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STUDIES IN RADAR CROSS SECTIONS XXV

DIFFRACTION BY AN IMPERFECTLY CONDUCTING WEDGE

by

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SUMMARY

The problem of the diffraction of a plane electromagnetic wave by an imperfectly conducting wedge is solved subject only to the physical approximation implied by the usual impedance-type boundary conditions imposed on the faces of the wedge. The method is based on one which was originally proposed by A. S. Peters for the treatment of a problem in hydrodynamics and leads to a difference equation for a function related to the Laplace transform of the field with respect to the radial distance from the edge. This is solved exactly to give an expression for the total field valid for any angle of wedge. Several particular cases are examined and some ramifications of the theory are discussed.

CHAPTER 1

INTRODUCTION

During the last few years considerable attention has been focussed on diffraction by structures whose conductivity is assumed to be infinite. In practice, however, the behaviour of actual metallic bodies may differ appreciably from that found theoretically in the ideal case, but attempts to take the conductivity into account have not met with great success.

One problem which is of some importance is diffraction by an imperfectly conducting wedge. The first satisfactory treatment of the effect of loss in the wedge faces was carried out by Raman and Krishnan (Ref. 1) who employed the ingenious device of multiplying by the appropriate Fresnel reflection coefficient that part of the two-dimensional Sommerfeld integral solution (for perfect conductivity) which gives rise asymptotically to a plane wave reflected from the wedge. This provides for the loss effect on the reflected plane wave, but though the solution reproduces some of the observed diffraction phenomena, it violates the reciprocity condition concerning the interchangeability of transmitter and receiver. Moreover, the method is inadequate theoretically in that it does not give the solution of the previously specified boundary-value problem.

In recent years more rigorous approaches to the problem have been taken by Jones and Pidduck (Ref. 2) and Felsen (Ref. 3), and the two methods are similar in many respects. Both adopt the usual approximate boundary conditions at the wedge faces and, using Green's functions, express the solution as a perturbation about that appropriate to a perfectly conducting wedge; this process leads to an integral form for the loss correction due to the finite conductivity.

Jones and Pidduck obtained a formal expression for the correction term when the incident field is that of an electric or magnetic line source parallel to the apex of the wedge, but considered in detail only the case of an incident plane wave. The integral was further simplified by assuming the point of observation to be far from the apex and well within the shadow region, but even then its form was not such as to be amenable to numerical calculation.

Felsen (Ref. 3), however, has shown that a reduction of the perturbation integral can be achieved without approximation in the case of an electric current source, the result being valid for arbitrary positions of source and observation points and for wedges of open angle less than π. A series representation of the correction term is given and this is rapidly convergent when the source or point of observation is near the wedge apex. For plane wave diffraction observed at large distances, an asymptotic calculation of the solution yields a set of plane waves reflected from the wedge faces with reflection coefficients appropriate to the material of the wedge, together with a cylindrical wave which appears to emanate from the apex, and this is analogous to the result obtained by Raman and Krishnan.

It should be noted that both the methods just described involve the assumption that the solution can be expanded as a series in ascending powers of η, where η is the reciprocal of the complex refractive index of the material comprising the wedge. Unfortunately this may not be justified and in the particular case of a metallic half-plane it has been shown (Ref. 4) that only for an H-polarized incident field (magnetic vector parallel to the edge) can the exact solution be so expanded, that for the other polarization containing a term $\eta \log \eta$. Accordingly, any treatment of the more general problem based upon this type of expansion is of doubtful validity, especially when the incident field is E-polarized.

In order to verify whether such an expansion is possible, it is necessary to solve the problem subject only to the (physical) approximation implied by the assumed boundary conditions. In the mathematical sense the solution is then exact, but the existing methods in diffraction theory are insufficient for its derivation. Whilst the Wiener - Hopf technique is capable of treating the half-plane, the corresponding integral equation is not susceptible to solution in this manner when the two surfaces of the diffracting structure are no longer parallel. If the structure is perfectly conducting the Sommerfeld method, based upon multi-valued solutions of the wave equation, applies equally well to the wedge and the half-plane; the procedure, however, is of a largely intuitive nature and does not lend itself to problems with mixed boundary conditions. Finally there is the method proposed by Kontorovich and Lebedev (Ref. 5). This is ideally suited to the wedge problem and when the conductivity is infinite the resulting integral equation can always be inverted directly using a special type of transform relation, whilst its logical nature suggests that it may also be applicable to an imperfectly conducting wedge. Unfortunately, this proves to be impossible since the integral equation is no longer capable of direct inversion. Moreover, to seek a solution by successive substitution is equivalent to expanding the unknown function as a series in powers of η, and even this technique is limited to the case of H-polarizati by the way in which η enters into the equation.

In his quest for alternative methods the author was led to consider that developed by Peters (Ref. 6) for the exact solution of a problem in hydrodynamics. The basis of the method is the expression of the differential equation and the boundary conditions as a difference equation for the determination of a regular function whose real part represents the velocity potential, and although the method has been extensively criticised (the integral solution obtained by Peters is not convergent), it can be modified to give the exact solution for diffraction by an imperfectly conducting wedge.

CHAPTER II

PETERS' METHOD

In reference 6, Peters derives the solution of a mixed boundary-value problem for the equation $(\nabla^2 - k^2)\phi = 0$ in a sector, the problem arising in connection with water waves on a sloping beach. In terms of polar coordinates (r, θ, z) the method enables solutions to be found for the two-dimensional equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} - k^2\phi = 0 \qquad (1)$$

in a sector $0 < \theta < \gamma$ under the boundary conditions

$$\phi_\theta = \lambda r \phi \qquad \text{for } \theta = 0 \qquad (2)$$

and

$$\phi_\theta = 0 \qquad \text{for } \theta = \gamma, \qquad (3)$$

where λ is a real constant and the suffices denote differentiation. The first condition is that for a free surface whilst the second is for a barrier and expresses the fact that there is no motion normal to it. Solutions of (1) representing progressive water waves are found by prescribing the required behaviour at the origin and at infinity.

The important stages in the method are (i) application to the differential equation of a Laplace transform with respect to the radial distance r; (ii) change of the transform variable from s to p to yield Laplace's equation for a function $\psi(p, \theta)$; (iii) identification of $\psi(p, \theta)$ with the real part of a complex function $f(\omega)$ regular in $0 < \text{Im. } \omega < \gamma$, where $\omega = p + i\theta$; (iv) expression of the boundary conditions as conditions upon $f(\omega)$ and their reduction to forms suitable for the use of an 'image' technique; and (v) derivation and subsequent solution of a difference equation for $f(\omega)$.

Although specifically developed for a discussion of water waves on a beach, the method can be applied to most problems which involve the solution of (1) in a sector of arbitrary angle γ; boundary conditions other than (2) and (3) can also be admitted, but the reality of these is essential for the success of the method as it stands.

The diffraction of an electromagnetic wave by a metallic wedge is a similar problem to that considered above and it is natural to attempt a solution in an analogous manner. Instead of (1) we now have the wave equation, but the change in sign of k^2 affects the method only as regards the transformation from s to p. The boundary conditions are also of the general form (2) but differ in that λ is no longer real. This appears to prevent the conditions from being cast into forms suitable for imaging, but the difficulty can be overcome by treating separately the real and imaginary parts of the function $\psi(p, \theta)$ and with these modifications the method serves to provide an exact solution of the diffraction problem.

CHAPTER III

FORMULATION OF THE ELECTROMAGNETIC PROBLEM

The wedge is defined in terms of the cylindrical polar coordinates (r, θ, z) by the equations $\theta = 0$ and $\theta = \Upsilon$, where Υ is the open angle of the wedge.

For simplicity the incident field is taken to be a plane wave normal to the apex of the wedge (z axis) and at an angle α $(0 \leq \alpha \leq \Upsilon)$ to one of the bounding faces, and if the magnetic vector is assumed to lie entirely in the z direction (H-polarized), it can be represented by

$$\phi^i(r, \theta) = e^{ikr\cos(\theta - \alpha)} \tag{4}$$

where the affix 'i' denotes the incident field. M.k.s. units are employed and a time factor e^{ikct} suppressed.

Since the diffracting structure extends to infinity in the z direction, the use of the plane wave (4) implies that the problem is two-dimensional, and hence the solution can be expressed in terms of a function $\phi(r, \theta)$ signifying the z component of the magnetic vector in the total (incident plus scattered) field.

The wedge itself is regarded as composed of a homogeneous material whose complex refractive index is

$$\frac{1}{\eta} = \sqrt{\frac{\epsilon}{\epsilon_0} - i\,\frac{\sigma}{kc\,\epsilon_0}}$$

where ϵ and σ are the permittivity and conductivity of the material and ϵ_0 is the permittivity of free space. Clearly

$$0 \leq \arg \eta \leq \pi/4 \quad ,$$

with the two extremes corresponding to a pure dielectric and a perfect conductor respectively.

At the faces of the wedge the usual approximate boundary conditions are imposed. These are

$$\phi_\theta - ik\eta \; r\phi = 0 \qquad \text{for } \theta = 0 \tag{5}$$

and

$$\phi_\theta + ik\eta \; r\phi = 0 \qquad \text{for } \theta = \Upsilon \tag{6}$$

(for their deriviation see, for example, Grünberg, Ref. 7) and the problem now confronting us is the exact solution of the wave equation

$$\phi_{rr} + \frac{1}{r}\,\phi_r + \frac{1}{r^2}\,\phi_{\theta\theta} + k^2\phi = 0 \tag{7}$$

under the conditions (5) and (6), together with certain additional conditions (to be discussed later) at the origin and at infinity.

CHAPTER IV

THE ADAPTATION OF PETERS' METHOD

Consider the Laplace transform

$$\phi^* (s,\theta) = \int_0^\infty e^{-sr} \phi(r,\theta)\ dr \tag{8}$$

at the outset s is regarded as a real parameter and the transform is supposed to converge for $s \geqslant \varepsilon$, where ε is a small positive number. If $r\phi(r,\theta) \rightarrow 0$ as $r \rightarrow 0$, application of the transform to equation (7) gives

$$(k^2 + s^2)\,\phi^*_{ss} + 3s\,\phi^*_s + \phi^* + \frac{\partial^2 \phi^*}{\partial \theta^2} = 0,$$

that is,

$$\frac{\partial}{\partial s}\left[(k^2 + s^2)\ \frac{\partial \phi^*}{\partial s}\right] + \frac{\partial}{\partial s}\ (s\phi^*) + \frac{\partial^2 \phi^*}{\partial \theta^2} = 0 \tag{9}$$

for $0 < \theta < \gamma$ and $s \geqslant \varepsilon$.

At first sight this does not appear to be simpler than (7), but when s is replaced by

$$s = k \sinh p \tag{10}$$

it is found that the function $\psi(p,\theta)$ defined as

$$\psi(p,\theta) = k \cosh p\ \phi^*(k \sinh p,\ \theta) \tag{11}$$

satisfies

$$\psi_{pp} + \psi_{\theta\theta} = 0 \tag{12}$$

Thus $\psi(p,\theta)$ is a harmonic function of p,θ in the semi-infinite strip $0 < \theta < \gamma$, $0 < \sinh^{-1} \varepsilon/k \leq p < \infty$. Furthermore, from equations (5) and (6) the boundary conditions on the function are

$$\psi_\theta \cosh^2 p + i\,\eta\,(\psi_p \cosh p - \psi \sinh p) = 0, \tag{13}$$

$$\psi_\theta \cosh^2 p - i\,\eta\,(\psi_p \cosh p - \psi \sinh p) = 0\,, \tag{14}$$

for θ and 0 and $\theta = \pi$ respectively.

As noted in Chapter II the technique which is adopted is to separate $\psi\,(p,\theta)$ into real and imaginary parts. Let

$$\psi\,(p,\theta) = \psi^1\,(p,\theta) + i\,\psi^2\,(p,\theta)$$

where ψ^1 and ψ^2 are real functions of the two real variables p and θ. These functions individually satisfy equation (12) and are subject to the boundary conditions

$$\psi_\theta^{\,1} \cosh^2 p - \left\{ (a\psi_p^2 + b\,\psi_p^{\,1}) \cosh p - (a\,\psi^2 + b\,\psi^1) \sinh p \right\} = 0, \tag{15}$$

$$\psi_\theta^{\,2} \cosh^2 p + \left\{ (a\psi_p^1 - b\,\psi_p^{\,2}) \cosh p - (a\,\psi^1 - b\,\psi^2) \sinh p \right\} = 0, \tag{16}$$

at $\theta = 0$, and

$$\psi_\theta^{\,1} \cosh^2 p + \left\{ (a\psi_p^{\,2} + b\,\psi_p^1) \cosh p - (a\,\psi^2 + b\,\psi^1) \sinh p \right\} = 0, \tag{17}$$

$$\psi_\theta^{2} \cosh^2 p - \left\{ (a\psi_p^1 - b\psi_p^{2}) \cosh p - (a\psi^1 - b\psi^2) \sinh p \right\} = 0, \tag{18}$$

at $\theta = \pi$, where $\eta = a + ib$ with a and b real.

Following Peters, p and θ are now regarded as the real and imaginary parts respectively of some complex variable ω. We shall write

$$\omega = p + i\theta, \tag{19}$$

but it must be emphasized that the complex notation is not the same as that which is implicit in the problem. Although the same symbol i is used in both cases to denote $\sqrt{-1}$, no confusion should arise since the two notations are never simultaneously employed. Indeed, the new 'i' is introduced solely for the purposes of the analysis and its presence implies that the analysis is carried out in a three-dimensional space which is in some respects analogous to the Riemann space used in the Sommerfeld method for treating diffraction by a perfectly conducting wedge.

Let f(ω) and g(ω) be two functions regular in the strip S: $0 < \theta < \gamma$, $0 < \sinh^{-1} \varepsilon/k \leq p < \infty$. The harmonic functions $\psi^1(p,\theta)$ and $\psi^2(p,\theta)$ can then be identified with the real parts of f(ω) and g(ω) respectively, so that

$$\psi^1(p,\theta) = \text{Re. } f(\omega) \tag{20}$$

and

$$\psi^2(p,\theta) = \text{Re. } g(\omega), \tag{21}$$

and in terms of f(ω) and g(ω) the boundary conditions (15) - (18) become

$$\text{Re.} \left[i \cosh^2 \omega f'(\omega) - \left\{ (ag'(\omega) + bf'(\omega)) \cosh \omega - (ag(\omega) + bf(\omega)) \sinh \omega \right\} \right] = 0 \tag{22}$$

$$\text{Re.} \left[i \cosh^2 \omega g'(\omega) + \left\{ (af'(\omega) - bg'(\omega)) \cosh \omega - (af(\omega) - bg(\omega)) \sinh \omega \right\} \right] = 0 \tag{23}$$

at $\theta = 0$;

$$\text{Re.} \left[i \cosh^2 (\omega - i\gamma) f'(\omega) + \left\{ (ag'(\omega) + bf'(\omega)) \cosh(\omega - i\gamma) - (ag(\omega) + bf(\omega)) \sinh(\omega - i\gamma) \right\} \right] = 0 \tag{24}$$

$$\text{Re.} \left[i \cosh^2 (\omega - i\gamma) g'(\omega) - \left\{ (af'(\omega) - bg'(\omega)) \cosh(\omega - i\gamma) - (af(\omega) - bg(\omega)) \sinh(\omega - i\gamma) \right\} \right] = 0 \tag{25}$$

at $\theta = \gamma$, where the primes denote differentiation with respect to ω. The problem remaining is therefore the determination of two functions regular in S and satisfying these conditions.

