\frac{\alpha z}{2ikN}\, e^{-ikNz} ,$$

giving

$$E_z' = e^{-ikNz}\left(A + \frac{\alpha z}{2ikN}\right) . \tag{53}$$

Hence

$$\frac{\partial E_z'}{\partial z} = -ikN E_z' + \frac{\alpha}{2ikN}\, e^{-ikNz}$$

and at $z = 0$ this reduces to

$$\frac{\partial E_z'}{\partial z} = -ikN E_z' + \frac{\alpha}{2ikN}$$

$$= -ik\eta E_z + \frac{1}{\eta}\left(E_x \frac{\partial \eta}{\partial x} + E_y \frac{\partial \eta}{\partial y}\right) \tag{54}$$

by using the expression for $\alpha$. If this is now inserted into equation (34) bearing in mind that $\frac{\partial \eta}{\partial z} = 0$, a boundary condition is obtained in the form

$$\frac{\partial E_z}{\partial z} = -ik\eta E_z - \frac{1}{\eta}\left(E_x \frac{\partial \eta}{\partial x} + E_y \frac{\partial \eta}{\partial y}\right) \tag{55}$$

at the interface $z = 0$. Apart from the presence of the tangential components $E_x$ and $E_y$ consequent upon the variation of $\epsilon$ throughout the medium this equation is the same as (20).

A boundary condition for the normal component of the magnetic field can be obtained by an analysis similar to the above. Since $\nabla \cdot \underline{H} = \nabla \cdot \underline{H}' = 0$, the continuity of the tangential components of $\underline{H}$ across the interface leads to the equation

$$\frac{\partial H_z}{\partial z} = \frac{\partial H_z'}{\partial z} \tag{56}$$

(cf equation 37) at $z = 0$. Inside the medium the field equations give

$$\underline{H}' = \frac{\mu_0 \epsilon_0}{\mu k^2} \nabla \wedge \left( \frac{1}{\epsilon} \nabla \wedge \underline{H}' \right)$$

$$= \frac{\mu_0 \epsilon_0}{\mu k^2} \left\{ \frac{1}{\epsilon} \nabla \wedge \nabla \wedge \underline{H}' + \nabla \frac{1}{\epsilon} \wedge (\nabla \wedge \underline{H}') \right\}$$

$$= -\frac{1}{k^2 N^2} \left( \nabla^2 \underline{H}' + 2ikN \frac{\mu_0}{\mu} Y \underline{E}' \wedge \nabla N \right)$$

and hence

$$\nabla^2 \underline{H}' + k^2 N^2 \underline{H}' + 2ikN \frac{\mu_0}{\mu} Y \underline{E}' \wedge \nabla N \; . \tag{57}$$

In particular, the z component of (57) is

$$\frac{\partial^2 H_z'}{\partial z^2} + k^2 N^2 H_z' + 2ikN \frac{\mu_0}{\mu} Y \left( E_x' \frac{\partial N}{\partial y} - E_y' \frac{\partial N}{\partial x} \right) = 0 \quad ,$$

where the x and y derivatives have been neglected in comparison with the z, and by using the expressions for $E_x'$ and $E_y'$ given by equations (49) and (50) respectively we arrive at the equation

$$\frac{\partial^2 H_z'}{\partial z^2} + k^2 N^2 H_z' + \beta \, e^{-ikNz} = 0 \tag{58}$$

(cf equation 51), where

$$\beta = 2ikN \frac{\mu_o}{\mu} Y \left( E_x \frac{\partial N}{\partial y} - E_y \frac{\partial N}{\partial x} \right) \tag{59}$$

(cf equation 52). The solution of equation (58) is

$$H_z' = e^{-ikNz} \left( B + \frac{\beta z}{2ikN} \right) \tag{60}$$

(cf equation 53), where B is some constant, and hence at $z = 0$

$$\begin{aligned} \frac{\partial H_z'}{\partial z} &= -ikNH_z' + \frac{\beta}{2ikN} \\ &= -ikN \frac{\mu_o}{\mu} H_z + \frac{\mu_o}{\mu} Y \left( E_x \frac{\partial N}{\partial y} - E_y \frac{\partial N}{\partial x} \right) \qquad (61) \\ &= -\frac{ik}{\eta} H_z + \frac{Y}{\eta} \left( E_y \frac{\partial \eta}{\partial x} - E_x \frac{\partial \eta}{\partial y} \right) . \end{aligned}$$

If this is substituted into (56), the boundary condition on the normal component of $\underline{H}$ at the interface is

$$\frac{\partial H_z}{\partial z} = -\frac{ik}{\eta} H_z + \frac{Y}{\eta} \left( E_y \frac{\partial \eta}{\partial x} - E_x \frac{\partial \eta}{\partial y} \right) \tag{62}$$

which is analogous to the condition (21). As with the condition (55), the variation of $\eta$ has introduced the tangential components $E_x$ and $E_y$ into a boundary condition which is otherwise the same as for a homogeneous medium.

From equations (55) and (62) boundary conditions can be derived involving only the tangential components of $\underline{E}$ and $\underline{H}$. Using the equation $\nabla \cdot \underline{E} = 0$ and the expression for $E_z$ in terms of $H_x$ and $H_y$, equation (55) becomes

$$\frac{\partial E_x}{\partial x} - \frac{1}{\eta} E_x \frac{\partial \eta}{\partial x} + \eta Z \frac{\partial H_y}{\partial x} = -\frac{\partial E_y}{\partial y} - \frac{1}{\eta} E_y \frac{\partial \eta}{\partial y} - \eta Z \frac{\partial H_x}{\partial y}$$

which reduces to

$$\frac{\partial}{\partial x}\left(\frac{E_x}{\eta} + Z\,H_y\right) = -\frac{\partial}{\partial y}\left(\frac{E_y}{\eta} - Z\,H_x\right) .$$

Similarly equation (62) can be written as

$$\frac{\partial}{\partial y}\left(\frac{E_x}{\eta} + Z\,H_y\right) = \frac{\partial}{\partial x}\left(\frac{E_y}{\eta} - Z\,H_x\right)$$

and by means of the analysis given in § 3 (equation 24 et seq.) it now follows that $\left(\frac{E_x}{\eta} + Z H_y\right)$ and $\left(\frac{E_y}{\eta} - Z H_x\right)$ are both identically zero at the interface. Hence,

$$E_x = -\eta Z\,H_y \tag{63}$$

$$E_y = \eta Z\,H_x \tag{64}$$

which are of precisely the same form as the conditions (25) and (26) for a homogeneous medium. In particular, the tangential derivatives of $\eta$ do not enter into these equations in spite of the fact that they appear in equations (55) and (62). Thus, the conditions (63) and (64) are relatively insensitive to changes in the medium, and any correction terms arising from the inhomogeneity must be of higher order than those considered here. Indeed, if $\eta$ is regarded as a function of z as well as x and y, it can be shown that

$$E_x = -\eta\,Z\,H_y\left\{1 + O\left(\frac{1}{k}\,\frac{\partial\eta}{\partial z}\right)\right\}$$

(see Rytov[10]), and by virtue of equation (40) $\frac{1}{k}\,\frac{\partial\eta}{\partial z} << \eta$.

In spite of the simplicity of equations (63) and (64), these boundary conditions are of little practical value as they stand. Although the coordinates x and y do not occur explicitly in these equations, the material parameter $\eta$ is itself a function of x and y, and accordingly the boundary conditions vary with position on the interface. This is a source of difficulty in any attempt to employ these conditions in the solution of an actual problem.

