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STUDY OF THE SCATTERING BEHAVIOR OF A SPHERE WITH AN ARBITRARILY PLACED CIRCUMFERENTIAL SLOT

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ABSTRACT

The electromagnetic scattering behavior of a metallic sphere loaded with a circumferential slot arbitrarily placed with respect to the direction of incidence is studied. Under the assumption that the slot is of small but nonzero width with a constant electric field across it, the analysis for the external fields is exact. Expressions for the scattered far field components, as well as for the total surface field components, are derived and then used to investigate the extent to which the scattering behavior can be controlled by varying the loading admittance and the slot position. An explicit formula for the loading of the zeroth mode to annul the back scattering cross section is derived, and from this the desired loading is obtained by means of a lumped load at the center of a radial cavity backing the slot. In particular, emphasis is placed on the case where the slot is in the plane of incidence and normal to the direction of the incident electric vector.

The numerical study is limited to the frequency range $0 < ka \leqslant 3.0$, where a is the radius of the sphere, and results are presented primarily for back scattering. To verify some of these results, a comparison is made with experimental data obtained using a metallic sphere with an equatorial slot backed by a radial cavity of adjustable depth.

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I
INTRODUCTION

In the recent investigation by Liepa and Senior (1964, 1966) of impedance loading applied to a perfectly conducting sphere, the narrow circumferential slot used to provide the loading was restricted to lie in the plane perpendicular to the direction of incidence. Analytically at least this leads to a considerable simplification since only the tesseral harmonics of the first order then appear, but it does create difficulties in any attempt to achieve the required loading. Thus, to control the scattering cross section over a specified bandwidth, it would seem that the loading must be synthemized either by a distributed network (or nonuniform transmission line) or by a large number of lumped two-port networks distributed around the sphere, and the sophistication of both methods does produce some difficulty in the practical realization of the loading.

The analytically simple case discussed above is, in fact, the most complicated one from the loading point of view, and in the hope of reducing the loading difficulties, we shall here extend the analysis to the general case of arbitrary incidence. The circumferential slot will now be located arbitrarily with respect to the direction of incidence and, in addition, the polarization will also be assumed arbitrary with respect to the plane of the slot. The analysis parallels in large measure that provided by Liepa and Senior (1964, 1966), the only major difference being the occurrence of doubly-infinite sets of modes. Analytical expressions for the field components and for the loading necessary to produce any desired form of cross section control are presented, and it is shown that if there exists a zeroth order mode excitation across the slot, the cross section control can be achieved by a lumped load at the center of a radial cavity.

For the particular case in which the incident electric vector is perpendicular to the plane of the slot, the expression for the loading required to give zero back scattering is obtained and computed. To check the theoretically predicted behavior, the back scattered field was measured using a sphere with an equatorial slot backed by a radial cavity of adjustable depth. The measured cross section as a function of depth is in excellent agreement with the predicted values.

II
THEORETICAL FORMULATION

Consider a perfectly conducting sphere of radius a whose center is located at the origin of a Cartesian coordinate system (x, y, z) and loaded with a narrow circumferential slot symmetrically placed with respect to the z-axis as shown in Fig. 2-1. The sphere is illuminated by a linearly-polarized plane electromagnetic wave incident in the direction of the negative z'-axis whose spherical angular coordinates with respect to the unprimed Cartesian coordinate system are θ_i and ϕ_i. If we assume that the incident electric field vector is oriented at an angle β to the x'-axis, where the primed Cartesian coordinate system (x', y', z') is obtained by a rotational transformation of the unprimed, namely by rotating through an angle ϕ_i with respect to the z-axis and then through an angle θ_i with respect to the y-axis, so that in matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_i & 0 & -\sin\theta_i \\ 0 & 1 & 0 \\ \sin\theta_i & 0 & \cos\theta_i \end{bmatrix} \begin{bmatrix} \cos\phi_i & \sin\phi_i & 0 \\ -\sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} ,$$

then the incident field can be written as

$$\underline{E}^i = (\hat{x}'\cos\beta + \hat{y}'\sin\beta)\,e^{ikz'} \tag{2.1a}$$

$$\underline{H}^i = -iY(-\hat{x}'\sin\beta + \hat{y}'\cos\beta)\,e^{ikz'} , \tag{2.1b}$$

where k is the propagation constant and Y the intrinsic admittance of free space. For convenience, the amplitude of the electric vector has been taken to be unity and the time factor $e^{i\omega t}$ suppressed.

The scattered field in any direction can be obtained by superposition of the field diffracted by an unloaded sphere and that radiated from an excited slot at the

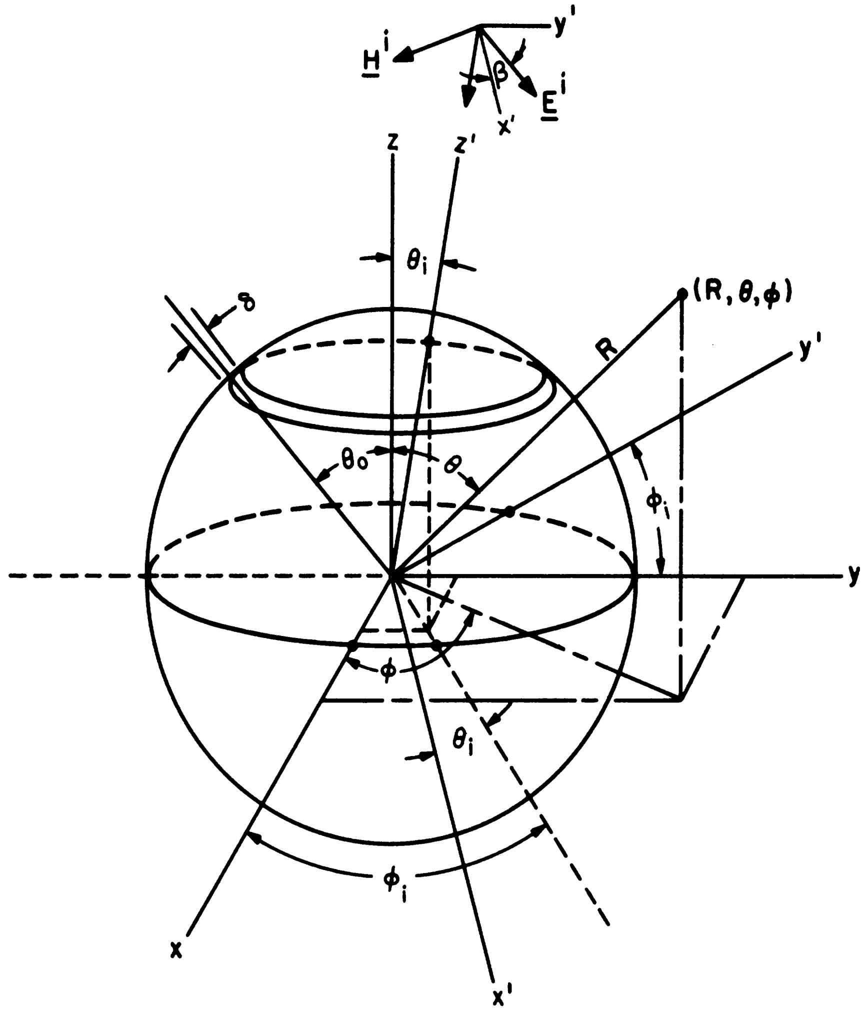

FIG. 2-1: COORDINATE SYSTEMS AND SPHERE GEOMETRY

position of the load. The radiation amplitude and phase are determined by the loading characteristics of the slot, and by controlling these a wide degree of scattering control can be exercised.

2.1 Diffraction by an Unloaded Sphere

To study the diffraction of a plane wave by a spherical object we must first find an expansion of the incident wave in terms of the spherical wave functions $\underline{M}_{\substack{e\\o}mn}$, $\underline{N}_{\substack{e\\o}mn}$.

For the field given by the equations (2.1) the representation in terms of spherical vector wave functions is (see Appendix A):

$$\underline{E}^i = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left\{ \left[\cos\beta \, A_{\substack{e\\o}mn} - \sin\beta \, B_{\substack{e\\o}mn} \right] \underline{M}^{(1)}_{\substack{e\\o}mn} - i \left[\sin\beta \, A_{\substack{e\\o}mn} + \cos\beta \, B_{\substack{e\\o}mn} \right] \underline{N}^{(1)}_{\substack{e\\o}mn} \right\} \tag{2.2a}$$

$$\underline{H}^i = iY \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left\{ \left[\cos\beta \, A_{\substack{e\\o}mn} - \sin\beta \, B_{\substack{e\\o}mn} \right] \underline{N}^{(1)}_{\substack{e\\o}mn} - i \left[\sin\beta \, A_{\substack{e\\o}mn} + \cos\beta \, B_{\substack{e\\o}mn} \right] \underline{M}^{(1)}_{\substack{e\\o}mn} \right\} \tag{2.2b}$$

where

$$A_{\substack{e\\o}mn} = \mp i^n \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \begin{matrix}\sin\\ \cos\end{matrix} \, m\phi_i \, \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i} \tag{2.3a}$$

$$B_{\substack{e\\o}mn} = i^n \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \begin{matrix}\cos\\ \sin\end{matrix} \, m\phi_i \, \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i} \tag{2.3b}$$

with

$$\epsilon_m = \begin{cases} 1\ , & m=0 \\ 2\ , & m=1,2,\dots \end{cases}$$

and

$$\underline{M}^{(1)}_{\substack{e\\o}mn} = \mp \frac{m}{\sin\theta}\,\frac{\psi_n(kR)}{kR}\,P_n^m(\cos\theta)\,{\sin \atop \cos} m\phi\,\hat{\theta}$$
$$-\frac{\psi_n(kR)}{kR}\,\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\,{\cos \atop \sin} m\phi\,\hat{\phi}$$

$$\underline{N}^{(1)}_{\substack{e\\o}mn} = n(n+1)\frac{\psi_n(kR)}{(kR)^2}\,P_n^m(\cos\theta)\,{\cos \atop \sin} m\phi\,\hat{R}$$
$$+\frac{\psi_n'(kR)}{kR}\left\{\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\,{\cos \atop \sin} m\phi\,\hat{\theta} \mp \frac{m}{\sin\theta}\,P_n^m(\cos\theta)\,{\sin \atop \cos} m\phi\,\hat{\phi}\right\}.$$

Here, $\psi_n(kR) = kR\,j_n(kR)$ where $j_n(kR)$ is the spherical Bessel function of order n, and the prime denotes differentiation with respect to the entire argument. $P_n^m(\cos\theta)$ is the Legendre function of degree n and order m as defined, for example, by Stratton (1941).

For the scattered field $(\underline{E}^s, \underline{H}^s)$, the requirement that it take the form of an outgoing wave at infinity, leads us to postulate the representation

$$\underline{E}^s = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(C_{\substack{e\\o}mn}\,\underline{M}^{(4)}_{\substack{e\\o}mn} + i\,D_{\substack{e\\o}mn}\,\underline{N}^{(4)}_{\substack{e\\o}mn}\right) \tag{2.4a}$$

$$\underline{H}^s = iY\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(C_{\substack{e\\o}mn}\,\underline{N}^{(4)}_{\substack{e\\o}mn} + i\,D_{\substack{e\\o}mn}\,\underline{M}^{(4)}_{\substack{e\\o}mn}\right)\ . \tag{2.4b}$$

in which the $M^{(4)}$ and $N^{(4)}$ differ from the $M^{(1)}$ and $N^{(1)}$ in having $\psi_n(kR)$ replaced by $\zeta_n(kR) = kR\, h_n^{(2)}(kR)$, where $h_n^{(2)}(kR)$ is the spherical Hankel function of the second kind. Application of the boundary condition

$$\hat{R} \times (\underline{E}^i + \underline{E}^s) = 0$$

at $R = a$ then gives

$$C_{\substack{e\\o}mn} = -\left[\cos\beta\, A_{\substack{e\\o}mn} - \sin\beta\, B_{\substack{e\\o}mn}\right] \frac{\psi_n(ka)}{\zeta_n(ka)} \tag{2.5a}$$

$$D_{\substack{e\\o}mn} = \left[\sin\beta\, A_{\substack{e\\o}mn} + \cos\beta\, B_{\substack{e\\o}mn}\right] \frac{\psi_n'(ka)}{\zeta_n'(ka)} \quad , \tag{2.5b}$$

and by substitution of (2.5) into (2.4) the scattered field is determined.