CHAPTER V

THE DIFFERENCE EQUATIONS

The form of the conditions (22) - (25) enables us to analytically continue the functions f(ω) and g(ω) across the lines Im. ω=0 and Im. ω= γ . The way in which this is carried out may be illustrated by considering equation (22) in which the expression in square brackets maps the lower boundary of the strip S (that is, the positive real axis of the ω plane) into part of an imaginary axis. Consequently, the expression can be continued across the real axis using Schwarz's reflection principle, and for ω in S we have*

$$i\cosh^2\bar{\omega}\bar{f}'(\omega) + \left\{(a\bar{g}'(\omega) + b\bar{f}'(\omega))\cosh\omega - (a\bar{g}(\omega) + b\bar{f}(\omega))\sinh\omega\right\}$$
$$= i\cosh^2\omega f'(\omega) - \left\{(ag'(\omega) + bf'(\omega))\cosh\omega - (ag(\omega) + bf(\omega))\sinh\omega\right\}. \qquad (26)$$

Similarly, from equations (23) - (25)

$$i\cosh^2\bar{\omega}\bar{g}'(\omega) - \left\{(a\bar{f}'(\omega) - b\bar{g}'(\omega))\cosh\omega - (a\bar{f}(\omega) - b\bar{g}(\omega))\sinh\omega\right\}$$
$$= i\cosh^2\omega g'(\omega) + \left\{(af(\omega) - bg(\omega))\cosh\omega - (af(\omega) - bg(\omega))\sinh\omega\right\}, \qquad (27)$$

$$i\cosh^2(\omega+i\gamma)\ \bar{f}'(\omega) - \left\{(a\bar{g}'(\omega) + b\bar{f}'(\omega))\cosh(\omega+i\gamma)\right.$$
$$\left. - (a\bar{g}(\omega) + b\bar{f}(\omega))\sinh(\omega+i\gamma)\right\} = i\cosh^2(\omega+i\gamma)\ f'(\omega+2i\gamma)$$
$$+ \left\{(ag'(\omega+2i\gamma) + bf'(\omega+2i\gamma))\cosh(\omega+i\gamma) - (ag(\omega+2i\gamma) + bf(\omega+2i\gamma))\sinh(\omega+i\gamma)\right\}, \qquad (28)$$

$$i\cosh^2(\omega+i\gamma)\ \bar{g}'(\omega) + \left\{(a\bar{f}'(\omega) - b\bar{g}'(\omega))\cosh(\omega+i\gamma)\right.$$
$$\left. - (a\bar{f}(\omega) - b\bar{g}(\omega))\sinh(\omega+i\gamma)\right\} = i\cosh^2(\omega+i\gamma)\ g'(\omega+2i\gamma) \qquad (29)$$
$$- \left\{(af'(\omega+2i\gamma) - bg'(\omega+2i\gamma))\cosh(\omega+i\gamma) - (af(\omega+2i\gamma) - bg(\omega+2i\gamma))\sinh(\omega+i\gamma)\right\}.$$

Together these provide the analytical continuation of f(ω) and g(ω) from the strip S into the

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* We write $\bar{\omega}$ = p-iθ for the conjugate of ω, $\overline{f(\omega)}$ for the conjugate of f(ω); $\bar{f}(\omega)$ is f(ω) with conjugate coefficients and $\bar{f}(\bar{\omega})$ is $\bar{f}(\omega)$ with ω replaced by $\bar{\omega}$, so that $\bar{f}(\bar{\omega}) = \overline{f(\omega)}$.

wider strip $-\gamma < \text{Im.}\ \omega < 3\gamma$.

After some re-arrangement, the above equations can be integrated directly, giving rise to four equations involving $f(\omega)$, $g(\omega)$ and their conjugates:

$$\left\{ i\cosh\omega - b \right\} f(\omega) - a g(\omega) = \left\{ i\cosh\omega + b \right\} \bar{f}(\omega) + a\bar{g}(\omega) + c_1 \tag{30}$$

$$\left\{ i\cosh\omega - b \right\} g(\omega) + a f(\omega) = \left\{ i\cosh\omega + b \right\} \bar{g}(\omega) - a\bar{f}(\omega) + c_2 \tag{31}$$

$$\left\{ i\cosh(\omega+i\gamma) + b \right\} f(\omega+2i\gamma) + a g(\omega+2i\gamma)$$
$$= \left\{ i\cosh(\omega+i\gamma) - b \right\} \bar{f}(\omega) - a\bar{g}(\omega) + c_3 \tag{32}$$

$$\left\{ i\cosh(\omega+i\gamma) + b \right\} g(\omega+2i\gamma) - a f(\omega+2i\gamma)$$
$$= \left\{ i\cosh(\omega+i\gamma) - b \right\} \bar{g}(\omega) + a\bar{f}(\omega) + c_4 \tag{33}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Elimination of the conjugate functions then leaves

$$\left[\left\{ i\cosh(\omega+i\gamma) - b \right\} \left\{ i\cosh\omega - b \right\} - a^2 \right] g(\omega) - \left[\left\{ i\cosh(\omega+i\gamma) - b \right\} \right.$$
$$\left. \left\{ i\cosh\omega - b \right\} - a^2 \right] g(\omega+2i\gamma) = -a \left[i \left\{ \cosh(\omega+i\gamma) + \cosh\omega \right\} - 2b \right] f(\omega)$$
$$- a \left[i \left\{ \cosh(\omega+i\gamma) + \cosh\omega \right\} + 2b \right] f(\omega+2i\gamma) + c_2 - a(c_1 + c_3)$$
$$+ (i\cosh\omega - b)\,(c_2 + c_4) \tag{34}$$

and

$$\left[\left\{ i\cosh(\omega+i\gamma) - b \right\} \left\{ i\cosh\omega - b \right\} - a^2 \right] f(\omega) - \left[\left\{ i\cosh(\omega+i\gamma) - b \right\} \right.$$
$$\left. \left\{ i\cosh\omega - b \right\} - a^2 \right] f(\omega+2i\gamma) = a \left[i \left\{ \cosh(\omega+i\gamma) + \cosh\omega \right\} - 2b \right] g(\omega)$$
$$+ a \left[i \left\{ \cosh(\omega+i\gamma) + \cosh\omega \right\} + 2b \right] g(\omega+2i\gamma) + c_1 + a(c_2 + c_4)$$
$$+ (i\cosh\omega - b)\,(c_1 + c_3). \tag{35}$$

If we now introduce the functions $F(\omega)$ and $G(\omega)$ defined by

$$F(\omega) = f(\omega) + i\, g(\omega) \qquad (36)$$

$$G(\omega) = f(\omega) - i\, g(\omega) \qquad (37)$$

we have, on multiplying equation (35) by i and subtracting from and adding to, equation (34),

$$\left\{\eta + \cosh(\omega + i\gamma)\right\}\left\{\eta + \cosh\omega\right\} F(\omega) = \left\{\eta - \cosh(\omega + i\gamma)\right\}\left\{\eta - \cosh\omega\right\}$$

$$F(\omega + 2i\gamma) - c_1 - ic_2 + (ai + b - i\cosh\omega)(c_1 + ic_2 + c_3 + ic_4), \qquad (38)$$

$$\left\{\bar{\eta} - \cosh(\omega + i\gamma)\right\}\left\{\bar{\eta} - \cosh\omega\right\} G(\omega) = \left\{\bar{\eta} + \cosh(\omega + i\gamma)\right\}\left\{\bar{\eta} + \cosh\omega\right\}$$

$$G(\omega + 2i\gamma) - c_1 + ic_2 - (ai - b + i\cosh\omega)(c_1 - ic_2 + c_3 - ic_4), \qquad (39)$$

respectively, where $\bar{\eta} = a - ib$.

These linear difference equations are necessary conditions which must be satisfied by the functions $F(\omega)$ and $G(\omega)$, but for our purposes it is sufficient to consider the corresponding homogeneous equations which follow from neglecting the arbitrary constants $c_1, \ldots\ldots, c_4$. The conditions upon $F(\omega)$ and $G(\omega)$ are therefore

$$\left\{\eta + \cosh(\omega + i\gamma)\right\}\left\{\eta + \cosh\omega\right\} F(\omega) = \left\{\eta - \cosh(\omega + i\gamma)\right\}\left\{\eta - \cosh\omega\right\} F(\omega + 2i\gamma), \qquad (40)$$

$$\left\{\bar{\eta} - \cosh(\omega + i\gamma)\right\}\left\{\bar{\eta} - \cosh\omega\right\} G(\omega) = \left\{\bar{\eta} + \cosh(\omega + i\gamma)\right\}\left\{\bar{\eta} + \cosh\omega\right\} G(\omega + 2i\gamma).$$

It is clear from the symmetry of equations (40) and (41) that if $X(\omega, \eta)$ is a solution of (40), a possible solution of (41) is $G(\omega) = X(\omega, -\bar{\eta})$, although this may not be the required solution. Nevertheless, from (30) and (31),

$$G(\omega) = -\frac{\bar{\eta} + \cosh\omega}{\bar{\eta} - \cosh\omega}\,\bar{F}(\omega) \qquad (42)$$

$$\bar{G}(\omega) = -\frac{\eta + \cosh\omega}{\eta - \cosh\omega}\,F(\omega) \qquad (43)$$

and thus the two cannot be chosen independently of one another. To specify (for example) $F(\omega)$ automatically determines $G(\omega)$ using equation (42) and for this reason it is sufficient to restrict attention to equation (40).

To reduce it is a form amenable to solution we make the substitution

$$\zeta = e^{\omega} \qquad (44)$$

which maps a strip of the ω plane on to a sector of the ζ plane. As it stands the transformation is multivalued, but a one-to-one correspondence can be established by inserting a cut into the ζ plane, the precise location of the cut being immaterial providing the strip $0 < \text{Im. } \omega < \gamma$ is mapped on to the sector $0 < \arg \zeta < \gamma$.

Let

$$F(\omega) = H(\zeta) \qquad (45)$$

so that

$$F(\omega + 2i\gamma) = H(\zeta e^{2i\gamma}).$$

Since

$$\eta + \cosh(\omega + i\gamma) = \frac{e^{i\gamma}}{2\zeta}\left(\zeta + \tau_1 e^{-i\gamma}\right)\left(\zeta + \tau_2 e^{-i\gamma}\right)$$

and

$$\eta - \cosh(\omega + i\gamma) = -\frac{e^{i\gamma}}{2\zeta}\left(\zeta - \tau_1 e^{-i\gamma}\right)\left(\zeta - \tau_2 e^{-i\gamma}\right),$$

where

$$\tau_1 = \eta + i\sqrt{1-\eta^2} \quad \text{and} \quad \tau_2 = \eta - i\sqrt{1-\eta^2}, \qquad (46)$$

equation (40) becomes

$$(\zeta + \tau_1 e^{-i\gamma})(\zeta + \tau_2 e^{-i\gamma})(\zeta + \tau_1)(\zeta + \tau_2)\, H(\zeta) \tag{47}$$

$$= (\zeta - \tau_1 e^{-i\gamma})(\zeta - \tau_2 e^{-i\gamma})(\zeta - \tau_1)(\zeta - \tau_2)\, H(\zeta e^{2i\gamma}) \, .$$

The solution of this equation is the crux of the problem now remaining.

CHAPTER VI

THE RELATION BETWEEN $\phi(r,\theta)$ AND $F(\omega)$

Before proceeding to the solution of the difference equation it is necessary to consider the way in which $\phi(r,\theta)$ is to be expressed in terms of $F(\omega)$. Only by doing so it is possible to formulate in their entirety the conditions to be imposed on the required function $F(\omega)$.

It will be remembered that the Laplace transform $\phi^*(s,\theta)$ led us to a function $\psi(p,\theta)$ whose real and imaginary parts were represented as the real parts of the functions $f(\omega)$ and $g(\omega)$ respectively. Moreover,

$$f(\omega) = \frac{1}{2}\left\{F(\omega) + G(\omega)\right\},$$

$$g(\omega) = \frac{i}{2}\left[G(\omega) - F(\omega)\right],$$

and if $F(\omega)$ and $G(\omega)$ are written as

$$F(\omega) = A(p,\theta) + iB(p,\theta), \qquad G(\omega) = C(p,\theta) + iD(p,\theta),$$

where A, B, C and D are real functions, then

$$\text{Re. } f(\omega) = \frac{1}{2}\left\{A(p,\theta) + C(p,\theta)\right\}$$

and

$$\text{Re. } g(\omega) = \frac{1}{2}\left[B(p,\theta) - D(p,\theta)\right].$$

But

$$\psi(p,\theta) = \psi^1(p,\theta) + \psi^2(p,\theta)$$

$$= \text{Re. } f(\omega) + i\,\text{Re. } g(\omega),$$

where the 'i' is that which implicit in the problem, and hence

$$\psi(p,\theta) = \frac{1}{2}\left\{A(p,\theta) + i\,B(p,\theta) + C(p,\theta) - i\,D(p,\theta)\right\}$$

$$= \frac{1}{2}\left\{F(\omega) + \bar{G}(\bar{\omega})\right\}. \qquad (48)$$

By using equation (43) we now have

$$\psi(p,\theta) = \frac{1}{2}\left\{F(\omega) - \frac{\eta + \cosh\bar{\omega}}{\eta - \cosh\bar{\omega}}\,F(\bar{\omega})\right\} \tag{49}$$

which gives the function $\psi(p,\theta)$ directly in terms of $F(\omega)$.

This result is important in enabling us to pass directly from the complex notation of Chapter IV to that of Chapter III without having to separate $F(\omega)$ and $G(\omega)$ into their real and imaginary parts. The nature of the function $\psi(p,\theta)$ is such that $F(\omega)$ and $G(\omega)$ can be taken over in their entirety and in going to equations (48) and (49) the analytical 'i' has been displaced by the naturally-occuring 'i' of Chapter III.

The form of equation (49) suggests that it may be possible to devise a more straight-forward derivation which would avoid the necessity for introducing the functions $\psi^1(p,\theta)$ and $\psi^2(p,\theta)$. This is indeed so and the new method turns out to be very much more concise. However, owing to the two complex notations which are now simultaneously employed, the steps in the argument are more difficult to follow, and for this reason the original treatment is to be preferred.

When a suitable expression for $F(\omega)$ has been found we have, from equations (11) and (49),

$$\phi^*(s,\theta) = \frac{1}{k\cosh p}\,\psi(p,\theta)$$

$$= \frac{1}{2\,k\cosh p}\left\{F(p+i\theta) - \frac{\eta + \cosh(p-i\theta)}{\eta - \cosh(p-i\theta)}\,F(p-i\theta)\right\} \tag{50}$$

where $k \sinh p = s$. So far s has been regarded as a real quantity, but from the theory of the Laplace transform $\phi^*(s,\theta)$ is known to be a regular function of s in the half-plane Re. $s \geqslant \epsilon$. Consequently, if s is complex, the right hand side of (50) is a function of the complex variable p which is regular in the domain corresponding to Re. $s > \epsilon$, and θ then takes on the role of a parameter.

The inversion formula for the Laplace transform gives

$$\phi(r,\theta) = \frac{1}{2\pi i} \int_{\ell} e^{sr} \phi^*(s,\theta)\, ds, \tag{51}$$

where the path of integration is a straight line parallel to the imaginary axis and to the right of all singularities of $\phi^*(s,\theta)$. The transformation to the p plane is effected by s=k sinh p, which maps the whole of the s plane on to each of the strips

$$(m-{}^1/_2)\,\pi \leq \text{Im. } p \leq (m+{}^1/_2)\,\pi, \qquad m = 0, \pm 1, \pm 2, \ldots.$$

but though the transformation is multivalued, the range of p cannot be chosen arbitrarily. The choice of strip is governed by the initial assumption that real s and p correspond, and this implies that the s plane is mapped on to

$$-\pi/_2 \leqslant \text{Im. } p \leqslant \pi/_2. \tag{52}$$

Since the right hand s half-plane corresponds to the right hand half-strip in the p plane,

$$\phi(r,\theta) = \frac{1}{4\pi i} \int_{\ell'} e^{kr \sinh p} \left\{ F(p+i\theta) - \frac{\eta + \cosh(p-i\theta)}{\eta - \cosh(p-i\theta)}\, F(p-i\theta) \right\} dp,$$

where the path of integration ℓ' in the p plane is a loop from $p = \infty - i\,\pi/_2$ to $p=\infty + i\,\pi/_2$.

If we now write $p=i(\pi/_2 - \nu)$ mapping the above strip of the p plane on to a displaced and rotated strip of the complex ν plane, then

$$\phi(r,\theta) = -\frac{1}{4\pi} \int_{\ell''} e^{ikr\cos\nu} \left\{ F\left[i(\frac{\pi}{2} - \nu + \theta)\right] - \frac{\eta + \sin(\nu+\theta)}{\eta - \sin(\nu+\theta)}\, F\left[i(\frac{\pi}{2} - \nu - \theta)\right] \right\} d\nu \tag{53}$$

where $\mathcal{D}''$ is a loop path from $\nu = \pi + i\infty$ to $\nu = i\infty$ whose precise form between these points is immaterial providing it lies above all singularities of the integrand.

