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SCATTERING BY A SPHERICAL SHELL WITH A CIRCULAR APERTURE

by

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ABSTRACT

A preliminary study of the scattering behavior of a spherical shell with a circular aperture is presented. The shell is assumed to be perfectly conducting and infinitesimally thin, and is illuminated by a plane electromagnetic wave symmetrically incident upon the aperture. Several different approaches to the solution of this problem are discussed, and the most promising one, the method of least square error, is described in detail. A numerical approach based on this scheme is devised, and values for the surface and back scattered far fields are given. In effect, the aperture and cavity provide a reactive load which modifies the scattering behavior of the complete shell (or sphere), and an example shows the cross section reduction achievable in this manner. Experimental confirmation was obtained, and the study is continuing.

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I
INTRODUCTION

In the general area of electromagnetic scattering, impedance loading has received much attention as one of the more promising techniques for radar cross section control, especially for the reduction of cross sections in the resonance region. In recent years a number of studies have been made of impedance loading applied to simple bodies such as the sphere (Liepa and Senior, 1964, 1966; Chang and Senior, 1967) and cylinder (Chen and Liepa, 1964a, 1964b; Chen, 1965a, 1965b; Sletten et al, 1964). From all of these studies, it would appear that the realization of the loading required to give (for example) zero backscattering cross section over a significant frequency range is very difficult to attain due to the peculiar frequency characteristics of the required loading.

The present work is directed at the problem of an infinitely thin spherical shell with a circular aperture of arbitrary size cut into the shell. In effect, therefore, we have a sphere loaded with a spherical cavity coupled through a circular aperture. By loading with such a cavity, it seems not unreasonable to expect that the bandwidth characteristics will be broader than in the cases studied by Liepa and Senior (1964, 1966) or Chang and Senior (1967) where the sphere was loaded with a narrow circumferential slot, which has an inherently narrow bandwidth. Moreover, it is felt that the loading with a spherical cavity is particularly appropriate to the present problem since the fields inside and outside the spherical shell are separable with the same spherical coordinate system, and hence the reasonance characteristics of the cavity and sphere are somewhat similar to each other.

The problem of a perfectly conducting spherical shell with a conical hole, which degenerates to our case when the shell thickness approaches zero, has already been rigorously formulated by Uslenghi and Zich (1965). However, the scattering amplitude coefficients were not evaluated and, in addition, the expected difficulties in truncating the infinite series representation for these coefficients, were not discussed. In the case when the spherical shell is less than a hemisphere, Blore and Musal (1965) and Raybin (1965) used a high frequency approximation by adding the edge-diffracted term to the physical optics value of the specular contribution, and it was shown that the results were reasonably close to those measured. To the author's knowledge, however, the electromagnetic problem of a spherical shell which is larger than a hemisphere has not been treated in the literature in either an exact or approximate sense as far as concrete answers are concerned. For the acoustic problem, Sommerfeld (1949) has used the method of least square error to obtain a system of linear equations for the diffraction coefficients; and more recently Thomas (1962) obtained a low frequency solution using an iteration method developed by Williams (1962).

In Section II we discuss three possible approaches to solving the given boundary value problem when the plane electromagnetic wave is incident symmetrically on the aperture. The first method is similar to that used by Uslenghi and Zich (1965), whereas in the second the boundary value problem is reduced to a system of two integral equations. The third is the method of least square error, and at the end of the section the system of linear equations resulting from this approach is solved by the Gauss-Seidel iterative technique. The following section presents some of the numerical results so far obtained using the least square method for different numbers of terms retained in the approximation, and experimental data is also presented for comparison with the results of the computations.

Section IV contains a brief discussion and summary of the results and concludes with an indication of the further work that is scheduled. In this connection it should be emphasized that the present report is preliminary inasmuch as the study has not yet been completed, and it is not necessarily felt that the behavior observed in the cases which have so far been treated, either experimentally or numerically, is typical of what can be obtained with a loading device of the type discussed here, especially when the cavity is filled with dielectric or magnetic material.

II

THEORETICAL FORMULATION

2.1 Geometry and Field Expressions

Let us consider an infinitely thin, perfectly conducting spherical shell with its center at the origin and the spherical segment given by $R = a$, $\theta_o < \theta \leq \pi$ as shown in Fig. 2-1. Assuming a plane electromagnetic wave is incident in the direction of the negative z-axis with its electric vector parallel to the x-axis, the incident field becomes

$$\underline{E}^i = \hat{x}\, e^{ikz} \,, \qquad \underline{H}^i = -\hat{y}\, Y\, e^{ikz} \,, \tag{2.1}$$

where k is the propagation constant, Y is the intrinsic admittance of free space, and a time factor $e^{i\omega t}$ has been suppressed.

The incident field of Eq. (2.1) can be expressed in terms of spherical vector wave functions as follows:

$$\underline{E}^i = \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left[\underline{M}^{(1)}_{o1n} - i\, \underline{N}^{(1)}_{e1n} \right] ,$$

$$\underline{H}^i = i\, Y \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left[\underline{N}^{(1)}_{o1n} - i\, \underline{M}^{(1)}_{e1n} \right] , \tag{2.2}$$

where $\underline{M}^{(1)}$ and $\underline{N}^{(1)}$ are the spherical wave functions (Stratton, 1941):

$$\underline{M}^{(1)}_{\substack{e\\o}mn} = \mp\, m \frac{\psi_n(kR)}{kR} \frac{P_n^m(\cos\theta)}{\sin\theta} \begin{matrix}\sin\\ \cos\end{matrix}\, m\phi\, \hat{\theta} - \frac{\psi_n(kR)}{kR} \frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\, \hat{\phi} \,,$$

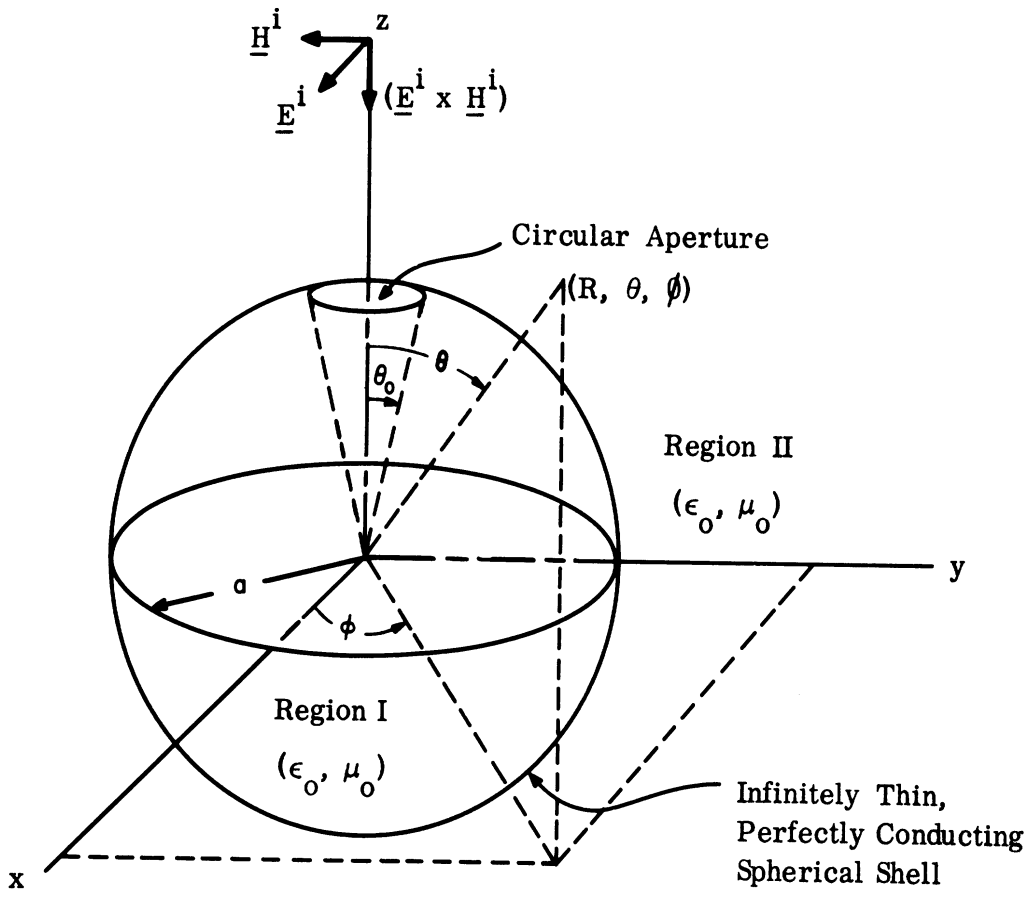

FIG. 2-1: COORDINATE SYSTEM

$$\underline{N}^{(1)}_{\substack{e\\o}mn} = n(n+1)\,\frac{\psi_n(kR)}{kR}\,P_n^m(\cos\theta)\,\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,\hat{R} \,+$$

$$+\,\frac{\psi_n'(kR)}{kR}\,\frac{\partial}{\partial\theta}\,P_n^m(\cos\theta)\,\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,\hat{\theta}\,\mp$$

$$\mp\, m\,\frac{\psi_n'(kR)}{kR}\,\frac{P_n^m(\cos\theta)}{\sin\theta}\,\begin{matrix}\sin\\ \cos\end{matrix}\, m\phi\,\hat{\phi}\;,$$

with

$$\psi_n(kR) = kR\, j_n(kR)\quad .$$

$j_n(kR)$ is the spherical Bessel function of order n, and the prime denotes differentiation with respect to the entire argument.

The total field due to the presence of the spherical cavity can now be expressed in forms similar to those for the incident field, but with unknown amplitude coefficients. Thus, in region I ($R \leq a$):

$$\underline{E}^I = \sum_{n=1}^{\infty}\left[A_n\,\underline{M}^{(1)}_{oln} - i\,B_n\,\underline{N}^{(1)}_{eln}\right] \;,$$

$$\underline{H}^I = i\,Y \sum_{n=1}^{\infty}\left[A_n\,\underline{N}^{(1)}_{oln} - i\,B_n\,\underline{M}^{(1)}_{eln}\right] \;; \qquad (2.3)$$

and in region II ($R \geq a$):

$$\underline{E}^{II} = \underline{E}^i + \underline{E}^s \quad , \qquad \underline{H}^{II} = \underline{H}^i + \underline{H}^s \quad , \qquad (2.4)$$

where

$$\underline{E}^s = \sum_{n=1}^{\infty} \left[C_n \underline{M}^{(4)}_{o1n} - i\, D_n \underline{N}^{(4)}_{e1n} \right] ,$$

$$\underline{H}^s = i\ Y \sum_{n=1}^{\infty} \left[C_n \underline{N}^{(4)}_{o1n} - i\, D_n \underline{M}^{(4)}_{e1n} \right] . \tag{2.5}$$

Here, the superscript s designates the scattered field, and $\underline{M}^{(4)}$ and $\underline{N}^{(4)}$ differ from $\underline{M}^{(1)}$ and $\underline{N}^{(1)}$ in having $\psi_n(kR)$ replaced by $\zeta_n(kR) = kR\, h^{(2)}_n(kR)$, where $h^{(2)}_n(kR)$ is the spherical Hankel function of the second kind. The coefficients A_n, B_n, C_n, and D_n are to be determined from the boundary conditions.

2.2 Boundary Conditions

The continuity of the tangential electric field through the aperture and the condition of zero tangential electric field at the surface of the perfectly conducting sphere segment require that

$$C_n\, \zeta_n(ka) = \left[A_n - i^n \frac{2n+1}{n(n+1)} \right] \psi_n(ka) , \tag{2.6}$$

$$D_n\, \zeta'_n(ka) = \left[B_n - i^n \frac{2n+1}{n(n+1)} \right] \psi'_n(ka) , \tag{2.7}$$

and, for $\theta_o < \theta \leq \pi$,

$$\sum_{n=1}^{\infty} \left[C_n\, \zeta_n(ka) \frac{P^1_n(\cos\theta)}{\sin\theta} - i\, D_n\, \zeta'_n(ka) \frac{\partial}{\partial\theta} P^1_n(\cos\theta) \right] = G_1(\theta), \tag{2.8}$$

$$\sum_{n=1}^{\infty} \left[C_n\, \zeta_n(ka) \frac{\partial}{\partial\theta} P^1_n(\cos\theta) - i\, D_n\, \zeta'_n(ka) \frac{P^1_n(\cos\theta)}{\sin\theta} \right] = G_2(\theta), \tag{2.9}$$

where

$$G_1(\theta) = - \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left[\psi_n(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} - \right.$$

$$\left. - i\, \psi'_n(ka) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \right] , \tag{2.10}$$

$$G_2(\theta) = - \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left[\psi_n(ka) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) - \right.$$

$$\left. -i\, \psi'_n(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} \right] . \tag{2.11}$$

Also, together with Eqs. (2.6) and (2.7), the continuity of the tangential magnetic field through the aperture requires that for $0 \leq \theta < \theta_o$:

$$\sum_{n=1}^{\infty} \left[C_n \frac{1}{\psi_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) - i\, D_n \frac{1}{\psi'_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \right] = 0, \tag{2.12}$$

$$\sum_{n=1}^{\infty} \left[C_n \frac{1}{\psi_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} - i\, D_n \frac{1}{\psi'_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \right] = 0 . \tag{2.13}$$

There seem to be various ways of approaching this kind of boundary value problem. Some of the possible ways, starting from Eqs. (2.8) through (2.13), are presented in the following.