On the other hand, if it is assumed that the variations of $\eta$ are random but uniform in some statistical sense, the difficulty can be overcome in a manner which is satisfactory for many practical applications. Such an assumption is, of course, additional to the restriction (40) and implies that if a large sample of the surface is chosen, the values of $\eta$ within this sample are substantially the same independently of the portion of the surface from which the sample is taken. Under these circumstances it is to be expected that the field will (in general) be a function of the statistical properties of the surface, rather than of individual features, and this leads us to consider an average field satisfying an averaged boundary condition. Such an average is obtained either by moving the transmitter and receiver whilst maintaining their positions relative to the plane z = 0 (so that different samples of surface appear beneath them), or by replacing the given surface by others of a family whose statistical properties are the same. The boundary conditions satisfied by the average field ($\underline{\bar{E}}$, $\underline{\bar{H}}$) can be found by the simple process of averaging equations (63) and (64). Bearing in mind that to the first order in $\eta$, $H_x$ and $H_y$ can be replaced by the components $H_x^o$ and $H_y^o$ for a perfectly conducting surface, (63) and (64) give

$$\overline{E}_x = -\overline{\eta} Z H_y^o$$

$$\overline{E}_y = \overline{\eta} Z H_x^o ,$$

which can be replaced by

$$\overline{E}_x = -\overline{\eta} Z \overline{H}_y \tag{65}$$

$$\overline{E}_y = \overline{\eta} Z \overline{H}_x \tag{66}$$

to the first order in $\eta$. Similarly, if the correction terms in (55) and (62) are neglected, the averaged versions are

$$\frac{\partial \overline{E}_z}{\partial z} = -ik\overline{\eta}\,\overline{E}_z \tag{67}$$

$$\frac{\partial \overline{H}_z}{\partial z} = -\frac{ik}{\overline{\eta}}\overline{H}_z \quad . \tag{68}$$

The above results are valid for statistically uniform surfaces whose refractive index $N = \frac{\mu}{\mu_o \eta}$ satisfies the restrictions (30) and (40). It will be observed that the average fields are determined by the average value of $\eta$, and not by the average values of $\epsilon$ or $\sigma$. This is in accordance with the conclusion reached by Feinberg[13] under the same restrictions but by a somewhat circuitous analysis.

These boundary conditions can be generalized so as to apply to a curved surface in the manner described in § 4. The restrictions under which this is valid are the same as in § 4, and will not be repeated here.

# PART III

# IMPEDANCE BOUNDARY CONDITIONS FOR STATISTICALLY ROUGH SURFACES

## Summary

It is shown that for an electromagnetic field incident on a perfectly conducting surface having small geometrical irregularities which are distributed at random but in a statistically uniform and isotropic manner, the boundary condition can be replaced by a generalized impedance condition applied at a neighboring mean surface. The surface impedance is a tensor function of the direction at which the field is incident as well as of the statistical properties of the irregularities, but simplifies in certain particular cases. Although the detailed analysis is carried out for a mean surface which is flat, the boundary condition is applicable to a curved surface providing the radii of curvature are large in comparison with the wavelength. It is believed that this approach is of value in studying the effect of minor surface roughnesses on the scattering of electromagnetic waves.

## §1. Introduction

In recent years an increasing amount of attention has been devoted to the effect of surface irregularities on the propagation and scattering of electromagnetic waves. In the course of this work many types of irregularity have been studied ranging from isolated bumps of simple mathematical form, through specific (or even periodic) arrangements of particular protuberances, to random distributions of general irregularities. Since the ultimate goal is a knowledge of the scattered field, most analyses have aimed at the direct calculation of this quantity, and this in turn has usually required that a separate mathematical treatment be provided for every shape of background surface on which the bumps are placed. In view of the complications associated with anything but a flat surface, an infinite plane background surface has been studied almost to the exclusion of any other shape.

The present paper is concerned only with the case of a surface having small geometrical irregularities which are distributed in a random but statistically uniform and isotropic manner, and is prompted by disagreements which have arisen about the influence of minor surface roughnesses in model scattering experiments. In order to achieve a degree of generality which is, perhaps, not otherwise obtainable, attention is directed at the boundary condition rather than at the scattered field. By taking the actual surface to be perfectly conducting, it is shown that the boundary condition can be replaced by a form of impedance boundary condition applied at a neighbouring (fictitious) mean surface. The effective surface impedance is a tensor function of the statistical properties of the irregularities and of the direction at which the field is incident. The analysis is given in detail for a mean surface which is flat, but the boundary condition is also applicable to a curved surface (and hence to a finite body) providing the radii of curvature ( and the minimum dimensions) are all large in comparison with the wavelength.

In certain cases the surface impedance can be taken as a scalar, and the boundary condition then reduces to one of the Leontovich type. This is the standard impedance boundary condition for a surface of large but finite conductivity (see, for example, Part II), and implies that the surface roughness has the same effect as changing the conductivity of the surface. Although this may seem strange at first sight, a direct consequence of the roughness is that the tangential components of the electric field at a nearby mean surface are related to the other field components through small parameters characteristic of the surface imperfections. If a suitable averaging process is applied, the conditions on the field at the mean surface reduce to a boundary condition of the type discussed in Part II, and to this degree of approximation the geometrical imperfections are therefore equivalent to a conductivity change.

A description of the surface which is considered is given in §2, and the appropriate boundary condition is derived in §3 through §5 for the particular case in which the mean surface is an infinite plane. The effective surface impedance is obtained explicitly in §6 and §7, and some numerical values are presented (§8). A general discussion which includes the application of these conditions to a curved surface is given in §9.

## §2. The Surface

The problem to be discussed is one in which an electromagnetic field is incident upon a perfectly conducting surface which varies in a statistically uniform manner about some mean surface. To begin with it is assumed for simplicity that the surface is infinite in extent and obtained by perturbation of a plane. This allows the mean surface to be taken as the plane z = 0 in a Cartesian system of coordinates (x, y, z), and only later is the problem generalized to the case of a mean surface which is curved.

The method which is used is based on one proposed by Feinberg[13, 14] for a study of ground wave propagation over a rough earth. The equation of the surface is taken as

$$z = \zeta (x, y) , \tag{1}$$

and the height and scale of the variation of $\zeta$ about its mean are denoted by the length parameters $\zeta_0$ and $\ell$ respectively. The first stage in the analysis is the expression of the boundary conditions on the actual surface as conditions upon the field components at the (fictitious) mean surface. This is accomplished by a Taylor expansion of the field about a point (x, y) on the mean surface, and it is clear that the expansion will only be valid if the behavior of the field at the mean surface differs but slightly from the behavior on the actual surface. This immediately places a restriction upon the type of surface which can be considered and also upon the location of the mean surface. In particular, large gradients or abrupt changes in gradient cannot be allowed since such perturbations may produce significant changes in the field in their immediate vicinity.

In the course of the Taylor expansion it is found that $\zeta$ and its first derivatives occur, and the typical (or root mean square) values of these make up the three parameters of smallness which are present in the problem. For a surface which is statistically isotropic, the typical values of $\frac{\partial \zeta}{\partial x}$ and $\frac{\partial \zeta}{\partial y}$ are equal ($= \gamma_o$, say), and only this case will be considered. The number of small parameters is now reduced to two, and the restriction to small surface gradients requires that

$$\gamma_o << 1 . \tag{2}$$

Moreover, in the practical case to be investigated here, the scale length will never exceed* the wavelength of the incident field, and if the mean surface is drawn so that $\zeta_o \sim \ell \gamma_o$, equation (2) gives

$$\zeta_o << \ell \tag{3}$$

which is a sufficient restriction on $\zeta_o$. It should be remarked, however, that (3) is not a necessary condition, and Feinberg[14] has shown that the higher terms in the Taylor expansion can still be neglected if

$$k \zeta_o << \sqrt{\frac{\ell}{\lambda}} . \tag{4}$$

By choosing $\ell / \lambda$ large compared with unity it becomes possible to allow surface imperfections which are not small in comparison with the wavelength providing the slopes are small. Such cases, however, will not be discussed.

## §3. First Order Boundary Conditions

Since the actual surface is perfectly conducting, the boundary condition at $z = \zeta$ is

$$\hat{n} \wedge \underline{E} = 0 \tag{5}$$

where $\hat{n}$ is a unit vector normal, and from this we obtain

*It will usually be considerably less than this.

$$E_x = - \frac{\partial \zeta}{\partial x} E_z, \tag{6}$$

$$E_y = - \frac{\partial \zeta}{\partial y} E_z. \tag{7}$$

The above equations specify relations which must be satisfied by the field components at $z = \zeta$, and once these conditions have been imposed the surface can be removed without affecting the field in the region above. The next task is to express equations (6) and (7) as conditions upon the field components at a mean surface, and this is done by expanding the field components in Taylor series. Since the field is finite and continuous everywhere throughout the free space region except at the source, we have

$$E_x(x, y, \zeta) = E_x(x, y, 0) + \zeta \frac{\partial}{\partial z} E_x(x, y, 0) + \frac{\zeta^2}{2} \frac{\partial}{\partial z^2} E_x(x, y, 0) + \dots .$$

where the differentiation must be carried out before z is put equal to zero. But if the incident field possesses a non-zero component $E_z$ (as will be assumed), $E_z$ for the total field will be O(1), and since equation (6) then shows that $E_x$ is of the first order in small quantities (denoted collectively by $\delta$),

$$E_x(x, y, \zeta) = E_x + \zeta \frac{\partial E_x}{\partial z} + O(\delta^3).$$

The field components on the right are evaluated at $z = 0$. Similarly,

$$E_y(x, y, \zeta) = E_y + \zeta \frac{\partial E_y}{\partial z} + O(\delta^3),$$

$$E_z(x, y, \zeta) = E_z + \zeta \frac{\partial E_z}{\partial z} + O(\delta^2).$$

Substituting into (6) and (7),

$$E_x = - \zeta_x E_z - \zeta \frac{\partial E_x}{\partial z} - \zeta\zeta_x \frac{\partial E_z}{\partial z} + O(\delta^3) \qquad (8)$$

$$E_y = - \zeta_y E_z - \zeta \frac{\partial E_y}{\partial z} - \zeta\zeta_y \frac{\partial E_z}{\partial z} + O(\delta^3) \qquad (9)$$

where $\zeta_x = \partial\zeta/\partial x$, etc, and these are the equivalent boundary conditions at the mean surface $z = 0$.