In general the transformation from s to ν using $s = ik \cos \nu$ will introduce branch points into the ν plane at $\nu = m\pi$, $m = 0, \pm 1, \pm 2, \ldots,$ and it is natural to take the resulting branch cuts parallel to the imaginary ν axis. The transformation, however, may serve to remove branch points, a multivalued function in the un-cut s plane becoming single-valued in the un-cut ν plane. This will certainly be true if the function $F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right]$ satisfies the following condition:

(I) In $-\pi \leqslant \text{Re}.\, \nu \leqslant \pi$, $F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right]$ has only a finite number of poles and is otherwise regular and free of singularities for all θ, $0 \leqslant \theta \leqslant \pi$.

Since the integrand of (53) is then single-valued in the un-cut ν plane, deformation of $\mathcal{D}''$ outside the strip to which it was originally confined is possible and, moreover, necessary if the integral is to converge. From a study of the regions in which the path may approach infinity, we are led to choose the path L of Figure 1, and whilst it is convenient to take the end points to be $\nu = 3\pi/2 + i\infty$ and $\nu = -\pi/2 + i\infty$, any path commencing at infinity in the strip $\pi < \text{Re}.\, \nu < 2\pi$ and ending at infinity in the strip $-\pi < \text{Re}.\, \nu < 0$ will do providing all singularities of the integrand lie below it.

If it is further assumed that

$$\text{(II)} \qquad \frac{\eta + \sin(\nu + \theta)}{\eta - \sin(\nu + \theta)}\, F\left[i\left(\frac{\pi}{2} - \nu - \theta\right)\right] = F\left[i\left(\frac{\pi}{2} + \nu + \theta\right)\right]$$

then

$$\phi(r,\theta) = \frac{1}{4\pi} \int_L e^{ikr\cos\nu} F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right] d\nu$$

$$= -\frac{1}{4\pi}\left[\int_L + \int_{L'}\right] e^{ikr\cos\nu} F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right] d\nu \qquad (54)$$

where L' is the path shown in Figure 1.

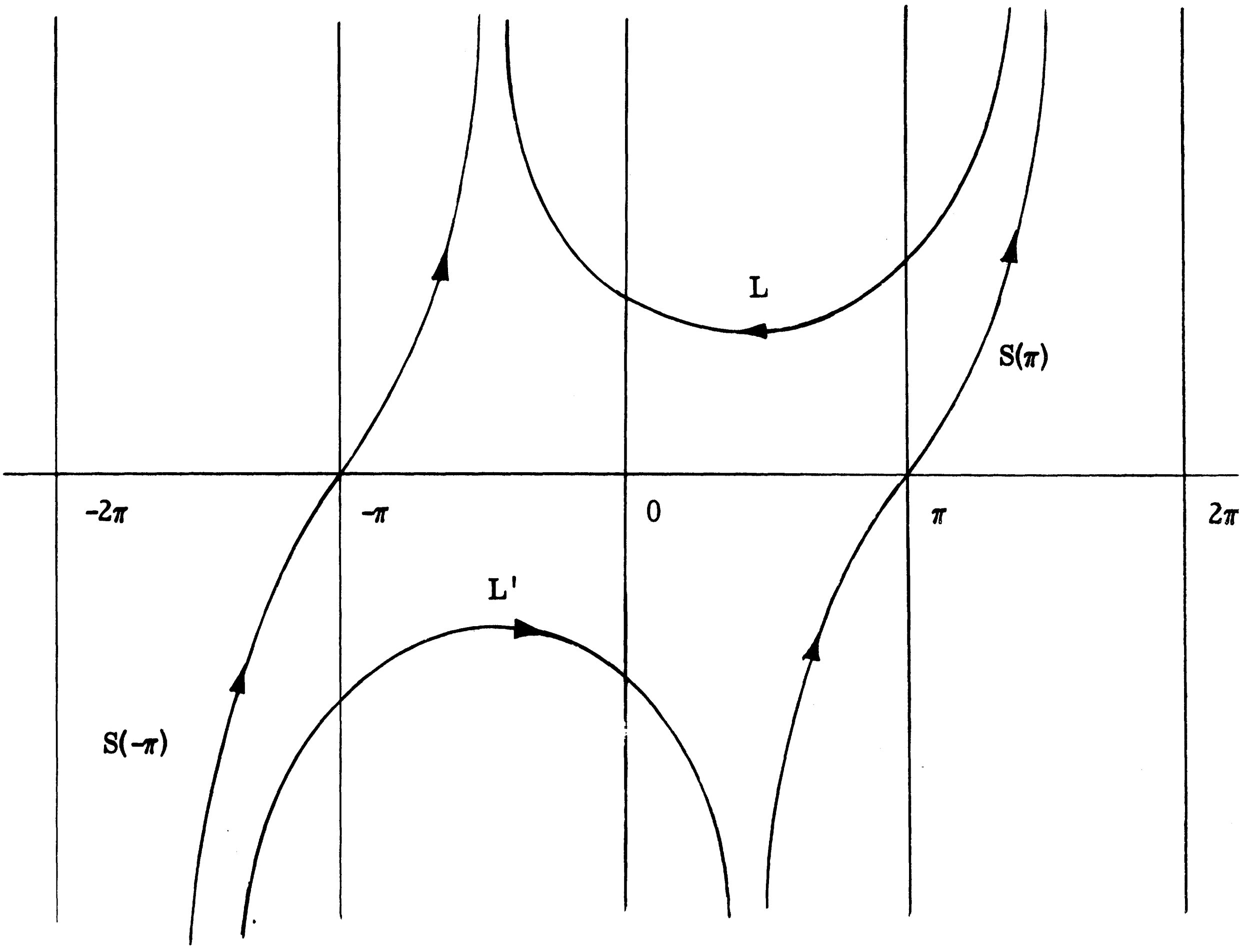


FIGURE 1

In view of the condition (I) the only singularities between the two paths are poles and hence

$$\phi(r,\theta) = \frac{1}{4\pi}\left[\int_{S(\pi)} - \int_{S(-\pi)}\right] e^{ikr\cos\nu}\, F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right] d\nu - \frac{i}{2}\,\phi_R$$

with ϕ_R denoting the sum of the residues at the included poles. It now only remains to make the substitutions $\beta = \nu + \pi$ on $S(\pi)$ and $\beta = \nu - \pi$ on $S(-\pi)$ to give

$$\phi(r,\theta) = \frac{1}{4\pi}\int_{S(0)} e^{-ikr\cos\beta}\left\{F\left[i\left(\frac{3\pi}{2} - \beta + \theta\right)\right] - F\left[i\left(\frac{-\pi}{2} - \beta + \theta\right)\right]\right\} d\beta - \frac{i}{2}\,\phi_R \tag{55}$$

where S(0) is a steepest descent path through the origin, and this is a solution of the desired form.

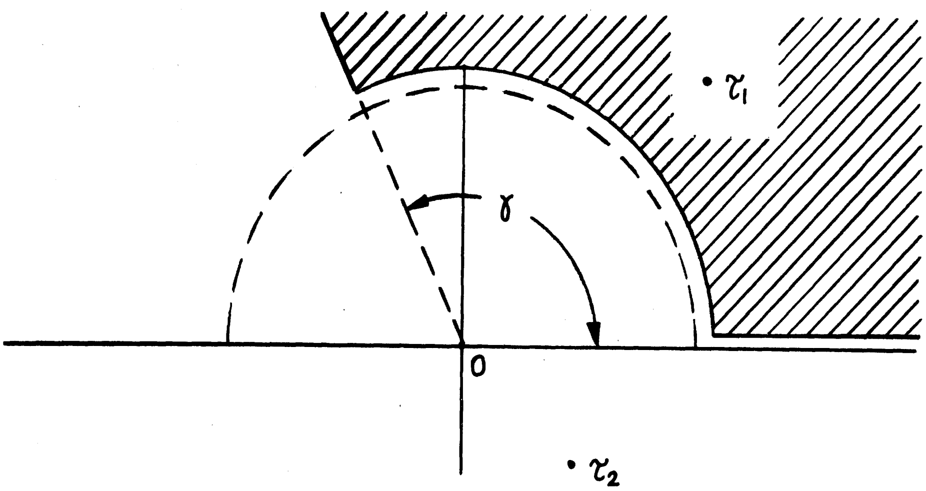
If the poles of $F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right]$ are chosen correctly, the term $\frac{i}{2}\,\phi_R$ will give the geometrical optics contribution and the integral can then be interpreted as the diffracted field. The expression for $\phi(r,\theta)$ as a whole will represent a possible solution of the diffraction problem if a function F can be found satisfying our assumptions, and it will be shown that the conditions (I) and (II) are not only consistent with those imposed on F in the original analysis but also serve to specify a unique solution of the problem.

CHAPTER VII

THE CONDITIONS ON H(ζ)

Let us first review the way in which the difference equations (40) and (41) were derived.

The functions f(ω) and g(ω) defined by (20) and (21) are, by virtue of the wave equation (7), regular in the strip $0 < \text{Im.}\ \omega < \gamma$ and consequently F(ω) and G(ω) are also regular in this strip. The boundary conditions on the faces of the wedge enable F(ω) and G(ω) to be analytically continued into a wider strip $-\gamma < \text{Im.}\ \omega < 3\gamma$, the values of the functions in the wider strip being given in terms of their values in $0 < \text{Im.}\ \omega < \gamma$ by the difference equations (40) and (41). These equations are directly analogous to the ones obtained by Peters (Ref. 6) in his discussion of the water wave problem.



Because of the relations (42) and (43) connecting F(ω) and G(ω) it is sufficient to consider only equation (40), and this can be converted into a difference equation for a function H(ζ) by means of the transformation (44). In the ζ plane the function H(ζ) must be regular in that region which corresponds to the strip S of the

ω plane; it must therefore be free of singularities in

$$\sum \quad : \quad 1 < \exp\left\{\sinh^{-1} \frac{\varepsilon}{k}\right\} \leq |\zeta| < \infty, \; 0 < \arg \zeta < \gamma$$

and since ε is small, this is essentially the sector $0 < \arg \zeta < \gamma$ with the inside and boundary of the unit circle excluded (Figure 2).

The behaviour of $H(\zeta)$ for large $|\zeta|$ is specified by the form of $\phi(r, \theta)$ when r is small. For a half plane ($\gamma = 2\pi$) of infinite conductivity the admissible edge condition has received considerable attention and it is now accepted that no field component can have a singularity of greater order than $r^{-\frac{1}{2}}$ as $r \rightarrow 0$. In itself, however, this does not serve to ensure uniqueness when the field is a two-dimensional H-polarized wave incident in the plane normal to the edge since all components are finite for $r=0$. This point has been brought out by Clemmow (Ref. 8), who shows that the condition becomes sufficient if the field is made three-dimensional and, in fact, for an H-polarized wave incident on a half-plane,

$$H_z \sim \text{constant as } r \rightarrow 0.$$

For the actual case of a wedge or corner the work of Jones (Ref. 9) has more direct application and leads us to impose the same condition on the wedge as on the half-plane. Although Jones was concerned only with perfect conductors, the presence of finite conductivity is unlikely to affect the general nature of the singularity, and we shall therefore assume that

$$\phi(r, \theta) \sim \text{constant as } r \rightarrow 0, \tag{56}$$

which implies

$$H(\zeta) \sim \text{constant as } |\zeta| \rightarrow \infty. \tag{57}$$

In passing it should be noted that the conditions: ϕ remains finite and $r \, \partial\phi/\partial r \to 0$ as $r \to 0$ used by Pauli (Ref. 10) in his analysis of the solution for a perfectly conducting wedge are satisfied by a function whose behaviour is given by (56). This is also true of the condition: $r\,\phi \to 0$ as $r \to 0$ assumed in the derivation of equation (9).

The difference equation (47) is valid independently of the field incident on the wedge, but the precise form of the solution is, of course, governed by the incident field. For the simple case of the two-dimensional plane wave (4) it can easily be verified that if the incident field is taken by itself

$$\psi\,(p,\theta) = \frac{\cosh p}{\sinh p - i\cos(\theta - \alpha)}$$

and this influences the choice of $H(\xi)$. To produce the required geometrical optics field (and, in particular, the plane wave 4), it is necessary that $H(\xi)$ should have one or more poles of suitable residues, a fact which is equivalent to specifying the behaviour of $\phi\,(r,\theta)$ for large r.

In addition to the physical conditions described above there are three others which are more intimately connected with the final reduction of the expression for $\phi\,(r,\theta)$. Two of these have been listed in Chapter VI and can be translated into the following conditions on $H(\xi)$:

(I) with the exception of geometrical optics poles, $H(\xi)$ is regular in $-\pi/2 \leq \arg \xi \leq 3\pi/2 + \gamma$ for all $|\xi|$, a fact which ensures the 'partial' regularity in $-\pi/2 + \theta \leq \arg \xi \leq 3\pi/2 + \theta$ for any θ, $0 \leq \theta \leq \gamma$;

(II)

$$\frac{(\bar{\xi} + \tau_1)(\bar{\xi} + \tau_2)}{(\bar{\xi} - \tau_1)(\bar{\xi} - \tau_2)} \quad H(\bar{\xi}) = H\left(\frac{e^{i\pi}}{\bar{\xi}}\right) \quad , \qquad (58)$$

where $\bar{\xi} = p - i\,\theta$.

The third condition concerns the residues at the geometrical optics poles and amounts to a restriction on the allowable zeros of $H(\xi)$. It guarantees that the solution is finite for all angles α at which the plane wave is incident, but is most conveniently formulated after an examination of the perfectly conducting wedge.

CHAPTER VIII

THE SOLUTION FOR INFINITE CONDUCTIVITY

When $\eta = 0$, τ_1 and τ_2 reduce to $\pm i$ respectively and the difference equation (47) collapses to

$$H(\zeta) = H(\zeta e^{2i\gamma}). \tag{59}$$

This represents the equation for a wedge of infinite conductivity and the solution $H(\zeta) = h_o(\zeta)$, say, must give rise to the Sommerfeld expression for $\phi(r, \theta)$.

For large and small r the general behaviour of $\phi(r, \theta)$ is independent of the conductivity of the wedge and hence the conditions to be imposed on $h_o(\zeta)$ can be deduced from Chapter VII. They are

(i) $h_o(\zeta) = h_o(\zeta e^{2i\gamma})$,

(ii) $h_o(\zeta)$ is regular in $\sum$,

(iii) $h_o(\zeta) \sim$ constant as $|\zeta| \to \infty$,

and (iv) $h_o(\zeta)$ has one or more poles whose residues serve to provide the geometrical optics field. With the exception of these poles, $h_o(\zeta)$ is regular in $-\pi/2 \leq \arg \zeta \leq 3\pi/2 + \gamma$.

There is also the condition

$$h_o(\bar{\zeta}) = h_o\left(\frac{e^{i\pi}}{\bar{\zeta}}\right) \tag{60}$$

(see equation 58), but this is satisfied automatically by a choice of $h_o(\zeta)$ in accordance with the above.

The solutions of equation (59) are of the type

$$h_o(\zeta) = \Lambda(\zeta^{\pi/\gamma}) \tag{61}$$

where Λ is a single-valued function of its argument, and this leads us to propose the following expression for $h_o(\zeta)$:

$$h_o(\zeta) = \frac{Q_1}{1 - e^{-q_1 \zeta^{\pi/\gamma}}} + \frac{Q_2}{1 - e^{-q_2 \zeta^{-\pi/\gamma}}} ; \qquad (62)$$

Q_1, Q_2, q_1 and q_2 are constants whose values have yet to be found and for this purpose a knowledge of the incident field alone is sufficient. The expression given by (62) certainly fulfills the conditions (ii) and (iii), and is of such a form as to enable (iv) to be satisfied.