2.3 Direct Conversion to Alebraic Equations

The first method yeilds an infinite set of linear equations by applying straightforward conversion techniques to Eqs. (2.8), (2.9), (2.12) and (2.13).

Multiplying Eqs. (2.8) and (2.9) by $P_m^1(\cos\theta)$ and $\frac{\partial}{\partial\theta} P_m^1(\cos\theta)$ respectively, adding both equations together, and integrating with respect to θ over θ_o to π, we have

$$\sum_{n=1}^{\infty} \left\{ C_n \zeta_n(ka) \left[\frac{2m^2(m+1)^2}{2m+1} \delta_{mn} - L_{mn}(\theta_o) \right] + i\, D_n \zeta_n'(ka) S_{mn}(\theta_o) \right\},$$

$$= \int_{\theta_o}^{\pi} \left[G_1(\theta) \frac{P_m^1(\cos\theta)}{\sin\theta} + G_2(\theta) \frac{\partial}{\partial\theta} P_m^1(\cos\theta) \right] \sin\theta\, d\theta, \quad (2.14)$$

$$(m = 1, 2, 3, \ldots)$$

where the following abbreviations have been introduced:

$$L_{mn}(\theta_o) = \int_0^{\theta_o} \left[\frac{\partial}{\partial\theta} P_m^1(\cos\theta) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) + \right.$$

$$\left. + \frac{P_m^1(\cos\theta)}{\sin\theta} \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \sin\theta\, d\theta\,, \quad (2.15)$$

$$S_{mn}(\theta_o) = \int_0^{\theta_o} \left[\frac{P_m^1(\cos\theta)}{\sin\theta} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) + \right.$$

$$\left. + \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{\partial}{\partial\theta} P_m^1(\cos\theta) \right] \sin\theta\, d\theta\,. \quad (2.16)$$

Hence, by making use of the orthogonal relations of the Legendre functions,

$$\int_{\theta_o}^{\pi} \left[\frac{\partial}{\partial\theta} P_m^1(\cos\theta) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) + \frac{P_m^1(\cos\theta)}{\sin\theta} \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \sin\theta\, d\theta$$

$$= \frac{2m^2(m+1)^2}{2m+1} \delta_{mn} - L_{mn}(\theta_o)\,,$$

$$\int_{\theta_o}^{\pi}\left[\frac{P_m^1(\cos\theta)}{\sin\theta}\frac{\partial}{\partial\theta}P_n^1(\cos\theta)+\frac{P_n^1(\cos\theta)}{\sin\theta}\frac{\partial}{\partial\theta}P_m^1(\cos\theta)\right]\sin\theta\,d\theta$$

$$= - S_{mn}(\theta_o) \quad .$$

Likewise, multiplying Eqs. (2.8) and (2.9) by $\frac{\partial}{\partial\theta}P_m^1(\cos\theta)$ and $P_m^1(\cos\theta)$ respectively, and integrating with respect to θ over θ_o to π, we have

$$\sum_{n=1}^{\infty}\left\{-C_n\zeta_n(ka)S_{mn}(\theta_o)-iD_n\zeta_n'(ka)\left[\frac{2m^2(m+1)^2}{2m+1}\delta_{mn}-L_{mn}(\theta_o)\right]\right\}$$

$$=\int_{\theta_o}^{\pi}\left[G_1(\theta)\frac{\partial}{\partial\theta}P_m^1(\cos\theta)+G_2(\theta)\frac{P_m^1(\cos\theta)}{\sin\theta}\right]\sin\theta\,d\theta \qquad (2\text{-}17)$$

$$(m = 1, 2, 3 \ldots)$$

Similarly, from Eqs. (2.12) and (2.13),

$$\sum_{n=1}^{\infty}\left[C_n\frac{1}{\psi_n(ka)}L_{mn}(\theta_o)-iD_n\frac{1}{\psi_n'(ka)}S_{mn}(\theta_o)\right]=0\,, \qquad (2.18)$$

$$\sum_{n=1}^{\infty}\left[C_n\frac{1}{\psi_n(ka)}S_{mn}(\theta_o)-iD_n\frac{1}{\psi_n'(ka)}L_{mn}(\theta_o)\right]=0 \qquad (2.19)$$

$$(m = 1, 2, 3 \ldots)$$

Equations (2.14), (2.17), (2.18) and (2.19), together with Eqs. (2.2) through (2.7) and (2.10) and (2.11), represent the formal solution of the problem. For a numerical solution, the scattering amplitude coefficients, C_n

and D_n, (n = 1, 2, ..., 2M) can be approximated by solving the linear equations (2.14), (2.17), (2.18) and (2.19) retaining only the first 2M terms of the infinite series. We then have to solve the following set of 4M simultaneous equations:

$$\sum_{n=1}^{2M}\left\{C_n\zeta_n(ka)\left[\frac{2m^2(m+1)^2}{2m+1}\delta_{mn}-L_{mn}(\theta_o)\right]+i\,D_n\zeta_n'(ka)S_{mn}(\theta_o)\right\}$$

$$=\int_{\theta_o}^{\pi}\left[G_1(\theta)\frac{P_m^1(\cos\theta)}{\sin\theta}+G_2(\theta)\frac{\partial}{\partial\theta}P_m^1(\cos\theta)\right]\sin\theta\,d\theta\,, \qquad (2.20)$$

$$\sum_{n=1}^{2M}\left\{-C_n\zeta_n(ka)S_{mn}(\theta_o)-i\,D_n\zeta_n'(ka)\left[\frac{2m^2(m+1)^2}{2m+1}\delta_{mn}-L_{mn}(\theta_o)\right]\right\}$$

$$=\int_{\theta_o}^{\pi}\left[G_1(\theta)\frac{\partial}{\partial\theta}P_m^1(\cos\theta)+G_2(\theta)\frac{P_m^1(\cos\theta)}{\sin\theta}\right]\sin\theta\,d\theta\,, \qquad (2.21)$$

$$\sum_{n=1}^{2M}\left[C_n\frac{1}{\psi_n(ka)}L_{mn}(\theta_o)-i\,D_n\frac{1}{\psi_n'(ka)}S_{mn}(\theta_o)\right]=0\,, \qquad (2.22)$$

$$\sum_{n=1}^{2M}\left[C_n\frac{1}{\psi_n(ka)}S_{mn}(\theta_o)-i\,D_n\frac{1}{\psi_n'(ka)}L_{mn}(\theta_o)\right]=0\,, \qquad (2.23)$$

for m = 1, 2, 3, . . . M .

In order to ensure that the values of the coefficients obtained from Eqs. (2.20) through (2.23) are reasonably accurate, it is necessary that M be as large as possible.

2.4 Integral Equations

In the second approach, we write the tangential components of the electric field in the aperture as follows:

$$E_\theta^{II}(a, \theta, \phi) = \frac{1}{ka} F_1(\theta) \cos \phi , \qquad 0 \leq \theta < \theta_o , \tag{2.24}$$

$$E_\phi^{II}(a, \theta, \phi) = - \frac{1}{ka} F_2(\theta) \sin \phi , \qquad 0 \leq \theta < \theta_o , \tag{2.25}$$

where $F_1(\theta)$ and $F_2(\theta)$ are unknown functions. We then seek to find $F_1(\theta)$ and $F_2(\theta)$ by solving the integral equations arising from Eqs. (2.8) through (2.13) and (2.24) and (2.25). Combining Eqs. (2.8), (2.9), (2.24) and (2.25), we obtain

$$\sum_{n=1}^{\infty} \left[C_n \zeta_n(ka) \frac{P_n^1(\cos \theta)}{\sin \theta} - i D_n \zeta_n'(ka) \frac{\partial}{\partial \theta} P_n^1(\cos \theta) \right]$$

$$= \begin{cases} G_1(\theta) + F_1(\theta) , & 0 \leq \theta < \theta_o \\ G_1(\theta) , & \theta_o < \theta \leq \pi \end{cases} , \tag{2.26}$$

$$\sum_{n=1}^{\infty} \left[C_n \zeta_n(ka) \frac{\partial}{\partial \theta} P_n^1(\cos \theta) - i D_n \zeta_n'(ka) \frac{P_n^1(\cos \theta)}{\sin \theta} \right]$$

$$= \begin{cases} G_2(\theta) + F_2(\theta) , & 0 \leq \theta < \theta_o \\ G_2(\theta) , & \theta_o < \theta \leq \pi \end{cases} . \tag{2.27}$$

By making use of the orthogonality of the Legendre functions, we have

$$C_n = \frac{1}{\zeta_n(ka)} \left\{ \frac{2n+1}{2n^2(n+1)^2} \int_0^{\theta_o} \left[F_1(\theta) \frac{P_n^1(\cos\theta)}{\sin\theta} + \right.\right.$$

$$\left.\left. + F_2(\theta) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \right] \sin\theta \, d\theta - i^n \frac{2n+1}{n(n+1)} \psi_n(ka) \right\}, \qquad (2.28)$$

$$-i D_n = \frac{1}{\zeta_n'(ka)} \left\{ \frac{2n+1}{2n^2(n+1)^2} \int_0^{\theta_o} \left[F_1(\theta) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) + \right.\right.$$

$$\left.\left. + F_2(\theta) \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \sin\theta \, d\theta + i^{n+1} \frac{2n+1}{n(n+1)} \psi_n'(ka) \right\}, \qquad (2.29)$$

and substitution into Eqs. (2.12) and (2.13) then gives

$$\sum_{n=1}^{\infty} \frac{2n+1}{2n^2(n+1)^2} \left\{ \int_0^{\theta_o} F_1(\theta') \left[\frac{1}{\psi_n(ka)\zeta_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \frac{P_n^1(\cos\theta')}{\sin\theta'} + \right.\right.$$

$$\left. + \frac{1}{\psi_n'(ka)\zeta_n'(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{\partial}{\partial\theta'} P_n^1(\cos\theta') \right] \sin\theta' \, d\theta' +$$

$$+ \int_0^{\theta_o} F_2(\theta') \left[\frac{1}{\psi_n(ka)\zeta_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \frac{\partial}{\partial\theta'} P_n^1(\cos\theta') + \right.$$

$$\left.\left. + \frac{1}{\psi_n'(ka)\zeta_n'(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{P_n^1(\cos\theta')}{\sin\theta'} \right] \sin\theta' \, d\theta' \right\}$$

$$= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left[\frac{1}{\zeta_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) - i \frac{1}{\zeta_n'(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \right],$$

(2.30)

$$\sum_{n=1}^{\infty} \frac{2n+1}{2n^2(n+1)^2} \left\{ \int_0^{\theta_o} F_1(\theta') \left[\frac{1}{\psi_n(ka)\zeta_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{P_n^1(\cos\theta')}{\sin\theta'} + \right.\right.$$

$$\left. + \frac{1}{\psi'_n(ka)\zeta'_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \frac{\partial}{\partial\theta'} P_n^1(\cos\theta') \right] \sin\theta' \, d\theta' +$$

$$+ \int_0^{\theta_o} F_2(\theta') \left[\frac{1}{\psi_n(ka)\zeta_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{\partial}{\partial\theta'} P_n^1(\cos\theta') + \right.$$

$$\left. + \frac{1}{\psi'_n(ka)\zeta'_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \frac{P_n^1(\cos\theta')}{\sin\theta'} \right] \sin\theta' \, d\theta'$$

$$= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left[\frac{1}{\zeta_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} - i \frac{1}{\zeta'_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \right] , \tag{2.31}$$

$$\text{for } 0 \leq \theta \leq \theta_o .$$

If we were to interchange the order of integration and summation in Eqs. (2.30) and (2.31), we could reduce them to two Fredholm integral equations. Unfortunately, however, the kernels represented by the infinite series diverge as O(1) or O(n), and the interchange of the orders of integration and summation is not therefore permissible.