2.1.1 Surface Fields

The total field is the sum of $(\underline{E}^i, \underline{H}^i)$ and $(\underline{E}^s, \underline{H}^s)$, and the only nonzero components at the surface of the sphere are E_R, H_θ and H_ϕ. Since the last two are related to the surface current density $\underline{J}$ via the equation

$$\underline{J} = \hat{R} \times \underline{H} \ ,$$

H_θ and H_ϕ are of direct concern to us. They are written, for convenience, in a notation similar to that of Kazarinoff and Senior (1962) as

$$J_\phi = H_\theta(a, \theta, \phi) = Y \sum_{m=0}^{\infty} \Big\{ \cos\beta \sin m(\phi - \phi_i) T_{1m}(\theta, \theta_i)$$
$$- \sin\beta \cos m(\phi - \phi_i) T_{1m}'(\theta, \theta_i) \Big\} \tag{2.6a}$$

$$-J_\theta = H_\phi(a,\theta,\phi) = Y \sum_{m=0}^{\infty} \left\{ \cos\beta \cos m(\phi-\phi_i) T_{2m}(\theta,\theta_i) \right.$$

$$\left. + \sin\beta \sin m(\phi-\phi_i) T'_{2m}(\theta,\theta_i) \right\} \qquad (2.6b)$$

where+

$$T_{1m}(\theta,\theta_i) = \frac{1}{ka} \sum_{n=m+\delta_m}^{\infty} i^{n+1} \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \frac{1}{\zeta'_n(ka)} \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i} \frac{mP_n^m(\cos\theta)}{\sin\theta} \right.$$

$$\left. + \frac{i}{\zeta_n(ka)} \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \right\}, \qquad (2.7a)$$

$$T'_{1m}(\theta,\theta_i) = \frac{1}{ka} \sum_{n=m+\delta_m}^{\infty} i^{n+1} \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \frac{1}{\zeta'_n(ka)} \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i} \frac{mP_n^m(\cos\theta)}{\sin\theta} \right.$$

$$\left. + \frac{i}{\zeta_n(ka)} \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \right\}, \qquad (2.7b)$$

$$T_{2m}(\theta,\theta_i) = \frac{1}{ka} \sum_{n=m+\delta_m}^{\infty} i^{n+1} \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \frac{1}{\zeta'_n(ka)} \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \right.$$

$$\left. + \frac{i}{\zeta_n(ka)} \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i} \frac{mP_n^m(\cos\theta)}{\sin\theta} \right\}, \qquad (2.7c)$$

+Note that the order of double summation has been changed from $\sum_{n=0}^{\infty} \sum_{m=0}^{n}$ to $\sum_{m=0}^{\infty} \sum_{n=m+\delta_m}^{\infty}$.

$$T'_{2m}(\theta,\theta_i) = \frac{1}{ka} \sum_{n=m+\delta_m}^{\infty} i^{n+1} \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \frac{1}{\zeta'_n(ka)} \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \frac{i}{\zeta_n(ka)} \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i} \frac{mP_n^m(\cos\theta)}{\sin\theta} \right\}, \qquad (2.7d)$$

with

$$\delta_m = \begin{cases} 1, & m=0 \\ 0, & m=1,2,3,\ldots \end{cases} .$$

In the special case $\theta_i = 0$, the fact that

$$\lim_{\theta\to 0} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} = \lim_{\theta\to 0} \frac{mP_n^m(\cos\theta)}{\sin\theta} = \begin{cases} 0, & m \neq 1 \\ \frac{n(n+1)}{2}, & m=1 \end{cases}$$

implies

$$T_{1m}(\theta,0) = T'_{1m}(\theta,0) = \begin{cases} 0, & m \neq 1 \\ T_1(\theta), & m = 1 \end{cases}$$

$$T_{2m}(\theta,0) = T'_{2m}(\theta,0) = \begin{cases} 0, & m \neq 1 \\ T_2(\theta), & m = 1 \end{cases}$$

where $T_1(\theta)$ and $T_2(\theta)$ are identical to the current components employed by Liepa and Senior (1964).

2.1.2 Scattered Far Fields

In the far zone the expressions for the scattered field can be obtained by replacing $\zeta_n(kR)$ and $\zeta'_n(kR)$ by the leading terms of their asymptotic expansions for large kR, viz.

$$\zeta_n(kR) \sim i^{n+1} e^{-ikR} ,$$

$$\zeta'_n(kR) \sim i^n e^{-ikR} .$$

The transverse components of the scattered electric field in the far zone then become

$$E^s_\theta = i \frac{e^{-ikR}}{kR} \sum_{m=0}^{\infty} \left\{ \cos\beta \cos m(\phi - \phi_i) S^s_{1m}(\theta, \theta_i) + \sin\beta \sin m(\phi - \phi_i) S'^s_{1m}(\theta, \theta_i) \right\} \tag{2.8a}$$

$$E^s_\phi = -i \frac{e^{-ikR}}{kR} \sum_{m=0}^{\infty} \left\{ \cos\beta \sin m(\phi - \phi_i) S^s_{2m}(\theta, \theta_i) - \sin\beta \cos m(\phi - \phi_i) S'^s_{2m}(\theta, \theta_i) \right\} , \tag{2.8b}$$

where

$$S^s_{1m}(\theta, \theta_i) = \sum_{n=m+\delta_m}^{\infty} (-1)^n \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \frac{\psi'_n(ka)}{\zeta'_n(ka)} \frac{\partial P^m_n(\cos\theta_i)}{\partial\theta_i} \frac{\partial P^m_n(\cos\theta)}{\partial\theta} - \frac{\psi_n(ka)}{\zeta_n(ka)} \frac{m P^m_n(\cos\theta_i)}{\sin\theta_i} \frac{m P^m_n(\cos\theta)}{\sin\theta} \right\}, \tag{2.9a}$$

$$S'^s_{1m}(\theta, \theta_i) = \sum_{n=m+\delta_m}^{\infty} (-1)^n \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \frac{\psi'_n(ka)}{\zeta'_n(ka)} \frac{m P^m_n(\cos\theta_i)}{\sin\theta_i} \frac{\partial P^m_n(\cos\theta)}{\partial\theta} - \frac{\psi_n(ka)}{\zeta_n(ka)} \frac{\partial P^m_n(\cos\theta_i)}{\partial\theta_i} \frac{m P^m_n(\cos\theta)}{\sin\theta} \right\}, \tag{2.9b}$$

$$S^s_{2m}(\theta,\theta_i) = \sum_{n=m+\delta_m}^{\infty} (-1)^n \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \frac{\psi_n'(ka)}{\zeta_n'(ka)} \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i} \frac{mP_n^m(\cos\theta)}{\sin\theta} - \frac{\psi_n(ka)}{\zeta_n(ka)} \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \right\}, \quad (2.9c)$$

$$S'^s_{2m}(\theta,\theta_i) = \sum_{n=m+\delta_m}^{\infty} (-1)^n \frac{\epsilon_m (2n+1)(n-m)!}{n(n+1)(m+m)!} \left\{ \frac{\psi_n'(ka)}{\zeta_n'(ka)} \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i} \frac{mP_n^m(\cos\theta)}{\sin\theta} - \frac{\psi_n(ka)}{\zeta_n(ka)} \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \right\}. \quad (2.9d)$$

In the back ($\phi = \phi_i$, $\theta = \theta_i$) and forward ($\phi = \pi + \phi_i$, $\theta = \pi - \theta_i$) directions it can be shown that

$$\sum_{m=0}^{\infty} S^s_{1m}(\theta_i,\theta_i) = \sum_{m=0}^{\infty} S'^s_{2m}(\theta_i,\theta_i) = S^s_1(0) \quad (2.10a)$$

and

$$\sum_{m=0}^{\infty} (-1)^m S^s_{1m}(\pi-\theta_i,\theta_i) = -\sum_{m=0}^{\infty} (-1)^m S'^s_{2m}(\pi-\theta_i,\theta_i) = S^s_1(\pi), \quad (2.10b)$$

where

$$S^s_1(\theta) = \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)}{n(n+1)} \left\{ \frac{\psi_n'(ka)}{\zeta_n'(ka)} \frac{\partial P_n^1(\cos\theta)}{\partial\theta} - \frac{\psi_n(ka)}{\zeta_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \right\}$$

is the far field amplitude defined by Liepa and Senior (1964). The component cross sections are therefore

$$\sigma_\theta(\theta_i, \phi_i) = \frac{\lambda^2}{\pi} \left| \cos\beta \, S_1^s(0) \right|^2 \tag{2.11a}$$

$$\sigma_\phi(\theta_i, \phi_i) = \frac{\lambda^2}{\pi} \left| \sin\beta \, S_1^s(0) \right|^2 \tag{2.11b}$$

for back scattering, and

$$\sigma_\theta(\pi - \theta_i, \pi + \phi_i) = \frac{\lambda^2}{\pi} \left| \cos\beta \, S_1^s(\pi) \right|^2 \tag{2.12a}$$

$$\sigma_\phi(\pi - \theta_i, \pi + \phi_i) = \frac{\lambda^2}{\pi} \left| \sin\beta \, S_1^s(\pi) \right|^2 \tag{2.12b}$$

for forward scattering.

2.2 Radiation by the Slot

Let us now consider the separate but related problem of a perfectly conducting sphere with a narrow zonal slot situated at $\theta = \theta_o$ as illustrated in Fig. 2-1. The slot occupies the region $\theta_o - \frac{\delta}{2} \leqslant \theta \leqslant \theta_o + \frac{\delta}{2}$ and its angular width δ is assumed very small (such that $ka\delta \ll 1$). Without any loss of generality, the direction of the incident field propagation can be assumed to lie in the x-z plane ($\phi_i = 0$).

Since the θ-component of the unperturbed current density at the surface can be represented as a summation of cosine and sine harmonics of ϕ (see equation (2.6b)), the following expressions for the tangential components of the electric field at the surface are assumed:

$$E_\theta(a, \theta, \phi) = \begin{cases} -\frac{1}{\delta a} \sum_{p=0}^{\infty} \left\{ V_p \cos\beta \cos p\phi + V'_p \sin\beta \sin p\phi \right\} , & \left|\theta - \theta_o\right| < \frac{\delta}{2} \\ 0 , & \text{otherwise} \end{cases} \tag{2.13a}$$

$$E_\phi(a, \theta, \phi) = 0 . \tag{2.13b}$$

$V_p \cos\beta$ and $V_p' \sin\beta$ are the amplitudes of the gap voltage due to the pth mode contribution corresponding to the cosine and sine variations, respectively, and they will be determined later.

Let the field radiated by the slot be

$$E^r = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(F_{\substack{e\\o}mn}\underline{M}^{(4)}_{\substack{e\\o}mn} + i\,G_{\substack{e\\o}mn}\underline{N}^{(4)}_{\substack{e\\o}mn}\right) , \qquad (2.14a)$$

$$H^r = iY\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(F_{\substack{e\\o}mn}\underline{N}^{(4)}_{\substack{e\\o}mn} + i\,G_{\substack{e\\o}mn}\underline{M}^{(4)}_{\substack{e\\o}mn}\right) , \qquad (2.14b)$$

where the coefficients $F_{\substack{e\\o}mn}$ and $G_{\substack{e\\o}mn}$ are to be determined from the above boundary conditions. From the θ component of the electric field we have, at $R = a$,

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{\mp F_{\substack{e\\o}mn}\frac{\zeta_n(ka)}{ka}\,\frac{mP_n^m(\cos\theta)}{\sin\theta}\,\begin{matrix}\sin\\ \cos\end{matrix}\,m\phi + i\,G_{\substack{e\\o}mn}\frac{\zeta_n'(ka)}{ka}\,\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\,\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi\right\}$$

$$= \begin{cases} -\dfrac{1}{\delta a}\displaystyle\sum_{p=0}^{\infty}\left\{V_p\cos\beta\cos p\phi + V_p'\sin\beta\sin p\phi\right\} , & |\theta-\theta_o| < \dfrac{\delta}{2} \\ 0 , & \text{otherwise} \end{cases} \qquad (2.15a)$$

and from the ϕ component

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{-F_{\substack{e\\o}mn}\frac{\zeta_n(ka)}{ka}\,\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\,\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi \mp i\,G_{\substack{e\\o}mn}\frac{\zeta_n'(ka)}{ka}\,\frac{mP_n^m(\cos\theta)}{\sin\theta}\,\begin{matrix}\sin\\ \cos\end{matrix}\,m\phi\right\}$$

$$= 0 . \qquad (2.15b)$$

But

$$\int_0^{\pi}\left\{P_n^m(\cos\theta)\,\frac{\partial P_{n'}^m(\cos\theta)}{\partial\theta} + P_{n'}^m(\cos\theta)\,\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\right\}d\theta = 0$$

and

$$\int_0^{\pi}\left\{\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\,\frac{\partial P_{n'}^m(\cos\theta)}{\partial\theta} + \frac{m^2}{\sin^2\theta}\,P_n^m(\cos\theta)P_{n'}^m(\cos\theta)\right\}\sin\theta\,d\theta$$

$$= \begin{cases} 0\,, & n \neq n' \\ \dfrac{2n(n+1)(n+m)!}{(2n+1)(n-m)!}\,, & n = n' \end{cases}$$

(Bailin and Silver, 1956). Hence

$$F_{\substack{e\\o}mn} = \pm\,\frac{(2n+1)(n-m)!}{2n(n+1)(n+m)!}\,\frac{k}{\zeta_n(ka)}\begin{pmatrix} V'_m\sin\beta \\ V_m\cos\beta\end{pmatrix} D_{nm}(\theta_o) \qquad (2.16a)$$

$$G_{\substack{e\\o}mn} = \frac{(2n+1)(n-m)!}{2n(n+1)(n+m)!}\,\frac{ik}{\zeta'_n(ka)}\begin{pmatrix} V_m\cos\beta \\ V'_m\sin\beta\end{pmatrix} C_{nm}(\theta_o) \qquad (2.16b)$$

where

$$C_{nm}(\theta_o) = \frac{1}{\delta}\int_{\theta_o-\frac{\delta}{2}}^{\theta_o+\frac{\delta}{2}} \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\,\sin\theta\,d\theta\,, \qquad (2.17a)$$

$$D_{nm}(\theta_o) = \frac{1}{\delta}\int_{\theta_o-\frac{\delta}{2}}^{\theta_o+\frac{\delta}{2}} mP_n^m(\cos\theta)\,d\theta\,, \qquad (2.17b)$$

and the required expressions for the components of the radiated field now follow from equations (2.14) upon inserting the above coefficients.

2.2.1 Surface Fields

At the surface R = a the components of the radiated magnetic field can be written in forms analogous to those in equation (2.6), viz.

$$H_\theta^r = Y \sum_{m=0}^{\infty} \left\{ V_m \cos\beta \sin m\phi - V_m' \sin\beta \cos m\phi \right\} T_{1m}^r(\theta, \theta_o) \tag{2.18a}$$

$$H_\phi^r = Y \sum_{m=0}^{\infty} \left\{ V_m \cos\beta \cos m\phi + V_m' \sin\beta \sin m\phi \right\} T_{2m}^r(\theta, \theta_o) \tag{2.18b}$$

where

$$T_{1m}^r(\theta, \theta_o) = \frac{i}{2a} \sum_{n=m+\delta_m}^{\infty} \frac{(2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ C_{nm}(\theta_o) \frac{\zeta_n(ka)}{\zeta_n'(ka)} \frac{mP_n^m(\cos\theta)}{\sin\theta} - D_{nm}(\theta_o) \frac{\zeta_n'(ka)}{\zeta_n(ka)} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \right\} , \tag{2.19a}$$

$$T_{2m}^r(\theta, \theta_o) = \frac{i}{2a} \sum_{n=m+\delta_m}^{\infty} \frac{(2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ C_{nm}(\theta_o) \frac{\zeta_n(ka)}{\zeta_n'(ka)} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - D_{nm}(\theta_o) \frac{\zeta_n'(ka)}{\zeta_n(ka)} \frac{mP_n^m(\cos\theta)}{\sin\theta} \right\} . \tag{2.19b}$$

For $\delta = 0$ the above series expressions for $T_{1m}^r(\theta, \theta_o)$ and $T_{2m}^r(\theta, \theta_o)$ are convergent, albeit slowly, with their nth terms for $n \gg m$ being $O(n^{-1})$ and $O(n^{-2})$ respectively. The terms in the series alternate in sign in groups of $2\pi/\delta$ terms, and therefore the series may be treated as an alternating series.