The constants will now be evaluated. From equations (53) and (62) we have, when $\eta = 0$,

$$\phi(r,\theta) = -\frac{1}{4\pi} \int_{\chi''} e^{ikr\cos\nu} \left\{ \frac{Q_1}{1 - e^{-q_1} e^{in/_2(\pi/_2 - \nu + \theta)}} + \frac{Q_2}{1 - e^{q_2} e^{-in/_2(\pi/_2 - \nu + \theta)}} \right.$$

$$\left. + \frac{Q_1}{1 - e^{-q_1} e^{in/_2(\pi/_2 - \nu - \theta)}} + \frac{Q_2}{1 - e^{q_2} e^{-in/_2(\pi/_2 - \nu - \theta)}} \right\} d\nu \qquad (63)$$

where $n = 2\pi/\gamma$. The poles of the integrand are at

$$\text{Re.}\,\nu = \left(\frac{4N}{n} + \frac{1}{2}\right)\pi \pm \theta - \frac{2}{n}\,\text{Im.}\,q_1 ; \quad \text{Im.}\,\nu = \frac{2}{n}\,\text{Re.}\,q_1 ; \qquad (64)$$

$$\text{Re.}\,\nu = \left(\frac{4N}{n} + \frac{1}{2}\right)\pi \pm \theta - \frac{2}{n}\,\text{Im.}\,q_2 ; \quad \text{Im.}\,\nu = \frac{2}{n}\,\text{Re.}\,q_2 ; \qquad (65)$$

with $N = 0, \pm 1, \pm 2, \ldots\ldots$ and since the incident field is

$$\phi^i(r,\theta) = e^{ikr\cos(\theta - \alpha)} , \qquad (4)$$

it is clear that the poles must occur for real values of ν, requiring that q_1 and q_2 be pure imaginary. The geometrical optics poles therefore lie on the real ν axis and will be included* in a positive sense in any deformation of the path of integration

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* As yet the path is confined to the strip $0 \leq \text{Re.}\,\nu \leq \pi$ and consequently only poles on this part of the axis can be absorbed.

into the lower half ν plane.

The incident field (4) will be provided by poles of the form

$$\nu = \pm (\theta - \alpha) + 2M\pi, \qquad M = 0, \pm 1, \pm 2, \ldots .$$

which coincide with the poles (64) of the first term in the integrand of (63) if

$$\frac{2}{n} \; \text{Im.}\, q_1 = 2 \left(\frac{2N}{n} - M + \frac{1}{4} \right) \pi + \theta \mp (\theta - \alpha) \; ;$$

the lower sign may be discounted since q_1 is independent of θ.

Hence

$$\text{Im.}\, q_1 = (2N - Mn + \frac{n}{4}) \pi + \frac{n\alpha}{2}$$

and the corresponding poles lie in the interval $(0, \pi)$ if

$$\alpha - 2M\pi \leqslant \theta \leqslant \alpha + \pi - 2M\pi .$$

Since $0 \leqslant \theta, \;\; \alpha \leqslant 2\pi/n$ with $n \geqslant 1$, the only admissible value of M is M = 0. We therefore take

$$q_1 = i \frac{n}{2} \left(\frac{\pi}{2} + \alpha \right) , \tag{66}$$

and if

$$Q_1 = -n \tag{67}$$

the resulting pole at $\nu = \theta - \alpha$ has residue $\frac{1}{2\pi i} \; e^{ikr \cos(\theta - \alpha)}$. With this choice of Q_1 and q_1 the third term in (63) provides no contribution to the incident field through the residue at its pole.

The illuminated region of geometrical optics is

$$0 \leqslant \theta \leqslant \min \;\; (\pi + \alpha, \; 2\pi/n) \tag{68}$$

and that part of the range which has not yet been accounted for is covered by the fourth term in (63). If the above process is repeated for this term, it is found that

$$q_2 = i\,\frac{n}{2}\left(\frac{\pi}{2} - \alpha\right) \tag{69}$$

and

$$Q_2 = n. \tag{70}$$

The poles of the first and fourth terms in the integrand of (63) then yield the required incident field over the entire range (68), whilst the second and third terms provide the rest of the geometrical optics field ——— a fact which can easily be verified. We thus have

$$F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right] = n\left\{\frac{1}{1 - e^{in/2(\nu - \theta - \alpha)}} - \frac{1}{1 - e^{-in/2(\nu - \theta + \alpha)}}\right\} \tag{71}$$

and the corresponding function $h_0(\xi)$ is

$$h_0(\xi) = n\left\{\frac{1}{1 - e^{in/2(\pi/2 - \alpha)}\,\xi^{-n/2}} - \frac{1}{1 - e^{-in/2(\pi/2 + \alpha)}\,\xi^{n/2}}\right\} \tag{72}$$

$$= n\,\frac{\xi^{n} - e^{in\pi/2}}{\xi^{n} - 2\,\xi^{n/2}\,e^{in\pi/4}\cos\frac{n\alpha}{2} + e^{in\pi/2}}\,. \tag{73}$$

This has the required properties and, in particular, satisfies the condition (60).

If any further poles had been included in the expression for $h_0(\xi)$ there would have been residues additional to those of geometrical optics which would have violated the condition upon the behaviour of $\phi(r, \theta)$ for large r. For this reason it is thought that the conditions (i) - (iv) in conjunction with (60) are sufficient to determine a unique function $h_o(\xi)$. Obviously it would have been possible to arrange for the incident field to be provided by the poles of the second and third terms in the integrand of (63) and yet another approach would have been to impose the condition (60) at the beginning, thereby relating Q_2 and q_2 with Q_1 and q_1. If desired the poles in the whole range $-\pi \leq \mathrm{Re.}\,\nu \leq \pi$ could now be absorbed and the incident field would then follow from the residue of the first term only. With both methods, however, the final expression for $h_o(\xi)$ is identical to that of equation (72).

The main reason for giving a detailed derivation of $h_o(\xi)$ is that this function is part of the complete solution of the difference equation (47). Nevertheless, $h_o(\xi)$ by itself does represent the solution appropriate to a perfectly conducting wedge and whilst this problem is hardly one for which the present method is well suited (for example, the method proposed in Ref. 5 is more simple and straight-forward), it is of interest to follow the solution through to its conclusion.

The geometrical optics field produced by the function $h_o(\xi)$ is of the expected form and need not be discussed, but for the diffracted field we have

$$\phi^D(r,\theta) = \frac{n}{4\pi}\int_{S(0)} e^{-ikr\cos\beta}\left\{\frac{1}{1-e^{in/2(\beta-\pi-\theta-\alpha)}} - \frac{1}{1-e^{-in/2(\beta-\pi-\theta+\alpha)}}\right\}d\beta$$

$$- \frac{n}{4\pi}\int_{S(0)} e^{-ikr\cos\beta}\left\{\frac{1}{1-e^{in/2(\beta+\pi-\theta-\alpha)}} - \frac{1}{1-e^{-in/2(\beta+\pi-\theta+\alpha)}}\right\}d\beta,$$

the terms of which can be combined in pairs to give

$$\phi^D(r,\theta) = i\,\frac{n}{4\pi}\,\sin\frac{n\pi}{2}\int_{S(0)} e^{-ikr\cos\beta}\left\{\frac{1}{\cos\frac{n(\beta+\theta+\alpha)}{2}-\cos\frac{n\pi}{2}}\right.$$

$$\left.+\frac{1}{\cos\frac{n(\beta+\theta-\alpha)}{2}-\cos\frac{n\pi}{2}}\right\}d\beta \qquad (74)$$

This is the well known Sommerfeld formula. Only in the particular case n = 1 (half-plane) is the integral capable of exact evaluation, and for other values of n it is necessary to use asymptotic methods assuming r large (see, for example, Ref. 10). It should be noted, however, that if $n/2$ is an integer m, $\phi^D(r,\theta)$ vanishes, which confirms that a wedge of angle π/m can be treated using a finite number of plane waves. This is true even if the wedge is imperfectly conducting, the complete solution being a set of plane waves whose directions can be obtained from image considerations.

CHAPTER IX

THE SOLUTION FOR FINITE CONDUCTIVITY: GENERAL THEORY

The singularities of $H(\xi)$ which are necessary in order to produce the required geometrical optics field are identical with those of $h_o(\xi)$ and this suggests that some simplification can be achieved by writing

$$H(\xi) = h_o(\xi)\, H_1(\xi) \tag{75}$$

where $H_1(\xi)$ also satisfies (47). $h_o(\xi)$ is the function derived in Chapter VIII and since this represents the solution for a wedge of infinite conductivity, the above separation implies that

$$H_1(\xi) \to 1 \text{ as } \eta \to 0.$$

The conditions governing the choice of $H_1(\xi)$ will be discussed in the next chapter, and for the moment we note only the requirement that $H_1(\xi)$ be regular in Σ. Let $h(\xi, c)$ be a function which is regular in the larger sector $0 < \arg \xi < 2\gamma$ for all $|\xi| > 0$ and which satisfies

$$(\xi + c)\, h(\xi, c) = (\xi - c)\, h(\xi e^{i\gamma}, c) \tag{76}$$

where c is some parameter. Then $h(\xi e^{i\gamma}, c)$ satisfies

$$(\xi + c e^{-i\gamma})\, h(\xi e^{i\gamma}, c) = (\xi - c e^{-i\gamma})\, h(\xi e^{2i\gamma}, c)$$

and hence, by multiplication, $h(\xi, c)$ is a solution of the difference equation

$$(\xi + c)(\xi + c e^{-i\gamma})\, h(\xi, c) = (\xi - c)(\xi - c e^{-i\gamma})\, h(\xi e^{2i\gamma}, c) \tag{77}$$

which is regular in $0 < \arg \xi < \gamma$ for all $|\xi| > 0$.

It is obvious that the problem of finding a suitable function $H_1(\xi)$ is tantamount to the solution of two equations of the general form (77) and this fact leads us to consider (76) in some detail.

The equation (76) is directly analogous to those obtained by Peters (Refs 6, 11) and while it can be treated by reduction to an ordinary difference equation, the more direct approach adopted by Peters is to be preferred.

If the ξ plane is transformed using $\xi = u^{1/n}$ and if the logarithms of both sides of equation (76) are taken, the function

$$\log\left\{h(\xi, c)\right\} = E(u)$$

is seen to satisfy

$$E(u) - E(ue^{2i\pi}) = \log\left\{\frac{u^{1/n} - c}{u^{1/n} + c}\right\}, \tag{78}$$

and a particular solution of this equation can be found in the following manner.

Let the initial point of a half-line $\mathcal{L}$ be the origin 0, the positive direction along $\mathcal{L}$ be that direction which is away from 0, with positive side on the left when $\mathcal{L}$ is traversed in the positive direction and let $\mathcal{L}$ make an angle Ω with the real axis of the complex u plane. Suppose that w(u) is a function regular in a region containing $\mathcal{L}$ and such that

$$X(u) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{w(t)}{t-u}\,dt \tag{79}$$

is regular in $\Omega < \arg u < 2\pi + \Omega$. It is well known that if t_0 is a point on $\mathcal{L}$ and $X(t_0)$, $X(t_0 e^{2i\pi})$ are the limits of X(u) as u approaches t_0 from the positive and negative sides of $\mathcal{L}$ respectively, then

$$X(t_0) = \frac{1}{2} w(t_0) + P(t_0)$$

and

$$X(t_o e^{2i\pi}) = -\frac{1}{2} w(t_o) + P(t_o),$$

where $P(t_o)$ is the principal value of the integral in (79) when $u=t_o$. A consequence of these formulae is that

$$X(t_o) - X(t_o e^{2i\pi}) = w(t_o),$$

and if this result is applied to (78) it is seen that the integral

$$\frac{1}{2\pi i}\int_0^\infty \frac{1}{t-u} \log\left\{\frac{t^{1/n}-c}{t^{1/n}+c}\right\} dt$$

represents a function which is regular in $0 < \arg u < 2\pi$ and which satisfies equation (78) at least when u approaches $\mathcal{L}$ from the positive side. In terms of ξ the integral is

$$I(\xi) = \frac{1}{2\pi i}\int_0^\infty \frac{1}{t-\xi^n} \log\left\{\frac{t^{1/n}-c}{t^{1/n}+c}\right\} dt$$

$$= -\frac{1}{n\pi i}\int_0^\infty \frac{ct^{1/n-1}}{t^{2/n}-c^2} \log\left\{1-\frac{t}{\xi^n}\right\} dt$$

and hence, by making the substitution $t^{1/n} = c\,\xi/v$,

$$I(\xi) = \frac{1}{\pi i}\int_0^{\infty\, c\xi} \frac{\xi}{v^2-\xi^2} \log\left\{1-\left(\frac{c}{v}\right)^n\right\} dv \tag{80}$$

The function $I(\xi)$ here defined is regular in $0 < \arg \xi < \gamma$ and $\exp\left\{I(\xi)\right\}$ satisfies (78) at least when $\xi - \rho \rightarrow +0$, where ρ is real and positive. Still more can be asserted. Since $\log\left\{1-\left(\frac{c}{v}\right)^n\right\} \longrightarrow 0$ as $v \rightarrow \infty$, $I(\xi)$ can be analytically continued into a region which contains $(0, \gamma)$ by moving the path of integration in (80), and the process will now be examined.

With reference to equation (80) we first remark that $\left\{1 - \left(\frac{c}{v}\right)^n\right\}$ has a branch point at the origin of the v plane and zeros at $v = \pm c e^{im\gamma}$ for all integer m, each of these giving rise to a branch point of

$$\log\left\{1 - \left(\frac{c}{v}\right)^n\right\}. \tag{81}$$

In order to make (81) single-valued it is necessary to insert certain cuts into the v plane and for convenience we choose radial cuts joining the origin to $v = \pm c\, e^{im\gamma}$, the branch of the logarithm being defined as that for which

$$-\pi < \arg \log \leq \pi.$$

A valid representation of $I(\xi)$ in the sector $0 < \arg \xi < \gamma$ is provided by equation (80). The path of integration is the half-ray from the origin making an angle $\arg(c\xi)$ with the positive real axis, and when $\arg \xi = \gamma/2$ the integral can be written as

$$I(\xi) = \frac{1}{2\pi i}\int_0^{\infty c e^{i\gamma/2}} \frac{1}{v-\xi}\log\left\{1 - \left(\frac{c}{v}\right)^n\right\} dv$$

$$+ \frac{1}{2\pi i}\int_{-\infty c e^{i\gamma/2}}^{0} \frac{1}{v-\xi}\log\left\{1 - \left(\frac{c}{-v}\right)^n\right\} dv \tag{82}$$

If $0 < \arg c < \pi$ the integrals in equation (82) represent $I(\xi)$ in the half-plane $\arg c + \gamma/2 - \pi < \arg \xi < \arg c + \gamma/2$, since they define a function which is regular in this region and which coincides with $I(\xi)$ when $\arg \xi = \gamma/2$.

The analytic continuation of $I(\xi)$ can be achieved as follows. If $\mathcal{C}$ is an arc of a circle of (large) radius R with centre at the origin in the v plane

$$\lim_{R\to\infty}\int_{\mathcal{C}} \frac{1}{v-\xi}\log\left\{1 - \left(\frac{c}{v}\right)^n\right\} dv = 0.$$

Moreover, for $\arg c + \gamma/2 - \pi < \arg \zeta < \arg c + \gamma/2$,

$$\frac{1}{v-\zeta} \log\left\{1 - \left(\frac{c}{v}\right)^n\right\}$$

has no singularities in the sector $\arg c + \gamma/2 < \arg \zeta < \arg c + \gamma$,

and hence, if C is any ray in this sector,

$$\int_C = \int_0^{\infty\, c e^{i\gamma/2}}$$

and

$$I(\zeta) = \frac{1}{2\pi i} \int_C \frac{1}{v-\zeta} \log\left\{1 - \left(\frac{c}{v}\right)^n\right\} dv$$

$$+ \frac{1}{2\pi i} \int_{-\infty\, c e^{i\gamma/2}}^{0} \frac{1}{v-\zeta} \log\left\{1 - \left(\frac{c}{-v}\right)^n\right\} dv \qquad (83)$$

for $\arg c + \gamma/2 - \pi < \arg \zeta < \arg c + \gamma/2$. These last integrals, however, define a function of ζ which is regular in $\arg c + \gamma/2 - \pi < \arg \zeta < \arg c + \gamma$ and which coincides with $I(\zeta)$ when $\arg c + \gamma/2 - \pi < \arg \zeta < \arg c + \gamma/2$. Since analytic continuation is unique, it can be concluded that $I(\zeta)$ is regular in $\arg c + \gamma/2 - \pi < \arg \zeta < \arg c + \gamma$.