Nevertheless, as in Section 2.3, Eqs. (2.30) and (2.31) can be converted to an infinite set of algebraic equations if we express $F_1(\theta)$ and $F_2(\theta)$ as convergent infinite series of functions orthogonal in the interval $0 \leq \theta \leq \theta_o$, for example

$$F_1(\theta) = \sum_{m=0}^{\infty} a_m \cos \frac{m\pi\theta}{\theta_o} , \quad 0 \leq \theta < \theta_o$$

$$F_2(\theta) = \sum_{m=0}^{\infty} b_m \cos \frac{m\pi\,\theta}{\theta_o} \quad , \qquad 0 \leq \theta < \theta_o$$

where the coefficients a_m, b_m are to be determined.

When in the course of computation each infinite series is approximated by a finite series, the final (approximated) results depend on the choice of the orthogonal functions used for the series expansions of $F_1(\theta)$ and $F_2(\theta)$. For a good approximation a proper function with an unknown constant may be introduced to represent each unknown function $F_1(\theta)$ or $F_2(\theta)$, in addition to the series of the proper orthogonal functions; and by appropriate choice, it would appear that the resulting series could be made to converge very rapidly. It is, however, not easy to find the optimum proper functions even though the method of trial and error might lead to ones which are adequate.

2.5 Method of Least Square Error

We here seek to find the scattering amplitude coefficients, C_n and D_n, by using the method of least squares. Two arbitrary weighting factors will be introduced and will later be replaced by quantities specifying the relative weights to be attached to the two boundary conditions - one for the tangential components of the electric field on the conducting surface and the other for the tangential components of the magnetic field through the aperture.

By introducing weighting factors $W_1(>0)$ in Eqs. (2.8) and (2.9), and $W_2(>0)$ in Eqs. (2.12) and (2.13), and by retaining only the first M terms in the infinite series, the total square error, $\mathcal{E}_M$ can be written as

$$\mathcal{E}_M = W_1 \int_{\theta_o}^{\pi} \Bigg| G_1(\theta) - \sum_{n=1}^{M} \Bigg[C_n \zeta_n(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} -$$

$$-iD_n \zeta_n'(ka)\frac{\partial}{\partial\theta} P_n^1(\cos\theta)\Bigg]\Bigg|^2 \sin\theta\, d\theta + W_1 \int_{\theta_o}^{\pi} \Bigg| G_2(\theta) -$$

$$- \sum_{n=1}^{M} \Bigg[C_n \zeta_n(ka)\frac{\partial}{\partial\theta} P_n^1(\cos\theta) - iD_n \zeta_n'(ka)\frac{P_n^1(\cos\theta)}{\sin\theta}\Bigg]\Bigg|^2 \sin\theta\, d\theta +$$

$$+ W_2 \int_0^{\theta_o} \Bigg| \sum_{n=1}^{M} \Bigg[C_n \frac{1}{\psi_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) -$$

$$- iD_n \frac{1}{\psi_n'(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta}\Bigg]\Bigg|^2 \sin\theta\, d\theta + W_2 \int_0^{\theta_o} \Bigg| \sum_{n=1}^{M} \times$$

$$\times \Bigg[C_n \frac{1}{\psi_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} - iD_n \frac{1}{\psi_n'(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta)\Bigg]\Bigg|^2 \sin\theta\, d\theta\,. \tag{2.32}$$

In order to arrive at weighting factors W_1 and W_2 having some physical significance, let us consider the following square error functions (by letting $C_n = D_n = 0$ for all $n > M$), normalized with respect to the corresponding incident field component. For the tangential electric field on the conducting surface ($R = a$, $\theta_o < \theta \leq \pi$):

$$\mathcal{E}_{ME} = \frac{\int_0^{2\pi}\int_{\theta_o}^{\pi} \left| \underline{E}_t^i + \underline{E}_t^s \right|^2_{R=a} a^2 \sin\theta\, d\theta\, d\phi}{\int_0^{2\pi}\int_{\theta_o}^{\pi} \left| \underline{E}_t^i \right|^2_{R=a} a^2 \sin\theta\, d\theta\, d\phi} \tag{2.33}$$

and for the tangential magnetic field through the aperture ($R = a,\ 0 < \theta < \theta_o$):

$$\mathcal{E}_{MH} = \frac{\int_0^{2\pi}\int_0^{\theta_o} \left|\underline{H}_t^I - \underline{H}_t^{II}\right|^2_{R=a} a^2 \sin\theta \, d\theta \, d\phi}{\int_0^{2\pi}\int_0^{\theta_o} \left|\underline{H}_t^i\right|^2_{R=a} a^2 \sin\theta \, d\theta \, d\phi} \quad . \tag{2.34}$$

Since

$$E_\theta^i\Big|_{R=a} = -\frac{1}{ka} \cos\phi \, G_1(\theta) = \cos\theta \cos\phi \, e^{ika\cos\theta} \tag{2.35}$$

$$E_\phi^i\Big|_{R=a} = \frac{1}{ka} \sin\phi \, G_2(\theta) = -\sin\phi \, e^{ika\cos\theta} \tag{2.36}$$

we have

$$\int_0^{2\pi}\int_{\theta_o}^{\pi} \left|\underline{E}_t^i\right|^2_{R=a} \sin\theta \, d\theta \, d\phi = \pi \left(\frac{4}{3} + \cos\theta_o + \frac{1}{3}\cos^3\theta_o\right) \; ; \tag{2.37}$$

and also, from Eq. (2.5) with $C_n = D_n = 0$ for all $n > M$, we have

$$\int_0^{2\pi}\int_{\theta_o}^{\pi} \left|\underline{E}_t^i + \underline{E}_t^s\right|^2 \sin\theta \, d\theta \, d\phi = \frac{\pi}{(ka)^2}\Bigg\{\int_{\theta_o}^{\pi} \Bigg| G_1(\theta) -$$

$$- \sum_{n=1}^{M}\left[C_n \zeta_n(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} - i D_n \zeta_n'(ka) \frac{\partial}{\partial\theta} P_n^1(\cos\theta)\right]\Bigg|^2 \sin\theta \, d\theta \; +$$

$$+ \int_{\theta_o}^{\pi} \left| G_2(\theta) - \sum_{n=1}^{M} \left[C_n \zeta_n(ka) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) - \right. \right.$$

$$\left. \left. - i D_n \zeta_n'(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \right|^2 \sin\theta \, d\theta \Bigg\} . \qquad (2.38)$$

Hence, from Eq. (2.33),

$$\mathcal{E}_{ME} = \frac{1}{(ka)^2} \frac{1}{\left(\frac{4}{3} + \cos\theta_o + \frac{1}{3}\cos^3\theta_o\right)} \left\{ \int_{\theta_o}^{\pi} \left| G_1(\theta) - \right. \right.$$

$$- \sum_{n=1}^{M} \left[C_n \zeta_n(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} - i D_n \zeta_n'(ka) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \right] \Bigg|^2 \sin\theta d\theta +$$

$$+ \int_{\theta_o}^{\pi} \left| G_2(\theta) - \sum_{n=1}^{M} \left[C_n \zeta_n(ka) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) - \right. \right.$$

$$\left. \left. - i D_n \zeta_n'(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \right|^2 \sin\theta \, d\theta \Bigg\} , \qquad (2.39)$$

and similarly, from Eq. (2.34),

$$\mathcal{E}_{MH} = \frac{1}{(ka)^2} \frac{1}{\left(\frac{4}{3} - \cos\theta_o - \frac{1}{3}\cos^3\theta_o\right)} \left\{ \int_0^{\theta_o} \left| \sum_{n=1}^{M} \left[C_n \frac{1}{\psi_n(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) - \right. \right. \right.$$

$$\left. \left. - iD_n \frac{1}{\psi_n'(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \right|^2 \sin\theta d\theta + \int_0^{\theta_o} \left| \sum_{n=1}^{M} \left[C_n \frac{1}{\psi_n(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} - \right. \right.$$

$$\left. \left. - i D_n \frac{1}{\psi_n'(ka)} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \right] \right|^2 \sin\theta \, d\theta \Bigg\} . \qquad (2.40)$$

It is clear that the smaller the quantities $\mathcal{E}_{ME}$ and $\mathcal{E}_{MH}$ become, the more accurate the solution. However, $\mathcal{E}_{ME}$ and $\mathcal{E}_{MH}$ cannot be minimised at the same time. If we let $(\mathcal{E}_{ME} + \mathcal{E}_{MH})$ be the total error which we would like to minimise, then, from Eqs. (2.32), (2.39) and (2.40) by letting

$$\mathcal{E}_M = \mathcal{E}_{ME} + \mathcal{E}_{MH} \quad , \tag{2.41}$$

we have

$$W_1 = \frac{1}{(ka)^2} \frac{1}{\left(\frac{4}{3} + \cos\theta_o + \frac{1}{3}\cos^3\theta_o\right)} \tag{2.42}$$

$$W_2 = \frac{1}{(ka)^2} \frac{1}{\left(\frac{4}{3} - \cos\theta_o - \frac{1}{3}\cos^3\theta_o\right)} \quad . \tag{2.43}$$

It is convenient to use the above weighting factors since we are dealing with a normalized error function and only a single constant need be given to determine the number of modes, M, regardless of the values of ka and θ_o, for the same degree of error criterion.

Returning to Eqs. (2.32), for least square error we require that

$$\frac{\partial \mathcal{E}_M}{\partial C_m} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{E}_M}{\partial D_m} = 0 \quad (m = 1, 2, 3, \ldots M), \tag{2.44}$$

from which we obtain

$$\sum_{n=1}^{M} \left\{ \frac{2m^2(m+1)^2}{2m+1} \left[\psi_m(ka)\right]^2 \delta_{mn} - \left[\psi_n(ka)\psi_m(ka) - \right.\right.$$

$$- \frac{W_2}{W_1} \frac{1}{\zeta_n(ka)\,\zeta_m^*(ka)} \Bigg] L_{mn}(\theta_o) \Bigg\} \frac{\zeta_n(ka)}{\psi_n(ka)} C_n + i \sum_{n=1}^{M} \Bigg[\psi_n'(ka)\psi_m(ka) -$$

$$- \frac{W_2}{W_1} \frac{1}{\zeta_n'(ka)\,\zeta_m^*(ka)} \Bigg] S_{mn}(\theta_o) \frac{\zeta_n'(ka)}{\psi_n'(ka)} D_n = - \psi_m(ka) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times \left\{ \psi_n(ka) \left[\frac{2m^2(2m+1)^2}{2m+1} \delta_{mn} - L_{mn}(\theta_o) \right] + i \psi_n'(ka) S_{mn}(\theta_o) \right\}, \qquad (2.45)$$

$$- \sum_{n=1}^{M} \left[\psi_n(ka)\psi_n'(ka) - \frac{W_2}{W_1} \frac{1}{\zeta_n(ka)\,\zeta_n'^*(ka)} \right] S_{mn}(\theta_o) \frac{\zeta_n(ka)}{\psi_n(ka)} C_n -$$

$$- i \sum_{n=1}^{M} \Bigg\{ \frac{2m^2(m+1)^2}{2m+1} \left[\psi_m'(ka) \right]^2 \delta_{mn} - \Bigg[\psi_n'(ka)\,\psi_m'(ka) -$$

$$- \frac{W_2}{W_1} \frac{1}{\zeta_n'(ka)\,\zeta_m'^*(ka)} \Bigg] L_{mn}(\theta_o) \Bigg\} \frac{\zeta_n'(ka)}{\psi_n'(ka)} D_n = \psi_m'(ka) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times \left\{ \psi_n(ka) S_{mn}(\theta_o) + i \psi_n'(ka) \left[\frac{2m^2(m+1)^2}{2m+1} \delta_{mn} - L_{mn}(\theta_o) \right] \right\} \qquad (2.46)$$

$$(m = 1, 2, 3, \ldots M)$$

where the asterisk denotes the complex conjugate. The 2M unknown scattering amplitude coefficients, C_n and D_n ($n = 1, 2, \ldots M$) can now be approximated in the least square sense by solving the above 2M linear algebraic equations obtained by imposing the boundary conditions and letting $C_n = D_n = 0$ for all $n > M$. It is clear that this approximation converges to the exact solution as M approaches infinity.

All the above three approaches to the solution of the scattering problem involve infinite systems of linear equations. In actual computation the number of linear equations to be solved must be large enough to give sufficient accuracy, but small enough to keep the computation time reasonable. It is well known that in solving large systems of linear equations, matrix iterative methods have distinct advantages, most notably with respect to the speed of execution, over conventional matrix inversion methods provided the corresponding iterative matrix has a reasonable rate of convergence. In the following, the application of the Gauss-Seidel iterative method (Varga, 1962; Cole, 1967) to Eqs. (2.45) and (2.46) will be considered. Besides its physical significance, the method of least square error guarantees that the Gauss-Seidel iteration will always converge if $\theta_o \neq 0$.