2.2.2 Radiated Far Fields

In the far zone, expressions for the radiated field components can be obtained from the equations (2.14) by replacing $\zeta_n(kR)$ and $\zeta_n'(kR)$ by the leading terms of their asymptotic expansion for large kR:

$$E_\theta^r = i\,\frac{e^{-ikR}}{kR}\sum_{m=0}^{\infty}\left\{V_m\cos\beta\cos m\phi + V'_m\sin\beta\sin m\phi\right\} S_{1m}^r(\theta,\theta_o) \tag{2.20a}$$

$$E_\phi^r = -i\,\frac{e^{-ikR}}{kR}\sum_{m=0}^{\infty}\left\{V_m\cos\beta\sin m\phi - V'_m\sin\beta\cos m\phi\right\} S_{2m}^r(\theta,\theta_o) \tag{2.20b}$$

where the mth mode radiated far field amplitudes are given by

$$S_{1m}^r(\theta,\theta_o) = \frac{k}{2}\sum_{n=m+\delta_m}^{\infty} i^{n+1}\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}\left\{\frac{C_{nm}(\theta_o)}{\zeta'_n(ka)}\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + i\,\frac{D_{nm}(\theta_o)}{\zeta_n(ka)}\frac{mP_n^m(\cos\theta)}{\sin\theta}\right\}, \tag{2.21a}$$

$$S_{2m}^r(\theta,\theta_o) = \frac{k}{2}\sum_{n=m+\delta_m}^{\infty} i^{n+1}\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}\left\{\frac{C_{nm}(\theta_o)}{\zeta'_n(ka)}\frac{mP_n^m(\cos\theta)}{\sin\theta} + i\,\frac{D_{nm}(\theta_o)}{\zeta_n(ka)}\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\right\}. \tag{2.21b}$$

2.3 Radiation Admittances

It is customary to define the admittance as the ratio of the current to the voltage for each given mode. This definition, however, is not well suited for our purpose since we have a nonuniform current density across the slot that does not permit a unique specification of the admittance.

We shall therefore use an alternative but equally acceptable definition of admittance, namely, twice the ratio of the complex power flow across the aperture to the square of the applied voltage.

From equations (2.13) and (2.18) the mth mode power radiated per unit length of the slot is

$$W_{rm} = -\frac{Y}{2\delta}\left\{V_m \cos\beta \cos m\phi + V_p' \sin\beta \sin m\phi\right\}^2 \int_{\theta_o - \frac{\delta}{2}}^{\theta_o + \frac{\delta}{2}} \widetilde{T^r_{2m}(\theta, \theta_o)} d\theta$$

where $\sim$ denotes the complex conjugate. The mth mode radiation admittance density is therefore

$$y_{rm} = \frac{2\widetilde{W}}{\left\{V_m \cos\beta \cos m\phi + V_p' \sin\beta \sin m\phi\right\}^2}$$

$$= -\frac{Y}{\delta} \int_{\theta_o - \frac{\delta}{2}}^{\theta_o + \frac{\delta}{2}} T^r_{2m}(\theta, \theta_o) d\theta \ , \tag{2.22}$$

and the total mth mode radiation admittance is

$$Y_{rm} = \int_0^{2\pi} y_{rm}\, a \sin\theta_o \, d\phi$$

$$= -i Y\pi \sin\theta_o \sum_{n=m+\delta_m}^{\infty} \frac{(2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ C_{nm}(\theta_o) E_{nm}(\theta_o) \frac{\zeta_n(ka)}{\zeta_n'(ka)} - D_{nm}(\theta_o) F_{nm}(\theta_o) \frac{\zeta_n'(ka)}{\zeta_n(ka)} \right\} , \tag{2.23}$$

where

$$E_{nm}(\theta_o) = \frac{1}{\delta}\int_{\theta_o-\frac{\delta}{2}}^{\theta_o+\frac{\delta}{2}} \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\, d\theta ,$$

$$F_{nm}(\theta_o) = \frac{1}{\delta}\int_{\theta_o-\frac{\delta}{2}}^{\theta_o+\frac{\delta}{2}} \frac{mP_n^m(\cos\theta)}{\sin\theta}\, d\theta .$$

The above expressions simplify somewhat if we assume $\epsilon \leqslant \theta_o \leqslant \pi - \epsilon$ with $\epsilon \gg \delta$, allowing us to neglect the variation of $\sin\theta$ over the slot. We then have

$$E_{nm}(\theta_o) \simeq \frac{C_{nm}(\theta_o)}{\sin\theta_o} ,$$

$$F_{nm}(\theta_o) \simeq \frac{D_{nm}(\theta_o)}{\sin\theta_o} ,$$

which lead to

$$Y_{rm} \simeq -iY\pi \sum_{n=m+\delta_m}^{\infty} \frac{(2n+1)(n-m)!}{n(n+1)(n+m)!} \left\{ \left[C_{nm}(\theta_o)\right]^2 \frac{\zeta_n(ka)}{\zeta_n'(ka)} - \left[D_{nm}(\theta_o)\right]^2 \frac{\zeta_n'(ka)}{\zeta_n(ka)} \right\} . \tag{2.24}$$

2.4 Complete Problem

For the complete problem in which the plane wave given in equations (2.1) is incident on the slotted sphere, the total field scattered in any direction can be obtained by superposition of the field diffracted by an unloaded sphere and the field radiated by an excited slot. If we assume that the same voltage

$$V = \sum_{m=0}^{\infty} \left\{ V_m \cos\beta \cos m\phi + V_m' \sin\beta \sin m\phi \right\}$$

is excited across the slot by the currents induced by the incident field, the transverse components of the total electric field in the far zone are the sums of those given in equations (2.8) and (2.20) and are

$$E_\theta = i\frac{e^{-ikR}}{kR}\sum_{m=0}^{\infty}\left\{\left[S^s_{1m}(\theta,\theta_i)+V_m S^r_{1m}(\theta,\theta_o)\right]\cos\beta\cos m\phi + \left[S'^s_{1m}(\theta,\theta_i)+V'_m S^r_{1m}(\theta,\theta_o)\right]\sin\beta\sin m\phi\right\} \quad (2.25a)$$

$$E_\phi = -i\frac{e^{-ikR}}{kR}\sum_{m=0}^{\infty}\left\{\left[S^s_{2m}(\theta,\theta_i)+V_m S^r_{2m}(\theta,\theta_o)\right]\cos\beta\sin m\phi - \left[S'^s_{2m}(\theta,\theta_i)+V'_m S^r_{2m}(\theta,\theta_o)\right]\sin\beta\cos m\phi\right\}. \quad (2.25b)$$

The components cross sections are therefore

$$\sigma_\theta(\theta,\phi) = \frac{\lambda^2}{\pi}\left|\sum_{m=0}^{\infty}\left\{\left[S^s_{1m}(\theta,\theta_i)+V_m S^r_{1m}(\theta,\theta_o)\right]\cos\beta\cos m\phi + \left[S'^s_{1m}(\theta,\theta_i)+V'_m S^r_{1m}(\theta,\theta_o)\right]\sin\beta\sin m\phi\right\}\right|^2, \quad (2.26a)$$

$$\sigma_\phi(\theta,\phi) = \frac{\lambda^2}{\pi}\left|\sum_{m=0}^{\infty}\left\{\left[S^s_{2m}(\theta,\theta_i)+V_m S^r_{2m}(\theta,\theta_o)\right]\cos\beta\sin m\phi - \left[S'^s_{2m}(\theta,\theta_i)+V'_m S^r_{2m}(\theta,\theta_o)\right]\sin\beta\cos m\phi\right\}\right|^2. \quad (2.26b)$$

The V_m and V'_m can now be related to the loading admittance of the slot for the corresponding mode. To derive these in terms of the loading admittance, we require the expression for the mth mode loading admittance density, $y_{\ell m}$. Applying the same technique as used above, we have

$$y_{\ell m} = \frac{1}{\delta\left[V_m \cos\beta \cos m\phi + V'_m \sin\beta \sin m\phi\right]} \int_{\theta_o - \frac{\delta}{2}}^{\theta_o + \frac{\delta}{2}} \left[H^i_\phi + H^s_\phi + H^r_\phi\right]_{R=a} d\theta$$

which can be written as

$$y_{\ell m} = -y_{rm} + \frac{Y}{\delta V_m} \int_{\theta_o - \frac{\delta}{2}}^{\theta_o + \frac{\delta}{2}} T_{2m}(\theta, \theta_i) d\theta \tag{2.27a}$$

or

$$y_{\ell m} = -y_{rm} + \frac{Y}{\delta V'_m} \int_{\theta_o - \frac{\delta}{2}}^{\theta_o + \frac{\delta}{2}} T'_{2m}(\theta, \theta_i) d\theta \tag{2.27b}$$

by using equations (2.6b) and (2.22) and, in addition, the relation

$$\frac{V_m}{V'_m} = \frac{T_{2m}(\theta, \theta_i)}{T'_{2m}(\theta, \theta_i)} \quad . \tag{2.28}$$

This last can be obtained from the linear property that the excited voltage is proportional to the current induced by the incident field.

Since the variations of $T_{2m}(\theta, \theta_i)$ and $T'_{2m}(\theta, \theta_i)$ across the slot can be neglected if δ is sufficiently small, equations (2.27) can be approximated as

$$y_{\ell m} = -y_{\ell m} + \frac{Y}{V_m} T_{2m}(\theta_o, \theta_i)$$

or

$$y_{\ell m} = -y_{rm} + \frac{Y}{V'_m} T'_{2m}(\theta_o, \theta_i) \ .$$

The total loading admittance for the mth mode is therefore

$$Y_{\ell m} = -Y_{rm} + \frac{Y}{V_m} 2\pi a \sin\theta_o T_{2m}(\theta_o, \theta_i) \tag{2.29a}$$

or

$$Y_{\ell m} = -Y_{rm} + \frac{Y}{V'_m} 2\pi a \sin\theta_o T'_{2m}(\theta_o, \theta_i) \tag{2.29b}$$

from which we have

$$V_m = \frac{Y}{Y_{\ell m} + Y_{rm}} 2\pi a \sin\theta_o T_{2m}(\theta_o, \theta_i) \tag{2.30a}$$

or

$$V'_m = \frac{Y}{Y_{\ell m} + Y_{rm}} 2\pi a \sin\theta_o T'_{2m}(\theta_o, \theta_i) \ . \tag{2.30b}$$

When the above equations are substituted into equations (2.26), the explicit expressions for the component cross sections are obtained as functions of ka, δ, θ_o, θ_i and $Y_{\ell m}$. Thus we expect some degree of scattering control—especially the reduction of the cross section which is of main interest to us—by varying the loading admittance.

To make the θ-component cross section zero in the direction $\theta = \theta'$ and $\phi = \phi'$ it is required that

$$\sum_{m=0}^{\infty} \left\{ \left[S^s_{1m}(\theta', \theta_i) + \frac{Y}{Y_{\ell m} + Y_{rm}} 2\pi a \sin\theta_o T_{2m}(\theta_o, \theta_i) S^r_{1m}(\theta', \theta_o) \right] \cos\beta \cos m\phi' \right.$$
$$\left. + \left[S'^s_{1m}(\theta', \theta_i) + \frac{Y}{Y_{\ell m} + Y_{rm}} 2\pi a \sin\theta_o T'_{2m}(\theta_o, \theta_i) S^r_{1m}(\theta', \theta_o) \right] \sin\beta \sin m\phi' \right\} = 0 \tag{2.31}$$

It is readily seen that for the left-hand side of equation (2.31), each mth term of the series may be made zero individually by choosing the proper value of the loading admittance for the corresponding mode if the loading for each mode is independently available. Unfortunately, this is almost a non-realizable case since all the mode admittances are, in general, dependent upon each other. If there exists only one mode excitation such that equation (2.31) is independent of all the other mode loading admittances, it is, of course, always possible to satisfy equation (2.31) by the proper loading of this mode.

On the other hand, if there is at least one mode whose loading admittance can be arbitrarily chosen without producing any significant change in the sum of all the other mode terms in equation (2.31), it is possible to select the loading admittance for that particular mode in such a way as to cancel out the effect of all the other mode loading admittances unless, in equation (2.31), the sum of the terms involving $Y/(Y_{\ell m}+Y_{rm})$ for that particular mode vanishes. This can be achieved by means of a radial cavity (see Appendix B) with a load, Y_b at the inner radius b when kb is very small ($kb \ll 1$). Here, $Y_{\ell o}$ alone can be controlled by varying Y_b without creating any significant change in the loading admittance of the higher order modes.

If we assume that

$$\cos\beta\, T_{20}(\theta_o, \theta_i) S^r_{10}(\theta', \theta_o) \neq 0 \;, \tag{2.32}$$

then, by solving equation (2.31) for the zeroth mode loading admittance, we have

$$\left\{Y_{\ell o}\right\}_o = -Y_{ro} - \frac{2\pi a Y \sin\theta_o \cos\beta\, T_{20}(\theta_o,\theta_i) S^r_{10}(\theta',\theta_o)}{\sum_{m=0}^{\infty} S^s_{1m}(\theta',\theta_i)\cos\beta\cos m\phi' + \sum_{m=1}^{\infty}\left\{V_m S^r_{1m}(\theta',\theta_o)\cos\beta\cos m\phi' + \left[S'^s_{1m}(\theta',\theta_i) + V'_m S^r_{1m}(\theta',\theta_i)\right]\sin\beta\sin m\phi'\right\}}. \tag{2.33}$$

If the condition (2.32) is not satisfied, σ_θ cannot be controlled using $Y_{\ell o}$. The obvious example of this is the case when the direction of the incident wave is perpendicular to the plane of the slot so that there is no zeroth mode excitation (in fact, only the first mode excitation exists).