In a similar manner we can shift the path in the second integral of (83) and so find that $I(\zeta)$ is regular in $\arg c - \pi < \arg \zeta < \arg c + \gamma$; if ζ is in this sector

$$I(\zeta) = \frac{1}{2\pi i} \int_C \frac{1}{v-\zeta} \log\left\{1 - \left(\frac{c}{v}\right)^n\right\} dv$$

$$+ \frac{1}{2\pi i} \int_D \frac{1}{v-\zeta} \log\left\{1 - \left(\frac{c}{-v}\right)^n\right\} dv, \qquad (84)$$

This process of rotating the paths of integration can be repeated, but care must be taken about the distribution of the singularities of the integrands. For example,

(81) is singular at $v=0$, $v=ce^{im\gamma}$ and if C is rotated in an anticlockwise direction the effect of the singularity at $ce^{i\gamma}$ must be considered. If C_1 is a ray in the sector $\arg c + \gamma < \arg \zeta < \arg c + 2\gamma$ and if $\arg c - \pi < \arg \zeta < \arg c + \gamma$, then

$$\int_C = \int_{C_1} + \int_\Gamma$$

where Γ is a loop path which circumscribes in a clockwise sense the branch cut from $ce^{i\gamma}$ to the origin. The integral along Γ is

$$\int_\Gamma \frac{1}{v-\zeta} \log\left\{1-\left(\frac{c}{v}\right)^n\right\} dv = -2\pi i \log\left(\frac{\zeta - ce^{i\gamma}}{\zeta}\right)$$

and hence, for the sector $\arg c - \pi < \arg \zeta < \arg c + 2\gamma$,

$$I(\zeta) = -\log\left(\frac{\zeta - ce^{i\gamma}}{\zeta}\right) + \frac{1}{2\pi i}\int_{C_1} \frac{1}{v-\zeta}\log\left\{1-\left(\frac{c}{v}\right)^n\right\} dv. \tag{85}$$

$$+ \frac{1}{2\pi i}\int_D \frac{1}{v-\zeta}\log\left\{1-\left(\frac{c}{-v}\right)^n\right\} dv$$

where the integrals represents regular functions in this sector.

By repeated use of this method the path C_1 can be moved into a path C' along the line $\arg v = \arg c + \theta_0$(say) and D can be moved into a path D' along $\arg v = \arg c + \theta_0 - 2\pi$. If these two integrals do not now cancel, C' and D' must then form the opposite banks of a branch cut, and in the sector $\arg c + \theta_0 - 2\pi < \arg \zeta < \arg c + \theta_0$ (which is to be regarded as a sheet of a Riemann surface), the function $I(\zeta)$ has logarithmic singularities only; with the exception of the one at the origin, these are situated on the circumference of a circle whose radius is $|c|$ with centre at the origin.

The above representations may be used (see Ref. 6) to verify that

$$\exp\left\{ I(\xi) \right\} = h(\xi, c) \tag{86}$$

satisfies equation (76) and we can therefore write

$$h(\xi, c) = \exp\left[\frac{1}{\pi i} \int_0^{\infty c \xi} \frac{\xi}{v^2 - \xi^2} \log\left\{ 1 - \left(\frac{c}{v}\right)^n \right\} dv \right]. \tag{87}$$

For small $|\xi|$,

$$h(\xi, c) \sim \xi^{n/2} \tag{88}$$

and if the branch point at $\xi = 0$ is excluded, the only singularities of $h(\xi, c)$ are poles outside and on the boundary of the sector $\arg c - \pi \leq \arg \xi \leq \arg c + \gamma$; in the interior of the sector $h(\xi, c)$ is regular and moreover, free of zeros.

Although these results have been obtained for the case in which $0 < \arg c < \pi$, other values of arg c can be treated in a similar manner. For example, if $-\pi < \arg c < 0$ it is still true that (87) satisfies equation (76) and is a regular zero-free function in $\arg c < \arg \xi < \arg c + \pi$, $|\xi| > 0$. Outside this sector, however, the behaviour is in direct contrast to that of $h(\xi, c)$ for $\arg c > 0$. For small $|\xi|$ it behaves as $\xi^{-n/2}$ and this branch point is the only singularity, the points which were poles of $h(\xi, c)$ for $\arg c > 0$ now being zeros. These facts are obvious when it is realized that the corresponding solution of equation (76) for $\arg c < 0$ is $\left\{ h(\xi, ce^{i\pi}) \right\}^{-1}$, where $h(\xi, c)$ is given by equation (87).

The apparent discontinuity in the solution when Im. c changes sign can be traced back to the derivation of equation (80) in which it was assumed that the line $\mathcal{L}$ lies in a region of the u plane where

$$\log\left\{ \frac{u^{1/n} - c}{u^{1/n} + c} \right\}$$

is regular. Since $\mathcal{L}$ has been taken to be the positive real axis the condition is violated when c is real and though such values can be taken account of by a deformation of the path $\mathcal{L}$, it is more convenient to exclude them from our consideration. If necessary, real values of c can always be treated as a limiting case.

The expression of $h(\xi, c)$ given by equation (87) is merely a particular solution of (76), but is sufficient to enable us to determine a function $H(\xi)$ in accordance with the required conditions. For this purpose c must be identified successively with τ_1 and τ_2, where

$$\tau_1 = \eta + i\sqrt{1-\eta^2}$$

and

$$\tau_2 = 1/\tau_1 = \eta - i\sqrt{1-\eta^2}$$

The branch of $\sqrt{1-\eta^2}$ can be chosen arbitrarily and taking that with positive sign we have, since $0 \leqslant \arg \eta \leqslant \pi/4$,

$$\pi/4 < \arg \tau_1 \leqslant \pi/2 \text{ and } -\pi/2 \leqslant \arg \tau_2 < -\pi/4$$

with $\qquad |\tau_1| \geqslant 1 \qquad$ and $|\tau_2| \leqslant 1$;

the moduli equal unity when η is real, corresponding to a pure dielectric wedge.

Since $|\tau_1| \geqslant 1$, the circle on which the main singularities of $h(\xi, \tau_1)$ lie intersects the region Σ and the assumption of regularity in Σ dictates the choice of function. On the other hand, the corresponding circle for $h(\xi, \tau_2)$ is everywhere outside Σ and the regularity condition in this case would have no effect, notwithstanding the fact that the 'singularities' are zeros and $h(\xi, \tau_2)$ is regular for all ξ, $|\xi| > 0$.

CHAPTER X

THE CHOSEN FUNCTION $H_1(\xi)$

The conditions upon the function $H_1(\xi)$ can be deduced from Chapter VII and are

(i) $H_1(\xi)$ satisfies the difference equation

$$(\xi + \tau_1 e^{-i\gamma})(\xi+\tau_2 e^{-i\gamma})(\xi + \tau_1)(\xi + \tau_2) \quad H_1(\xi)$$
$$= (\xi - \tau_1 e^{-i\gamma})(\xi - \tau_2 e^{-i\gamma})(\xi - \tau_1)(\xi - \tau_2) \; H_1(\xi \; e^{2i\gamma}). \tag{89}$$

(ii) $H_1(\xi)$ is regular in $\sum$: $0 < \arg \xi < \gamma$, $|\xi| > 1$.

(iii) In $-\pi/2 \leq \arg \xi \leq 3\pi/2 + \gamma$ $H_1(\xi)$ has, at most, only such singularities as are cancelled out by $h_0(\xi)$. Since $h_0(\xi)$ is proportional to $\xi^n - e^{in\pi/2}$ (equation 73), the only allowable singularities of $H_1(\xi)$ are those arising from a pole factor $(\xi^n - e^{in\pi/2})^{-1}$.

(iv) In order for the residue at the geometrical optics pole $\xi = \frac{\pi}{2} + \alpha$ to produce the required incident field (4), it is necessary for $H_1(\xi)$ to have the value unity at this point for all α, $0 \leq \alpha \leq \gamma$. This can be achieved by normalising with the factor $\left\{ H_1\left[e^{i(\pi/2+\alpha)}\right] \right\}^{-1}$ and if this is to be finite for all α in the range, $H_1(\xi)$ must be free of zeros in $\pi/2 \leq \arg \xi \leq \pi/2 + \gamma$.

(v) $H_1(\xi) \sim$ constant as $|\xi| \to \infty$

(vi) $H_1(\xi) \to 1$ as $\eta \to 0$.

(vii) $$\frac{(\bar{\xi} + \tau_1)(\bar{\xi} + \tau_2)}{(\bar{\xi} - \tau_1)(\bar{\xi} - \tau_2)} \; H_1(\bar{\xi}) = H_1\left(\frac{e^{i\pi}}{\bar{\xi}}\right).$$

In seeking to derive a function $H_1(\xi)$ in accordance with these conditions we first observe that to multiply a solution of the difference equation by either $\xi^n - \tau_1^{\ n}$ or $\xi^n - \tau_2^{\ n}$ does not affect the satisfaction of the equation but does change the location of the singularities. This fact is most important and is made use of in the following way.

In Chapter IX it was shown that equation (89) could be split up into two equations of the form (76) which can be solved in a relatively simple manner to yield a function $h(\xi, c)$ regular in $0 < \arg \xi < \gamma$ for all $|\xi| > 0$ and whose reciprocal is also regular in this region. When $c = \tau_1$ the sector of regularity is $\delta - \pi < \arg \xi < \delta + \gamma$, where $\pi/4 < \delta = \arg \tau_1 \leq \pi/2$, and is bounded by poles at $\xi = \tau_1 e^{-i\pi - im\gamma}$, $\xi = \tau_1 e^{i(m+1)\gamma}$, $m = 0, 1, 2, \ldots$. By itself this function certainly contributes to the solution of (89), but whilst the region of regularity is more than sufficient to satisfy (ii), it is not in agreement with condition (iii). However, the interfering singularities can be annulled by multiplying the function by $\xi^n - \tau_1^{\ n}$ without affecting the satisfaction of the difference equation and the regularity is now in accordance with the conditions (ii) and (iii).

When $c = \tau_2$ the situation is somewhat different in that the bounding 'singularities' are here zeros, which allows $h(\xi, \tau_2)$ to be used by itself. Hence, consider

$$(\xi^n - \tau_1^{\ n})\ h(\xi, \tau_1)\ h(\xi, \tau_2).$$

This reduces to $\xi^n - e^{in\pi/2}$ when $\eta = 0$, and to comply with the limit condition

(vi) it is necessary to divide by $\xi^n - e^{in\pi/2}$, a process which is permissible in view of the identical factor contained in $h_0(\xi)$. The new solution is now

$$J(\xi) = \frac{\xi^n - \tau_1^n}{\xi^n - e^{in\pi/2}} \; h(\xi, \tau_1)\, h(\xi, \tau_2) \tag{90}$$

$$= \frac{\xi^n - \tau_1^n}{\xi^n - e^{in\pi/2}} \exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{1}{t - \xi^n} \log\left\{\left(\frac{t^{1/n} - \tau_1}{t^{1/n} + \tau_1}\right)\left(\frac{t^{1/n} - \tau_2}{t^{1/n} + \tau_2}\right)\right\} dt\right]$$

which clearly satisfies conditions (i) - (iii). Since $h(\xi, \tau_1)$ is entirely free of zeros and $h(\xi, \tau_2)$ is free in $-\delta < \arg\xi < \pi - \delta + \gamma$, the condition (iv) can be satisfied (but only by the smallest possible margin) by using the constant normalising factor $J\left[e^{i(\pi/2 + \alpha)}\right]$, and this gives

$$H_1(\xi) = \frac{J(\xi)}{J\left[e^{i(\pi/2+\alpha)}\right]}, \tag{91}$$

which represents the required solution of the difference equation (89).

For large $|\xi|$, $\xi \log\left\{J(\xi)\right\} \sim$ constant (see Ref. 6) and hence

$$H_1(\xi) \sim \text{constant as } |\xi| \to \infty.$$

Since the condition (vi) is obviously complied with, there now remains only the condition (vii) and it is shown in the Appendix that this also is satisfied. The analysis given there strongly suggests that (vii) is vital for the unique specification of $H_1(\xi)$. If any factor additional to those of equations (90) and (91) were to be included in the expression for $H_1(\xi)$ it would have to transform into itself when $\xi \to e^{i\pi}/\xi$; if, then, it were to appear in the denominator of $H_1(\xi)$ its own singularities would conflict with condition (iii) and if it appeared in the numerator the normalising factor demanded by condition (vi) would violate (iv).

For this reason it is believed that the conditions (i) - (vii) are not only essential, but also enable $H_1(\xi)$ to be uniquely determined. Whether or not this is true, they have certainly led to the true solution of the present, quite general, problem. With the exception of (vii) all the conditions have a physical origin and whilst it has not been possible to attach a direct meaningful interpretation to (vii), it is obviously necessary for the reduction of the solution in the manner described in Chapter VI.

The final check upon the determination of $H(\xi)$ and, indeed, upon the method as a whole, must come from an examination of the resulting expression for $\phi(r,\theta)$. From equations (73), (75), (90) and (91) we have

$$F\left[i\left(\frac{\pi}{2}-\nu+\theta\right)\right] = \frac{n}{2}\;\frac{e^{-in/2(\nu-\theta)}-\left[\tau_1\,e^{-i/2(\pi-\nu+\theta)}\right]^n}{\cos\frac{n(\nu-\theta)}{2}-\cos\frac{n\alpha}{2}}\left\{J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1}$$

$$\exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{1}{t-e^{in(\pi/2-\nu+\theta)}}\log\left\{\left(\frac{t^{1/n}-\tau_1}{t^{1/n}+\tau_1}\right)\left(\frac{t^{1/n}-\tau_2}{t^{1/n}+\tau_2}\right)\right\}dt\right] \tag{92}$$

and the expression for $\phi(r,\theta)$ then is

$$\phi(r,\theta) = -\frac{1}{4\pi}\int_L e^{ikr\cos\nu}\,F\left[i\left(\frac{\pi}{2}-\nu+\theta\right)\right]d\nu$$

$$+\frac{1}{4\pi}\int_L e^{ikr\cos\nu}\,\frac{\eta+\sin(\nu+\theta)}{\eta-\sin(\nu+\theta)}\,F\left[i\left(\frac{\pi}{2}-\nu-\theta\right)\right. \tag{93}$$

where L is the path shown in Figure 1. It is now a simple matter to verify that the boundary conditions (5) and (6) are fulfilled.

For small values of θ we make the substitutions $\mu = \nu - \theta$ in the first integral of equation (93). This eliminates θ from F and displaces the path an amount θ to the left, but since the integral is free* of singularities in the intervening region, the path can be shifted back to its original position. With the second integral in (93) the corresponding substitution is $\mu = \nu + \theta$ and hence

$$\phi(r,\theta) = -\frac{1}{4\pi}\int_L \left\{ e^{ikr\cos(\mu+\theta)} - \frac{\eta + \sin\mu}{\eta - \sin\mu}\, e^{ikr\cos(\mu-\theta)} \right\} F\left[i\left(\frac{\pi}{2}-\mu\right)\right] d\mu$$

which can be inserted into the boundary condition on the lower face, $\theta=0$, of the wedge to give

$$\frac{\partial\phi}{\partial\theta} - ik\eta\, r\phi = 0$$

as required.

When $\gamma - \theta$ is small the appropriate substitutions in equation (93) are $\mu = \nu \pm (\gamma - \theta)$, leading to an expression for $\phi(r,\theta)$ in the form

$$\phi(r,\theta) = -\frac{1}{4\pi}\int_L \left\{ e^{ikr\cos(\mu+\theta-\gamma)}\, F\left[i\left(\frac{\pi}{2}-\mu+\gamma\right)\right] - \frac{\eta + \sin(\mu+\gamma)}{\eta - \sin(\mu+\gamma)}\, e^{ikr\cos(\mu-\theta+\gamma)}\, F\left[i\left(\frac{\pi}{2}-\mu-\gamma\right)\right]\right\} d\mu \qquad (94)$$

This can be written alternatively as

$$\phi(r,\theta) = -\frac{1}{4\pi}\int_L \left\{ e^{ikr\cos(\mu+\theta-\gamma)} - \frac{\eta - \sin\mu}{\eta + \sin\mu}\, e^{ikr\cos(\mu-\theta+\gamma)} \right. \left. F\left[i\left(\frac{\pi}{2}-\mu+\gamma\right)\right]\right\} d\mu$$

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*The fact that θ can be made as small as we please ensures that no geometrical optics poles are included.

by using the difference equation (40) for $F(\omega)$, and when $\theta = \gamma$ we now have

$$\frac{\partial \phi}{\partial \theta} + ik\eta\, r\, \phi = 0$$

in accordance with the boundary condition on the upper face of the wedge.

The only remaining conditions are for large and small r. For large r the radiation condition is satisfied by the solution for infinite conductivity and consequently, by virtue of the decomposition of $H(\xi)$, for all conductivities. And similarly with the edge condition. Since all the requirements are met, the uniqueness theorem for electromagnetic problems enables us to conclude that the solution which has been obtained is, indeed, the true and correct solution of the problem formulated in Chapter III.