A little simplification can be achieved by using the fact that the divergence relation

$$\frac{\partial E_z}{\partial z} = - \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} \qquad (10)$$

holds at all points including those on the mean surface. If the expressions for $E_x$ and $E_y$ on $z = 0$ are inserted, it is seen that

$$\frac{\partial E_z}{\partial z} = O(\delta)$$

and hence equations (12) and (13) can be written as

$$E_x = - \zeta_x E_z - \zeta \frac{\partial E_x}{\partial z} + O(\delta^3) \qquad (11)$$

$$E_y = - \zeta_y E_z - \zeta \frac{\partial E_y}{\partial z} + O(\delta^3) \qquad (12)$$

for $z = 0$. In combination with equation (10) we now have

$$\frac{\partial E_z}{\partial z} = \frac{\partial}{\partial x} \left\{ \zeta_x E_z + \zeta \frac{\partial E_x}{\partial z} \right\} + \frac{\partial}{\partial y} \left\{ \zeta_y E_z + \zeta \frac{\partial E_y}{\partial z} \right\} + O(\delta^3)$$

(13)

which is identical to the equation obtained by Feinberg.

The significance of the above results becomes apparent on using the field equation.*

$$\nabla \wedge \underline{E} = ik\, Z\, \underline{H}$$

to eliminate the normal derivatives from equations (11) and (12), which then become

$$E_x = ik\, Z\, \zeta\, H_y - \frac{\partial}{\partial x}(\zeta E_z) + O(\delta^3) \tag{14}$$

$$E_y = -\, ik\, Z\, \zeta\, H_x - \frac{\partial}{\partial y}(\zeta E_z) + O(\delta^3) \tag{15}$$

where $Z = 1/Y$ is the intrinsic impedance of free space. In this form the equations differ from the Leontovich boundary condition only in the presence of the terms involving $E_z$ on the right hand sides, and these terms apart, the equations are the same as for an imperfectly conducting material having a surface impedance $-ik\,\zeta$. For a statistically rough surface, however, such an interpretation is dependent on the choice of the mean surface. To order $\delta$ the field components for a smooth surface can be inserted into the right hand sides of equations (14) and (15), and if the mean surface is chosen so that $\bar{\zeta} = 0$, where the bar denotes an average taken over the whole xy plane, the boundary conditions satisfied by the average fields are

$$E_x = E_y = 0 + O(\delta^2).$$

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* M.k.s. units are employed with a time factor $e^{-i\omega t}$

These are the conditions for a perfectly conducting smooth surface at $z = 0$, showing that the terms of order $\delta$ produce no conductivity effect. This conclusion, however, is a consequence of choosing a particular mean surface. If this does not coincide with the "average" surface, the roughness will produce a first order effect, as is to be expected since the boundary conditions are then being applied on a surface which is displaced even in the limit of zero roughness.

In the practical case of a surface having a statistical type of roughness, it is natural to choose a mean surface which coincides with the average, and if the resulting boundary conditions are to take this roughness into account, it is necessary to retain the second order terms in, for example, (11) and (12). This in turn requires us to obtain expressions for $E_z$, $\frac{\partial E_x}{\partial z}$ and $\frac{\partial E_y}{\partial z}$ on the surface accurate to $O(\delta)$, which expressions can be substituted into (11) and (12) to make explicit the terms of order $\delta^2$.

## §4. Second Order Boundary Conditions

At any point in space the electric and magnetic fields can be written as integrals involving the field components on the surface. Thus, from Stratton[3] we have

$$\underline{E}(x, y, z) = \frac{1}{2}\,\underline{E}_A + \frac{1}{4\pi}\int_S \left\{ ikZ(\hat{n}_\wedge \underline{H})\,\phi + \hat{n}_\wedge \underline{E}_\wedge \nabla\,\phi + (\hat{n}\cdot\underline{E})\nabla\phi \right\} dS \qquad (16)$$

where the differentiation is with respect to the surface coordinates $(x_1, y_1, z_1)$ and $\hat{n}$ is a unit vector normal to the mean surface S drawn inwards as regards free space. The symbol $\underline{E}_A$ denotes a surface integral over an infinite hemisphere if the incident field is a plane wave, or over a small sphere surrounding the source if this is at a finite distance; $\underline{E}_A$ is therefore a function of the incident field alone and is independent of the characteristics of the surface S. $\phi$ is the free space Green's function

$$\phi = \frac{e^{ik\rho}}{\rho} ,$$

with

$$\rho = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} .$$

Taking the surface S to be the plane $z = 0$, we have $\hat{n} = (0, 0, 1)$ and hence

$$\underline{E}(x, y, z) = \frac{1}{2} \underline{E}_A(x, y, z) + \frac{1}{4\pi} \int\int \left\{ ikZ(-H_y, H_x, 0)\phi + (E_x \frac{\partial \phi}{\partial z_1}, E_y \frac{\partial \phi}{\partial z_1}, -E_x \frac{\partial \phi}{\partial x_1} - E_y \frac{\partial \phi}{\partial y_1}) + E_z (\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial y_1}, \frac{\partial \phi}{\partial z_1}) \right\} dx_1 \, dy_1 . \qquad (17)$$

In particular,

$$E_z(x, y, z) = \frac{1}{2} E_{Az}(x, y, z) + \frac{1}{4\pi} \int\int (-E_x \frac{\partial \phi}{\partial x_1} - E_y \frac{\partial \phi}{\partial y_1} + E_z \frac{\partial \phi}{\partial z_1}) \, dx_1 \, dy_1$$

and by applying partial integration to the first two terms of the integrand, the equation becomes

$$E_z(x, y, z) = \frac{1}{2} E_{Az} + \frac{1}{4\pi} \int\int \left\{ (\frac{\partial E_x}{\partial x_1} + \frac{\partial E_y}{\partial y_1})\phi + E_z \frac{\partial \phi}{\partial z_1} \right\} dx_1 \, dy_1$$

$$= \frac{1}{2} E_{Az} - \frac{1}{4\pi} \int\int \cdot (\frac{\partial E_x}{\partial z_1} \phi - E_z \frac{\partial \phi}{\partial z_1}) \, dx_1 \, dy_1 .$$

If the observation point is now allowed to approach the surface S, the fact that

$$\lim_{z \to 0} \iint E_z \frac{\partial \phi}{\partial z_1} \, dx_1 \, dy_1 = 2\pi \, E_z(x, y, 0)$$

leads to the result

$$E_z(x, y, 0) = E_{Az}(x, y, 0) - \frac{1}{2\pi} \iint \frac{\partial E_z}{\partial z_1} \phi \, dx_1 \, dy_1 \, . \quad (18)$$

The final step is to use the expression for $\frac{\partial E_z}{\partial z}$ at a point on the mean surface. From equation (13)

$$\frac{\partial E_z}{\partial z_1} = ikZ \left\{ \frac{\partial}{\partial x_1} (\zeta_1 H_y) - \frac{\partial}{\partial y_1} (\zeta_1 H_x) \right\} + \left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y_1{}^2}\right) (\zeta_1 E_z)$$

$$= ikZ \left\{ \zeta_{x_1} H_y - \zeta_{y_1} H_x \right\} + \left(k^2 + \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y_1{}^2}\right) (\zeta_1 E_z)$$

which can be substituted into equation (18) to give

$$E_z = E_{Az} - \frac{1}{2\pi} \iint \Big[ ikZ (\zeta_{x_1} H_y - \zeta_{y_1} H_x)$$

$$+ \left(k^2 + \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y_1{}^2}\right) (\zeta_1 E_z) \Big] \phi \, dx_1 \, dy_1 + O(\delta^3) \quad (19)$$

at any point (x, y) on the surface z = 0.