Similarly, if we assume that

$$\sin\beta\, T'_{20}(\theta_o,\theta_i) S^r_{20}(\theta',\theta_o) \neq 0 \ , \tag{2.34}$$

to make the ϕ-component of the cross section zero in the same direction ($\theta = \theta'$ and $\phi = \phi'$), we have from equation (2.26b)

$$\left\{Y_{\ell o}\right\}_o = -Y_{ro} - \frac{2\pi a Y \sin\theta_o \sin\beta\, T'_{20}(\theta_o,\theta_i) S^r_{20}(\theta',\theta_o)}{\sum_{m=0}^{\infty} S'_{2m}(\theta',\theta_i)\sin\beta\cos m\phi' + \sum_{m=1}^{\infty}\left\{V'_m S^r_{2m}(\theta',\theta_o)\sin\beta\cos m\phi' - \left[S^s_{2m}(\theta',\theta_i) + V_m S^r_{2m}(\theta',\theta_o)\right]\cos\beta\sin m\phi'\right\}} \ . \tag{2.35}$$

Of course, the θ-component and the ϕ-component cross sections cannot be reduced to zero simultaneously unless equations (2.33) and (2.35) yield the same value of $\left\{Y_{\ell o}\right\}_o$.

For the case of back scattering ($\theta = \theta_i$ and $\phi = \phi_i = 0$) the equations (2.26) become

$$\sigma_\theta = \frac{\lambda^2}{\pi} \left| S_1^s(0) + \sum_{m=0}^{\infty} V_m S_{1m}^r(\theta_i, \theta_o) \right|^2 \cos^2\beta \tag{2.36a}$$

$$\sigma_\phi = \frac{\lambda^2}{\pi} \left| S_1^s(0) + \sum_{m=0}^{\infty} V'_m S_{2m}^r(\theta_i, \theta_o) \right|^2 \sin^2\beta \tag{2.36b}$$

and by using the relations

$$aS_{10}^r(\theta_i, \theta_o) = \frac{(ka)^2}{2} \sin\theta_o \, T_{20}(\theta_o, \theta_i) \; , \tag{2.37a}$$

$$aS_{20}^r(\theta_i, \theta_o) = \frac{(ka)^2}{2} \sin\theta_o \, T'_{20}(\theta_o, \theta_i) \; , \tag{2.37b}$$

valid for $\epsilon \leqslant \theta_o \leqslant \pi - \epsilon$, $\epsilon \gg \delta$, it follows that under the essential assumption that

$$T_{20}(\theta_o, \theta_i) \neq 0$$

the zeroth mode loading required to make σ_θ zero is

$$\left\{ Y_{\ell o} \right\}_o = -Y_{ro} - \frac{\pi Y \left[ka \sin\theta_o \, T_{20}(\theta_o, \theta_i) \right]^2}{S_1^s(0) + \sum_{m=1}^{\infty} V_m S_{1m}^r(\theta_i, \theta_o)} \; , \tag{2.38a}$$

and it also follows that under the assumption

$$T'_{20}(\theta_o, \theta_i) \neq 0$$

the zeroth mode loading required to make σ_ϕ zero is

$$\left\{ Y_{\ell o} \right\}_o = -Y_{ro} - \frac{\pi Y \left[ka \sin\theta_o T'_{20}(\theta_o, \theta_i) \right]^2}{S_1^s(0) + \sum_{m=1}^{\infty} V'_m S_{1m}^r(\theta_i, \theta_o)} . \tag{2.38b}$$

Again, the total back scattering cross section cannot be reduced to zero except in certain selected cases. When the incident magnetic field is parallel to the plane of the slot ($\beta = 0$), σ_ϕ becomes zero regardless of the loading, and zero back scattering can be achieved by the loading given in equation (2.38a) if $T_{20}(\theta_o, \theta_i) \neq 0$. Similarly, when the incident electric field is parallel to the plane of the slot ($\beta = \pi/2$), it can be achieved by the loading given in equation (2.38b) if $T'_{20}(\theta_o, \theta_i) \neq 0$.

2.5 Low Frequency Approximations

The low frequency characteristics of the above formulae can be obtained by expanding each expression in a series of increasing positive powers of ka. If we consider only the case when $\theta_i = \theta_o = \pi/2$, $\phi_i = 0$, and $\beta = 0$, then from equation (2.38a) the expansion for the zeroth mode loading admittance required for zero back scattering is

$$\left\{ Y_{\ell o} \right\}_o \sim -iY\pi \left\{ \left[\sum_{n=3}^{\infty} \frac{2n+1}{n^2(n+1)} C_{no}^2(\pi/2) \right] ka \right.$$

$$\left. + \left[\sum_{n=1}^{\infty} \frac{2n+1}{n^3(n+1)(2n-1)} C_{no}^2(\pi/2) - \frac{131}{9} \right] (ka)^3 \right\} + Y\pi \frac{\delta^2}{2} (ka)^4 + O\left[(ka)^5\right] , \tag{2.39}$$

and for the corresponding loading admittance Y_b at $r = b$ we have, from (B.12),

$$\mathrm{Re} \left\{ Y_b \right\} \sim \frac{Y\pi\delta^2}{2} (ka)^4 + O\left[(ka)^5\right] \tag{2.40a}$$

$$\mathrm{Im}\left\{Y_b\right\} \sim \left[\sum_{n=3}^{\infty} \frac{2n+1}{n^2(n+1)} C_{no}^2(\pi/2)\right] Y\pi ka + O\left[(ka)^2\right] . \qquad (2.40b)$$

Furthermore, when the sphere is loaded to give zero back scattering, the equations (2.26) can be reduced to

$$\sigma_\theta(\theta, \phi) \sim \frac{\lambda^2}{4\pi} (ka)^6 (\sin\theta - \cos\phi)^2 \qquad (2.41a)$$

$$\sigma_\phi(\theta, \phi) \sim \frac{\lambda^2}{4\pi} (ka)^6 \cos^2\theta \sin^2\phi \qquad (2.41b)$$

and hence the total cross section is

$$\sigma(\theta, \phi) = \sigma_\phi(\theta, \phi) + \sigma_\phi(\theta, \phi) \sim \frac{\lambda^2}{4\pi} (ka)^6 \left[(\sin\theta - \cos\phi)^2 + \cos^2\theta \sin^2\phi\right] . \qquad (2.42a)$$

If we now express this in terms of the primed coordinate by transformation, we have

$$\sigma(\theta', \phi') \sim \frac{\lambda^2}{4\pi} (ka)^6 (1 - \cos\theta')^2 \qquad (2.42b)$$

or, by normalizing with respect to the back scattering cross section σ_o of an unloaded sphere,

$$\frac{\sigma(\theta', \phi')}{\sigma_o} \sim \frac{1}{9} (1 - \cos\theta')^2 \qquad (2.42c)$$

Comparison with the results given by Liepa and Senior (1966) shows that the scattering pattern is the same as when the slot is positioned in the plane perpendicular to the direction of incidence apart from a reduction in amplitude by a factor 4.

III
NUMERICAL COMPUTATIONS

A computer program was written to calculate either the loading impedances, Z_b (= $1/Y_b$), required for zero back scattering, or the relative back scattering cross section, $\sigma(0)/\sigma_o$, for a given value of the loading impedance when the slot is in the plane of the incident propagation vector and normal to the incident electric field vector ($\theta = \theta_i = \theta_o = 90^o$, $\phi = \phi_i = \phi = 0$ and $\beta = 0$).

In the formulas for computation, the functions $Y_{\ell m}$, Y_{rm}, $S_1^s(0)$, $T_{20}(\pi/2, \pi/2)$, V_m and $S_{1m}^r(\pi/2, \pi/2)$ are involved through equations (2.30a), (2.36a), (2.38a), (B.10) and (B.12). In all cases the infinite series were approximated by neglecting all the terms for m and n greater than M and N respectively, with the numbers M and N chosen sufficiently large for four digit accuracy. In practice the computation was restricted to the frequency range corresponding to $0 < ka \leqslant 3.0$, where the maximum value of N was 35 for $S_1^s(0)$ and $T_{20}(\pi/2, \pi/2)$, 100 for $Y_{rm}(m \neq 0)$, V_m and $S_{1m}^r(\pi/2, \pi/2)$, and 1080 for Y_{ro}; and throughout the computation the value of M was 5. The coefficients $C_{nm}(\pi/2)$ and $D_{nm}(\pi/2)$ in the expressions for Y_{rm} and $S_{1m}^r(\pi/2, \pi/2)$ were approximated by using equations (C.7) and (C.8) for $n \leqslant 15$ and by equation (C.9) and (C.10) for $n > 15$. To see how fast the series

$$\sum_{m=1}^{\infty} V_m S_{1m}^r(\pi/2, \pi/2)$$

converges, some typical values of V_m, $S_{1m}^r(\pi/2, \pi/2)$ and $V_m S_{1m}^r(\pi/2, \pi/2)$ were computed and these are shown in Table 3-1. It is seen that the summation of the terms only through $m = 5$ gives a good approximation to the infinite series in this case. More terms are, however, required for $ka > 3.0$.

If Fig. 3-1, the real and imaginary parts of Y_{rm}/Y are plotted as functions of ka, $0 < ka \leqslant 3.0$ for $m = 0$ to 5. The real parts are zero for ka = 0, and rise

TABLE 3-1: NUMERICAL VALUES OF $V_m S^r_{1m}(\pi/2, \pi/2)$

$Re[S^s_1(0)]$	$Im[S^s_1(0)]$	$Re[T_{20}(\pi/2, \pi/2)]$	$Im[T_{20}(\pi/2, \pi/2)]$
0.738229	-0.139549	-0.492003	-0.536931

For: ka = 1.9074
$Z_b = 0$
b/a = 0.05
$\delta = 0.0399$

m	$Re[V_m]$	$Im[V_m]$	$Re[S^r_{1m}(\pi/2, \pi/2)]$	$Im[S^r_{1m}(\pi/2, \pi/2)]$	$Re[V_m S^r_{1m}]$	$Im[V_m S^r_{1m}]$
1	0.552345	0.068610	0.628829	-1.048052	0.419237	0.535755
2	0.026019	0.010616	0.183804	-0.430829	0.009356	-0.009258
3	-4.317376×10^{-4}	0.012744	0.419609	0.015120	-0.000373	0.005341
4	-3.438531×10^{-3}	-7.298828×10^{-6}	-3.87993×10^{-4}	0.169496	0.000026	-0.000582
5	5.779219×10^{-8}	-6.995535×10^{-4}	-0.045191	-4.071185×10^{-6}	-5.5×10^{-9}	0.000031

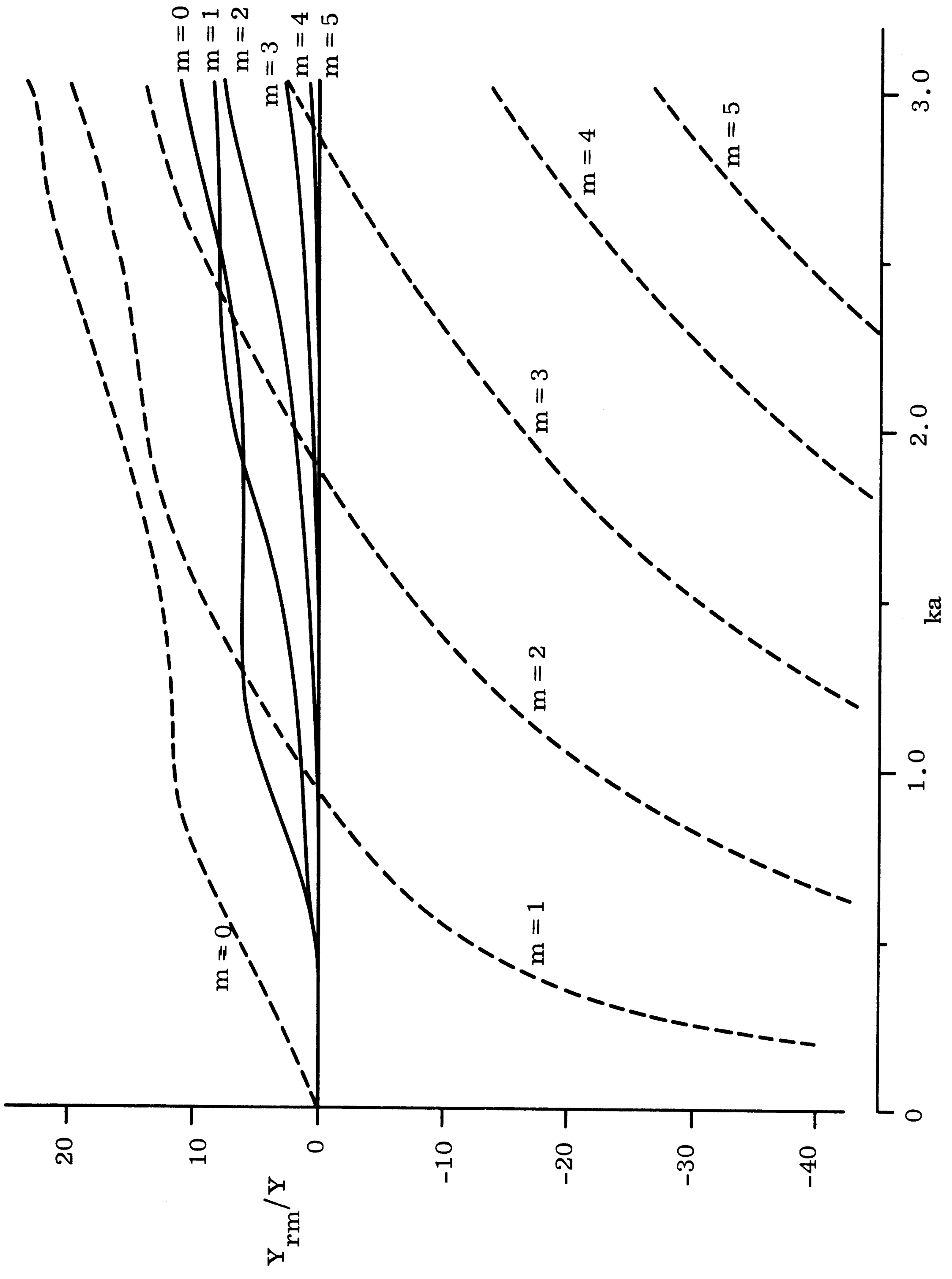

FIG. 3-1: REAL (——) AND IMAGINARY (– – –) PARTS OF NORMALIZED RADIATION ADMITTANCES FOR $\theta_o = 90°$ AND $\delta = 0.0399$.

through positive values with a small but regular oscillation as ka increases. On the other hand, the imaginary parts are O(ka) for m = 0 and $O\left[(ka)^{-1}\right]$ for $m \neq 0$ when ka is small, and also increase with a small regular oscillation as ka increases while their signs change from negative to positive at a value of ka near 0.95 m for the mth mode excitation.