CHAPTER XI

SOME ELEMENTARY EXAMPLES

The degree of complication of the final expression for $\phi(r,\theta)$ is determined by the form of $H_1(\xi)$ and it is of interest, therefore, to consider $H_1(\xi)$ for certain particular values of wedge angle.

From equation (90) we have

$$h(\xi, c) = \exp\left[\frac{1}{2\pi i}\int_0^{\infty}\frac{1}{t-\xi^n}\log\left\{\frac{t^{1/n}-c}{t^{1/n}+c}\right\}dt\right]$$

and hence

$$\frac{\partial}{\partial\xi}\left\{\log h(\xi,c)\right\} = \frac{n}{2\pi i}\int_0^{\infty}\frac{\xi^{n-1}}{(t-\xi^n)^2}\log\left\{\frac{t^{1/n}-c}{t^{1/n}+c}\right\}dt$$

$$= -\frac{n}{2\pi i}\left[\frac{\xi^{n-1}}{t-\xi^n}\log\left\{\frac{t^{1/n}-c}{t^{1/n}+c}\right\}\right]_0^{\infty}$$

$$+\frac{1}{\pi i}\int_0^{\infty}\frac{\xi^{n-1}}{t-\xi^n}\,\frac{ct^{1/n-1}}{t^{2/n}-c^2}\,dt$$

$$= \pm\frac{n}{2\xi} + \frac{1}{\pi i}\int_0^{\infty}\frac{\xi^{n-1}}{t-\xi^n}\,\frac{ct^{1/n-1}}{t^{2/n}-c^2}\,dt \qquad (95)$$

with the upper or lower sign according as $\arg c \gtrless 0$ respectively.

If γ is of the form $p\pi/q$ (that is, $n = 2q/p$) where p, q are integers $\geqslant 1$, the integral is capable of considerable reduction and, in some instances, of complete expression in terms of trigonometrical factors. The simplest cases are those for which p=1 and whilst the analysis is then superfluous in that $\phi(r,\theta)$ reduces to the geometrical optics field alone (see Chapter VIII) and is independent of $h(\xi, c)$, some trivial examples can be quoted.

Thus, for q = 1 we have $\gamma = \pi$ and

$$h(\xi, \tau_1) = \frac{\xi}{\xi + \tau_1}, \qquad h(\xi, \tau_2) = \frac{\xi - \tau_2}{\xi},$$

giving

$$J(\xi) = \frac{(\xi - \tau_1)(\xi - \tau_2)}{\xi^2 + 1}$$

Similarly, when $\gamma = \pi/2$ (q = 2)

$$J(\xi) = \frac{(\xi - \tau_1)(\xi - \tau_2)(\xi + i\tau_1)(\xi + i\tau_2)}{\xi^4 - 1}$$

and when $\gamma = \pi/3$ (q = 3),

$$J(\xi) = \frac{(\xi - \tau_1)(\xi - \tau_2)(\xi - \tau_1 e^{-i\pi/3})(\xi - \tau_2 e^{-i\pi/3})(\xi - \tau_1 e^{-2i\pi/3})(\xi - \tau_2 e^{-2i\pi/3})}{\xi^6 + 1}$$

The generalisation to a wedge of any angle π/q is now obvious, and it can easily be verified that all these functions satisfy the conditions (i) - (vii) of Chapter X.

CHAPTER XII

THE HALF-PLANE

When $p \neq 1$ the solution becomes more complicated and this can be illustrated by taking the simplest wedge of this type, namely, the half-plane, for which $p = 2$ and $q = 1$.

For $n = 1$ equation (95) becomes

$$\frac{\partial}{\partial \xi}\left\{\log h(\xi, c)\right\} = \pm \frac{1}{2\xi} + \frac{1}{\pi i}\int_0^\infty \frac{1}{t-\xi}\,\frac{c}{t^2-c^2}\,dt$$

$$= \pm \frac{1}{2\xi} + \frac{1}{2\pi i}\left[\frac{1}{\xi - c}\left\{\log(t-\xi) - \log(t-c)\right\} - \frac{1}{\xi + c}\left\{\log(t-\xi) - \log(t+c)\right\}\right]_0^\infty$$

$$= \pm \frac{1}{2}\left(\frac{1}{\xi} - \frac{1}{\xi - c}\right) + \frac{1}{2}\left(\frac{1}{\xi - c} - \frac{1}{\xi + c}\right) + \frac{1}{2\pi i}\left(\log c - \log \xi\right)\left(\frac{1}{\xi - c} - \frac{1}{\xi + c}\right)$$

giving

$$\log h(\xi, c) = \int_{-\infty}^{\xi} \frac{\partial}{\partial \xi}\left\{\log h(\xi, c)\right\} d\xi$$

$$= \pm \frac{1}{2}\log\frac{\xi}{\xi - c} + \frac{1}{2}\log\frac{\xi - c}{\xi + c} + \frac{1}{2\pi i}\log c \log\frac{\xi - c}{\xi + c} - \frac{1}{2\pi i}\log \xi \log\frac{\xi - c}{\xi + c} + \frac{1}{2\pi i}\int_0^{\log \xi}\log\frac{e^x - c}{e^x + c}\,dx,$$

where the lower limit of integration was chosen in view of the known behaviour of $h(\xi, c)$ for large $|\xi|$. Hence

$$h(\xi, c) = \left(\frac{\xi}{\xi - c}\right)^{\pm 1/2}\left(\frac{\xi - c}{\xi + c}\right)^{1/2}\left(\frac{\xi - c}{\xi + c}\right)^{\frac{1}{2\pi i}(\log c - \log \xi)} \exp\left[\frac{1}{2\pi i}\int_0^{\log \xi}\log\frac{e^x - c}{e^x + c}\,dx\right]$$

with the upper or lower sign for arg c $\gtrless$ 0 respectively, and from this we obtain

$$J(\xi) = \frac{\{(\xi-\tau_1)(\xi-\tau_2)\}^{1/2}}{\xi - i} \left\{ \frac{(\xi-\tau_1)(\xi+\tau_2)}{(\xi+\tau_1)(\xi-\tau_2)} \right\}^{\frac{1}{2\pi i}\log\tau_1} \tag{96}$$

$$\left\{ \frac{(\xi+\tau_1)(\xi+\tau_2)}{(\xi-\tau_1)(\xi-\tau_2)} \right\}^{\frac{1}{2\pi i}\log(\xi e^{i\pi})} \exp\left[\frac{1}{2\pi i} \int_0^{\log\xi} \log\left\{ \left(\frac{e^x-\tau_1}{e^x+\tau_1}\right)\left(\frac{e^x-\tau_2}{e^x+\tau_2}\right)\right\} dx \right]$$

The pole at $\xi = i$ is cancelled out by the corresponding zero of $h_0(\xi)$ and if this point is excluded the behaviour of $J(\xi)$ is as follows:

arg ξ = $\delta - \pi$:	pole	arg ξ = $-\delta + 2\pi$:	R and N-Z
$-\delta$:	zero	$\delta + 2\pi$:	R and N-Z
δ :	zero	$-\delta + 3\pi$:	zero
$-\delta + \pi$:	R and N-Z	$\delta + 3\pi$:	zero
$\delta + \pi$:	R and N-Z	$-\delta + 4\pi$:	pole

where $\delta = \arg \tau_1$ and 'R and N-Z' means 'regular and non-zero'. Thus, with the exception of the pole at $\xi = i$, $J(\xi)$ is regular in $\delta - \pi < \arg\xi < -\delta + 4\pi$, thereby satisfying conditions (ii) and (iii) of Chapter X, and since it is also free of zeros in $\delta < \arg\xi < -\delta + 2\pi$, which region includes $(\pi/2, \pi/2 + \gamma)$, condition (iv) is fulfilled. It is no more difficult to show that the remaining conditions are satisfied.

The particular case of an imperfectly conducting half-plane can be treated by the Wiener-Hopf techniqe (Ref. 12) and the solution can be used to provide a check on the present method. For this purpose it is necessary to work through to the final expression for $\phi(r, \theta)$.

The function $H_1(\xi)$ is given in terms of $J(\xi)$ by equation (91) and since

$$h_o(\xi) = \frac{\xi - i}{\xi - 2(i\xi)^{1/2}\cos\frac{\alpha}{2} + i}$$

(equation 73), we have

$$H(\xi) = \left\{J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1} \frac{\{(\xi-\tau_1)(\xi-\tau_2)\}^{1/2}}{\xi - 2(i\xi)^{1/2}\cos\frac{\alpha}{2} + i} \left\{\frac{(\xi-\tau_1)(\xi+\tau_2)}{(\xi+\tau_1)(\xi-\tau_2)}\right\}^{\frac{1}{2\pi i}\log\tau_1}$$

$$\left\{\frac{(\xi+\tau_1)(\xi+\tau_2)}{(\xi-\tau_1)(\xi-\tau_2)}\right\}^{\frac{1}{2\pi i}\log(\xi e^{-i\pi})} \exp\left[\frac{1}{2\pi i}\int_0^{\log\xi} \log\left\{\left(\frac{e^x-\tau_1}{e^x+\tau_1}\right)\left(\frac{e^x-\tau_2}{e^x+\tau_2}\right)\right\} dx\right]$$

(97)

giving

$$F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right] = -\sqrt{\frac{\tau_1}{2}}\, e^{3i\pi/4} \frac{\left\{J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1}}{\cos\frac{\alpha}{2} - \cos\frac{\nu-\theta}{2}}$$

$$\frac{\eta - \sin(\nu-\theta)}{\{\eta + \sin(\nu-\theta)\}^{1/2}} \left\{\frac{\sqrt{1-\eta^2} - \cos(\nu-\theta)}{\sqrt{1-\eta^2} + \cos(\nu-\theta)}\right\}^{\frac{1}{2\pi i}\log\tau_1}$$

$$\left\{\frac{\eta + \sin(\nu-\theta)}{\eta - \sin(\nu-\theta)}\right\}^{1/4-(\nu-\theta)/(2\pi)} \exp\left[\frac{1}{2\pi}\int_0^{\pi/2-(\nu-\theta)} \log\left(\frac{\eta - \cos x}{\eta + \cos x}\right) dx\right]$$

(98)

An expression for the total field now follows by inserting this result into equation (55).

If ν is replaced by $\pi + \beta$, equation (98) becomes

$$F\left[i\left(-\frac{\pi}{2}-\beta+\theta\right)\right] = -\sqrt{\frac{\tau_1}{2}}\, e^{3i\pi/4}\, \frac{\left\{J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1}}{\cos\frac{\alpha}{2}+\sin\frac{\beta-\theta}{2}} \left\{\eta+\sin(\beta-\theta)\right\}^{1/2}$$

$$\left\{\frac{\sqrt{1-\eta^2}+\cos(\beta-\theta)}{\sqrt{1-\eta^2}-\cos(\beta-\theta)}\right\}^{\frac{1}{2\pi i}\log\tau_1} \left\{\frac{\eta+\sin(\beta-\theta)}{\eta-\sin(\beta-\theta)}\right\}^{3/4+(\beta-\theta)/(2\pi)}$$

$$\exp\left[\frac{1}{2\pi}\int_0^{\pi/2-(\beta-\theta)} \log\left(\frac{\eta+\cos x}{\eta-\cos x}\right) dx\right]$$

$$= \frac{1}{2}\sqrt{\frac{\tau_1}{\eta}}\, e^{3i\pi/4}\, \frac{\left\{J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1}}{\cos\frac{\alpha}{2}+\sin\frac{\beta-\theta}{2}}\, \frac{\eta+\sin(\beta-\theta)}{\sin\frac{\beta-\theta}{2}}\, L(-\beta+\theta),$$

say, where

$$L(\beta) = \sqrt{2\eta}\, \frac{\sin\frac{\beta}{2}}{\left\{\eta+\sin\beta\right\}^{1/2}} \left\{\frac{\sqrt{1-\eta^2}+\cos\beta}{\sqrt{1-\eta^2}-\cos\beta}\right\}^{\frac{1}{2\pi i}\log\tau_1} \left\{\frac{\eta-\sin\beta}{\eta+\sin\beta}\right\}^{1/4-\beta/(2\pi)}$$

$$\exp\left[\frac{1}{2\pi}\int_0^{\pi/2-\beta} \log\left(\frac{\eta+\cos x}{\eta-\cos x}\right) dx\right]. \qquad (99)$$

Also

$$F\left[i\left(\frac{3\pi}{2}-\beta+\theta\right)\right] = \frac{1}{2}\sqrt{\frac{\tau_1}{\eta}}\, e^{3i\pi/4}\, \frac{\left\{J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1}}{\cos\frac{\alpha}{2}-\sin\frac{\beta-\theta}{2}}\, \frac{\eta-\sin(\beta-\theta)}{\sin\frac{\beta-\theta}{2}}\, L(-\beta+\theta)$$

and hence, from (55), the diffracted field is given by

$$\phi^D(r,\theta) = -\frac{1}{8\pi}\left(\sqrt{\frac{\tau_1}{\eta}}\, e^{3i\pi/4}\right)\int_{S(0)} \frac{L(-\beta+\theta)}{J\left[e^{i(\pi/2+\alpha)}\right]}\left\{\frac{\eta+\sin(\beta-\theta)}{\cos\frac{\alpha}{2}+\sin\frac{\beta-\theta}{2}} - \frac{\eta-\sin(\beta-\theta)}{\cos\frac{\alpha}{2}-\sin\frac{\beta-\theta}{2}}\right\}\frac{e^{-ikr\cos\beta}}{\sin\frac{\beta-\theta}{2}}\, d\beta$$

$$= -\frac{1}{2\pi i}\left(e^{i\pi/4}\sqrt{\tau_1\eta}\right)\int_{S(\theta)} \frac{L(\beta)}{J\left[e^{i(\pi/2+\alpha)}\right]}\,\frac{1-\frac{2}{\eta}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\cos\alpha+\cos\beta}\, e^{-ikr\cos(\beta-\theta)}\, d\beta. \tag{100}$$

The geometrical optics terms are of the expected form and need not be considered.

Equation (100) is identical to the solution obtained by the Wiener-Hopf method (Ref. 12) and this fact provides striking confirmation of the present theory. Indeed, $L(\beta)$ is merely the 'split' function $\left\{L_+(k\cos\beta)\right\}^{-1}$ introduced in that paper, with

$$\frac{e^{-i\pi/4}}{\sqrt{\tau_1\eta}}\; J\left[e^{i(\pi/2+\alpha)}\right]$$

equal to the complementary function $\left\{L_-(-k\cos\alpha)\right\}^{-1}$.

CHAPTER XIII

WEDGES OF ANGLE $3\pi/2$ AND $2\pi/3$

There is obviously no limit to the number of examples that could be worked out and for each wedge angle the solution can be expected to reveal some new features. However, to illustrate the general nature of the results it is sufficient to consider particular wedge angles, and the two that have been selected are $\gamma = 3\pi/2$ and $\gamma = 2\pi/3$.

When $\gamma = 3\pi/2$ ($n = 4/3$) the integral expression for $h(\xi, c)$ can be evaluated completely to give

$$h(\xi, \tau_1) = \frac{\xi^{4/3} - \tau_1^{4/3} e^{4i\pi/3}}{(\xi + \tau_1)(\xi + i\tau_2)}$$

and

$$h(\xi, \tau_2) = \frac{(\xi - \tau_2)(\xi - i\tau_2)}{\xi^{4/3} - \tau_2^{4/3} e^{2i\pi/3}},$$

from which we obtain

$$J(\xi) = \frac{(\xi - \tau_1)(\xi - \tau_2)(\xi - i\tau_1)(\xi - i\tau_2)}{(\xi^{4/3} - e^{2i\pi/3})(\xi^{4/3} - \tau_1^{4/3} e^{2i\pi/3})(\xi^{4/3} - \tau_2^{4/3} e^{2i\pi/3})} \tag{101}$$

With the exception of the poles provided by the first factor in the denominator (and these are cancelled out by the corresponding zeros of $h_0(\xi)$), $J(\xi)$ is regular in $-\pi + \delta < \arg \xi < 7\pi/2 - \delta$, and since it is also free of zeros in $\delta < \arg \xi < 5\pi/2 - \delta$, the conditions (ii), (iii) and (iv) of Chapter X are satisfied.