2.6 Gauss-Seidel Iteration Method

After some re-arrangement, Eqs. (2.45) and (2.46) can be written in matrix form as

$$\left[a_{pq}\right]\left[\chi_q\right] = \left[f_p\right] \quad , \quad (p,\ q = 1,\ 2,\ .\ .\ .,\ 2M) \qquad (2.47)$$

where the following notations have been introduced;

$$a_{2m-1,2n-1} = \frac{2m^2(m+1)^2}{2m+1}\left[\psi_m(ka)\right]^2 \delta_{mn} - \left[\psi_n(ka)\psi_m(ka) - \right.$$

$$\left. - \frac{W_2}{W_1} \frac{1}{\zeta_n(ka)\zeta_m^*(ka)}\right] L_{mn}(\theta_o) \quad , \qquad (2.48)$$

$$a_{2m,2n} = \frac{2m^2(m+1)^2}{2m+1}\left[\psi'_m(ka)\right]^2 \delta_{mn} - \left[\psi'_n(ka)\psi'_m(ka) - \right.$$

$$\left. - \frac{W_2}{W_1} \frac{1}{\zeta'_n(ka)\zeta'^*_n(ka)}\right] L_{mn}(\theta_o) \quad , \qquad (2.49)$$

$$a_{2m-1,2n} = -\left[\psi_n'(ka)\psi_m(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_n'(ka)\zeta_m^*(ka)}\right] S_{mn}(\theta_o) \qquad (2.50)$$

$$a_{2m,2n-1} = -\left[\psi_n(ka)\psi_n'(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_n'(ka)\zeta_m'^*(ka)}\right] S_{mn}(\theta_o) \ , \qquad (2.51)$$

$$\chi_{2n-1} = \frac{\zeta_n(ka)}{\psi_n(ka)} C_n \ , \qquad (2.52)$$

$$\chi_{2n} = -i\,\frac{\zeta_n'(ka)}{\psi_n'(ka)} D_n \ , \qquad (2.53)$$

$$f_{2m-1} = -\psi_m(ka)\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\left\{\psi_n(ka)\left[\frac{2m^2(m+1)^2}{2m+1}\delta_{mn} - L_{mn}(\theta_o)\right] + i\,\psi_n'(ka)\, S_{mn}(\theta_o)\right\} \ , \qquad (2.54)$$

$$f_{2m} = \psi_m'(ka)\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\left\{\psi_n(ka) S_{mn}(\theta_o) + i\,\psi_n'(ka)\left[\frac{2m^2(m+1)^2}{2m+1}\delta_{mn} - L_{mn}(\theta_o)\right]\right\} \ . \qquad (2.55)$$

It is interesting to observe that after some manipulation the coefficients f_{2m-1} and f_{2m} given in Eqs. (2.54) and (2.55) respectively can be expressed in terms of finite series as follows:

$$f_{2m-1} = i\,\psi_m(ka)\cdot m(m+1)\, g_m(\theta_o) \qquad (2.56)$$

$$f_{2m} = i\,\psi_m'(ka)\frac{m(m+1)}{2m+1}\left[(m+1)g_{m+1}(\theta_o) + m\,g_{m-1}(\theta_o)\right] \qquad (2.57)$$

where

$$g_m(\theta_o) = \left[e^{ika\chi} \sum_{r=0}^{m} \left(\frac{i}{ka}\right)^r \frac{\partial^r}{\partial\chi^r} P_m(\chi) \right]_{\chi = -1}^{\chi = \cos\theta_o} . \qquad (2.58)$$

It should be noted that for the special case in which $\psi_r(ka) = 0$, Eq. (2.6) gives $C_r = 0$ and the expression $\left[\zeta_r(ka)/\psi_r(ka)\right] C_r$ must be replaced by

$$\left[A_r - i^r \frac{2r + 1}{r(r + 1)}\right] .$$

Similarly, if $\psi_r'(ka) = 0$, we have $D_r = 0$, and $\left[\zeta_r'(ka)/\psi_n'(ka)\right] D_r$ must be replaced by

$$\left[B_r - i^r \frac{2r + 1}{r(r + 1)}\right] .$$

In order to solve iteratively the matrix Eq. (2.47), we express the matrix $\mathbb{A} = \left[a_{pq}\right]$ as the matrix sum

$$\mathbb{A} = \mathbb{D} - \mathbb{L} - \mathbb{U} \qquad (2.59)$$

where $\mathbb{D} = \text{diag}\left\{a_{11}, a_{22}, \ldots, a_{2M,2M}\right\}$, and $\mathbb{L}$ and $\mathbb{U}$ are respectively strictly lower and upper triangular 2M x 2M matrices, whose entries are the negatives of the entries of $\mathbb{A}$ respectively below and above the main diagonal of $\mathbb{A}$.

Using the notation

$$\mathbb{X} \equiv \begin{bmatrix} \chi_1 \\ \chi_2 \\ \cdot \\ \cdot \\ \cdot \\ \chi_{2M} \end{bmatrix} \quad \text{and} \quad \mathbb{F} \equiv \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{2M} \end{bmatrix}, \qquad (2.60)$$

Equation (2.47) can be written as

$$(\mathbb{D} - \mathbb{L})\,\mathbb{X} = \mathbb{U}\,\mathbb{X} + \mathbb{F} \tag{2.61}$$

which leads to the Gauss-Seidel iterative method[+]

$$(\mathbb{D} - \mathbb{L})\mathbb{X}^{(r+1)} = \mathbb{U}\,\mathbb{X}^{(r)} + \mathbb{F}\ , \qquad r \geq 0 \tag{2.62}$$

which can be written in termsof components as

$$a_{pp}\chi_p^{(r+1)} = -\sum_{q=1}^{p-1} a_{pq}\chi_q^{(r+1)} - \sum_{q=p+1}^{2M} a_{pq}\chi_q^{(r)} + f_p, \quad r \geq 0 \tag{2.63}$$

where

$$\mathbb{X}^{(r)} \equiv \begin{bmatrix} \chi^{(r)} \\ \chi_2^{(r)} \\ \cdot \\ \cdot \\ \cdot \\ \chi_{2M}^{(r)} \end{bmatrix}, \qquad r \geq 0 \ . \tag{2.64}$$

$\mathbb{X}^{(o)}$ is an arbitrary, initial complex matrix approximation of the solution matrix $\mathbb{X}$ of Eq. (2.47). If $\mathbb{X}^{(o)}$ is given, a sequence of matrix iterates $\mathbb{X}^{(r)}$ is successively defined by Eq. (2.62) or (2.63).

For a given Hermitian matrix $\mathbb{H} = \left[h_{pq}\right]$ in the matrix equation $\mathbb{H}\mathbb{X} = \mathbb{Y}$, the necessary and sufficient condition for the Gauss-Seidel iterative method to be convergent, i.e. $\lim_{r\to\infty} \mathbb{X}^{(r)} = \mathbb{X}$, is that the matrix $\mathbb{H}$ be positive de-

[+]Strictly speaking, the method is the point Gauss-Seidel iterative one.

finite (Varga, 1962). From Eqs. (2.48) through (2.51) it is readily seen that $a_{pq} = a^*_{pq}$, and the positive definiteness of the matrix $\mathbb{A}$ can be proved by showing that for any given matrix $\mathbb{Y}] = y_p]$ for p = 1, 2, ..., 2M,

$$\overline{y^*_p}\left[a_{pq}\right]y_q\Big] = W_1 \int_{\theta_o}^{\pi} \Bigg| \sum_{n=1}^{M} \left[y_{2n-1}\psi_n(ka)\frac{P_n^1(\cos\theta)}{\sin\theta} + \right.$$

$$\left. + y_{2n}\psi'_n(ka)\frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right]\Bigg|^2 \sin\theta\, d\theta + W_1 \int_{\theta_o}^{\pi} \Bigg| \sum_{n=1}^{M} \times$$

$$\times \left[y_{2n-1}\psi_n(ka)\frac{\partial}{\partial\theta}P_n^1(\cos\theta) + y_{2n}\psi'_n(ka)\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\Bigg|^2 \sin\theta\, d\theta +$$

$$+ W_2 \int_0^{\theta_o} \Bigg| \sum_{n=1}^{M} \left[y_{2n-1}\frac{1}{\zeta_n(ka)}\frac{\partial}{\partial\theta}P_n^1(\cos\theta) + y_{2n}\frac{1}{\zeta'_n(ka)} \times \right.$$

$$\left. \times \frac{P_n^1(\cos\theta)}{\sin\theta}\right]\Bigg|^2 \sin\theta\, d\theta + W_2 \int_0^{\theta_o} \Bigg| \sum_{n=1}^{M} \left[y_{2n-1}\frac{1}{\zeta_n(ka)}\frac{P_n^1(\cos\theta)}{\sin\theta} + \right.$$

$$\left. + y_{2n}\frac{1}{\zeta'_n(ka)}\frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right]\Bigg|^2 \sin\theta\, d\theta \ , \qquad (2.65)$$

and whose right hand side is always greater than zero for $\theta_o > 0$ unless $y_1 = y_2 = \ldots = y_{2M} = 0$ (See Appendix B).

Since the matrix $\mathbb{A}$ is positive defihte, the lower triangular matrix $(\mathbb{D} - \mathbb{L})$ in Eq. (2.62) is non-singular. We can therefore write

$$\mathbb{X}^{(r+1)} = (\mathbb{D} - \mathbb{L})^{-1}\,\mathbb{U}\mathbb{X}^{(r)} + (\mathbb{D} - \mathbb{L})^{-1}\,\mathbb{F}\ , \ r \geq 0\ . \qquad (2.66)$$

If we define error matrices $\mathbb{\Delta}^{(r)} = \left.\delta_p^{(r)}\right]$, (p = 1, 2, ..., 2M), by

$$\mathbb{\Delta}^{(r)} \equiv \mathbb{X}^{(r)} - \mathbb{X}\ , \qquad r \geq 0 \tag{2.67}$$

then from Eq. (2.66) we obtain

$$\mathbb{\Delta}^{(r)} = \mathbb{G}\,\mathbb{\Delta}^{(r-1)} = \ldots = \mathbb{G}^r\,\mathbb{\Delta}^{(o)}\ , \quad r \geq 0 \tag{2.68}$$

where the matrix $\mathbb{G}$, which is called the Gauss-Seidel matrix associated with matrix $\mathbb{A}$, is defined as

$$\mathbb{G} = (\mathbb{D} - \mathbb{L})^{-1}\ \mathbb{U} \quad . \tag{2.69}$$

The matrix $\mathbb{G}$ is of course convergent (to zero) since $\lim_{r \to \infty} \mathbb{X}^{(r)} = \mathbb{X}$. However, the rate of convergence is a function of ka, θ_o and M, and due to its complicated expression, it is difficult to see its behavior until some actual results of the numerical computation are available. As we can see from Eq. (2.68), the error matrix for r iterations depends not only on the matrix $\mathbb{G}$ but also on the initial error matrix $\mathbb{\Delta}^{(o)}$, which is unknown. Equation (2.68) is not therefore convenient for determining the number of iterations to give a reasonably good approximation to the solution of Eq. (2.47).

To determine the number of iterations desired for a certain error criterion, we consider the total square error $\mathcal{E}_M$. By using the same notation as in Eq. (2.47), Eq. (2.32) can be manipulated to give

$$\mathcal{E}_M = W_1 \left\{ (ka)^2 \left(\frac{4}{3} + \cos\theta_o + \frac{1}{3}\cos^3\theta_o\right) - \sum_{p=1}^{2M} \times \right.$$

$$\left. \times\ \mathrm{Re}\left[\chi_p^* \left(2f_p - \sum_{q=1}^{2M} a_{pq}\,\chi_q\right)\right] \right\} \quad . \tag{2.70}$$

When the solution of Eq. (2.47) is approximated by the r^{th} iterative solution, the corresponding square error $\mathcal{E}_M^{(r)}$ follows from Eq. (2.70) on replacing the χ_p's by $\chi_p^{(r)}$'s, and is

$$\mathcal{E}_M^{(r)} = W_1 \left\{ (ka)^2 \left(\frac{4}{3} + \cos\theta_o + \frac{1}{3}\cos^3\theta_o\right) - \sum_{p=1}^{2M} \mathrm{Re}\left[\chi_p^{(r)*}\left(2f_p - \sum_{q=1}^{2M} a_{pq}\chi_q^{(r)}\right)\right]\right\} \quad . \tag{2.71}$$

We are now able to specify the number of iterations by applying an error criterion to $\mathcal{E}_M^{(r)}$. Thus, for example, we might allow

$$0 \leq \left(\mathcal{E}_M^{(r)} - \mathcal{E}_M^{(r+1)}\right) \Big/ \mathcal{E}_M^{(r)} < \alpha_1 \tag{2.72}$$

where α_1 is a given small quantity by which the accuracy of the computed results can be decided.