In Table 3-2 the numerical values of Y_{rm} for ka = 2.0 with $\delta = 0.0349$ (approximately 2^o) and $\theta_o = 90^o$ are given for comparison with the results obtained by Mushiake and Webster (1957). Their values for $m > 0$ have been multiplied by a factor 2 to account for a difference in the definition of admittance and are shown in parentheses. Although the real parts are very close to each other, the imaginary parts differ by as much as 10 percent. It is believed that the present values are much closer to the exact ones, and that the discrepancies in Mushiake and Webster's data are attributable to their retention of an insufficient number of terms in the slowly converging series for $\text{Im}\left[Y_{rm}\right]$.

The real and imaginary parts of the optimum loading impedance, Z_{bo}, at R = b for zero back scattering, are shown in Figs. 3-2a and 3-2b, respectively, as function of ka for the range $0 \leqslant ka \leqslant 3.0$. The real part is zero for ka = 0 and increases slowly with ka up to ka = 1.6 (approx.). Above ka = 1.6 the curve for the real part is quite irregular in structure. It crosses the zero axis at ka = 1.685, 1.710 and 1.907, and as ka increases further above ka = 1.907, it decreases through negative values out to at least ka = 3.0. In the region where $\text{Re}\left[Z_{bo}\right] < 0$ it is apparent that zero back scattering is achievable only by using an active load. The curve for the imaginary part of Z_{bo} has a singularity of order $(ka)^{-1}$ at ka = 0, and decreases rapidly as ka increases, crossing the zero axis at ka = 0.845. This curve is somewhat similar to that obtained by negatively inverting the impedance of an L-C parallel circuit or transmission line.

Finally, the relative back scattering cross sections were computed for a certain range of Z_b within which a complete suppression of the back scattering

TABLE 3-2: RADIATION ADMITTANCES FOR ka =2.0, $\delta = 2^o$ AND $\theta_o = 90^o$

m	$Re\left[Y_{rm}\right] \times 10^3$	$Im\left[Y_{rm}\right] \times 10^{-3}$
0	16.60434 (16.601)	43.88532 (38.629)
1	17.96643 (17.9600)	37.42239 (34.310)
2	5.55220 (5.5502)	4.74812 (5.080)
3	1.58845 (1.5876)	-44.52784 (-39.516)
4	0.20727 (0.2072)	-103.57280 (-93.082)
5	0.01362 (0.0136)	-147.56936 (-134.756)

( ): Values given by Mushiake and Webster (1957).

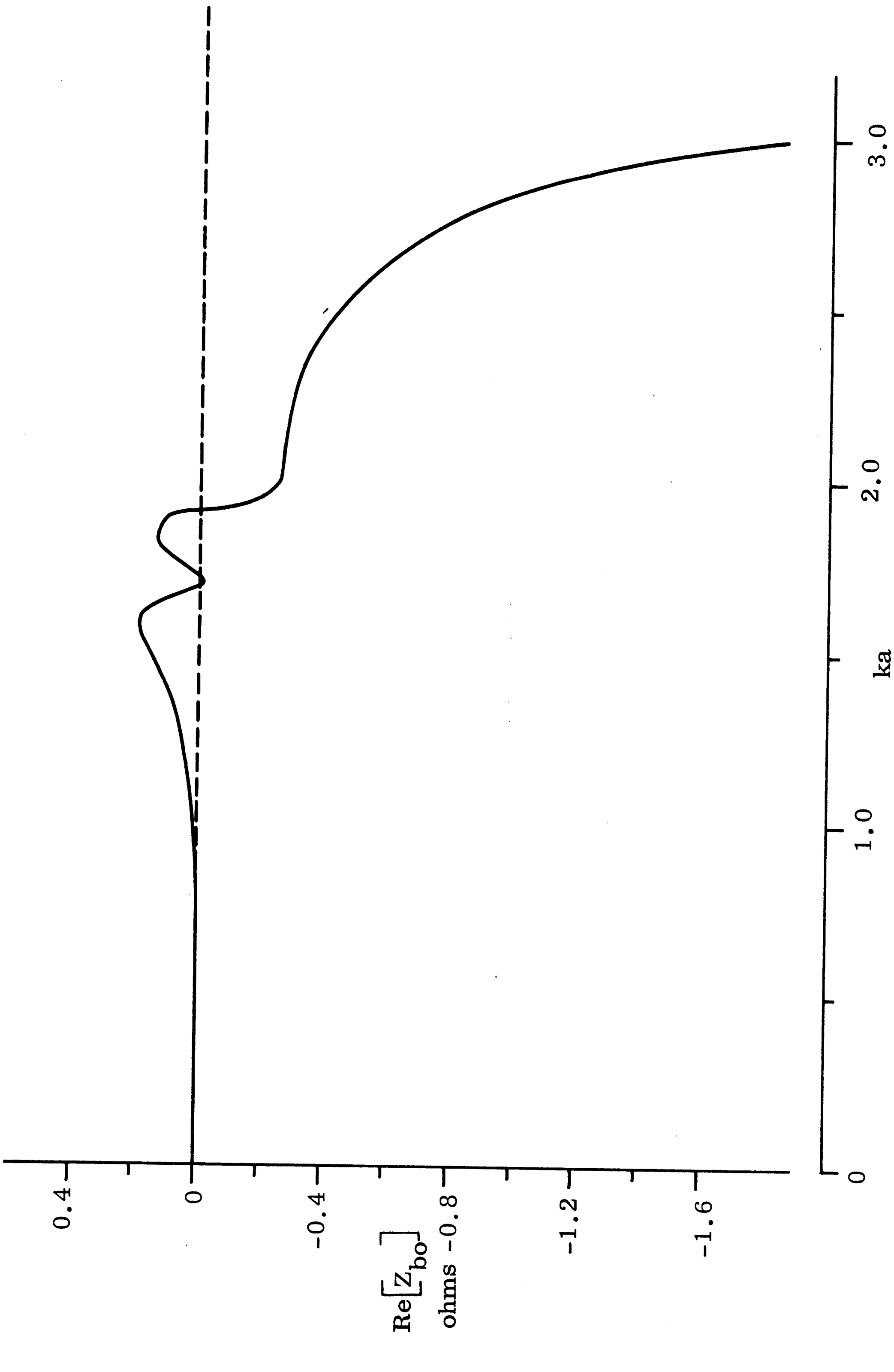

FIG. 3-2a: REAL PART OF LOADING IMPEDANCE FOR ZERO BACK SCATTERING WITH $\theta_o = \theta_i = 90^o$, $\phi_o = \phi_i = 0$ AND $\beta = 0$. ($b/a = 0.04$, $\delta = 0.0399$)

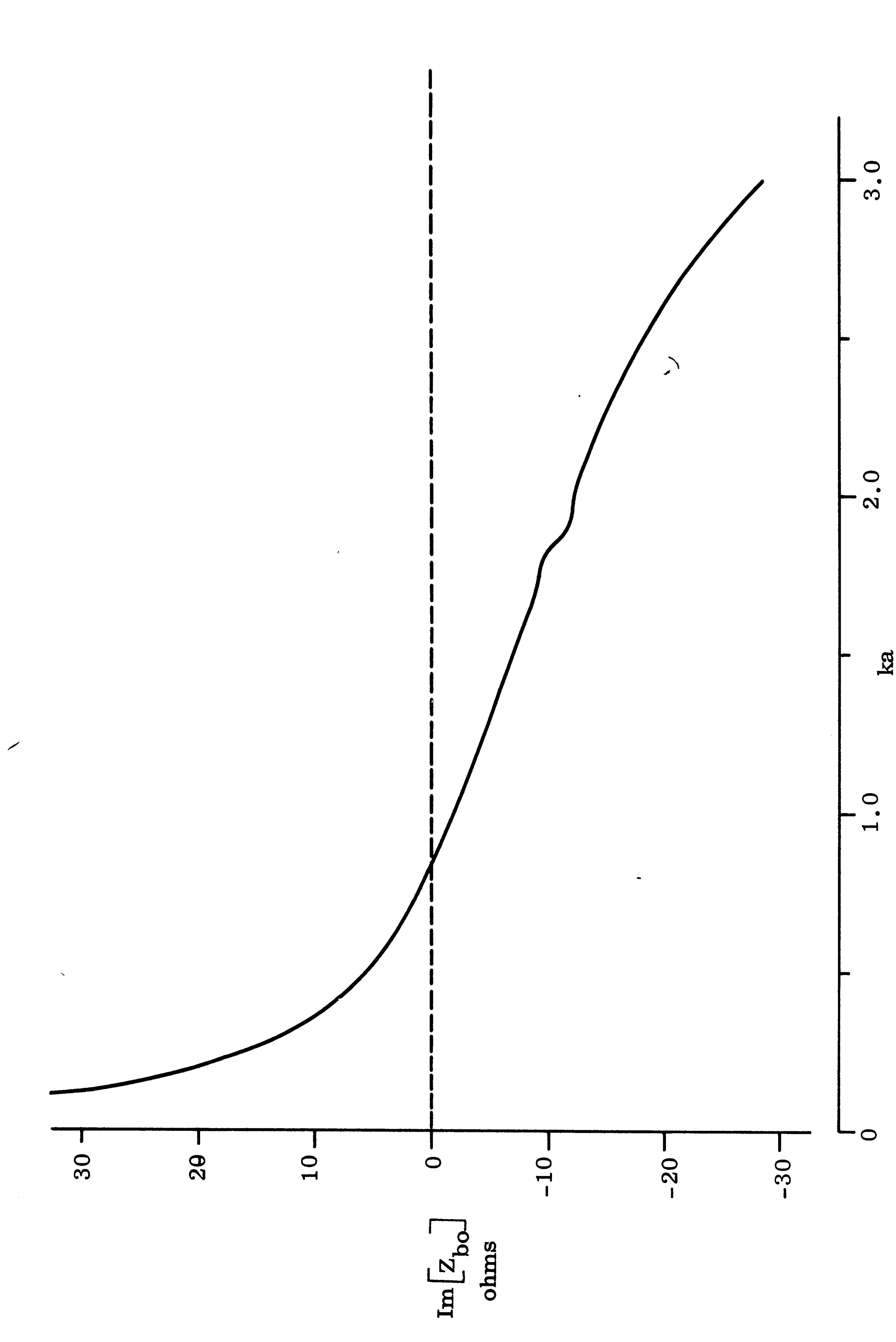

FIG. 3-2b: IMAGINARY PART OF LOADING IMPEDANCE FOR ZERO BACK SCATTERING WITH $\theta_o = \theta_i = 90^o$, $\phi_o = \phi_i = 0$ AND $\beta = 0$. (b/a = 0.04, $\delta = 0.0399$)

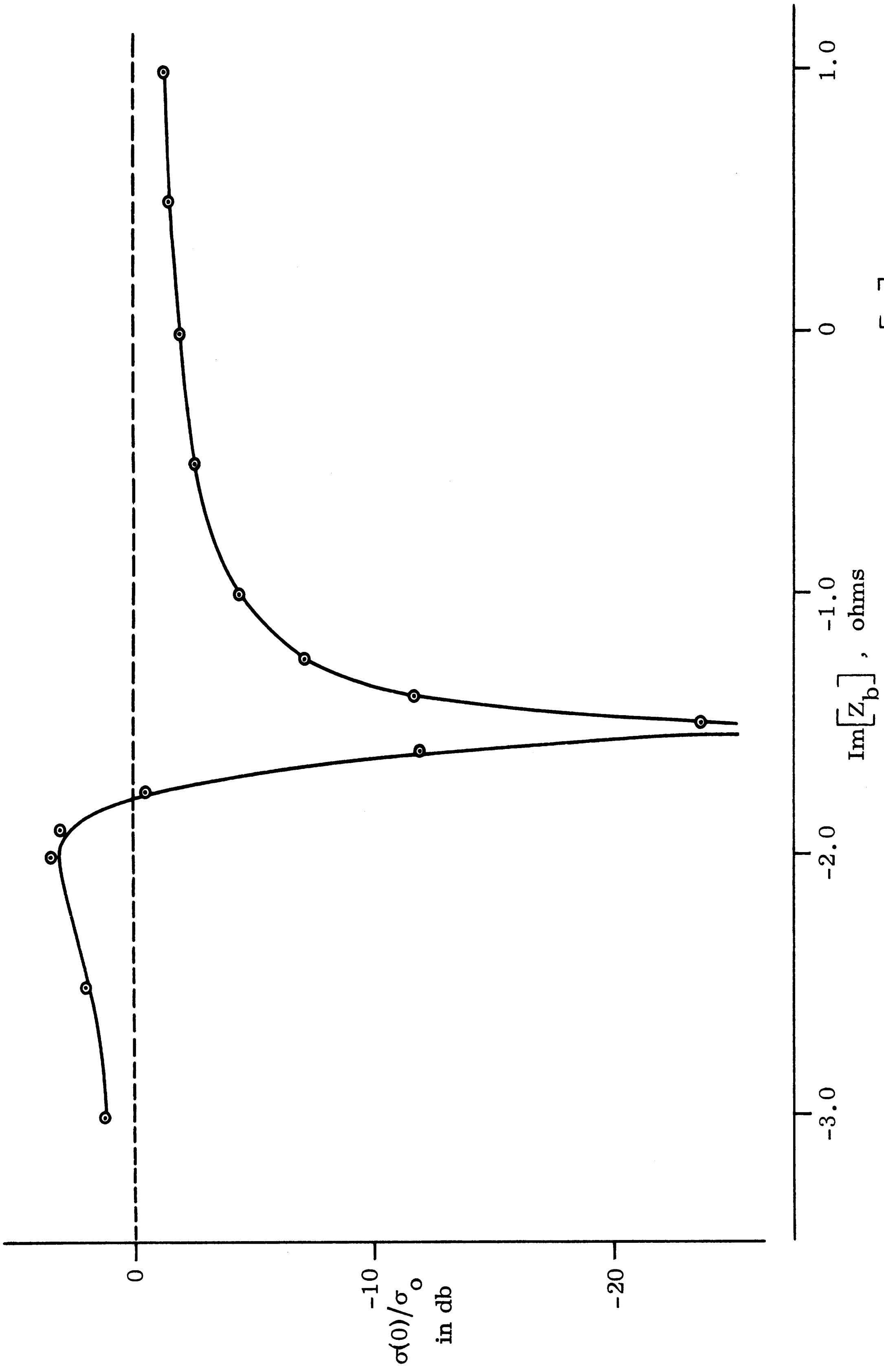

FIG. 3-3: RELATIVE BACK SCATTERING CROSS SECTION WITH Re$[Z_b]$ = 0.0129 Ω (——) AND Re$[Z_b]$ = 0 (⊙ ⊙ ⊙) FOR ka = 1.0, b/a = 0.04, δ = 0.0349 and $\theta_o = 90^o$. (Z_{bo} = 0.0129 - i1.5219 Ω)