Moreover, from equation (73),

$$h_0(\xi) = \frac{4}{3} \; \frac{\xi^{4/3} - e^{2i\pi/3}}{\xi^{4/3} - 2\xi^{2/3} e^{i\pi/3} \cos\frac{2\alpha}{3} + e^{2i\pi/3}}$$

and hence

$$H(\xi) = \frac{4}{3} \frac{\left\{ J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1} (\xi - \tau_1)(\xi - \tau_2)(\xi - i\tau_1)(\xi - i\tau_2)}{\left(\xi^{4/3} - 2\xi^{2/3} e^{i\pi/3} \cos\frac{2\alpha}{3} + e^{2i\pi/3}\right)\left(\xi^{4/3} - \tau_1^{4/3} e^{2i\pi/3}\right)\left(\xi^{4/3} - \tau_2^{4/3} e^{2i\pi/3}\right)}$$

where

$$J\left[e^{i(\pi/2+\alpha)}\right] = - \frac{(\eta + \sin\alpha)(\eta - \cos\alpha)}{\sin\frac{2\alpha}{3}\left(\cos\frac{4\alpha}{3} - T\right)}$$

with

$$T = \frac{1}{2}\left(\tau_1^{4/3} + \tau_2^{4/3}\right),$$

giving

$$F\left[i\left(\frac{\pi}{2} - \nu + \theta\right)\right] = \frac{4i}{3} \frac{\{\eta - \sin(\nu-\theta)\}\{\eta - \cos(\nu-\theta)\}}{(\eta + \sin\alpha)(\eta - \cos\alpha)} \frac{\left(\cos\frac{4\alpha}{3} - T\right)\sin\frac{2\alpha}{3}}{\left\{\cos\frac{4}{3}(\nu-\theta) - T\right\}\left\{\cos\frac{2}{3}(\nu-\theta) - \cos\frac{2\alpha}{3}\right\}}$$

If this is inserted into (55), the equation for the diffracted field can be reduced to the form

$$\phi^D(r,\theta) = \frac{1}{\sqrt{3}\pi i}\int_{S(\theta)} \left(\frac{\eta - \sin\beta}{\eta + \sin\alpha}\right)\left(\frac{\eta + \cos\beta}{\eta - \cos\alpha}\right) \frac{\left(2\cos\frac{2\alpha}{3}\cos\frac{2\beta}{3} + \frac{1}{2} - T\right)}{\cos\frac{4}{3}(\pi - \beta) - T}$$

$$\frac{\left(\cos\frac{4\alpha}{3} - T\right)}{\left\{\cos\frac{4}{3}(\pi+\beta) - T\right\}} \frac{\sin\frac{2\alpha}{3}\sin\frac{2\beta}{3}}{\left\{\cos\frac{2}{3}(\pi-\beta) - \cos\frac{2\alpha}{3}\right\}\left\{\cos\frac{2}{3}(\pi+\beta) - \cos\frac{2\alpha}{3}\right\}}$$

$$e^{-ikr\cos(\beta-\theta)}\, d\beta \qquad (102)$$

It will be observed that the integrand is composed entirely of trigonometrical factors and as a result the solution is not very much more complicated than that for the corresponding perfectly conducting wedge.

When $\gamma = 2\pi/3$ (n = 3), the evaluation of $h(\xi, c)$ is more tedious and ultimately leads to the following expression for $J(\xi)$:

$$J(\xi) = \frac{(\xi - \tau_1)(\xi - \tau_2)}{\xi^3 + i} \left[\frac{(\xi - \tau_1 e^{2i\pi/3})(\xi - \tau_2 e^{2i\pi/3})(\xi - \tau_1 e^{4i\pi/3})(\xi - \tau_2 e^{4i\pi/3})}{(\xi + \tau_1)(\xi + \tau_2)} \right]^{1/2}$$

$$\left\{ \frac{(\xi + \tau_1 e^{2i\pi/3})(\xi + \tau_2 e^{2i\pi/3})(\xi - \tau_1 e^{4i\pi/3})(\xi - \tau_2 e^{4i\pi/3})}{(\xi - \tau_1 e^{2i\pi/3})(\xi - \tau_2 e^{2i\pi/3})(\xi + \tau_1 e^{4i\pi/3})(\xi + \tau_2 e^{4i\pi/3})} \right\}^{1/6} \left\{ \frac{(\xi^3 - \tau_1^3)(\xi^3 + \tau_2^3)}{(\xi^3 + \tau_1^3)(\xi^3 - \tau_2^3)} \right\}^{1/(2\pi i)\log \tau_1}$$

$$\left\{ \frac{(\xi^3 + \tau_1^3)(\xi^3 + \tau_2^3)}{(\xi^3 - \tau_1^3)(\xi^3 - \tau_2^3)} \right\}^{1/(2\pi i)\log \xi} \exp\left[\frac{1}{2\pi i} \int_0^{\log \xi} \log \left\{ \left(\frac{e^{3x} - \tau_1^3}{e^{3x} + \tau_1^3} \right) \left(\frac{e^{3x} - \tau_2^3}{e^{3x} + \tau_2^3} \right) \right\} dx \right]$$

With the exception of the poles arising from the factor $(\xi^3 + i)^{-1}$, $J(\xi)$ is regular in $\delta - \pi < \arg \xi < 8\pi/3 - \delta$ and is free of zeros in $\delta < \arg \xi < 5\pi/3 - \delta$.

Also

$$h_0(\xi) = \frac{\xi^3 + i}{\xi^3 - 2\xi^{3/2} e^{3i\pi/4} \cos\frac{3\alpha}{2} - i}$$

and hence

$$F\left[i\left(\frac{\pi}{2}-\nu+\theta\right)\right]=6\,i\,\tau_1^{3/2}\left\{J\left[e^{i(\pi/2+\alpha)}\right]\right\}^{-1}\frac{\eta-\sin(\nu-\theta)}{\cos\frac{3}{2}\alpha-\cos\frac{3}{2}(\nu-\theta)}$$

$$\left[\frac{\{\eta-\sin(\nu-\theta+2\pi/3)\}\{\eta-\sin(\nu-\theta-2\pi/3)\}}{\eta+\sin(\nu-\theta)}\right]^{1/2}\left[\frac{\{\eta+\sin(\nu-\theta+2\pi/3)\}\{\eta-\sin(\nu-\theta-2\pi/3)\}}{\{\eta-\sin(\nu-\theta+2\pi/3)\}\{\eta+\sin(\nu-\theta-2\pi/3)\}}\right]^{1/6}$$

$$\left[\frac{\{\sqrt{1-\eta^2}-\cos(\nu-\theta)\}\{\sqrt{1-\eta^2}-\cos(\nu-\theta+2\pi/3)\}\{\sqrt{1-\eta^2}-\cos(\nu-\theta-2\pi/3)\}}{\{\sqrt{1-\eta^2}+\cos(\nu-\theta)\}\{\sqrt{1-\eta^2}+\cos(\nu-\theta+2\pi/3)\}\{\sqrt{1-\eta^2}+\cos(\nu-\theta-2\pi/3)\}}\right]^{\frac{1}{2\pi i}\log\tau_1}$$

$$\left[\frac{\{\eta+\sin(\nu-\theta)\}\{\eta+\sin(\nu-\theta+2\pi/3)\}\{\eta+\sin(\nu-\theta-2\pi/3)\}}{\{\eta-\sin(\nu-\theta)\}\{\eta-\sin(\nu-\theta+2\pi/3)\}\{\eta-\sin(\nu-\theta-2\pi/3)\}}\right]^{1/4-(\nu-\theta)/(2\pi)}$$

$$\exp\left[\frac{1}{2\pi}\int_0^{\pi/2-(\nu-\theta)}\log\left\{\left(\frac{\eta-\cos x}{\eta+\cos x}\right)\left(\frac{\eta-\cos x+2\pi/3}{\eta+\cos x+2\pi/3}\right)\left(\frac{\eta-\cos x-2\pi/3}{\eta+\cos x+2\pi/3}\right)\right\}dx\right]$$

$$=\frac{\{\eta-\sin(\nu-\theta)\}\{\eta-\sin(\nu-\theta+2\pi/3)\}\{\eta-\sin(\nu-\theta-2\pi/3\}}{\cos\frac{3}{2}(\nu-\theta)\left\{\cos\frac{3}{2}\alpha-\cos\frac{3}{2}(\nu-\theta)\right\}}\;\frac{T(-\nu+\pi+\theta)}{J\left[e^{i(\pi/2+\alpha)}\right]}$$

say, where

$$T(\beta) = -6i\,\tau_1^{3/2}\;\frac{\sin\frac{3}{2}\beta}{\left[\{\eta+\sin\beta\}\{\eta-\sin(\beta+2\pi/3)\}\{\eta-\sin(\beta-2\pi/3)\}\right]^{1/2}}$$

$$\left[\frac{\{\eta+\sin(\beta-2\pi/3)\}\{\eta-\sin(\beta+2\pi/3)\}}{\{\eta-\sin(\beta-2\pi/3)\}\{\eta+\sin(\beta+2\pi/3)\}}\right]^{1/6}$$

$$\left[\frac{\{\sqrt{1-\eta^2}+\cos\beta\}\{\sqrt{1-\eta^2}+\cos(\beta+2\pi/3)\}\{\sqrt{1-\eta^2}+\cos(\beta-2\pi/3)\}}{\{\sqrt{1-\eta^2}-\cos\beta\}\{\sqrt{1-\eta^2}-\cos(\beta+2\pi/3)\}\{\sqrt{1-\eta^2}-\cos(\beta-2\pi/3)\}}\right]^{\frac{1}{2\pi i}\log\tau_1}$$

$$\left[\frac{\{\eta-\sin\beta\}\{\eta-\sin(\beta+2\pi/3)\}\{\eta-\sin(\beta-2\pi/3)\}}{\{\eta+\sin\beta\}\{\eta+\sin(\beta+2\pi/3)\}\{\eta+\sin(\beta-2\pi/3)\}}\right]^{\frac{1}{4}-\beta/(2\pi)} \qquad (103)$$

$$\exp\left[\frac{1}{2\pi}\int_0^{\pi/2-\beta}\log\left\{\left(\frac{\eta+\cos x}{\eta-\cos x}\right)\left(\frac{\eta+\cos x+2\pi/3}{\eta-\cos x-2\pi/3}\right)\left(\frac{\eta+\cos x-2\pi/3}{\eta-\cos x-2\pi/3}\right)\right\}dx\right]$$

With this definition of $T(\beta)$,

$$J\left[e^{i(\pi/2+\alpha)}\right] = \frac{2i}{3}\;\frac{\{\eta+\sin\alpha\}\{\eta+\sin(\alpha+2\pi/3)\}\{\eta+\sin(\alpha-2\pi/3)\}}{\sin 3\alpha}\;T(\pi+\alpha),$$

$$F\left[i\left(\frac{3\pi}{2}-\beta+\theta\right)\right] = \frac{\{\eta-\sin(\beta-\theta)\}\{\eta-\sin(\beta-\theta+2\pi/3)\}\{\eta-\sin(\beta-\theta-2\pi/3)\}}{\sin\frac{3}{2}(\beta-\theta)\left\{\cos\frac{3}{2}\alpha+\sin\frac{3}{2}(\beta-\theta)\right\}}\;\frac{T(\theta-\beta)}{J\left[e^{i(\pi/2+\alpha)}\right]}$$

$$F\left[i\left(-\frac{\pi}{2}-\beta+\theta\right)\right] = \frac{\{\eta+\sin(\beta-\theta)\}\{\eta+\sin(\beta-\theta+2\pi/3)\}\{\eta+\sin(\beta-\theta-2\pi/3)\}}{\sin\frac{3}{2}(\beta-\theta)\left\{\cos\frac{3}{2}\alpha-\sin\frac{3}{2}(\beta-\theta)\right\}}\;\frac{T(\theta-\beta)}{J\left[e^{i(\pi/2+\alpha)}\right]}$$

and if these are inserted into equation (55) we obtain

$$\phi^D(r,\theta) = -\frac{1}{4\pi}\int_{S(0)} \frac{\left\{\eta(4\eta^2-3) - 2\cos\frac{3}{2}\alpha\cos\frac{3}{2}(\beta-\theta)\right\}}{\cos 3(\beta-\theta) + \cos 3\alpha} \; \frac{T(\theta-\beta)}{J\left[e^{i(\pi/2+\alpha)}\right]} \; e^{-ikr\cos\beta}\, d\beta$$

$$= -\frac{3}{8\pi i}\int_{S(0)} \frac{\left\{\eta(4\eta^2-3) - 2\cos\frac{3}{2}\alpha\cos\frac{3}{2}(\beta-\theta)\right\}}{\left\{\eta+\sin\alpha\right\}\left\{\eta+\sin(\alpha+2\pi/3)\right\}\left\{\eta+\sin(\alpha-2\pi/3)\right\}}$$

$$\frac{\sin 3\alpha}{\cos 3(\beta-\theta)+\cos 3\alpha} \; \frac{T(\theta-\beta)}{T(\pi+\alpha)} \; e^{-ikr\cos\beta}\, d\beta.$$

A final substitution of β in place of $\theta-\beta$ then gives an expression for the scattered field in the form

$$\phi^D(r,\theta) = \frac{3}{8\pi i}\int_{S(\theta)} \frac{\left\{\eta(4\eta^2-3) - 2\cos\frac{3}{2}\alpha\cos\frac{3}{2}\beta\right\}}{\left\{\eta+\sin\alpha\right\}\left\{\eta+\sin(\alpha+2\pi/3)\right\}\left\{\eta+\sin(\alpha-2\pi/3)\right\}} \tag{104}$$

$$\frac{\sin 3\alpha}{\cos 3\beta+\cos 3\alpha} \; \frac{T(\beta)}{T(\pi+\alpha)} \; e^{-ikr\cos(\beta-\theta)}\, d\beta\,.$$

The fact that $H_1(\xi)$ is more complicated for $\gamma = 2\pi/3$ than for $\gamma = 3\pi/2$ has led to a more involved solution and, indeed, equations (103) and (104) are typical of those wedge angles for which the expression for $h(\xi, c)$ is capable of any reduction.

The function $T(\beta)$ contains an irreducible integral which can be split up into a product of separate integrals of the type

$$\int_0^y \log\left(\frac{\eta + \cos x}{\eta - \cos x}\right) dx \tag{105}$$

and this seems to be true of most values of n of the form 2q/p, where p and q are integers. And only for these n is it possible to simplify $h(\zeta, c)$ at all.

Many wedges of angle $\gamma = p\pi/q$ have been examined and for all of them $h(\zeta, c)$ can be written as a product of linear factors or as a product of such factors with integrals of the form (105). This suggests that if (105) were to be evaluated numerically for $0 < y \leq \pi$ it would cover all wedges of homogeneous refractive index $1/\eta$ and angle $p\pi/q$, and would enable the appropriate $h(\zeta, c)$ to be calculated in a relatively straightforward manner.

CHAPTER XIV

SOME RAMIFICATIONS

In the preceding chapters a method has been developed for the exact solution of second order differential equations of the form (7) under boundary conditions imposed on radial surfaces. The method has been illustrated by application to the two-dimensional problem of a plane electromagnetic wave diffracted by a metallic wedge, but differential equations other than (7) can be treated, as can boundary conditions more complex than (5) and (6). Indeed, it is obvious that boundary conditions containing derivatives of higher order than the first can be treated in a similar manner.

Our main interest is centred on the electromagnetic problem and whilst only an H-polarized incident plane wave has been considered, it should be emphasized that the method can be applied to more general types of incident field.

Thus, for an E-polarized plane wave having E_z^i given by equation (4), the boundary conditions (5) and (6) are replaced by

$$\phi_\theta - i\left(k/\eta\right) \ r\,\phi = 0 \tag{106}$$

for θ=0 and

$$\phi_\theta + i\left(k/\eta\right) r\,\phi = 0 \tag{107}$$

for $\theta = \gamma$, respectively. If, then, the difference equation is retained in its original form, the parameters τ_1 and τ_2 must be interpreted as

$$\frac{1}{\eta}\left(1 + \sqrt{1-\eta^2}\right) \quad \text{and} \quad \frac{1}{\eta}\left(1 - \sqrt{1-\eta^2}\right)$$

and the formulae for $H(\xi)$ and $\phi(r, \theta)$ now differ from those for H-polarization in having $1/\eta$ in place of η.

This result agrees with that found for an imperfectly conducting half-plane (Ref. 12) and is an important advantage attached to a (mathematically) exact solution of the boundary-value problem. In particular, it enables the solution for one polarization to be deduced from that for the other (and opposite) polarization by means of the transformation

$$Y\underline{E} \rightarrow Z\underline{H}, \quad \underline{H} \rightarrow -\underline{E}, \quad \eta \rightarrow 1/\eta ,$$

a duality which is a direct consequence of the impedance-type boundary conditions and which corresponds to the invariance of Maxwell's equations under the transformation $Y\underline{E} \rightarrow Z\underline{H}$, $\underline{H} \rightarrow -\underline{E}$.