The number M can be determined according to the accuracy we require. If, for a given small positive quantity α_2, we require the total error $\mathcal{E}_M$ to be less than α_2, we start with M = M', a relatively small number (10 ~ 20) and keep increasing M by a certain increment until the condition $\mathcal{E}_M < \alpha_2$ is satisfied. In the process, when M is changed by an increment, it would seem attractive to use the previously approximated solution as the initial approximation to begin the iterative procedure for the increased M.

Finally, it should be noted that if the matrix Eq. (2.47) is partitioned in the form

$$\begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} & \cdot & \cdot & \cdot & \mathbb{A}_{1N} \\ \mathbb{A}_{21} & \mathbb{A}_{22} & \cdot & \cdot & \cdot & \mathbb{A}_{2N} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \mathbb{A}_{N1} & \mathbb{A}_{N2} & \cdot & \cdot & \cdot & \mathbb{A}_{NN} \end{bmatrix} \begin{bmatrix} \mathbb{X}_1 \\ \mathbb{X}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{X}_N \end{bmatrix} = \begin{bmatrix} \mathbb{F}_1 \\ \mathbb{F}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{F}_N \end{bmatrix} \qquad (2.73)$$

where the diagonal submatrices $\mathbb{A}_{pp}$, $1 \leq p \leq N$, are square, the associated partitioned Gauss-Seidel iterative method

$$\mathbb{A}_{pp}\mathbb{X}_p^{(r+1)} = -\sum_{q=1}^{p-1} \mathbb{A}_{pq}\mathbb{X}_q^{(r+1)} - \sum_{q=p+1}^{N} \mathbb{A}_{pq}\mathbb{X}^{(r)} + \mathbb{F}_p \,, \qquad (2.74)^{+}$$

$$r \geq 0$$

is also convergent (Varga, 1962). If each diagonal submatrix $\mathbb{A}_{pp}$ is a 1 x 1 matrix, Eq. (2.74) reduces to Eq. (2.63). Each different partitioning of the matrix $\mathbb{A}$ gives rise to a different rate of convergence, and we are therefore interested in finding the optimum partitioning of the matrix $\mathbb{A}$ for which the rate of convergence is fastest. At present, however, it is difficult (if not impossible) to find the optimum partitioning.

+We assume that matrix equations $\mathbb{A}_{pp}\mathbb{X}_p = \mathbb{F}_p$ can be solved directly.

III

PRELIMINARY RESULTS

In the first attempt to compute the scattering amplitude coefficients C_n and D_n, Eqs. (2.47) through (2.55) were programmed in a single precission mode for the IBM 7090 digital computer. It was found that the amplitudes of the matrix coefficients, a_{pq}, are rapidly decreasing functions of p and q, and exceeded the machine capacity (10^{-37}) for p and q $>$ 20. For the time being, the number M was therefore limited to $M \leq 10$.

In order to examine the accuracy of the solution obtained by the method of least square error, the tangential components of the total electric field, E_θ^{II} and E_ϕ^{II}, were computed from the approximate scattering amplitude coefficients C_n and D_n (n = 1, 2, ... M) for M = 8, 9 and 10 with $\theta_o = 30^o$ and ka = 1.0. Their magnitudes are plotted in Figs. 3-1 and 3-2. Knowing that the exact tangential components of the total electric field must vanish at the conducting surface R = a and $30^o < \theta \leq 180^o$, we see that the results for M = 10 give a better approximation than those for M = 8. In addition, we also know that the field E_θ^{II} at the edge, $\theta = 30^o$, of the shell has a singularity of order $\rho^{-1/2}$, where ρ is the distance from the edge. For M = 8, such a singularity is not evident but for M = 10 there are signs of its emergence. However, the error is still significant and much larger values of M (at least 20 or 30, say) may be required to attain a reasonable approximation. The error $\mathscr{E}_M$ and the normalized backscattering cross section $\sigma^{(o)}/\sigma_o$ for M = 8, 9 and 10 are as follows:

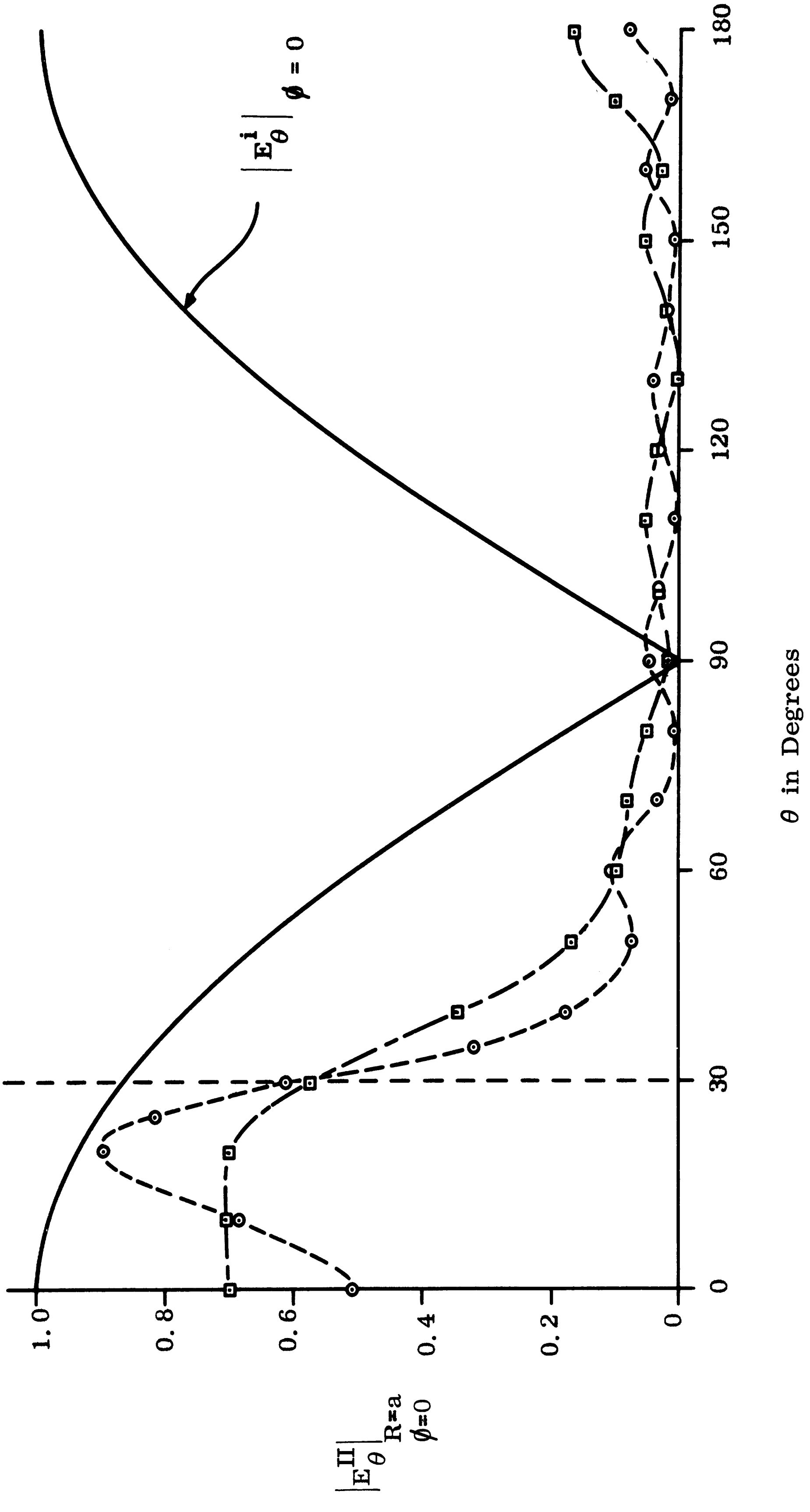

FIG. 3-1: COMPUTED θ-COMPONENTS OF THE TOTAL ELECTRIC FIELD AT THE SURFACE R = a WITH M = 8 (⊡⊡⊡) AND 10 (⊙⊙⊙), FOR ka = 1.0, $\theta_o = 30^o$, AND $\phi = 0$.

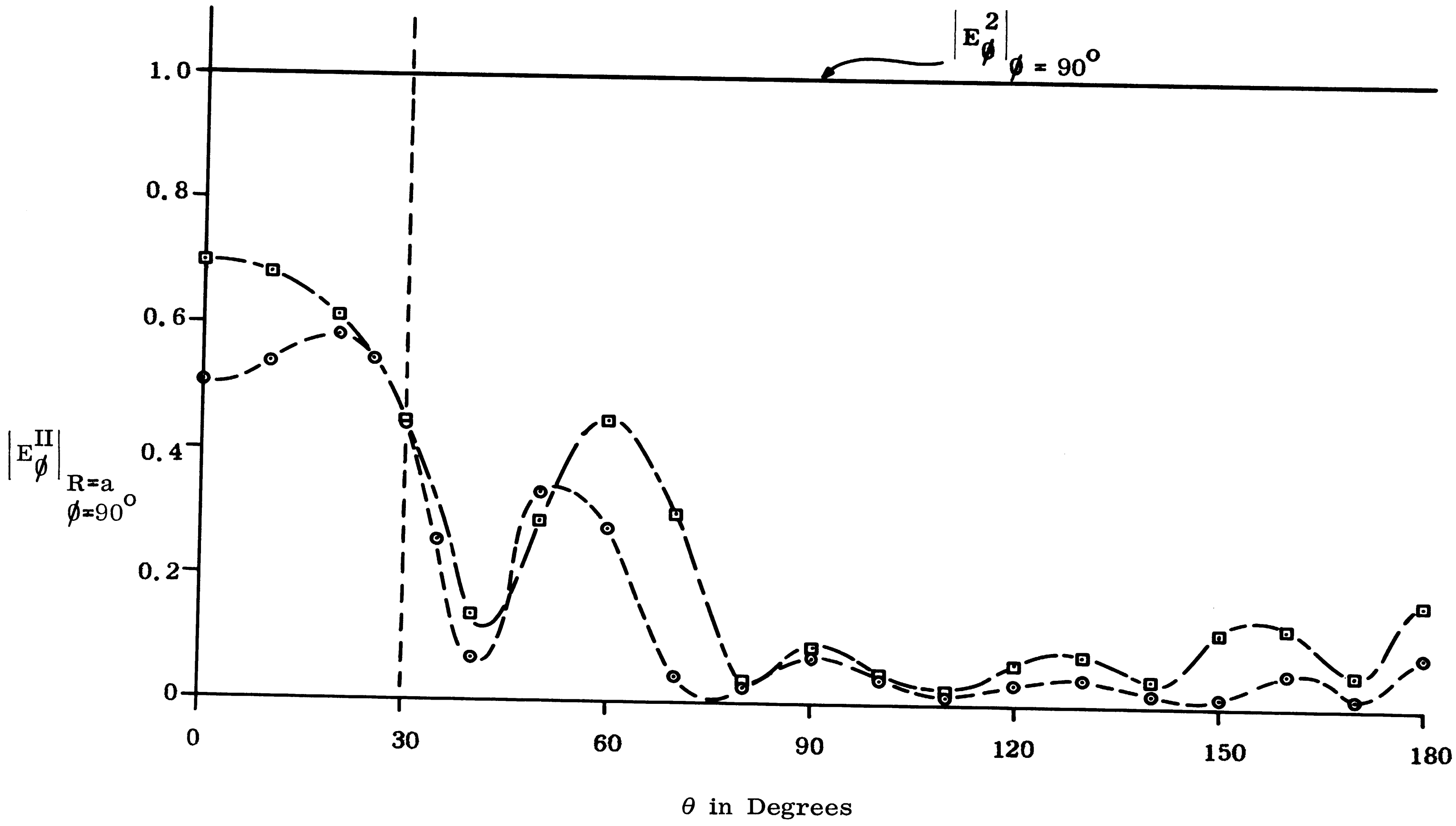

FIG. 3-2: COMPUTED ϕ-COMPONENTS OF THE TOTAL ELECTRIC FIELD AT THE SURFACE R = a WITH M = 8 (⊡⊡⊡) AND M = 10 (⊙⊙⊙), ka = 1.0, $\theta_o = 30^o$, AND $\phi = 90^o$.