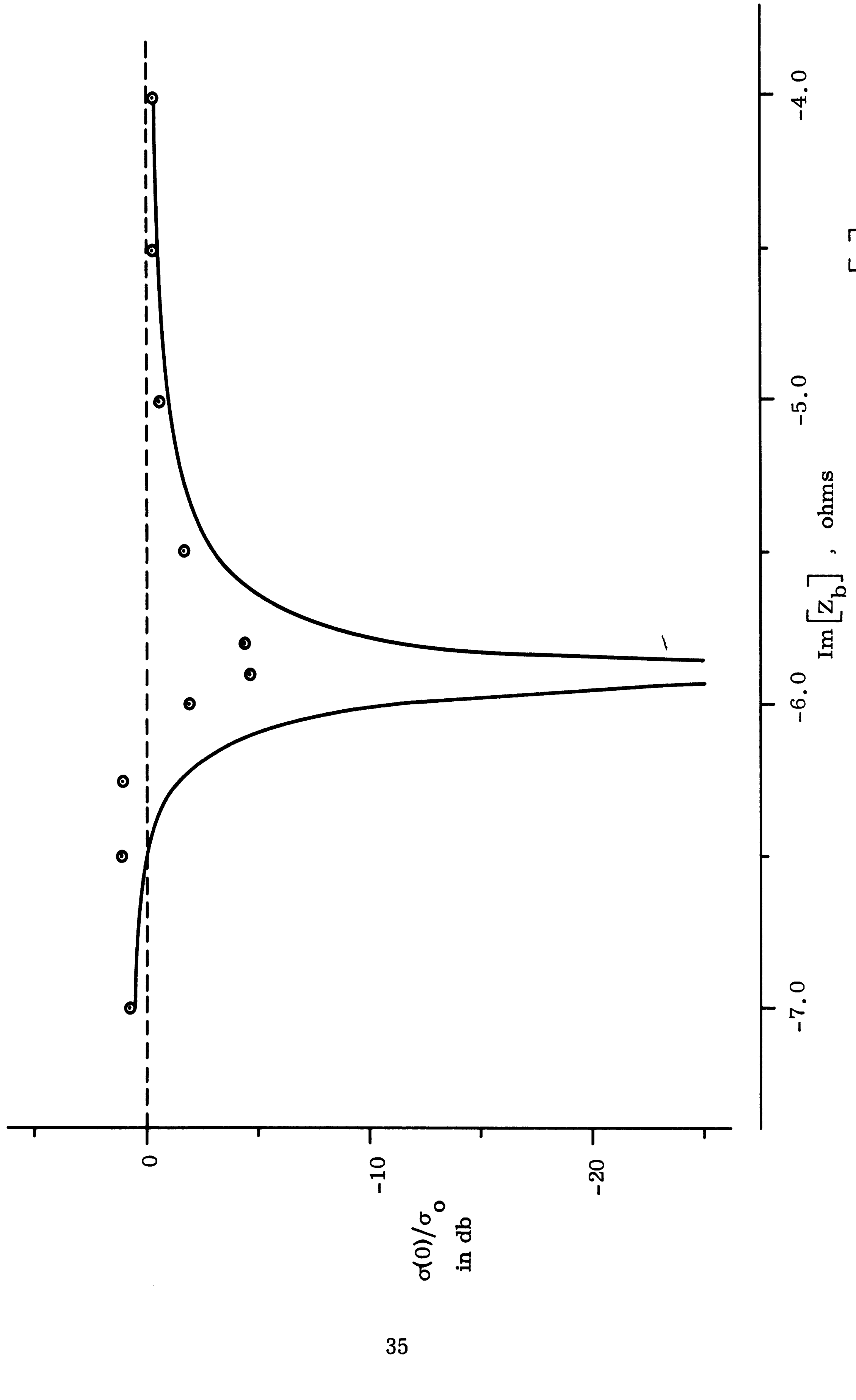

FIG. 3-4: RELATIVE BACK SCATTERING CROSS SECTION WITH $Re[Z_b] = 0.1263\,\Omega$ (——) AND $Re[Z_b] = 0$ (⊙ ⊙ ⊙) FOR ka = 1.5, b/a = 0.04, $\delta = 0.0349$ and $\theta_o = 90°$. ($Z_{bo} = 0.1263 - i5.9\text{-}16\,\Omega$).

field occurs. The curves are plotted in Figs. 3-3 and 3-4 for $ka = 1.0$ and $ka = 1.5$, respectively, as functions of the imaginary parts of Z_b while the real parts are fixed. The cross sections are so sensitive to the loading that only a one or two percent deviation of the loading impedance from the optimum, Z_{bo}, is sufficient to decrease the cross section reduction to only 20 db. Figure 3-3 shows that for $ka = 1.0$ neglecting $\mathrm{Re}\left[Z_{bo}\right] = 0.0129$, which is about 0.8 percent of $\left|Z_{bo}\right|$, has little effect. On the other hand, for the case $ka = 1.5$, only a 5 db reduction would be possible (see Fig. 3-4) if a purely reactive loading were used; that is, neglecting $\mathrm{Re}\left[Z_{bo}\right] = 0.1263$, which is about 2 percent of $\left|Z_{bo}\right|$.

IV
EXPERIMENT

To confirm the theoretical predictions, the back scattering cross sections were measured using the model constructed for the earlier experiments (Liepa and Senior, 1966). The model consists of two identical solid aluminum caps joined together by means of a partially threaded shaft at the center, but spaced 1/16 inch apart to form a radial cavity of the same width. The cavity is shorted at the center and the diameter of this short (2b) can be varied from 0.1253 inches to 3.133 inches (the diameter, 2a, of the sphere) by changing the size of disc employed.

The measurements were carried out using conventional cw equipment in an anechoic room. The distance from the antenna to the support pedestal was 24 feet and the model was placed on the pedestal with the plane of the slot perpendicular to the incident electric field (which was horizontally polarized).

The back scattering cross section was measured for a series of shorting discs at a frequency 2.2873 GHz, corresponding to ka = 1.9074, and the results, normalized relative to the cross section of the unloaded sphere,are given in Table 4-1. A comparison with the theoretical values, computed from equation (2.35) for $\theta_i = \theta_o = 90^{\circ}$ and $\beta = 0$ is shown in Fig. 4-1 and it will be observed that the agreement is excellent.

By using the above described model, shorted at the center of the cavity, $Y_{\ell m}$ is always purely susceptive and consequently it is not, in general, possible to obtain a complete cross section reduction at any frequency. For the experimental verification, however, it appeared reasonable to choose frequencies at which the back scattering could be made quite small using only susceptive loading even though perfect cancellation was not attainable. From Fig. 3-2a, b it is seen that four such choices of frequency are 1.0133, 2.0206, 2.0506 and 2.2873 GHz, corresponding to ka = 0.845, 1.685, 1.710 and 1.9074, respectively. At ka = 0.845 the imaginary

TABLE 4-1: EXPERIMENTAL DATA, BACK SCATTERING CROSS SECTION NORMALIZED WITH RESPECT TO THE UNSLOTTED SPHERE. (2a = 3.133", ka = 1.9074, f = 2.2873 GHz)

2b, inches	b/a	$\sigma(0)/\sigma_o(0)$, db
0.1253	0.0399	-8.1
0.3125	0.0991	-10.3
0.445	0.1420	-15.3
0.500	0.1596	-13.0
0.657	0.2097	+2.8
0.750	0.2394	+7.1
0.875	0.2793	+6.4
1.000	0.3192	+5.5
1.125	0.3591	+5.5
1.250	0.3990	-4.3
1.370	0.4389	-0.7
1.500	0.4788	+0.1
1.625	0.5187	+0.5
1.750	0.5586	+0.5
2.000	0.6384	+0.3
2.250	0.7182	+0.3
2.500	0.7980	+0.2
2.874	0.9177	+0.2
3.133	1.000	0

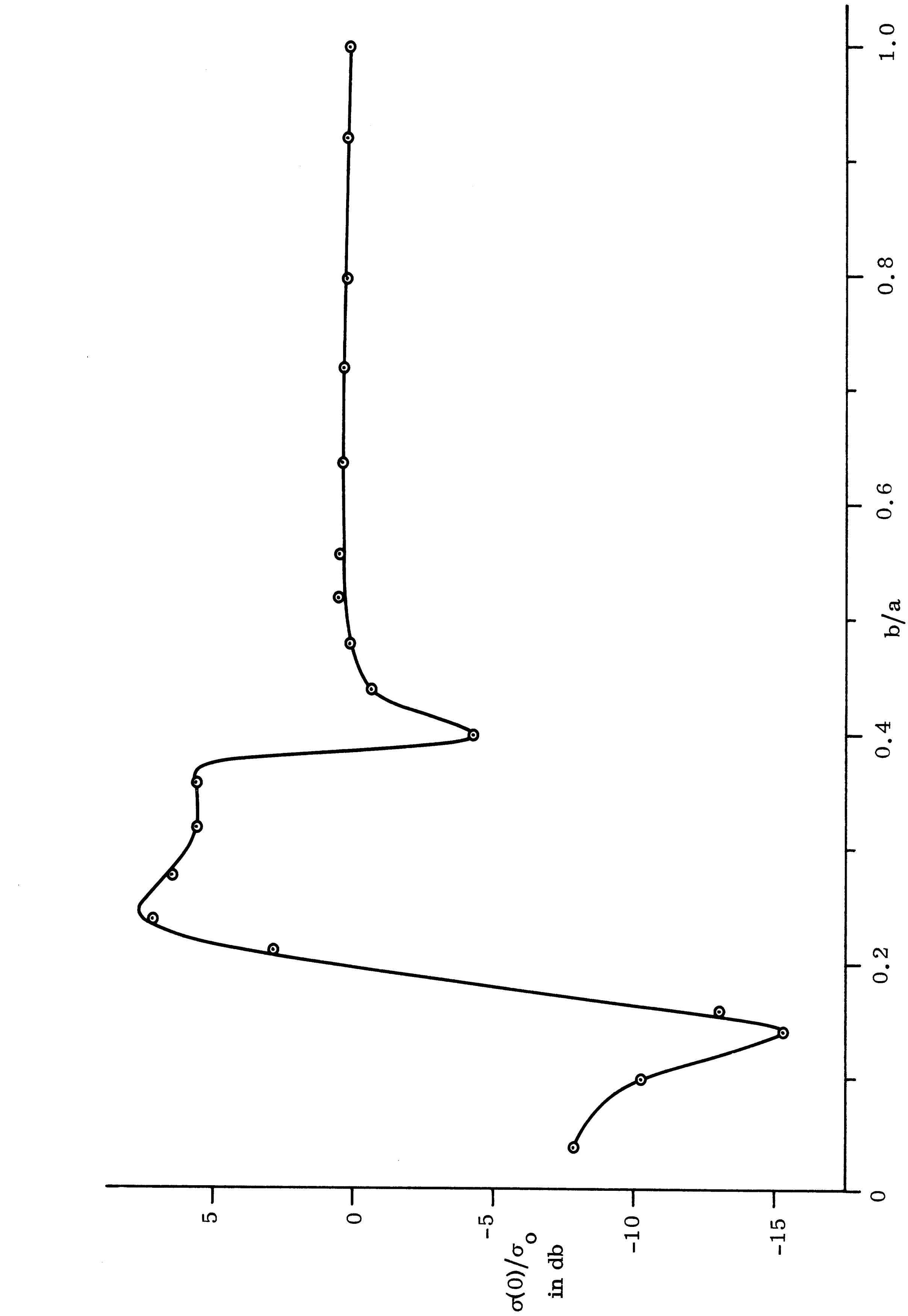

FIG. 4-1: THEORETICAL (——) AND EXPERIMENTAL (⊙ ⊙ ⊙) RELATIVE BACK SCATTERING CROSS SECTIONS WITH VARIOUS CAVITY DEPTHS: ka = 1.9074, $\theta_i = \theta_o = 90^o$ and $\delta = 0.0399$.

part of Z_{bo} crosses the zero axis and the real part is very close to zero, while at ka = 1.685, 1.710 or 1.9074 Z_{bo} is purely reactive and there is a possibility that Z_{bo} can be made zero (or almost zero) by choosing the cavity depth appropriately. Since the preferred frequencies for the available experimental facilities were above 2 GHz, the frequency f = 2.2873 GHz was selected for the experiment, although the minimum cross section obtainable at this frequency was only 15.2 db below that of the unslotted sphere. To achieve this requires b/a = 0.1420, as shown in Fig. 4-1. The incomplete cancellation of the cross section that then results is attributable to the fact that when the cavity depth increases in the region $b/a > 0.05$, not only $Y_{\ell o}$ changes, but also $Y_{\ell m}$, $m > 0$.

V
CONCLUSIONS

This report has been devoted to the study of the scattering behavior of a metallic sphere loaded with a narrow circumferential slot arbitrarily placed with respect to the direction of incidence. General expressions for the scattered far field components, the total surface field components and the scattered cross section components were derived as functions of the loading admittances and the position of the slot, and it was shown that the modification of the scattering behavior, primarily for zero back scattering, can be achieved by means of a lumped load at the center of a radial cavity backing the slot. For simplicity, emphasis has been placed on the case where the direction of the incident wave is such that its electric vector is perpendicular to an equatorial slot backed by a radial cavity, and the numerical computations have been limited to the frequency range corresponding to $0 < ka \leqslant 3.0$.