If the plane wave is incident at an oblique angle the problem is no longer two-dimensional, but the solution can be found from the above in the manner described in Ref. 13. By regarding modified forms of the two-dimensional solutions as single-component Hertz vectors, two fundamental fields are obtained which can be used to determine the solution for a general three-dimensional plane wave, and hence the solution for any incident plane wave can be deduced from that given in Chapter X by performing only simple transformations.

For an incident field produced by either an electric or magnetic line source parallel to the apex of the wedge, the solution can be derived by integrating the corresponding plane wave solution with respect to its angle of incidence α. If the incident field is that of a point source the analogous technique is to integrate the appropriate three-dimensional plane wave solution with respect to the component angles of incidence, and this method was used in Ref. 12 to determine the field diffracted by a perfectly conducting half-plane when excited by a dipole.

Finally there is the generalization to the case of a partially inhomogeneous wedge. If the two faces are composed of different materials having unequal refractive indices, the method can still be used although the simplification to the difference equation represented by equation (76) no longer holds.

To conclude this chapter we shall examine briefly the solutions which have been obtained for particular wedge angles. In Chapter I reference was made to the dangers involved in attempting an approximate solution by assuming a series expansion in powers of η, and for a half-plane the expansion of $L(\beta)$ gives

$$L(\beta) = \eta^{1/2} \left\{ 1 - \frac{\eta}{\sin\beta} \left(1 - \frac{\beta}{\pi} \right) + 0\left(\eta^2\right) \right\}$$

(see equation 99). For large η, however,

$$L(\beta) = \sqrt{2\sin\frac{\beta}{2}} \left[1 - \frac{1}{\eta} \left\{ \sin\beta \left(1 - \frac{\beta}{\pi}\right) - \frac{1}{\pi}\cos\beta(1 + \log 2\eta) \right\} + 0\left(\frac{1}{\eta^2}\right) \right]$$

and the correspondence between the two polarizations now shows that a series expansion is not valid for E-polarization owing to the term in log η.

For a wedge of angle $3\pi/2$ it is obvious that a series expansion is always possible (see equation 102), and this, surprisingly enough, is true also for $\gamma = 2\pi/3$ (equation 104). No attempt has been made to expand the solution for arbitrary wedge angle, but it seems unlikely that the term in log η is peculiar to the solution for a half-plane.

CHAPTER XV

CONCLUSIONS

As developed in this paper the method is, in many respects, similar to that of Wiener and Hopf, of which it may be regarded as an extension; the use of different transformations, however, makes difficult any detailed comparison between the two.

The crux of the Weiner-Hopf technique is the splitting of a function into two parts with overlapping regions of regularity (see, for example, Ref. 14) and as applied to the problem of an H-polarized plane wave incident on a metallic sheet the task is the determination of a function $L_+(\xi)$ such that

$$L_+(\xi)\, L_+(-\xi) = \frac{1}{\eta} + \frac{k}{\sqrt{k^2 - \xi^2}} ,$$

where $L_+(\xi)$ is regular in the upper half-plane of the Fourier transform variable ξ. By treating the ξ plane as a Riemann surface, the above equation can be written as

$$\frac{L_+(\xi)}{L_+(\xi e^{2i\pi})} = \frac{k\eta + \sqrt{k^2 - \xi^2}}{k\eta - \sqrt{k^2 - \xi^2}} \tag{108}$$

which is, to all intents and purposes, a difference equation for $L_+(\xi)$. The equation has a 'periodicity' 2π, equal to the open angle of the 'wedge', and gives the value of the function at the point ξ in terms of its value at the corresponding point on a lower sheet of the Riemann surface. Moreover, the right-hand side of (108) closely resembles the Fresnel reflection coefficient of the diffracting half-plane, a fact which is brought out more clearly on making the transformation $\xi = k\cos\beta$.

The essential feature of Peters' method is the derivation of a function $F(\omega)$ satisfying the difference equation

$$\frac{F(\omega)}{F(\omega+2i\gamma)} = \left\{\frac{\eta - \cosh\omega}{\eta + \cosh\omega}\right\}\left\{\frac{\eta - \cosh(\omega+i\gamma)}{\eta + \cosh(\omega+i\gamma)}\right\}, \tag{109}$$

the right-hand side of which is related to the product of the Fresnel reflection coefficients for the two faces of the wedge. Equation (109) can be written alternatively as

$$\frac{F(\omega)}{F(\omega+i\gamma)} = \frac{\eta - \cosh\omega}{\eta + \cosh\omega}, \tag{110}$$

where γ is the open angle of the wedge (c.f. equation 76), and the analogy with the Wiener-Hopf method is now obvious.

The above comparison shows that the two techniques are of essentially the same type. The analysis involved in similar and in each case the Fresnel reflection coefficient plays a dominant role, notwithstanding the fact that the final solution contains the coefficient explicitly only in its geometrical optics terms. The method originated by Peters represents a new and powerful tool for the exact solution of diffraction problems and is particularly suited to such problems as the imperfectly conducting wedge where the physical discontinuities lie in inclined planes. By comparison the Wiener-Hopf approach and others (for example, that of Clemmow, Ref. 8) based upon it are more limited in scope, being virtually restricted to cases which involve discontinuities in parallel planes.

The present method has much of the elegance associated with the older technique, although its simplicity is somewhat marred by the necessity for introducing complex variables additional to those implicit in the problem, and this is liable to cause confusion in converting conditions upon $\phi(r, \theta)$ into conditions upon the required solution of the difference equation. With both treatments, however, most of the analytical difficulties are concentrated in the solution of equations of the form (108) or (109).

Note added in proof:

Dr. L. B. Felsen has drawn the attention of the author to a paper by Malyuzinec (Ref. 15) dealing with the radiation of sound by a vibrating wedge. In an appendix the solution is stated for a problem which is equivalent to the diffraction of an H-polarized electromagnetic wave by a wedge whose faces have different refractive indices. Although no details of the method are given (Malyuzinec refers the reader to previous papers by himself: Refs 16-19), a preliminary examination suggests that the results are in agreement with those which we have obtained.

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APPENDIX

The chosen function $H_1(\xi)$ is

$$H_1(\xi) = \frac{\xi^n - \tau_1^n}{\xi^n - e^{in\pi/2}} \quad h(\xi, \tau_1)\, h(\xi, \tau_2),$$

where $h(\xi, c)$ is given by equation (87), and for $H_1(\xi)$ to satisfy the conditions stated in Chapter X it is necessary that

$$\left(\frac{\bar{\xi} + \tau_1}{\bar{\xi} - \tau_1}\right)\left(\frac{\bar{\xi} + \tau_2}{\bar{\xi} - \tau_2}\right) \quad H_1(\bar{\xi}) = H_1\left(\frac{e^{i\pi}}{\bar{\xi}}\right) .$$

To prove the relationship we consider first $h(\bar{\xi}, c)$. This is defined in terms of $h(\xi, c)$ and if ξ is replaced by $\bar{\xi}$ in the integrand of the expression for $h(\xi, c)$, we have

$$\log\left\{h(\bar{\xi}, c)\right\} = \frac{1}{\pi i}\int_0^{\infty c \bar{\xi}} \frac{\bar{\xi}}{v^2 - \bar{\xi}^2} \log\left\{1 - \left(\frac{c}{v}\right)^n\right\} dv \qquad \text{(A1)}$$

$$= \log\left(\frac{\bar{\xi} - c}{\bar{\xi} + c}\right) + \frac{1}{\pi i}\int_0^{\infty c \bar{\xi}} \frac{\bar{\xi}}{v^2 - \bar{\xi}^2} \log\left\{1 - \left(\frac{c}{v}\right)^n\right\} dv$$

$$= \log\left(\frac{\bar{\xi} - c}{\bar{\xi} + c}\right) + I(\bar{\xi}, c)$$

where

$$I(\bar{\xi}, c) = -\frac{1}{n\pi i}\int_0^{\infty} \frac{c t^{1/n - 1}}{t^{2/n} - c^2} \log\left(1 - \frac{t}{\bar{\xi}^n}\right) dt$$

When $c = \tau_1$ the path of integration in equation (A2) can lie anywhere in the sector $\{n(\delta - \pi), n\delta\}$, where $\delta = \arg \tau_1$, and if its inclination to the positive real axis is denoted by $n\Delta_1$, the integral defines a function which is regular in $-\gamma + \Delta_1 < \arg \bar{\xi} < \Delta_1$. Since $\arg \bar{\xi} - \Delta_1 < 0$, all the allowed values of $\bar{\xi}$ lie below the path and this fact enables us to rotate the path in an anticlockwise direction so as to collect a contribution from the residue of the integrand at its pole $t = \tau^n$, giving

$$I(\bar{\xi}, \tau_1) = -\log\left\{1 - \left(\frac{\tau_1}{\bar{\xi}}\right)^n\right\} - \frac{1}{n\pi i}\int_0^{\infty e^{in(\Delta_1 + \pi)}} \frac{\tau_1 t^{1/n - 1}}{t^{2/n} - \tau_1^2} \log\left(1 - \frac{t}{\bar{\xi}^n}\right) dt.$$

The integral now defines a regular function of $\bar{\xi}$ in $-\gamma + \pi + \Delta_1 < \arg \bar{\xi} < \pi + \Delta_1$ and the transformation $t = (\tau_1 \bar{\xi} s)^n$ reduces it to the form

$$\frac{1}{\pi i}\int_0^{\infty e^{i\Delta_1'}} \frac{e^{i\pi}\bar{\xi}}{\bar{\xi}^2 s^2 - e^{2\pi i}} \log\left[-\left(\frac{s}{\tau_2}\right)^n\left(1 - \left[\frac{\tau_2}{s}\right]^n\right)\right] ds,$$

where $-\delta < \Delta_1' < -\delta + \gamma$. But this represents a regular function of $\frac{e^{i\pi}}{\bar{\xi}}$ in $\Delta_1' < \arg\left(\frac{e^{i\pi}}{\bar{\xi}}\right) < \Delta_1' + \pi$ and hence, from equation (A1), it can be written as

$$\log\left\{h\left(\frac{e^{i\pi}}{\bar{\xi}}, \tau_2\right)\right\} + \frac{1}{\pi i}\int_0^{\infty e^{i\Delta_1'}} \frac{e^{i\pi}\bar{\xi}}{\bar{\xi}^2 s^2 - e^{2i\pi}} \log\left\{-\left(\frac{s}{\tau_2}\right)^n\right\} ds.$$

The remaining integral can be simplified by introducing a new variable x, where $x = s\tau_1\bar{\xi}e^{-i\pi}$, and since the new path lies in the sector $(-\pi + \delta, \delta)$ we can now write

$$I(\bar{\xi}, \tau_1) = -\log\left\{1 - \left(\frac{\tau_1}{\bar{\xi}}\right)^n\right\} + \log\left\{h\left(\frac{e^{i\pi}}{\bar{\xi}}, \tau_2\right)\right\}$$
$$+ \frac{1}{\pi i}\int_0^{\infty} \frac{\tau_1}{x^2 - \tau_1^2} \log\left\{-\left(\frac{xe^{i\pi}}{\bar{\xi}}\right)^n\right\} dx. \qquad (A3)$$

The case $c = \tau_2$ can be treated in a similar manner. The path of integration is initially confined to the sector $\{-n\delta, n(\pi-\delta)\}$, but if it is rotated in an anticlockwise direction we obtain

$$I(\bar{\xi}, \tau_2) = \log\left\{1-\left(\frac{\tau_2 e^{i\pi}}{\bar{\xi}}\right)^n\right\} - \frac{1}{n\pi i}\int_0^{\infty e^{in(\Delta_2+\pi)}} \frac{\tau_2 t^{1/n-1}}{t^{2/n}-\tau_2^2}\log\left(1-\frac{t}{\bar{\xi}^n}\right) dt, \tag{A4}$$

where $-\delta < \Delta_2 < \pi - \delta$. The above sequence of transformations then gives

$$I(\bar{\xi}, \tau_2) = \log\left\{1-\left(\frac{\tau_2 e^{i\pi}}{\bar{\xi}}\right)^n\right\} + \log\left\{h\left(\frac{e^{i\pi}}{\bar{\xi}}, \tau_1\right)\right\} + \frac{1}{\pi i}\int_0^{\infty} \frac{\tau_2}{x^2-\tau_2^2}\log\left\{-\left(\frac{x e^{i\pi}}{\bar{\xi}}\right)^n\right\} dx. \tag{A5}$$

Hence,

$$\log\left[\left(\frac{\bar{\xi}+\tau_1}{\bar{\xi}-\tau_1}\right)\left(\frac{\bar{\xi}+\tau_2}{\bar{\xi}-\tau_2}\right) H_1(\bar{\xi})\right] = \log\left\{\frac{\bar{\xi}^n - (\tau_2 e^{i\pi})^n}{\bar{\xi}^n - e^{in\pi/2}}\right\}$$

$$+ \log\left\{h\left(\frac{e^{i\pi}}{\bar{\xi}}, \tau_2\right)\right\} + \log\left\{h\left(\frac{e^{i\pi}}{\bar{\xi}}, \tau_1\right)\right\}$$

$$+ \frac{1}{\pi i}\int_0^{\infty}\left(\frac{\tau_1}{x^2-\tau_2^2} \quad \frac{\tau_2}{x^2-\tau_2^2}\right)\log\left\{-\left(\frac{x e^{i\pi}}{\bar{\xi}}\right)^n\right\} dx$$

and since

$$\left\{\frac{\bar{\xi}^n - (\tau_2 e^{i\pi})^n}{\bar{\xi}^n - e^{in\pi/2}}\right\} = (\tau_2 e^{i\pi/2})^n \left\{\frac{\left(\frac{e^{i\pi}}{\bar{\xi}}\right)^n - \tau_1^n}{\left(\frac{e^{i\pi}}{\bar{\xi}}\right)^n - e^{in\pi/2}}\right\},$$

we have

$$\left(\frac{\bar{\xi} \quad \tau_1}{\bar{\xi}-\tau_1}\right)\left(\frac{\bar{\xi} \quad \tau_2}{\bar{\xi}-\tau_2}\right) H_1(\bar{\xi}) = W(\bar{\xi})\, H_1\left(\frac{e^{i\pi}}{\bar{\xi}}\right), \tag{A6}$$

where

$$\log\left\{W(\bar{\xi})\right\} = n\log(\tau_2 e^{i\pi/2}) + \frac{1}{\pi i}\int_0^\infty \left(\frac{\tau_1}{x^2-\tau_1^2} + \frac{\tau_2}{x^2-\tau_2^2}\right)\log\left\{-\left(\frac{xe^{i\pi}}{\bar{\xi}}\right)^n\right\} dx$$

$$= n\log(\tau_2 e^{i\pi/2}) - \frac{n}{2\pi i}\int_0^\infty \log\left\{\left(\frac{x-\tau_1}{x+\tau_1}\right)\left(\frac{x-\tau_2}{x+\tau_2}\right)\right\}\frac{dx}{x} .$$

In the complex x plane the branch cuts radiate from the orgin to $\pm\tau_1$, $\pm\tau_2$ and if the path of integration is rotated through an angle π, contributions are obtained from the integrations along the branch cuts to τ_1 and $\tau_2 e^{i\pi}$, giving

$$\int_0^\infty = \int_0^{-\infty} + 2\pi i\left[\log x\right]_0^{\tau_2 e^{i\pi}} - 2\pi i\left[\log x\right]_0^{\tau_1}$$

$$= \int_0^{-\infty} + 2\pi i \log\left(\frac{\tau_2 e^{i\pi}}{\tau_1}\right).$$

An identical result would have been obtained by moving the path in a clockwise direction, and using the fact that

$$\int_0^{-\infty} = -\int_0^\infty ,$$

we now have

$$\int_0^\infty = \pi i \log\left(\frac{\tau_2 e^{i\pi}}{\tau_1}\right):$$

Thus

$$\log\left\{W(\bar{\xi})\right\} = n\log(\tau_2 e^{i\pi/2}) - \frac{n}{2}\log\left(\frac{\tau_2 e^{i\pi}}{\tau_1}\right) \qquad \text{(A7)}$$

$$= 0,$$

which confirms that $H_1(\bar{\xi})$ has the required property.