Turning now to the x component of equation (17) we have

$$E_x(x, y, z) = \frac{1}{2} E_{Ax}(x, y, z) + \frac{1}{4\pi} \iint \left(- ikZ \, H_y \phi + E_x \frac{\partial \phi}{\partial z_1} + E_z \frac{\partial \phi}{\partial x_1}\right) dx_1 \, dy_1$$

$$= \frac{1}{2} E_{Ax}(x, y, z) + \frac{1}{4\pi} \iint \left(\frac{\partial E_z}{\partial x_1} \phi + E_z \frac{\partial \phi}{\partial x_1} + E_x \frac{\partial \phi}{\partial z_1} - \frac{\partial E_x}{\partial z_1} \phi\right) dx_1 \, dy_1$$

and since the first two terms of the integrand integrate to zero,

$$E_x(x,y,z) = \frac{1}{2} E_{Ax}(x,y,z) + \frac{1}{4\pi}\iint \left(E_x \frac{\partial \phi}{\partial z_1} - \frac{\partial E_x}{\partial z_1}\phi\right) dx_1\, dy_1 .$$

Hence,

$$\frac{\partial}{\partial z} E_x(x,y,z) = \frac{1}{2}\frac{\partial}{\partial z} E_{Ax}(x,y,z) - \frac{1}{4\pi}\iint \left(E_x \frac{\partial^2 \phi}{\partial z_1{}^2} - \frac{\partial E_x}{\partial z_1}\frac{\partial \phi}{\partial z_1}\right) dx_1\, dy_1$$

and in the limit of an observation point on the mean surface,

$$\frac{\partial}{\partial z} E_x(x,y,0) = \frac{\partial}{\partial z} E_{Ax}(x,y,0) - \frac{1}{2\pi} \lim_{z\to 0} \iint E_x \frac{\partial^2 \phi}{\partial z_1{}^2}\, dx_1\, dy_1$$

$$= \frac{\partial}{\partial z} E_{Ax}(x,y,0) + \frac{1}{2\pi}\iint \left(k^2 E_x + \frac{\partial^2 E_x}{\partial x_1^2} + \frac{\partial^2 E_x}{\partial y_1{}^2}\right)\phi\, dx_1\, dy_1 .$$

If the expression for $E_x$ given by (14) is substituted into this integral, the boundary value of $\frac{\partial E_x}{\partial z}$ is found to be

$$\frac{\partial E_x}{\partial z} = \frac{\partial E_{Ax}}{\partial z} - \frac{1}{2\pi}\iint \left[\left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1{}^2}\right)\left\{-ikZ\,\zeta_1\, H_y + \frac{\partial}{\partial x_1}(\zeta_1\, E_z)\right\}\right]\phi\, dx_1\, dy_1 + O(\delta^3) \tag{20}$$

which can be combined with equations (14) and (19) to give

$$E_x = -\zeta_x E_{Az} - \zeta \frac{\partial E_{Ax}}{\partial z} + \frac{1}{2\pi}\iint P_x\, \phi\, dx_1\, dy_1 + O(\delta^3) \tag{21}$$

where

$$P_x = ikZ(\zeta_x \zeta_{x_1} H_y - \zeta_x \zeta_{y_1} H_x) + ikZ\left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1{}^2}\right)(\zeta\zeta_1 H_y)$$

$$+ \left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1}\right) (\zeta \zeta_1 E_z) . \qquad (22)$$

By an analysis similar in all respects to the above it can also be shown that on the mean surface

$$\frac{\partial E_y}{\partial z} = \frac{\partial E_{Ay}}{\partial z} - \frac{1}{2\pi} \iint \left[ \left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) \left\{ ikZ \; \zeta_1 H_x \right.\right.$$

$$\left.\left. + \frac{\partial}{\partial y_1} (\zeta_1 E_z) \right\} \right] \phi \, dx_1 \, dy_1 + O(\delta^3) \qquad (23)$$

and by using equations (15), (19) and (23) we have

$$E_y = - \zeta_y E_{Az} - \zeta \frac{\partial E_{Ay}}{\partial z} + \frac{1}{2\pi} \iint P_y \phi \, dx_1 \, dy_1 + O(\delta^3) \qquad (24)$$

where

$$P_y = ikZ(\zeta_y \zeta_{x_1} H_y - \zeta_x \zeta_{y_1} H_x) - ikZ \left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) (\zeta \zeta_1 H_x)$$

$$+ \left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial y_1}\right) (\zeta \zeta_1 E_z) . \qquad (25)$$

These equations express the tangential components of the electric field on the mean surface in a way which makes explicit the factors of order $\delta^2$. To this order, it is sufficient to insert into the formulae for $P_x$ and $P_y$ the field components for a perfectly conducting (smooth) surface at z = 0, but before doing so we shall consider the averaging processes which must be applied to equations (21) and (25) if the actual surface is defined in a statistical manner.

## §5 Averaged Boundary Conditions

In order to discuss the effect of roughness with any degree of generality, it is necessary to assume that the surface is known only as regards its statistical properties. Let us therefore consider a surface which is statistically uniform and isotropic, but which is otherwise defined by its statistical parameters alone. The field behavior near to the surface can now be determined only in some average sense, leading to the concept of averaged boundary conditions. Such conditions can be obtained from equations (21) and (25) by applying either of two averaging processes. In the first of these the points $(x, y)$, $(x_1, y_1)$ are allowed to roam over a surface having the required statistical properties, the relative positions of the two points being kept constant. At every point the field is evaluated and the results are then averaged over all x and y. This is essentially a "space average" applied to one particular surface.

The second type of average is obtained by keeping the points $(x, y)$, $(x_1, y_1)$ fixed and introducing different samples of surface into the region between them. All the surfaces are, of course, members of the same statistical family and, in consequence, the averages are here "ensemble averages." Although the two averaging processes are equivalent in most practical cases, the second kind proves most convenient in the present work and will be used throughout the subsequent analysis.

The surface parameters which appear in equations (22) and (25) are $\zeta$, $\zeta \zeta_1$ and their derivatives. In specifying their average values we first observe that $\bar{\zeta}$ involves the location of the mean surface, and by choosing this such that the departure of the actual surface is zero on the average, we have

$$\bar{\zeta} = \bar{\zeta}_x = \bar{\zeta}_y = 0 , \tag{26}$$

thereby justifying the description "mean". For $\zeta \zeta_1$ the average value represents an effective correlation function, and since the surface is uniform and isotropic, this will be defined as

$$\overline{\zeta(x, y)\ \zeta(x_1, y_1)} = \zeta_o^2\ F(\rho) \tag{27}$$

where $\zeta_o$ is the standard deviation (or root mean square departure from the mean), and $F(\rho)$ is real and a function only of the distance $\rho$ separating the two points $(x, y)$ and $(x_1, y_1)$. $F(\rho)$ has a maximum value of unity at $\rho = 0$ (at which point $\partial F / \partial \rho = 0$), and falls rapidly to zero for increasing $\rho > \ell$, where $\ell$ is typical of the roughness scale. It is assumed that $\zeta_o^2$ and $F(\rho)$ are known for the surface under consideration.

If the averaging process is now applied to equations (21) and (25), the boundary conditions become

$$E_x = \frac{\zeta_o^2}{2\pi} \iint \bar{P}_x \phi \ dx_1\ dy_1 + O(\delta^3) \tag{28}$$

where

$$\bar{P}_x = ikZ \left( \frac{\partial^2 F}{\partial x \partial x_1} H_y - \frac{\partial^2 F}{\partial x \partial y_1} H_x \right)$$

$$+ ikZ \left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) (F H_x)$$

$$+ \left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1}\right) (F E_z), \tag{29}$$

and similarly for $E_y$. To the required order in $\delta$, the field components $H_x$, $H_y$ and $E_z$ can be replaced by the corresponding components for a smooth surface, and for this reason they have been excluded from the averaging process.