TABLE III-1

$\mathcal{S}_8$	= 0.052900	$\sigma^{(o)}/\sigma_o \Big\vert_{M=8}$	= 0.35 db
$\mathcal{S}_9$	= 0.038767	$\sigma^{(o)}/\sigma_o \Big\vert_{M=9}$	= 0.43 db
$\mathcal{S}_{10}$	= 0.033010	$\sigma^{(o)}/\sigma_o \Big\vert_{M=10}$	= 0.47 db

(ka = 1.0, $\theta_o = 30^o$)

Here, $\sigma(o)$ is the backscattering cross section of a spherical shell with $\theta_o = 30^o$, and σ_o is that of a solid sphere with the same radius as the spherical shell.

To compare with the numerical results and to obtain some insight into the scattering behavior of a spherical shell, the backscattering cross sections were measured using a spherical shell model. The spherical shell was made by joining two steel hemispheres (cold drawn from thin steel sheet) and then cutting a hole of appropriate size ($\theta_o = 30^o$) in the top of one of the hemispheres. The diameter of the resulting model was 3.09 inches and the shell thickness about 0.030 inches.

The backscatter measurements were carried out using conventional CW equipment in an anechoic room. The distance from the antenna to the supporting pedestal was 9 feet and the model was placed so that the plane of the aperture was parallel to the axis of the supporting pedestal. The backscattering cross section was measured as a function of θ in the E-plane for a series of L- and S-band frequencies corresponding to ka = 0.9 to 1.4 and ka = 2.0 to 3.3 respectively.[+] When the aperture of the model was directed at the antenna

[+]Measurements in the range ka = 1.4~2.0 were not possible because the corresponding frequencies were not available.

(θ = 0), the measured backscattering cross sections, $\sigma(o)$, were as shown in Figs. 3-3a and 3-3b.

Four complete cross section patterns for ka = 1.0, 2.0, 2.75 and 3.0 are given in Fig. 3-4, and it is interesting to observe that for ka = 1.0 the spherical shell model has almost the same cross section as a solid sphere of the same size. As ka increases, however, the effect of the aperture becomes more apparent and, as seen from Figs. 3-3a and 3-3b, the relative cross section is in general enhanced except in the region near ka = 2.75, where it is reduced to about - 14 db.

At ka = 2.75 the diameter of the aperture is $2b = 2a \sin\theta_o = 0.4377\lambda$. This suggests that the cross section reduction at ka = 2.75 is mainly due to a resonance effect of the aperture since the resonance of a disc (or a hole in a screen) occurs when the disc diameter is approximately this size. On the other hand, the lowest resonant mode in the spherical cavity occurs at the first root of $\psi_1'(ka) = 0$, which also corresponds to ka $\simeq$ 2.75. Hence, the cross section reduction obtained near ka = 2.75 may also be attributable to a cavity resonance. In retrospect, the choice of aperture size was undoubtedly an unfortunate one as far as pin pointing the physical phenonema is concerned, but it is expected that the numerical program, when completed, will serve to separate these two effects. So far we have been able to compare the measured values with computed ones only for backscattering with ka = 1.0. There the measured value of the ratio $\sigma(o)/\sigma_o$ was 0.1 db, compared with the computed value (Table III-1 with M = 10) of 0.47 db.

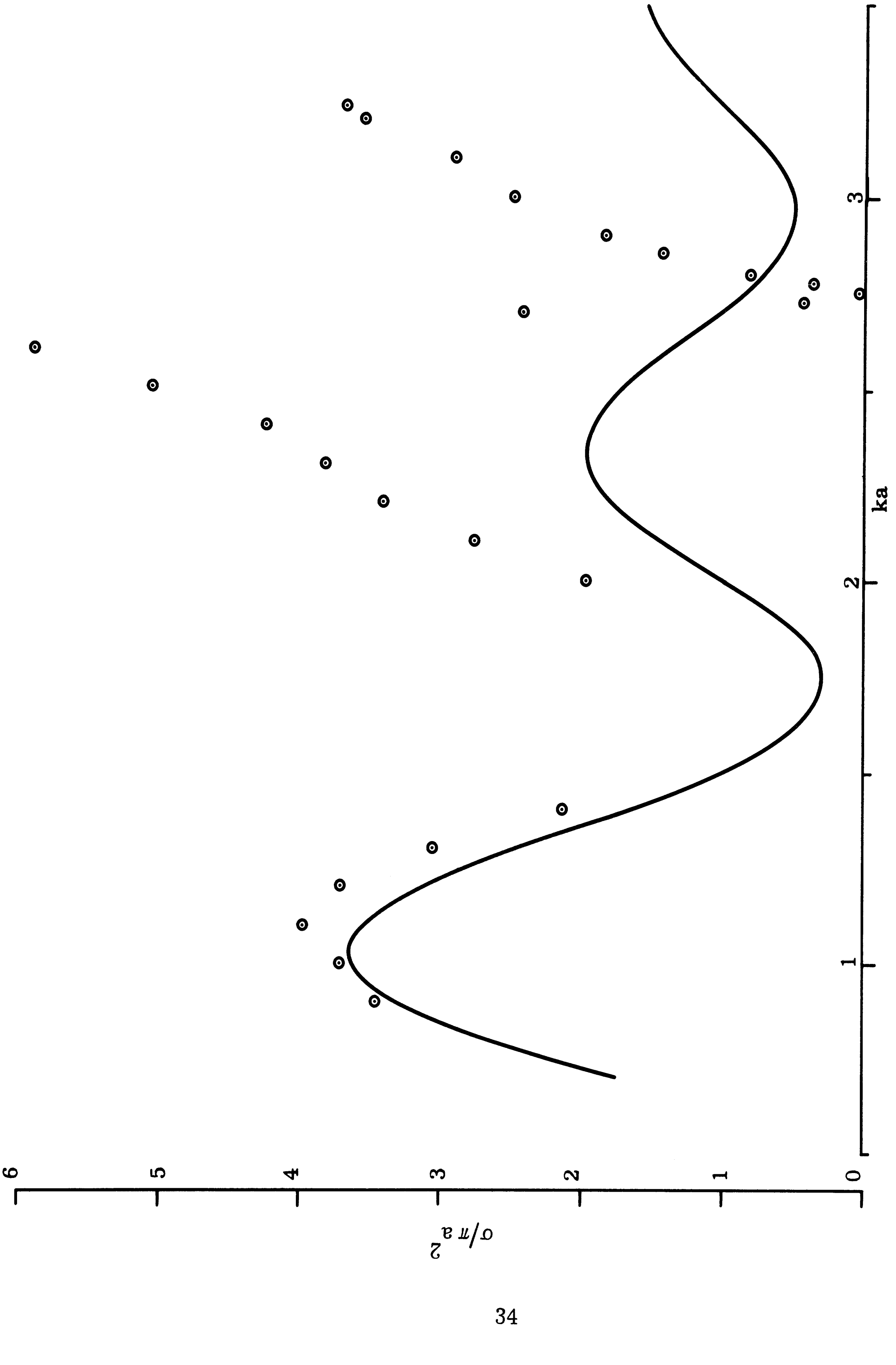

FIG. 3-3a: MEASURED BACKSCATTERING CROSS SECTIONS (⊙⊙⊙) OF A SPHERICAL SHELL WITH $\theta_o = 30^o$ AND THEORETICAL BACKSCATTERING CROSS SECTION (——) OF A SPHERE.

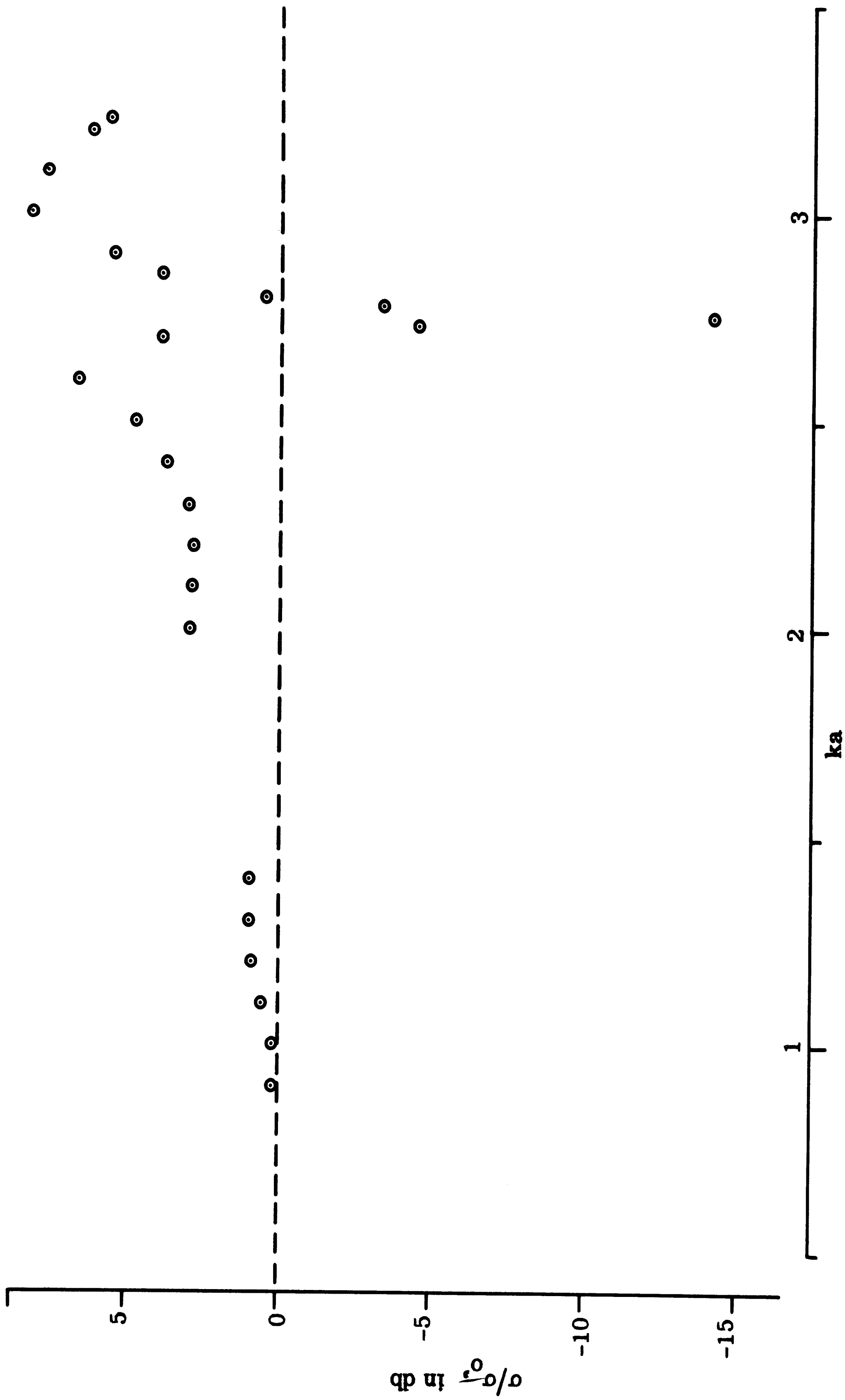

FIG. 3-3b: MEASURED RELATIVE BACKSCATTERING CROSS SECTIONS OF A SPHERICAL SHELL WITH $\theta_o = 30^o$.