To confirm the theoretical predictions, the back scattering cross sections were measured using a metallic sphere with an equatorial slot backed by a radial cavity of adjustable depth. When the measured data were compared with the numerical data, excellent agreement was found. Unfortunately, complete reduction in cross section was not possible using this model since the optimum loading impedance, Z_{bo}, at the center of the cavity for zero back scattering is, in general, complex, while the model was shorted there. The minimum cross section obtained with the model was, however, about 15 db below that of the unloaded sphere.

In the realization of the required loading for zero back scattering over a certain frequency band, it is clear that simple passive loading alone will not suffice, and that more sophisticated loading techniques must be developed. The loading impedance required tends to have an imaginary part whose behavior as a function of frequency is more or less the reverse of what we would expect from a passive impedance, while its real parts are very small in comparison with its imaginary parts (see Fig. 3-2a, b). This suggests the use of active loading using negative impedance converters (NIC's). Another variational method would be to use an appropriate frequency-dependent medium in the cavity so as to modify the frequency-dependent characteristics of the optimum loading, and, hopefully, make them similar to those of a passive impedance. Some materials composed of dipolar molecules have their dielectric constant decreasing as the frequency increases in the region we are interested in (Smyth, 1955; M.I.T., 1945). The use of this kind of material medium would make the propagation constant ($k = \omega\sqrt{\mu\epsilon}$) not directly proportional to the frequency, but less dependent on it. Although it is obvious that the realization of the required loading by means of NIC's or by a material medium will be rather difficult, these approaches appear to have sufficient promise to warrent further investigation.

APPENDIX A
EXPANSION OF A VECTOR PLANE WAVE IN TERMS OF SPHERICAL VECTOR WAVE FUNCTIONS

Consider a plane wave represented by

$$\underline{E}^i = \hat{x}' e^{ikz'} \tag{A.1}$$

$$\underline{H}^i = -\hat{y}' Y e^{ikz'} \tag{A.2}$$

where a time factor $e^{i\omega t}$ has been suppressed and the coordinate system is as shown in Fig. 2-1. (We assume $\beta = 0$ for the time being.) Since the field must be finite at the origin, the expression for the above vector plane wave can be written in terms of the spherical vector functions of the first kind $\underline{M}^{(1)}_{\substack{e\\o}mn}$, $\underline{N}^{(1)}_{\substack{e\\o}mn}$ as

$$\underline{E}^i = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(A_{\substack{e\\o}mn} \underline{M}^{(1)}_{\substack{e\\o}mn} - i B_{\substack{e\\o}mn} \underline{N}^{(1)}_{\substack{e\\o}mn} \right) \tag{A.3}$$

$$\underline{H}^i = i Y \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(A_{\substack{e\\o}mn} \underline{N}^{(1)}_{\substack{e\\o}mn} - i B_{\substack{e\\o}mn} \underline{M}^{(1)}_{\substack{e\\o}mn} \right) \tag{A.4}$$

where $\underline{M}^{(1)}_{\substack{e\\o}mn}$ and $\underline{N}^{(1)}_{\substack{e\\o}mn}$ are given in Section 2.1.

To determine the coefficients $A_{\substack{e\\o}mn}$ and $B_{\substack{e\\o}mn}$ we multiply equation (A.3) by $\underline{M}_{\substack{e\\o}m'n'}$ and $\underline{N}_{\substack{e\\o}m'n'}$, respectively and integrate over the surface R = constant. By expressing equation (A.1) in terms of the spherical polar coordinates (R, θ, ϕ) (Stratton, 1941):

$$\underline{E}^i = \left\{ \left[\cos\theta_i \sin\theta \cos(\phi - \phi_i) - \sin\theta_i \cos\theta \right] \hat{R} + \left[\cos\theta_i \cos\theta \cos(\phi - \phi_i) + \sin\theta_i \sin\theta \right] \hat{\theta} - \left[\cos\theta_i \sin(\phi - \phi_i) \right] \hat{\phi} \right\} \left\{ \sum_{n=0}^{\infty} i^n (2n+1) j_n(kR) \left[P_n(\cos\theta_i) P_n(\cos\theta) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_i) P_n^m(\cos\theta) \cos m(\phi - \phi_i) \right] \right\}, \quad (A.5)$$

and by using the orthogonality relations for the trigonometric and Legendre functions, followed by the recurrence relations for the latter, we obtain

$$\int_0^{\pi} \int_0^{2\pi} \underline{E}^i \cdot \underline{M}_{\substack{e \\ o}m'n'} \sin\theta \, d\phi d\theta = \mp i^{n'} 4\pi \frac{m' P_{n'}^{m'}(\cos\theta_i)}{\sin\theta_i} \begin{matrix} \sin \\ \cos \end{matrix} m'\phi_i \cdot \left[j_{n'}(kR) \right]^2 \quad (A.6)$$

$$\int_0^{\pi} \int_0^{2\pi} \underline{E}^i \cdot \underline{N}_{\substack{e \\ o}m'n'} \sin\theta \, d\phi d\theta = i^{n'} 4\pi \frac{\partial P_{n'}^{m'}(\cos\theta_i)}{\partial\theta_i} \begin{matrix} \cos \\ \sin \end{matrix} m'\phi_i \cdot \frac{1}{2n'+1} \left\{ (n'+1) \left[j_{n'-1}(kR) \right]^2 + n' \left[j_{n'+1}(kR) \right]^2 \right\}. \quad (A.7)$$

On the other hand, from the right hand side of equation (A.3), by using the orthogonal relations of $\underline{M}$, $\underline{N}$ functions, we have

$$\int_0^{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(A_{\substack{e\\o}mn}\underline{M}^{(1)}_{\substack{e\\o}mn} - iB_{\substack{e\\o}mn}\underline{N}^{(1)}_{\substack{e\\o}mn}\right)\cdot\underline{M}_{\substack{e\\o}m'n'}\sin\theta\, d\phi d\theta$$

$$= A_{\substack{e\\o}m'n'}(1+\delta_o)\frac{2\pi(n'+m')!}{(2n'+1)(n'-m')!}n'(n'+1)\left[j_{n'}(kR)\right]^2 \tag{A.8}$$

$$\int_0^{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(A_{\substack{e\\o}mn}\underline{M}_{\substack{e\\o}mn} - iB_{\substack{e\\o}mn}\underline{N}_{\substack{e\\o}mn}\right)\cdot\underline{N}_{\substack{e\\o}m'n'}\sin\theta\, d\phi d\theta$$

$$= B_{\substack{e\\o}m'n'}(1+\delta_o)\frac{2\pi(n'+m')!}{(2n'+1)^2(n'-m')!}\, n'(n'+1)\left\{(n'+1)\left[j_{n'}(kR)\right]^2 + n'\left[j_{n'+1}(kR)\right]^2\right\} \tag{A.9}$$

where

$$\delta_o = \begin{cases} 0\,, & m \neq 0 \\ 1\,, & m = 0 \end{cases} .$$

By equating equations (A.6) and (A.8), and (A.7) and (A.9), we then have

$$A_{\substack{e\\o}mn} = \mp i^n \frac{\epsilon_m(2n+1)(n-m)!}{n(n+1)(n+m)!}\begin{matrix}\sin\\ \cos\end{matrix} m\phi_i \frac{mP_n^m(\cos\theta_i)}{\sin\theta_i}\,, \tag{A.10}$$

$$B_{\substack{e\\o}mn} = i^n \frac{\epsilon_m(2n+1)(n-m)!}{n(n+1)(n+m)!}\begin{matrix}\cos\\ \sin\end{matrix} m\phi_i \frac{\partial P_n^m(\cos\theta_i)}{\partial\theta_i}\,, \tag{A.11}$$

where

$$\epsilon_m = \begin{cases} 1\,, & m = 0 \\ 2\,, & m \neq 0 \end{cases} .$$

Now, for an arbitrarily polarized plane wave whose electric field vector makes an angle β with respect to the x'-axis, its field can be expressed as

$$\underline{E}^i = (\hat{x}' \cos\beta + \hat{y}' \sin\beta)\, e^{ikz'} \quad , \tag{A.12}$$

$$\underline{H}^i = -iY(-\hat{x}' \sin\beta + \hat{y}' \cos\beta)\, e^{ikz'} \quad . \tag{A.13}$$

Hence, using the relations given in equations (A.1) through (A.4), we finally obtain

$$\underline{E}^i = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left\{ \left[\cos\beta\, A_{\substack{e\\o}mn} - \sin\beta\, B_{\substack{e\\o}mn} \right] \underline{M}^{(1)}_{\substack{e\\o}mn} - i \left[\sin\beta\, A_{\substack{e\\o}mn} + \cos\beta\, B_{\substack{e\\o}mn} \right] \underline{N}^{(1)}_{\substack{e\\o}mn} \right\} \quad , \tag{A.14}$$

$$\underline{H}^i = iY \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left\{ \left[\cos\beta\, A_{\substack{e\\o}mn} - \sin\beta\, B_{\substack{e\\o}mn} \right] \underline{N}^{(1)}_{\substack{e\\o}mn} - i \left[\sin\beta\, A_{\substack{e\\o}mn} + \cos\beta\, B_{\substack{e\\o}mn} \right] \underline{M}^{(1)}_{\substack{e\\o}mn} \right\} \quad . \tag{A.15}$$

APPENDIX B
THE INPUT ADMITTANCE OF A RADIAL CAVITY

Consider a radial transmission line with input at its outer radius $(r = a)$

$$V = V_m \cos m\phi \ , \qquad \text{(m integer)} \tag{B.1}$$

and with load admittance at its inner radius $(r = b)$ where the load admittance is assumed uniformly distributed about the circumference as shown in Fig. B-1. If we assume $kd \ll 1$ and $d \ll a$, the field components are

$$H_z = E_r = E_\phi = 0$$

and

$$E_z(r,\phi) = \left\{A J_m(kr) + B N_m(kr)\right\} \cos m\phi \tag{B.2}$$

$$H_\phi(r,\phi) = -iY \left\{A J'_m(kr) + B N'_m(kr)\right\} \cos m\phi \tag{B.3}$$

$$H_r(r,\phi) = -i \frac{Ym}{kr} \left\{A J_m(kr) + B N_m(kr)\right\} \sin m\phi \tag{B.4}$$

(Ramo and Whinnery, 1944) where $J_m(kr)$ and $N_m(kr)$ are cylindrical Bessel functions of the first and second kinds respectively.

To determine the constants A and B, the boundary conditions at the inner and outer surfaces must be considered.

1. Inner Boundary Condition:

By using the same definition of the admittance as given in section 2.3, the load admittance density at $r = a$ is expressed as

$$y_b = \frac{2\widetilde{W}_b}{\left[V(b)\right]^2} = \frac{H_\phi\big|_{r=b}}{dE_z\big|_{r=b}} \ , \tag{B.5}$$

where

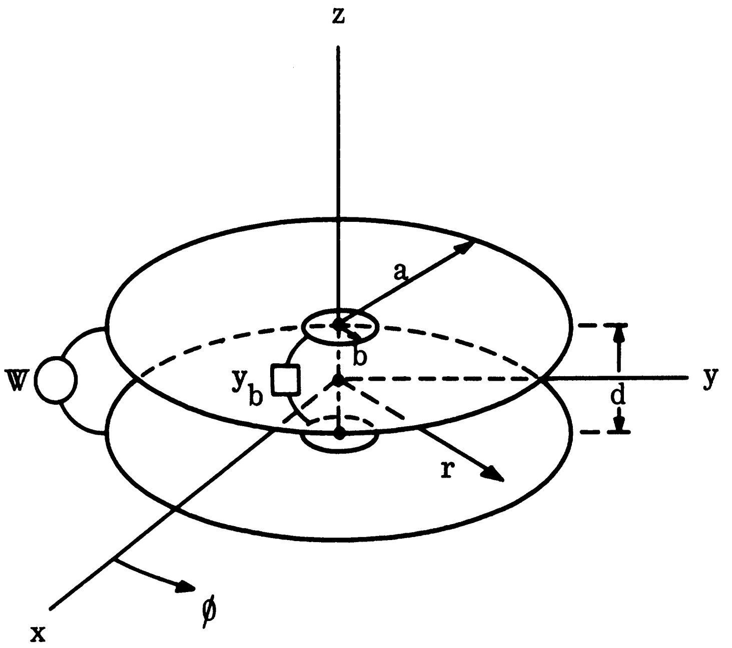

FIG. B-1: GEOMETRY OF A RADIAL CAVITY.

$$W_b = -\frac{1}{2}\int_{-d/2}^{d/2} (\underline{E} \times \underline{\tilde{H}}) \cdot \hat{r}\, dz \Big|_{r=b}$$

and

$$V(b) = -d E_z \Big|_{r=b} \quad ,$$

from which we have

$$\left[Y_b J_m(kb) + \beta J'_m(kb)\right] A + \left[Y_b N_m(kb) + \beta N'_m(kb)\right] B = 0 \tag{B.6}$$

where $Y_b = 2\pi b y_b$ is the total load admittance at $r = b$, and

$$\beta = iY \frac{2\pi b}{d} \quad .$$

2. Outer Boundary Condition:

At the outer radius, $r = a$, equation (B.1) gives

$$E_z \Big|_{r=a} = E_a \cos m\phi$$

where

$$E_a = -V_m/d \, .$$

Hence, from equation (B.2),

$$A J_m(ka) + B N_m(ka) = E_a \tag{B.7}$$

After determining the coefficients A and B from equations (B.6) and (B.7), substitution into equations (B.2) and (B.3) gives

$$E_z = \frac{J_m(kr)\left[Y_b N_m(kb) + \beta N'_m(kb)\right] - N_m(kr)\left[Y_b J_m(kb) + \beta J'_m(kb)\right]}{J_m(ka)\left[Y_b N_m(kb) + \beta N'_m(kb)\right] - N_m(ka)\left[Y_b J_m(kb) + \beta J'_m(kb)\right]} E_a \cos m\phi, \tag{B.8}$$

$$H_\phi = -iY \frac{J'_m(kr)\left[Y_b N_m(kb) + \beta N'_m(kb)\right] - N'_m(kr)\left[Y_b J_m(kb) + \beta J'_m(kb)\right]}{J_m(ka)\left[Y_b N_m(kb) + \beta N'_m(kb)\right] - N_m(ka)\left[Y_b J_m(kb) + \beta J'_m(kb)\right]} E_a \cos m\phi . \tag{B.9}$$