Since $F(\rho)$ is a function only of the variable $\rho$, it follows that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1}\right) F = 0$$

and hence

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1}\right)(FE_z) = F\,\frac{\partial E_z}{\partial x_1} = i\,\frac{Z}{k}\,F\left(\frac{\partial^2 H_y}{\partial x_1} - \frac{\partial^2 H_x}{\partial x_1\,\partial y_1}\right),$$

which enables $P_x$ to be expressed as a function of the components $H_x$ and $H_y$, in the form

$$\bar{P}_x = ikZ\left[\left\{\frac{\partial^2 F}{\partial x \partial x_1} + \Gamma\left(1 + \frac{1}{k^2}\,\frac{\partial^2}{\partial x_1^2}\right)\right\} H_y - \left\{\frac{\partial^2 F}{\partial x \partial y_1} + \frac{\Gamma}{k^2}\,\frac{\partial^2}{\partial x_1\,\partial y_1}\right\} H_x\right] \tag{30}$$

with

$$\Gamma = F\left(k^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) + \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial y_1^2} + 2\,\frac{\partial F}{\partial x_1}\,\frac{\partial}{\partial x_1} + 2\,\frac{\partial F}{\partial y_1}\,\frac{\partial}{\partial y_1}\,. \tag{31}$$

Similarly,

$$\bar{P}_y = ikZ\left[\left\{\frac{\partial^2 F}{\partial x \partial y_1} + \frac{\Gamma}{k^2}\,\frac{\partial^2}{\partial x_1\,\partial y_1}\right\} H_y - \left\{\frac{\partial^2 F}{\partial y \partial y_1} + \Gamma\left(1 + \frac{1}{k^2}\,\frac{\partial}{\partial y_1^2}\right)\right\} H_x\right] \tag{32}$$

and these results give rise to the following matrix equation

$$(\bar{P}_x, \bar{P}_y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} -Z H_y \\ Z H_x \end{pmatrix} \tag{33}$$

where

$$a_{11} = -ik \left\{ \frac{\partial^2 F}{\partial x \partial x_1} + \Gamma \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial x_1^2}\right) \right\} \tag{34}$$

$$\begin{aligned} a_{12} &= a_{21} \\ &= -ik \left\{ \frac{\partial^2 F}{\partial x \partial y_1} + \frac{\Gamma}{k^2} \frac{\partial^2}{\partial x \partial y_1} \right\} \end{aligned} \tag{35}$$

$$a_{22} = -ik \left\{ \frac{\partial^2 F}{\partial y \partial y_1} + \Gamma \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial y_1^2}\right) \right\} . \tag{36}$$

Although the elements $a_{ij}$ are differential operators, we note the interesting fact that as written above the matrix is symmetric.

## §6. The Surface Impedance Matrix

In order to evaluate the matrix representing the effective surface impedance, it is necessary to integrate the elements $a_{ij}$ in the manner shown in equation (28). This in turn requires us to insert into (28) the dependence of the components $H_x$ and $H_y$ on the surface coordinates $x_1$, $y_1$ in a neighbourhood of the point $(x, y)$.

Since the surface can be regarded as smooth as far as these components are concerned, we can write

$$H_x(x_1, y_1, 0) = H_x(x, y, 0)\, e^{i\left\{k_x(x_1 - x) + k_y(y_1 - y)\right\}}$$

$$H_y(x_1, y_1, 0) = H_y(x, y, 0)\, e^{i\left\{k_x(x_1 - x) + k_y(y_1 - y)\right\}}$$

where $k_x$ and $k_y$ can be assumed constant throughout the integration. If the incident field is produced by a point source at a finite distance from the surface,

$k_x$ and $k_y$ are the direction cosines of the source relative to the point (x, y) and are therefore functions of x and y. Equations (37) and (38) are then valid unless the source is within a wavelength or so of the surface, and even in this case the equations fail only for that portion of the surface which is in the immediate vicinity of the source. If, on the other hand, the incident field is a plane wave (corresponding to a source at infinity), $k_x$ and $k_y$ are the tangential components of the propagation vector and are the same at all points of the surface. For a plane wave incident in a direction making angles $\alpha$ and $\beta$ with the positive x and negative z axes respectively,

$$k_x = k \cos\alpha \sin\beta \tag{39}$$

$$k_y = k \sin\alpha \sin\beta, \tag{40}$$

and we note in passing that

$$k^2 - k_x^2 - k_y^2 \neq 0$$

except for grazing incidence ($\beta = \pm\ \pi/2$).

Using the above expressions for $H_x$ and $H_y$, equations (34) through (36) become

$$a_{11} = ik \left\{ \frac{\partial^2 F}{\partial x \partial x_1} + \left(1 - \frac{k_x^2}{k^2}\right)\Gamma \right\}$$

$$a_{12},\ a_{21} = -ik \left\{ \frac{\partial^2 F}{\partial x \partial y_1} - \frac{k_x k_y}{k^2}\Gamma \right\}$$

$$a_{22} = -ik \left\{ \frac{\partial^2 F}{\partial y \partial y_1} + \left(1 - \frac{k_y^2}{k^2}\right) \Gamma \right\}$$

with

$$\Gamma = (k^2 - k_x^2 - k_y^2) F + \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial y_1^2} + 2i \left(k_x \frac{\partial F}{\partial x_1} + k_y \frac{\partial F}{\partial y_1}\right),$$

and to the second order in $\zeta$ the boundary condition on the mean surface can now be written as

$$(E_x, E_y) = \begin{pmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{pmatrix} \begin{pmatrix} -ZH_y \\ ZH_x \end{pmatrix} \tag{41}$$

where

$$A_{ij} = \frac{\zeta_0^2}{2\pi} \iint a_{ij} \, e^{i\left\{k_x(x_1 - x) + k_y(y_1 - y)\right\}} \phi \, dx_1 \, dy_1 \tag{42}$$

The matrix $(A_{ij})$ is, of course, symmetric, and equation (41) represents a generalized form of the usual impedance boundary condition.

The integration in equation (42) is most easily carried out by introducing the polar coordinates $(\rho, \theta)$ where

$$x_1 = x + \rho \cos\theta$$

$$y_1 = y + \rho \sin\theta .$$

If, in addition, we place

$$k_x = \tau \cos\alpha$$

$$k_y = \tau \sin\alpha$$

with $\quad \gamma = \sqrt{k_x^2 + k_y^2}$ , then

$$A_{ij} = \frac{\zeta_o^2}{2\pi} \int_0^\infty \int_0^{2\pi} a_{ij}\, e^{i\rho\left\{k + \gamma\cos(\theta - \alpha)\right\}} d\theta\, d\rho$$

and since F is only a function of $\rho$, the $\theta$ integration can be carried out immediately to give

$$A_{11} = -ik\zeta_o^2 \int_0^\infty \left\{(1 - \frac{k_x^2}{k^2}) B - \frac{1}{2} (\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho}) J_o(\gamma\rho)\right.$$

$$\left. + \frac{1}{2} (\frac{\partial^2 F}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial F}{\partial\rho}) \cos 2\alpha\, J_2(\gamma\rho)\right\} e^{ik\rho} d\rho \qquad (43)$$

$$A_{12},\ A_{21} = i\,\frac{k_x k_y \zeta_o^2}{k} \int_0^\infty \left\{B - (\frac{k}{\gamma})^2 (\frac{\partial^2 F}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial F}{\partial\rho}) J_2(\gamma\rho)\right\} e^{ik\rho} d\rho$$

$$(44)$$

$$A_{22} = -ik\zeta_o^2 \int_0^\infty \left\{(1 - \frac{k_y^2}{k^2}) B - \frac{1}{2} (\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho}) J_o(\gamma\rho)\right.$$

$$\left. - \frac{1}{2} (\frac{\partial^2 F}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial F}{\partial\rho}) \cos 2\alpha\, J_2(\gamma\rho)\right\} e^{ik\rho} d\rho \qquad (45)$$

where

$$B = \left\{\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho} + (k^2 - \gamma^2)\right\} J_o(\gamma\rho) - 2\gamma\frac{\partial F}{\partial\rho} J_1(\gamma\rho)$$

$$= (\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + k^2)(FJ_o) . \qquad (46)$$

These are the elements of the effective surface impedance matrix, and it is seen that they depend on the direction of the incident field as specified by the factors $k_x$, $k_y$ and $k_z = \sqrt{k^2 - \tau^2}$.

## §7. A Study of Equation (41)

The properties of the boundary condition (41) are best described in terms of the impedance condition which obtains at the (smooth) surface of a material of large but finite refractive index. This condition is usually attributed to Leontovich, and can be written as

$$\underline{E} - (\hat{n} \cdot \underline{E})\,\hat{n} = \eta Z\, \hat{n}_\wedge \underline{H}, \tag{47}$$

where ($\underline{E}$, $\underline{H}$) is the field in the region outside the material (which region is regarded as free space), and $\hat{n}$ is a unit vector normal in the outward direction. The parameter $\eta$ is proportional to the reciprocal of the complex refractive and is defined by the equation

$$\eta = \left\{ \frac{\mu_0}{\mu} \left( \frac{\epsilon}{\epsilon_0} + i\,\frac{\sigma}{\omega\epsilon_0} \right) \right\}^{-1/2}, \tag{48}$$

where $\epsilon$, $\mu$ and $\sigma$ are respectively the permittivity, permeability and conductivity of the material; the suffix 'o' denotes the same quantities for free space. Equation (47) is valid for surfaces of varying curvature as well as materials whose refractive index differs from point to point providing the tangential variation of the field is relatively slow, and with this restriction the boundary condition is accurate to the first order in $\eta$. A full discussion is given in Part II.