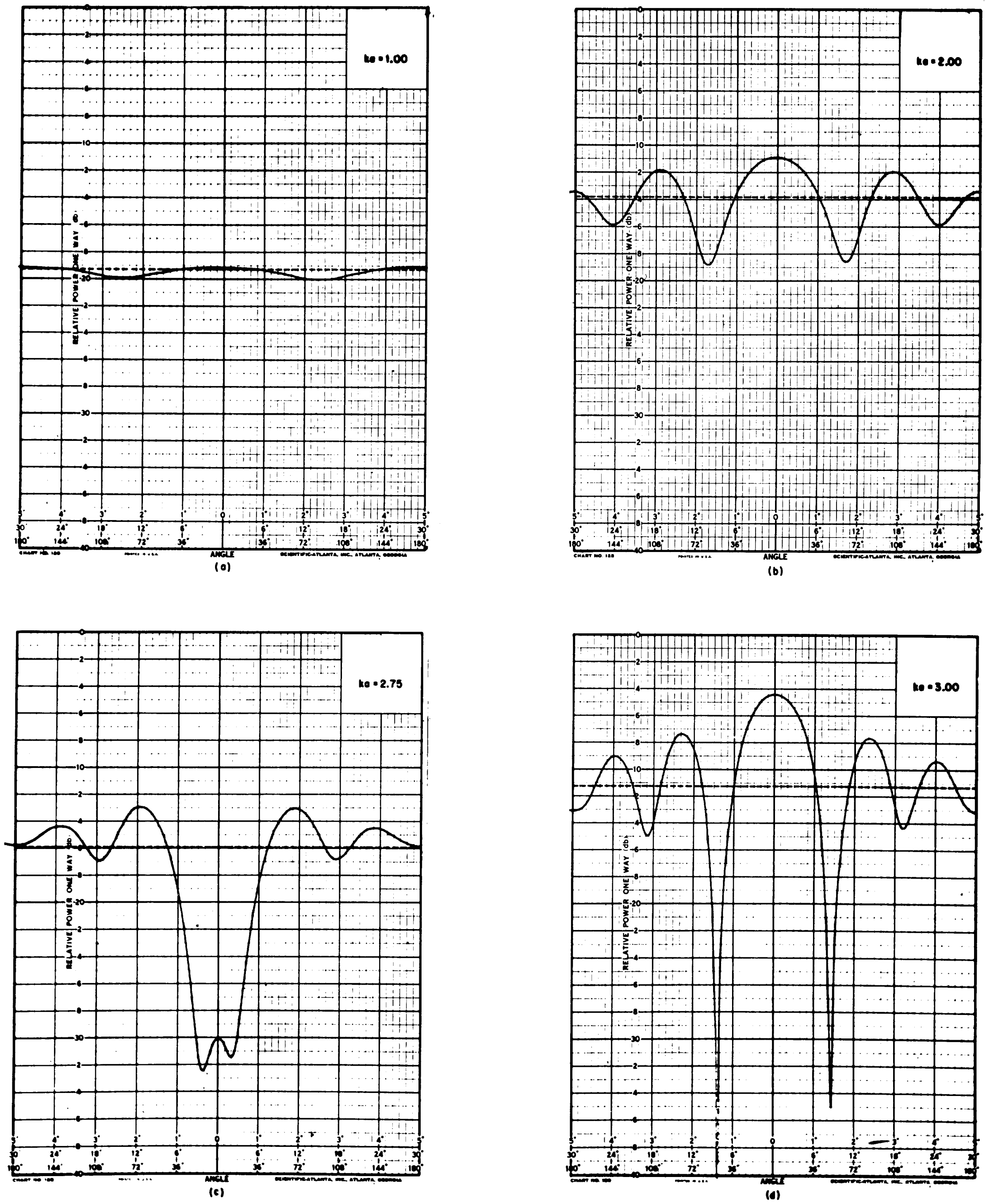

FIG. 3-4: BACKSCATTERING PATTERNS OF A SPHERICAL SHELL WITH $\theta_o = 30^o$ (——) AND A SOLID SPHERE (- - - -). Horizontal Polarization.

IV

AN ALTERNATIVE APPROACH

As pointed out in Section III, the range of magnitudes of the matrix elements a_{pq} exceeds that available on the IBM 7090 computer when p and q are large. Because of this, the computations that we have described were obtained by applying the least squares method to a maximum of 10 terms (or modes) in the field expansions, and even for ka as small as 1.0, this is not sufficient to provide a reasonable approximation to the fields in the aperture and on the surface of the shell.

The reason for the underflow is evident on examining the expressions for the matrix elements given in Eqs. (2.48) through (2.51). These involve products of Bessel or Hankel functions, and their derivatives, and since for large n,

$$\psi_n(ka) \sim \frac{(ka)^{n+1}}{(2n+1)(2n-1)\ldots 5\cdot 3\cdot 1} ,$$

$$\psi'_n(ka) \sim \frac{(n+1)(ka)^{n}}{(2n+1)(2n-1)\ldots 5\cdot 3\cdot 1} ,$$

$$\frac{1}{\zeta_n(ka)} \sim -i\,\frac{(ka)^{n}}{(2n-1)\ldots 5\cdot 3\cdot 1} ,$$

$$\frac{1}{\zeta'_n(ka)} \sim -i\,\frac{(ka)^{n+1}}{n\left[(2n-1)\ldots 5\cdot 3\cdot 1\right]} ,$$

the underflow is unavoidable with the present scheme. Nevertheless, the difficulty can be overcome by a reformulation of the problem.

Consider the following diagonal matrix $\mathbb{T}$.

$$\mathbb{T} = \text{diag}\left\{ \zeta_1(ka), \zeta_1'(ka), \zeta_2(ka), \zeta_2'(ka), \ldots \zeta_7(ka), \zeta_7'(ka), \frac{\zeta_8(ka)}{\psi_8(ka)}, \right.$$
$$\left. \frac{\zeta_8'(ka)}{\psi_8'(ka)}, \frac{\zeta_9(ka)}{\psi_9(ka)}, \frac{\zeta_9'(ka)}{\psi_9'(ka)}, \ldots, \frac{\zeta_M(ka)}{\psi_M(ka)}, \frac{\zeta_M'(ka)}{\psi_M'(ka)} \right\}. \tag{4.1}$$

If we now define

$$\left[b_{pq}\right] \equiv \mathbb{B} \equiv \mathbb{A}\,\mathbb{T} \quad , \tag{4.2}$$

$$\left.y_q\right] \equiv \mathbb{Y} \equiv \mathbb{T}^{-1}\mathbb{X} \quad , \quad (pq = 1, 2, \ldots, 2M) \tag{4.3}$$

Eq. (2.47) becomes

$$\left[b_{pq}\right] \left.y_q\right] = \left.f_p\right] \quad , \quad (pq = 1, 2, \ldots 2M) \tag{4.4}$$

in which b_{pq} and y_q can be written as

$$b_{2m-1,2n-1} = \psi_n(ka)\left\{ \frac{2m^2(m+1)^2}{2m+1}\, \psi_m(ka)\zeta_m(ka)\, \delta_{mn} - \right.$$
$$\left. - \left[\psi_m(ka)\, \zeta_n(ka) - \frac{W_2}{W_1}\, \frac{1}{\zeta_m^*(ka)\, \psi_n(ka)}\right] L_{mn}(\theta_o) \right\} \quad , \tag{4.5}$$

$$b_{2m,2n} = \psi_n'(ka)\left\{ \frac{2m^2(m+1)^2}{2m+1}\, \psi_m'(ka)\,\zeta_m'(ka)\, \delta_{mn} - \right.$$
$$\left. - \left[\psi_m'(ka)\,\zeta_n'(ka) - \frac{W_2}{W_1}\, \frac{1}{\zeta_m'^*(ka)\psi_n'(ka)}\right] L_{mn}(\theta_o) \right\} \quad , \tag{4.6}$$

$$b_{2m-1,2n} = -\psi_n'(ka)\left\{\psi_m(ka)\zeta_n'(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_m^*(ka)\psi_n'(ka)}\right\}S_{mn}(\theta_o)\,, \quad (4.7)$$

$$b_{2m,2n-1} = -\psi_n(ka)\left\{\psi_m'(ka)\zeta_n(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_m'^*(ka)\psi_n(ka)}\right\}S_{mn}(\theta_o)\,, \quad (4.8)$$

$$y_{2n-1} = \zeta_n(ka)\,\chi_{2n-1}\,, \quad (4.9)$$

$$y_{2n} = \zeta_n'(ka)\,\chi_{2n}\,, \quad (4.10)$$

for $n \leq 7$; and

$$b_{2m-1,2n-1} = \left\{\frac{2m^2(m+1)^2}{2m+1}\psi_m(ka)\,\zeta_m(ka)\,\delta_{mn} - \left[\psi_m(ka)\,\zeta_n(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_m^*(ka)\psi_n(ka)}\right]L_{mn}(\theta_o)\right\}, \quad (4.11)$$

$$b_{2m,2n} = \left\{\frac{2m^2(m+1)^2}{2m+1}\psi_m'(ka)\zeta_m'(ka)\,\delta_{mn} - \left[\psi_m'(ka)\zeta_n'(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_m'^*(ka)\psi_n'(ka)}\right]L_{mn}(\theta_o)\right\}, \quad (4.12)$$

$$b_{2m-1,2n} = -\left[\psi_m(ka)\zeta_n'(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_m^*(ka)\psi_n'(ka)}\right]S_{mn}(\theta_o)\,, \quad (4.13)$$

$$b_{2m,2n-1} = -\left[\psi_m'(ka)\,\zeta_n(ka) - \frac{W_2}{W_1}\frac{1}{\zeta_m'^*(ka)\,\psi_n(ka)}\right]S_{mn}(\theta_o), \quad (4.14)$$

$$y_{2n-1} = \frac{\zeta_n(ka)}{\psi_n(ka)} \, x_{2n-1} \quad , \qquad (4.15)$$

$$y_{2n} = \frac{\zeta_n'(ka)}{\psi_n'(ka)} \, x_{2n} \quad , \qquad (4.16)$$

for $n > 7$. For the most part, each Hankel or Bessel function now occurs in product form with a corresponding Bessel or Hankel function, and when n and m are large, the magnitudes of the functions are such as to compensate each other, leading to matrix elements, b_{pq}, whose magnitudes span only a reasonable range. The differing expressions for the elements when $n \leq 7$ and when $n > 7$ stem from the imposed requirement that the transformation matrix $\mathbb{T}$ and its inverse $\mathbb{T}^{-1}$ be non-singular to the range $0 < ka \leq 10$. It can be verified (see the NBS Tables, 1962) that neither $\psi_n(ka)$ nor $\psi_n'(ka)$ has zeros in this range if $n > 7$, and can also be shown that $\zeta_n(ka)$ and $\zeta_n'(ka)$ have no real zeros. Hence, neither $\mathbb{T}$ nor $\mathbb{T}^{-1}$ is singular for $0 < ka \leq 10$.

It is interesting to note that the Gauss-Seidel method still converges after the transformation. In place of Eq. (2.62), we now have

$$\mathbb{Y} = \mathbb{G}' \, \mathbb{Y} + \mathbb{T}^{-1} (\mathbb{D} - \mathbb{L})^{-1} \mathbb{F} \qquad (4.17)$$

where

$$\mathbb{G}' = \mathbb{T}^{-1} \, \mathbb{G} \, \mathbb{T} \qquad (\mathbb{T} \text{ non-singular})$$

and the Gauss-Seidel matrix $\mathbb{G}'$ associated with the matrix $\mathbb{B}$ has the same eigenvalues as the matrix $\mathbb{G}$. Thus, the spectral radius (maximum absolute value of the eigenvalues) of the matrix $\mathbb{G}'$ is less than unity, and the approach indicated in Eq. (4.17) converges.[+]

[+] The Gauss-Seidel method converges if and only if the spectral radius of its associated matrix is less than unity (see Varga, 1962).

V

CONCLUSIONS

This report has been devoted to a preliminary study of the scattering behavior of a spherical shell with a circular hole. Having postulated expansions of the field components in terms of spherical modes, it is necessary to satisfy the boundary conditions in the aperture and on the surface of the shell, and three numerical schemes for doing so have been discussed. Of the three, the method of least square error appears most promising, and this is the one that was used for the derivation of the numerical results.

In order to verify some of the theoretical findings, an experimental model was constructed consisting of a 3 inch diameter spherical shell with a 30^{o} (half angle) hole cut in it. Backscattering measurements were made at frequencies corresponding to ka = 0.9 to 3.3. For ka = 1.0, the cut shell has almost the same return as a complete shell (or solid sphere) of the same size, but as the frequency is increased, the effect of the aperture becomes apparent, and for ka near 2.75, the relative cross section experiences a sharp reduction of (about) 14 db.

The study of this particular geometry is still continuing and when an adequate selection of numerical and experimental data has been produced, it is our intention to extend the treatment to more complex structures such as those obtained by filling the cavity with a dielectric material or by inserting a concentric inner core. The prime objective in so doing is to transfer the main effect of the aperture and cavity to the vicinity of ka = 1.0.

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APPENDIX A

EVALUATION OF $L_{mn}(\theta_o)$ AND $S_{mn}(\theta_o)$.