Analogously to equation (B.5), the input admittance density at $r = a$ is

$$y_{\ell m} = \frac{H_\phi \big|_{r=a}}{d E_z \big|_{r=a}} ,$$

and thus the total input admittance is

$$Y_{\ell m} = 2\pi a y_{\ell m} = -\alpha \frac{J'_m(ka)\left[Y_b N_m(kb) + \beta N'_m(kb)\right] - N'_m(ka)\left[Y_b J_m(kb) + \beta J'_m(kb)\right]}{J_m(ka)\left[Y_b N_m(kb) + \beta N'_m(kb)\right] - N_m(ka)\left[Y_b J_m(kb) + \beta J'_m(kb)\right]} \tag{B.10}$$

where

$$\alpha = iY \frac{2\pi a}{d} .$$

When kb is very small the asymptotic expression for $Y_{\ell m}$ is as follows:

$$Y_{\ell m} \simeq \begin{cases} -\alpha \dfrac{1}{J_o(ka)} \left\{ J'_o(ka) - \dfrac{1}{J_o(ka)\, ka\, \ell n(kb)} \left[1 + \dfrac{N_o(ka)\pi}{J_o(ka)\, 2\, \ell n(kb)} - i \dfrac{2\pi Y}{kd\, Y_b\, \ell n(kb)} \right] \right\} , & \text{for } m = 0 \\[2ex] -\alpha \dfrac{1}{J_m(ka)} \left\{ J'_m(ka) + (2/ka) \dfrac{1}{m\left[(m-1)!\right]^2} \dfrac{kbY_b + i\, 2\pi m Y}{kbY_b - i 2\pi m Y} \left(\dfrac{1}{2} kb\right)^{2m} \right\} . & \text{for } m \geqslant 1 \end{cases}$$

The above expression shows that when kb approaches zero, the term involving Y_b approaches zero as $\left[1/\ln kb\right]^2$ and $(kb)^{2m}$ for $m=0$ and $m \geqslant 1$ respectively.* Therefore the proper choice of kb would make it possible to change $Y_{\ell o}$ within some range by varying Y_b without changing significantly the higher mode admittance values. Moreover, if kb is sufficiently small, we may consider the loading admittance Y_b at $r=b$ as a lumped one.

Solving equation (B.10) for Y_b in terms of $Y_{\ell o}$ we obtain

$$Y_b = -\beta \frac{N_o'(kb)\left[Y_{\ell o}J_o(ka) + \alpha J_o'(ka)\right] - J_o'(kb)\left[Y_{\ell o}N_o(ka) + \alpha N_o'(ka)\right]}{N_o(kb)\left[Y_{\ell o}J_o(ka) + \alpha J_o'(ka)\right] - J_o(kb)\left[Y_{\ell o}N_o(ka) + \alpha N_o'(ka)\right]} \tag{B.12}$$

and in particular, for kb small,

$$Y_b \simeq -iY\frac{2\pi}{kd\ln(kd)}\left[1 + \frac{\pi}{2\ln(kb)}\,\frac{Y_{\ell o}N_o(ka) + \alpha N_o'(ka)}{Y_{\ell o}J_o(ka) + \alpha J_o'(ka)}\right] \tag{B.13}$$

while the higher mode input admittances are given by

$$Y_{\ell m} \simeq -\alpha\frac{J_m'(ka)}{J_m(ka)} \quad . \tag{B.14}$$

To see the feasibility of the above approximations when kb is small, the input admittances for several modes have been computed using the exact formula (B.10) for $(b/a) = 0.05(0.05)0.95$ with $ka = 1.9074$, $Y/Y_b = \frac{10^{-5}}{120\pi} + i0$ and $\delta = 0.0399$, and are plotted in Fig. B-2. It will be observed that the higher the modal

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*Note that $(1/\ln kb)^2$ approaches zero more slowly than kb;

$$\lim_{x\to 0} \frac{x}{(1/\ln x)^2} = \lim_{x \to 0} 2x = 0 \ .$$

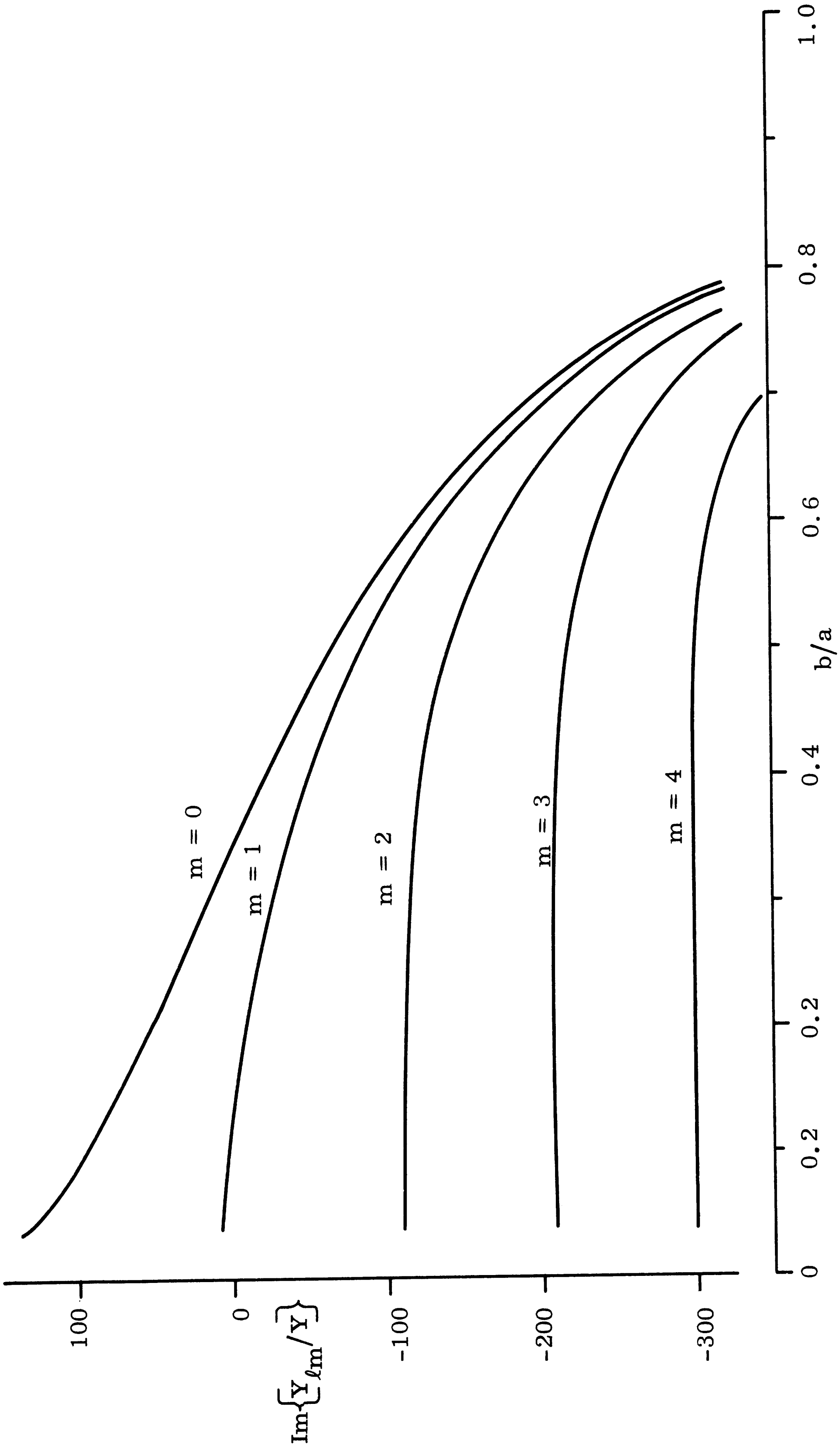

FIG. B-2: NORMALIZED INPUT ADMITTANCE OF A RADIAL CAVITY FOR ka = 1.9074, $Y/Y_b = 10^{-5}/120\pi + i0$, AND $\delta = 0.399$

number, the slower the variation of the admittance as a function of b/a. In particular, for b/a near 0.05, $Y_{\ell o}$ changes rapidly while the others remain almost constant, and when the cavity depth approaches zero ($b/a \rightarrow 1.0$) $Y_{\ell m}$ approaches $-i\infty$ for all m, as expected.

If the cavity were filled with a medium of refractive index n, the expressions for the corresponding $Y_{\ell m}$ and Y_b would follow immediately from the above equations on replacing k by nk and Y by the intrinsic admittance of the medium. Thus, for real n, numerical values can be obtained by scaling those for an air-filled cavity.

APPENDIX C
EVALUATION OF $C_{nm}(\theta_o)$ AND $D_{nm}(\theta_o)$

From equation (2.17a) we have

$$C_{nm}(\theta_o) = \int_{\theta_o - \frac{\delta}{2}}^{\theta_o + \frac{\delta}{2}} \sin\theta \, \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \, d\theta \, . \tag{C.1}$$

If we limit the position of the slot to be such that $\epsilon \lessdot \theta_o \lessdot \pi - \epsilon$, with $\epsilon \gg \delta$, an adequate approximation to the above is

$$C_{nm}(\theta_o) = \frac{\sin\theta_o}{\delta} \left\{ P_n^m\left[\cos(\theta_o + \frac{\delta}{2})\right] - P_n^m\left[\cos(\theta_o - \frac{\delta}{2})\right] \right\} \, . \tag{C.2}$$

For the case in which $n \gg m$, each Legendre function can then be replaced by the leading term of its asymptotic expansion, viz.

$$P_n^m(\cos\theta) \sim (-n)^m \sqrt{\frac{2}{n\pi\sin\theta}} \cos\left[(n+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right] \, ,$$

to give

$$C_{nm}(\theta_o) \sim -(-n)^m \sqrt{\frac{2n\sin\theta_o}{\pi}} \sin\theta\left[(n+\frac{1}{2})\theta_o - \frac{\pi}{4} + \frac{m\pi}{2}\right] \frac{\sin\frac{n\delta}{2}}{\frac{n\delta}{2}} \, . \tag{C.3}$$

Similarly, from equation (2.17b),

$$D_{nm}(\theta_o) = \frac{1}{\delta} \int_{\theta_o - \frac{\delta}{2}}^{\theta_o + \frac{\delta}{2}} m P_n^m(\cos\theta) d\theta \, , \tag{C.4}$$

but unfortunately an approximation analogous to that given for $C_{nm}(\theta_o)$ in equation (C.2) is not possible here. There are, however, the following asymptotic approximations to $D_{nm}(\theta_o)$:

$$D_{nm}(\theta_o) \sim mP_n^m(\cos\theta_o) \tag{C.5}$$

valid for $n\delta \ll \pi$, and

$$D_{nm}(\theta_o) \sim (-n)^m m\sqrt{\frac{2}{n\pi\sin\theta_o}}\cos\left[(n+\frac{1}{2})\theta_o - \frac{\pi}{4} + \frac{m\pi}{2}\right]\frac{\sin\frac{n\delta}{2}}{\frac{n\delta}{2}} \tag{C.6}$$

valid for $n \gg m$, and these are quite effective for most numerical purposes. Indeed, for $n \leqslant 15$, the maximum error in using equation (C.5) is only about one percent, and this is true also for equation (C.6) when $n > 15$ with $m \leqslant 5$. Not surprisingly, the error is greatest in the crossover region.

In the particular case $\theta_o = \pi/2$ equations (C.2) and (C.5) reduce to

$$C_{nm}(\pi/2) = \begin{cases} 0 & , \quad (n-m) \text{ even} \\ \frac{2}{\delta}P_n^m\left[\cos(\frac{\pi}{2}+\frac{\delta}{2})\right] & , \quad (n-m) \text{ odd}, \; n\delta \ll \pi \end{cases} \tag{C.7}$$

$$D_{nm}(\pi/2) = \begin{cases} 0 & , \quad (n-m) \text{ odd} \\ (-1)^{\frac{n-m}{2}} m \dfrac{(n+m-1)(n+m-3)\ldots 3\cdot 1}{(n-m)(n-m-2)\ldots 4\cdot 2} & , \quad (n-m) \text{ even}, \; n\delta \ll \pi \end{cases} \tag{C.8}$$

respectively, and equations (C.3) and (C.6) reduce to

$$C_{nm}(\pi/2) \simeq \begin{cases} 0 & , \quad (n-m) \text{ even} \\ (-1)^{\frac{n-m+1}{2}} n^m \sqrt{\frac{2n}{\pi}} \dfrac{\sin \frac{n\delta}{2}}{\frac{n\delta}{2}} & , \quad (n-m) \text{ odd}, \; n \gg m \end{cases} \tag{C.9}$$

$$D_{nm}(\pi/2) \simeq \begin{cases} 0 & , \quad (n-m) \text{ odd} \\ (-1)^{\frac{n-m}{2}} m n^m \sqrt{\frac{2}{n\pi}} \dfrac{\text{din} \frac{n\delta}{2}}{\frac{n\delta}{2}} & , \quad (n-m) \text{ even}, \; n \gg m \end{cases} \tag{C.10}$$

respectively.

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13. ABSTRACT

The electromagnetic scattering behavior of a metallic sphere loaded with a circumferential slot arbitrarily placed with respect to the direction of incidence is studied. Under the assumption that the slot is of small but nonzero width with a constant electric field across it, the analysis for the extermal fields is exact. Expressions for the scattered far field components, as well as for the total surface field components, are derived and then used to investigate the extent to which the scattering behavior can be controlled by varying the loading admittances and the slot position. An explicit formula for the loading of the zeroth mode to annul the back scattering cross section is derived, and from this the desired loading is obtained by means of a lumped load at the center of a radial cavity backing the slot. In particular, emphasis is placed on the case where the slot is in the plane of incidence and normal to the direction of the incident electric vector.

The numerical study is limited to the frequency range $0 < ka < 3.0$, where a is the radius of the sphere, and results are presented primarily for back scattering. To verify some of these results, a comparison is made with experimental data obtained using a metallic sphere with an equatorial slot backed by a radial cavity of adjustable depth.

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