In recent years this type of boundary condition has been increasingly used in the analysis of propagation and scattering problems. Because of the restriction to small values of $\eta$, it is natural to regard it as a means for obtaining a perturbation about the solution for perfect conductivity, but in addition solutions which are mathematically exact and subject only to the (physical) approximation implied by equation (47) have been obtained for certain simple shapes of body. Examples are the sphere, the circular cylinder, the half plane and the wedge of arbitrary angle.

For the particular case in which the imperfectly conducting material occupies the half space $z < 0$, so that the interface is an infinite plane, equation (47) reduces to

$$(E_x, E_y) = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} -ZH_y \\ ZH_x \end{pmatrix} \tag{49}$$

and this is the most elementary form of the impedance boundary condition. If equations (41) and (49) are now compared, it is seen that the boundary condition for the rough surface is only equivalent to a Leontovich condition if $A_{11} = A_{22}$ and $A_{12} = A_{21} = 0$, and although this is true for selected angles of incidence, it is not true in general.

The fact that the elements $A_{ij}$ are functions of the incidence angle is a direct consequence of the nature of the surface and represents a fundamental difference between imperfectly conducting and rough surfaces. This is in spite of the roughness being small and isotropic. As long as the mean scattering surface is a plane, the dependence is not a severe handicap, but it does mean that the tensor surface impedance is a variable function of position on the surface unless the incident field is a plane wave. For this reason the boundary condition will seldom permit an exact solution of the boundary value problem, and the usefulness of the condition then rests on the degree to which it facilitates a perturbation solution.

For certain angles of incidence the boundary condition (41) takes on a simpler form, and to demonstrate this fact we shall consider the example of a plane wave incident in a direction specified by the angles $\alpha$ and $\beta$ defined in §6. If the incidence is normal to the mean surface ($\beta = 0$), then $k_x = k_y = \tau = 0$ and equations (43) through (45) give

$$A_{11},\ A_{22} = -\frac{ik\,\zeta_o^2}{2}\int_0^\infty \left\{\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho} + 2k^2 F\right\} e^{ik\rho}\, d\rho,$$

$$A_{12},\ A_{21} = 0.$$

In this case equation (41) becomes

$$(E_x,\ E_y) = \begin{pmatrix} \eta_\perp & 0 \\ 0 & \eta_\perp \end{pmatrix}\begin{pmatrix} -ZH_y \\ ZH_x \end{pmatrix} \tag{50}$$

where

$$\eta_\perp = -\frac{ik\,\zeta_o^2}{2}\int_0^\infty \left\{\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho} + 2k^2 F\right\} e^{ik\rho}\, d\rho, \tag{51}$$

and this is now the boundary condition for the surface z = 0. The condition is of the standard Leontovich type and is accurate to the first order in the (small) parameter $\eta_\perp$, where $\eta_\perp$ is the effective surface impedance.

If the incidence is not normal, the true situation becomes apparent on writing the expressions for the $A_{ij}$ as

$$A_{11} = ik\zeta_0^2 \left[ \frac{1}{2} \frac{k_x^2 - k_y^2}{k^2} Q - \int_0^\infty \left\{ \left(1 - \frac{1}{2}\frac{\tau^2}{k^2}\right) B - \frac{1}{2}\left(\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho}\right) J_0(\tau\rho) \right\} e^{ik\rho} d\rho \right] , \tag{52}$$

$$A_{12}, A_{21} = ik\zeta_0^2 \frac{k_x k_y}{k^2} Q , \tag{53}$$

$$A_{22} = ik\zeta_0^2 \left[ -\frac{1}{2} \frac{k_x^2 - k_y^2}{k^2} Q - \int_0^\infty \left\{ \left(1 - \frac{1}{2}\frac{\tau^2}{k^2}\right) B - \frac{1}{2}\left(\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho}\right) J_0(\tau\rho) \right\} e^{ik\rho} d\rho \right] , \tag{54}$$

where

$$Q = \int_0^\infty \left\{ B - \left(\frac{k}{\tau}\right)^2 \left(\frac{\partial^2 F}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial F}{\partial \rho}\right) J_2(\tau\rho) \right\} e^{ik\rho} d\rho , \tag{55}$$

and from these it is seen that the Leontovich form of impedance condition is only obtained if $k_x k_y$ and $(k_x^2 - k_y^2)$ are both zero (as in the case of normal incidence) or $Q = 0$. It is a trivial matter to show that Q is not identically zero nor, in general, is it small compared with the other terms common to $A_{11}$ and $A_{22}$.

Nevertheless, there is another situation in which the boundary condition simplifies. In many problems involving rough surfaces it is sufficient if the approximate magnitude of the roughness effect can be determined, and for the purposes of such analyses it is only necessary that the boundary condition employed reveal the main roughness effect. Under these circumstances it seems probable that the dependence on the angle of incidence will not be of prime importance, and can be suppressed without destroying the efficacy of the boundary condition. One way in which the suppression can be achieved is to average the condition over all angles of incidence.

To this end we recall that in the right hand side of (41) the field components $H_x$ and $H_y$ can be replaced by the corresponding components for a perfectly conducting surface at $z = 0$, and accordingly

$$H_x = 2H_x^i$$

$$H_y = 2H_y^i ,$$

where the affix 'i' denotes the incident field. If $\beta \neq \pi/2$, $H_x^i$ and $H_y^i$ can be assigned values on $z = 0$ which are independent of one another, and which are independent of $\alpha$ and $\beta$ providing the strength and polarization of the equivalent source are suitably adjusted. If equation (41) is now averaged over all $\alpha$ and $\beta$ with $H_x$ and $H_y$ kept constant, then since $0 \leq \alpha \leq 2\pi$ and $0 \leq \beta \leq \pi/2$,

$$\text{av.} \quad k_x,\ k_y,\ k_x k_y = 0$$

$$\text{av.} \quad k_x^2,\ k_y^2 = \frac{1}{4} k^2$$

$$\text{av.} \quad \tau^2 = \frac{1}{2} k^2 .$$

This process has the effect of making zero* the coefficients of the unwanted terms in (52) through (54) and produces a boundary condition at z = 0 of the type shown in equation (49). The surface impedance is

$$\eta' = - \frac{ik\,\zeta_o^2}{4} \int_0^\infty \left\{ \left[ \left( \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + 2k^2 \right] \left( F\, J_o\left(\frac{k\rho}{\sqrt{2}}\right) \right) - \frac{k}{\sqrt{2}} \frac{\partial F}{\partial \rho} J_1\left(\frac{k\rho}{\sqrt{2}}\right) \right\} e^{ik\rho}\, d\rho \qquad (56)$$

and can be regarded as the average for a field incident at any angle. The evaluation of the integral is described in §8.

Before leaving this discussion of equation (41), a few words should be said about the exceptional case of grazing incidence ($\beta = \pi/2$). This is the case treated by Feinberg[13, 14] and its relevance to the present work is that it leads to a "bastard" form of the Leontovich condition if the coordinate system is suitably chosen. When $\beta = \pi/2$, $H_x^i$ and $H_y^i$ cannot be assigned values on the

* Note that equations (52) through (54) only involve even powers of $\tau$.

surface independently of one another and, indeed,

$$k_x H_x^i = - k_y H_y^i \tag{57}$$

which introduces an apparent ambiguity into equation (41) as long as $H_x$ and $H_y$ are given the values of the incident field components.

The difficulty, however, can be overcome by considering separately fields which propagate in the x and y directions. If the propagation is in the x direction, $k_y = 0$ and equation (57) shows that $H_x^i$ is then zero. Since $k_x = \tau = k$,

$$A_{11} = - \frac{ik\,\zeta_o^2}{2} \int_0^\infty \left\{ \left( \frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F}{\partial \rho} \right) J_o(k\rho) - \left( \frac{\partial^2 F}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial F}{\partial \rho} \right) J_2(k\rho) \right\} e^{ik\rho} \, d\rho$$

$$A_{21} = 0$$

and equation (41) gives

$$E_x = - \eta_{\parallel} \; Z H_y \tag{58}$$

$$E_y = 0$$

with

$$\eta_{\parallel} = \frac{ik\,\zeta_o^2}{2} \int_0^\infty \left\{ \left( \frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F}{\partial \rho} \right) J_o(k\rho) \right.$$

$$- \left(\frac{\partial^2 F}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial F}{\partial \rho}\right) J_2\,(k\rho)\Big\}\quad e^{ik\rho}\, d\rho\,. \tag{60}$$

Equations (58) and (59) are consistent with a Leontovich boundary condition for a field having $H_x = 0$. Similarly, if $k_x = 0$

$$E_x = 0 \tag{61}$$

$$E_y = \eta_{\parallel}\quad ZH_x \tag{62}$$

on $z = 0$ where $\eta_{\parallel}$ is again* given by (60), and this value for the surface impedance is equivalent to the one obtained by Feinberg who likewise assumed propagation in the direction of a coordinate axis. But if neither $k_x$ nor $k_y$ is zero, the impedance reverts to a tensor form and no reduction of (41) is then possible.

## §8. Values for the Surface Impedance

We shall now examine in rather more detail the integral expressions for $\eta_{\perp}$ , $\eta'$ and (briefly) $\eta_{\parallel}$ . Although the precise form of $F(\rho)$ is left unspecified to begin with, it should be noted that $F(0) = 1$ and $\left(\frac{\partial F}{\partial \rho}\right)_{\rho=0} = 0$ by virtue of the type of rough surface under consideration.

If integration by parts is applied to equation (51), it is found that

$$\eta_{\perp} = - \frac{ik\zeta_0^2}{2}\left[ik + \int_0^{\infty}\left(\frac{1}{\rho}\frac{\partial F}{\partial \rho} + k^2 F\right) e^{ik\rho} d\rho\right]$$

* The fact that the impedances are the same in both cases is a consequence of the isotropy of the surface.

and similarly

$$\eta' = -\frac{ik\zeta_o^2}{4}\left[ik + \int_0^\infty \left\{\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + k^2\right)\left(FJ_o\left(\frac{k\rho}{\sqrt{2}}\right)\right)\right.\right.$$

$$\left.\left. - \frac{k}{\sqrt{2}}\frac{\partial F}{\partial\rho} J_1\left(\frac{k\rho}{\sqrt{2}}\right)\right\} e^{ik\rho}\, d\rho\right] . \qquad (63)$$

For roughnesses whose scale is such that $k\ell << 1$, these equations further reduce to

$$\eta_\perp = -\frac{ik\zeta_o^2}{2}\int_0^\infty \frac{1}{\rho}\frac{\partial F}{\partial\rho} e^{ik\rho}\, d\rho$$

$$\eta' = -\frac{ik\zeta_o^2}{4}\int_0^\infty \frac{1}{\rho}\frac{\partial F}{\partial\rho} e^{ik\rho}\, d\rho , \qquad (64)$$

which may be compared with the value

$$\eta_{\|} = \frac{ik\zeta_o^2}{2}\int_0^\infty \frac{1}{\rho}\frac{\partial F}{\partial\rho} e^{ik\rho}\, d\rho$$

deduced from equation (60) under the same restriction. Hence, for small $k\ell$,

$$\eta_\perp = 2\eta' = -\eta_{\|} . \qquad (65)$$

To proceed further with the evaluation of these integrals it is necessary to insert an expression for the function $F(\rho)$. In practice, this expression should be determined by a study of the actual surface, but for small scale roughness at least it is unlikely that the impedance will depend critically on the choice of F. One of the simplest cases to consider is a Gaussian function, and for convenience this is assumed throughout the subsequent analysis. It is believed that the results obtained are typical.

If $F(\rho)$ is defined as

$$F(\rho) = e^{-4\rho^2/\ell^2} \tag{66}$$

where $\ell$ is interpreted as the scale of roughness, then

$$\frac{\partial F}{\partial \rho} = -\frac{8\rho}{\ell^2} F$$

and equation (64) now gives

$$\eta' \sim i \frac{\sqrt{\pi}}{2} \frac{k\zeta_o^2}{\ell} . \tag{67}$$

The corresponding value for $\eta_{\parallel}$ (see equation 65) is in agreement with Feinberg's result for small scale roughness.

If desired, the surface impedance can be associated with an equivalent conductivity by using equation (48) and attributing the non-zero value of $\eta$ to a conductivity $\sigma$ rather than to a permittivity $\epsilon$. Providing the conductivity term is dominant,

$$\sigma = - i Y \frac{k}{\eta^2} \text{ mhos/m}$$

where, for simplicity, $\mu$ has been put equal to $\mu_0$, and by inserting the above expression for $\eta'$ we arrive at the equivalent conductivity

$$\sigma' = i \, \frac{4}{\pi} \, Y \, \frac{\ell^2}{k \, \zeta_0^4} \quad \text{mhos/m} . \tag{68}$$

Taking, for example, $k\ell = 1/5$ and $k\zeta_0 = 1/100$,

$$\left| \sigma' \right| \sim \frac{10^5}{\lambda} \quad \text{mhos/m}$$

and at x band frequencies this is similar to the conductivity of ordinary metals.

For larger values of $k\ell$ (but still not large compared with unity), the approximations made in going from equation (63) to (64) are no longer valid, and it becomes necessary to employ the full expression for $\eta'$ given in equation (63). And similarly for $\eta_\perp$ and $\eta_\parallel$ . As long as $k\ell$ is less than (about) 2.5, however, an analytic evaluation is still possible, and has been used to compute the formula for $\eta'$. For this purpose, equation (63) is written as

$$\eta' = \left( \frac{k \zeta_0}{2} \right)^2 (L + i\, M) \tag{69}$$

where L and M are real and functions only of the parameter $u = \left( \frac{k\ell}{2} \right)^2$ .
The expressions for L and M are

$$L = 1 - \frac{\sqrt{2}}{u} \int_0^\infty \left\{ (1 - \frac{3}{4} u) J_0(x) + \frac{1}{4} u J_2(x) \right.$$

$$\left. - 2x J_1(x) \right\} e^{-2x^2/u} \sin(x\sqrt{2}) \, dx$$

$$M = \frac{\sqrt{2}}{u} \int_0^\infty \left\{ (1 - \frac{3}{4} u) J_o (x) + \frac{1}{4} u J_2 (x) - 2x J_1 (x) \right\} e^{-2x^2/u} \cos (x \sqrt{2})\, dx$$

and if the series expansions for the Bessel functions are inserted, each integral can be reduced to a sum of Fresnel integrals. These in turn can be replaced by their expansions for small argument, leading to the expression of L and M as series in ascending powers of u, which series are convergent for u less than (about) 1.5. Based on these formulae, numerical values of L and M have been computed for roughness scales up to 0.39 $\lambda$, and are plotted in Figure 1. It is seen that the imaginary part of $\eta'$ does not depart significantly from the value indicated by equation (67) until $k\ell$ exceeds 0.8, by which time the real part of $\eta$ is also becoming important. The real and imaginary parts are equal for $k\ell = 2.06$ (approx).

## §9. General Discussion

In the preceding sections it has been shown that for a perfectly conducting plane which is perturbed, or roughened, in a random sort of manner, the boundary condition can be expressed as a form of impedance condition at a neighbouring mean surface. This is valid for a wide variety of small and statistically uniform perturbations, and if the higher order effects are ignored, the boundary condition is as shown in equation (41). The result will be regarded as exact for the purposes of the following remarks.

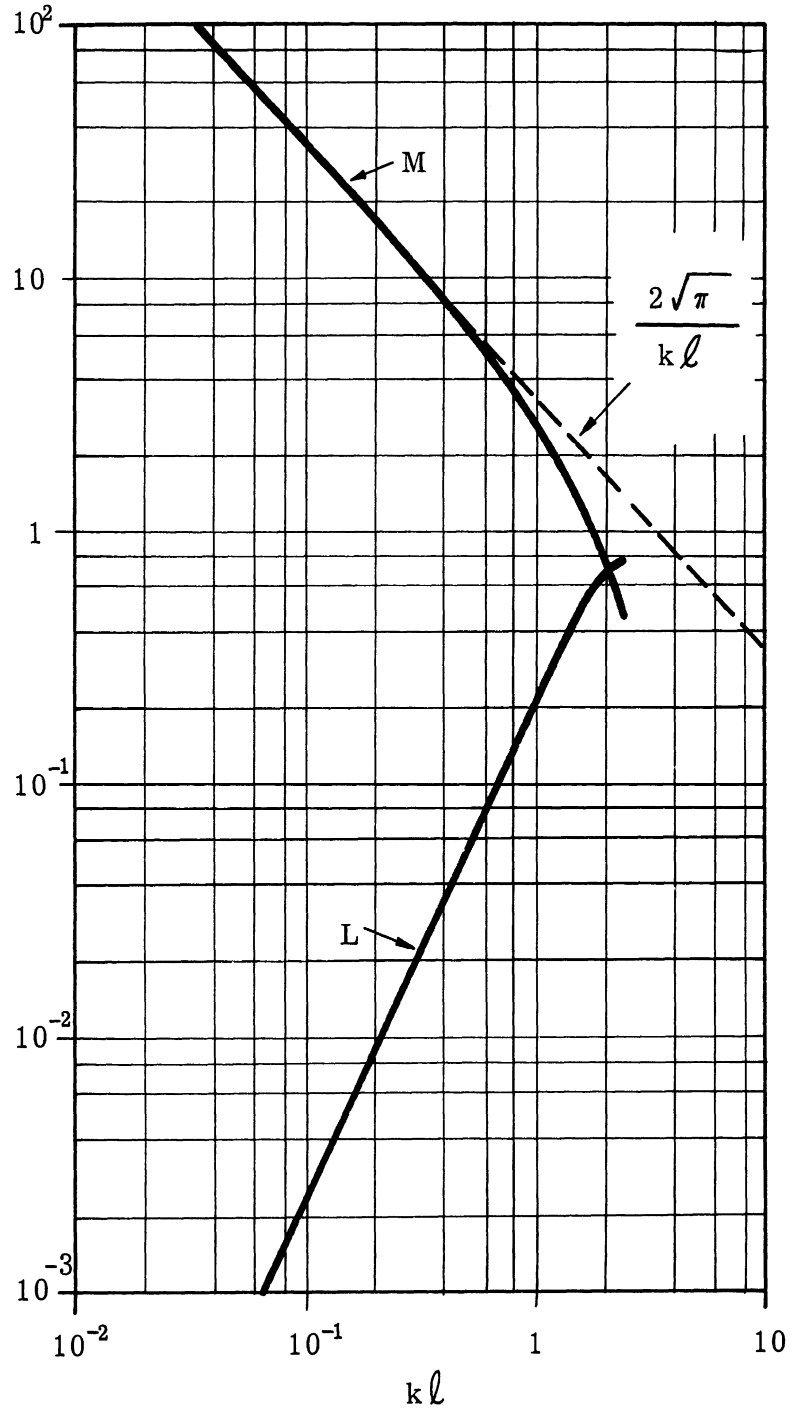


Figure 10

If the boundary condition were only applicable to a flat mean surface it would be of little practical value, and we shall now consider how it can be generalized to a mean surface which is curved. By means of a local analysis it is not difficult to see that under certain circumstances the boundary condition can be taken over as it stands. Although a rigorous proof of this fact is difficult, the extension can be justified in part by the semi-intuitive argument which appears in Part I, and this indicates that a sufficient restriction on the type of surface is for the radii of curvature (and, if the surface is closed, dimensions of the body) to be large in comparison with the wavelength. The requirement is therefore taken to be

$$R >> \lambda,$$

where R is the smallest length parameter associated with the mean surface, and if this is satisfied the curvature enters into the boundary condition only in the higher order terms. We observe in passing that for the roughness scales considered here the restriction also ensures that $R >> \ell$.

In equation (41) the elements of the impedance matrix $(A_{ij})$ are functions of the direction of the incident field relative to the surface, and although this is not a serious drawback to the use of this condition for analysing the scattering of a plane wave by an infinite perturbed plane, it does mean that when the same condition is applied to a mean surface which is curved, or to a flat surface under point source illumination, the effective surface impedance becomes a function of position on the surface. This complication is additional to the one posed by the tensor nature of the impedance. Few (if any) mathematical techniques are available for treating problems with boundary conditions of this type, and consequently there is little hope of using the condition (41) to obtain exact solutions for scattering by rough bodies.

On the other hand, the boundary condition (41) is well suited to the method of successive approximations. Knowing the field of the smooth body at all points in space and, in particular, on the mean surface itself, the boundary values of the tangential magnetic field can be inserted into the right hand side of (41), thereby specifying the components of the tangential electric field at the mean surface. These in turn can be fed into the radiation integral to give the field of the rough body at all points.

In the above method the values of $k_x$ and $k_y$ are obtained from the direction of the incident field, and consequently $k_x$ and $k_y$ will, in general, vary over the surface. A difficulty arises, however, if a portion of the body is in shadow, since it is then unlikely that the phase behaviour of the field over the dark portion of the body will be determined to a sufficient degree of approximation by the incident field alone. In this case it may be necessary to also calculate $k_x$ and $k_y$ from the smooth body solution and, in effect, regard equation (28), and the corresponding equation for $E_y$, as the fundamental equations representing the boundary condition.

In many instances, however, the accuracy provided by these boundary conditions may not be fully required, and a simpler form of condition may then prove sufficient. If, for example, the body and all its radii of curvature are <u>very</u> large in comparison with the wavelength, the field in the shadow region is unlikely to exert a profound effect on the return at angles less than (say) $60^{o}$ from back scattering, which suggests that a precise statement of the phase dependence of the smooth body field on the unilluminated side may be unnecessary. The parameters $k_x$ and $k_y$ can then be found from the incident field alone.

A further, and more striking, simplification is possible if the bulk of the return is provided by either a surface at constant inclination to the incident field, or by a relatively small portion of the whole surface. The latter case is one in which the smooth body has a specular point, and here it may be sufficient to use equation (41) with the incident field direction appropriate to this point. The same condition would then be applied regardless of position on the body, and this is particularly valuable in back scattering since the surface impedance reduces to a scalar at normal incidence.

With all these simplifications, however, approximations are introduced additional to those inherent in the boundary condition itself, and each body must therefore be considered on its merits to see which approximations (if any) are warranted.

From the above remarks it will be appreciated that a rigorous discussion of scattering by even the simplest rough body remains a problem of considerable complexity in spite of the assistance provided by the boundary condition (41). On the other hand, if the aim of the analysis is only to determine the approximate magnitude of the roughness effect, it may be sufficient to average (41) over all directions of the incident field, and this produces a tremendous simplification. The boundary condition is now the one shown in equation (49), and is seen to be of the standard Leontovich type with a surface impedance $\eta'$ given by equation (63). This condition has been used with some success to calculate the effects of minor surface roughnesses on the back scattering cross section of a large sphere. The results obtained are in reasonable agreement with experiment and are described in Part I.

A normalized form of the surface impedance $\eta'$ is plotted as a function of $k\ell$ in Figure 10, and changes from being purely reactive for small $k\ell$ to part resistive and part reactive for values of $k\ell$ near to unity. Since this impedance can be interpreted in terms of the physical properties associated with the equivalent scattering surface, it may be of interest to examine in more detail the variation with $k\ell$. In the first place, a pure imaginary $\eta'$ corresponds to a displacement of the surface parallel to itself, the displacement being in the outwards direction when the imaginary part is positive. The fact that the imaginary part is always positive in the present case is a direct consequence of the random nature of the irregularities and the chosen location of the mean surface, the ensemble averaging having removed the first order displacement (which may be either positive or negative) leaving a positive second order effect. In general, such a displacement can be expected to increase the scattered field, and this is clearly seen in the case of scattering by a large sphere. As distinct from this a portion of both the real and imaginary parts of $\eta'$ can be attributed to a true surface resistivity, and if the conduction current is large compared with the displacement current the corresponding impedance has argument $-\pi/4$. This portion of $\eta'$ is associated with a dissipation (or storage) of energy by the surface, and may be expected to decrease the scattered field in the direction of observation.

From Figure 10 it is now seen that for roughnesses of very small scale the dominant effect is a straightforward displacement of the surface which will usually lead to an increase in the scattering, but as the scale increases the resistivity increases and introduces an opposing trend. With any given value of $k\ell$, the question as to whether the scattering cross section is increased or decreased depends upon the associated amplitude of the roughness, and this is apparent from the formula for the back scattering cross section of a rough sphere (see Part I). For this body at least the resistivity will often outweigh the effect of surface displacement as the roughness increases in scale, and the back scattered field will then be less than that of a smooth sphere of radius equal to the radius of the mean surface.

## §10. Conclusions

The method by which the features of surface roughness are incorporated in the boundary conditions would appear to have advantages not only where precise solutions are required for particular types of body, but also in those cases where the desire is merely to estimate the approximate magnitude of the surface roughness effects.

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