From Eqs. (2.15) and (2.16) we have

$$L_{mn}(\theta_o) = \int_0^{\theta_o} \left(\frac{\partial P_m^1}{\partial \theta} \frac{\partial P_n^1}{\partial \theta} + \frac{P_m^1}{\sin\theta} \frac{P_n^1}{\sin\theta} \right) \sin\theta \, d\theta \quad , \qquad (A.1)$$

$$S_{mn}(\theta_o) = \int_0^{\theta_o} \left(\frac{P_m^1}{\sin\theta} \frac{\partial P_n^1}{\partial \theta} + \frac{P_n^1}{\sin\theta} \frac{\partial P_m^1}{\partial \theta} \right) \sin\theta \, d\theta \quad , \qquad (A.2)$$

which, on integrating by parts, become

$$L_{mn}(\theta_o) = \sin\theta \, P_m^1 \frac{\partial P_n^1}{\partial \theta} \Bigg]_{\theta=\theta_o} + n(n+1) \int_0^{\theta_o} P_m^1 P_n^1 \sin\theta \, d\theta \qquad (A.3)$$

$$S_{mn}(\theta_o) = P_n^1 P_m^1 \Bigg]_{\theta=\theta_o} \quad . \qquad (A.4)$$

The integral in Eq. (A.3) can be written as (Tsu, 1961)

$$\int_0^{\theta_o} P_m^1 P_n^1 \sin\theta \, d\theta = \begin{cases} \dfrac{\sin\theta_o}{(n-m)(n+m+1)} \left[P_n^1 \dfrac{\partial P_m^1}{\partial \theta} - P_m^1 \dfrac{\partial P_n^1}{\partial \theta} \right]_{\theta=\theta_o} & \text{for } m \neq n \qquad (A.5a) \\[2ex] \dfrac{\sin\theta_o}{(2n+1)} \left\{ \left(\dfrac{\partial P_\lambda^1}{\partial \lambda} \right)_{\lambda=n} \dfrac{\partial P_n^1}{\partial \theta} - P_n^1 \left[\dfrac{\partial}{\partial \lambda} \left(\dfrac{\partial P_\lambda^1}{\partial \theta} \right) \right]_{\lambda=n} \right\}_{\theta=\theta_o} & \text{for } m = n \qquad (A.5b) \end{cases}$$

and in the second of these the derivatives with respect to order, namely

$$\left(\frac{\partial P^1_\lambda}{\partial \lambda}\right)_{\lambda=n} \quad \text{and} \quad \left[\frac{\partial}{\partial \lambda}\left(\frac{\partial P^1_\lambda}{\partial \theta}\right)\right]_{\lambda=n} \quad ,$$

can be evaluated in closed form by using the recurrence relations for the Legendre functions, starting with (Tsu, 1961)

$$\left[\frac{\partial}{\partial \lambda} P_\lambda (\cos \theta)\right]_{\lambda=0} = \ell n \left(\cos^2 \frac{\theta}{2}\right)$$

and

$$\left[\frac{\partial}{\partial \lambda} P_\lambda (\cos \theta)\right]_{\lambda=1} = \cos \theta \left[\ell n \left(\cos^2 \frac{\theta}{2}\right) + 1\right] - 1 \, .$$

However, in the following it is shown that a more direct and convenient recurrence relation for the integral itself can be found without evaluating the derivatives of the Legendre functions with respect to order.

Let us define

$$I^\ell_n(\theta_o) = \int_0^{\theta_o} \left[P^\ell_n(\cos \theta)\right]^2 \sin \theta \, d\theta \qquad \text{(A.6)}$$

$$(n, \ell = 0, 1, 2, \ldots, \ell \leq n) \, ,$$

and seek a recurrence relation for the $I^\ell_n(\theta_o)$. If we also define a function

$$Q^\ell_n(\theta) = \sin \theta \left\{ \left(\frac{\partial P^\ell_\lambda}{\partial \lambda}\right)_{\lambda=n} \frac{\partial P^\ell_n}{\partial \theta} - P^\ell_n \left[\frac{\partial}{\partial \lambda}\left(\frac{\partial P^\ell_\lambda}{\partial \theta}\right)\right]_{\lambda=n} \right\} \, , \qquad \text{(A.7)}$$

then

$$I_n^{\ell}(\theta_o) = \frac{1}{2n+1} \quad Q_n^{\ell}(\theta_o) \tag{A.8}$$

and from the well known recurrence relation (Stratton, 1941)

$$\sin\theta \frac{\partial P_n^{\ell}}{\partial\theta} = n \cos\theta \, P_n^{\ell} - (n+\ell) \, P_{n-1}^{\ell} \tag{A.9}$$

by differentiating with respect to the order n, we obtain

$$\sin\theta \left[\frac{\partial}{\partial\lambda}\left(\frac{\partial P_\lambda^{\ell}}{\partial\theta}\right)\right]_{\lambda=n} = (\cos\theta \, P_n^{\ell} - P_{n-1}^{\ell}) + n\cos\theta\left(\frac{\partial P_\lambda^{\ell}}{\partial\lambda}\right)_n - (n+\ell)\left(\frac{\partial P_\lambda^{\ell}}{\partial\lambda}\right)_{n-1} \, . \tag{A.10}$$

Substitution of Eqs. (A.9) and (A.10) into (A.7) now gives

$$Q_n^{\ell}(\theta) = (n+\ell)\left[P_n^{\ell}\left(\frac{\partial P_\lambda^{\ell}}{\partial\lambda}\right)_{n-1} - P_{n-1}^{\ell}\left(\frac{\partial P_\lambda^{\ell}}{\partial\lambda}\right)_n\right] - \cos\theta\left[P_n^{\ell}\right]^2 + P_n^{\ell} P_{n-1}^{\ell} \, . \tag{A.11}$$

Similarly, starting with the recurrence relation

$$\sin\theta \frac{\partial P_n^{\ell}}{\partial\theta} = (n-\ell+1) \, P_{n+1}^{\ell} - (n+1)\cos\theta \, P_n^{\ell} \quad ,$$

we have

$$Q^{\ell}_{n-1}(\theta) = (n-\ell)\left[P^{\ell}_{n}\left(\frac{\partial P^{\ell}_{\lambda}}{\partial\lambda}\right)_{n-1} - P^{\ell}_{n-1}\left(\frac{\partial P^{\ell}_{\lambda}}{\partial\lambda}\right)_{n}\right] -$$

$$- P^{\ell}_{n-1}\,P^{\ell}_{n} + \cos\theta\left[P^{\ell}_{n-1}\right]^{2} \quad , \tag{A.12}$$

and hence, from Eqs. (A.11) and (A.12) with (A.8) the following recurrence relation for $I^{\ell}_{n}(\theta_o)$ results:

$$I^{\ell}_{n}(\theta_o) = \frac{1}{(2n+1)(n-\ell)}\left\{(2n-1)(n+\ell)\,I^{\ell}_{n-1}(\theta_o) - \right.$$

$$\left. - \cos\theta\left[(n+\ell)(P^{\ell}_{n-1})^{2} + (n-\ell)(P^{\ell}_{n})^{2}\right] + 2n\,P^{\ell}_{n-1}\,P^{\ell}_{n}\right\} \tag{A.13}$$

for $n > \ell$.

When $\ell = 1$, Eq. (A.13) becomes

$$I^{1}_{n}(\theta_o) = \frac{1}{(2n+1)(n-1)}\left\{(2n-1)(n+1)\,I^{1}_{n-1}(\theta_o)\right.$$

$$\left. - \cos\theta\left[(n+1)(P^{1}_{n-1})^{2} + (n-1)(P^{1}_{n})^{2}\right] + 2n\,P^{1}_{n-1}\,P^{1}_{n}\right\} \tag{A.14a}$$

for $n > 1$,

or, using Eq. (A.4),

$$I^{1}_{n}(\theta_o) = \frac{1}{(2n+1)(n-1)}\left\{(2n-1)(n+1)\,I^{1}_{n-1}(\theta_o)\right.$$

$$\left. - \cos\theta\left[(n+1)\,S_{n-1,n-1} + (n-1)S_{n,n}\right] + 2n\,S_{n,n-1}\right\} \tag{A.14b}$$

for $n > 1$.

Since

$$I_1^1(\theta_o) = 1/3\,(2 - 3\cos\theta_o + \cos^3\theta_o) \qquad \text{(A.15)}$$

by direct integration of Eq. (A.6), $I_n^1(\theta_o)$ can be evaluated for all n from the recurrence relation (A.14).

In summary, therefore, $L_{mn}(\theta_o)$ can be written as follows:

1) When $m \neq n$

$$L_{mn}(\theta_o) = \frac{\sin\theta_o}{(n-m)(n+m+1)}\left[n(n+1)\,P_n^1\,\frac{\partial P_m^1}{\partial\theta} - m(m+1)\,P_m^1\,\frac{\partial P_n^1}{\partial\theta}\right]_{\theta=\theta_o} \qquad \text{(A.16)}$$

2) When $m = n$

$$L_{mn}(\theta_o) = \left[\sin\theta\,P_n^1\,\frac{\partial P_n^1}{\partial\theta}\right]_{\theta=\theta_o} + n(n+1)\,I_n^1(\theta_o) \qquad \text{(A.17)}$$

where $I_n^1(\theta_o)$ is given by Eqs. (A.14) and (A.15).

APPENDIX B

PROOF OF POSITIVE DEFINITENESS OF THE MATRIX $\mathbb{A}$

In order to show that the right-hand side of Eq. (2.65) is always positive for $\theta_o \neq 0$ and $\mathbb{Y} = \left\{y_1, y_2, \ldots, y_{2M}\right\} \neq 0$, it is sufficient to show that it becomes zero only if $y_1 = y_2 = \ldots y_{2M} = 0$, since it is never negative. Therefore, we shall here show that for $\theta_o \neq 0$,

$$y_1 = y_2 = \ldots = y_{2M} = 0$$

is the only solution of the equations:

$$\sum_{n=1}^{M}\left[y_{2n-1}\psi_n(ka)\frac{P_n^1(\cos\theta)}{\sin\theta} + y_{2n}\psi_n'(ka)\frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right] = 0, \qquad \text{(B.1)}$$

$$\theta_o < \theta < \pi ,$$

$$\sum_{n=1}^{M}\left[y_{2n-1}\psi_n(ka)\frac{\partial}{\partial\theta}P_n^1(\cos\theta) + y_{2n}\psi_n'(ka)\frac{P_n^1(\cos\theta)}{\sin\theta}\right] = 0 , \qquad \text{(B.2)}$$

$$\theta_o < \theta < \pi ,$$

$$\sum_{n=1}^{M}\left[y_{2n-1}\frac{1}{\zeta_n(ka)}\frac{\partial}{\partial\theta}P_n^1(\cos\theta) + y_{2n}\frac{1}{\zeta_n'(ka)}\frac{P_n^1(\cos\theta)}{\sin\theta}\right] = 0 , \qquad \text{(B.3)}$$

$$0 < \theta < \theta_o ,$$

$$\sum_{n=1}^{M}\left[y_{2n-1}\frac{1}{\zeta_n(ka)}\frac{P_n^1(\cos\theta)}{\sin\theta} + y_{2n}\frac{1}{\zeta_n'(ka)}\frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right] = 0 , \qquad \text{(B.4)}$$

$$0 < \theta < \theta_o ,$$

which are the necessary and sufficient condtions for Eq. (2.65) to be zero.

By adding Eqs. (B.1) to (B.2), and (B.3) to (B.4), we have

$$\sum_{n=1}^{M}\left[y_{2n-1}\psi_n(ka) + y_{2n}\psi_n'(ka)\right]\left[\frac{P_n^1(\cos\theta)}{\sin\theta} + \frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right] = 0 \,, \quad \text{(B.5)}$$

$$\theta_o < \theta < \pi ,$$

$$\sum_{n=1}^{M}\left[y_{2n-1}\frac{1}{\zeta_n(ka)} + y_{2n}\frac{1}{\zeta_n'(ka)}\right]\left[\frac{P_n^1(\cos\theta)}{\sin\theta} + \frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right] = 0 \,, \quad \text{(B.6)}$$

$$0 < \theta < \theta_o ,$$

respectively.

Let

$$\phi_n(\theta) = \frac{P_n^1(\cos\theta)}{\sin\theta} + \frac{\partial}{\partial\theta}P_n^1(\cos\theta) \,.$$

All $\phi_n(\theta)$, n = 1, 2, ... are continuous everywhere and their derivatives of all orders, $\frac{\partial}{\partial\theta}\phi_n(\theta)$, $(\partial^2/\partial\theta^2)\phi_n(\theta)$, ..., exist and are also continuous. Hence each of the left-hand sides of Eqs. (B.5) and (B.6), which is a finite sum of $\phi_n(\theta)$'s, has the same properties as the $\phi_n(\theta)$'s. Consequently, Eqs. (B.5) and (B.6) must be satisfied not only in each of the given intervals, but also throughout the entire domain of θ. Furthermore, $\left\{\phi_1(\theta), \phi_2(\theta), \ldots, \phi_M(\theta)\right\}$ is a set of orthogonal, linear independent functions.

Therefore, we must have that each of the coefficients in Eqs. (B.5) and (B.6) vanishes. Thus, we have

$$y_{2n-1}\psi_n(ka) + y_2\psi_n'(ka) = 0 \quad \text{(B.7)}$$

$$y_{2n-1}\frac{1}{\zeta_n(ka)} + y_{2n}\frac{1}{\zeta_n'(ka)} = 0 \quad \text{(B.8)}$$

for n = 1, 2, . . . , M.

Similarly, by subtracting (B.2) from (B.1), and (B.3) from (B.4), we obtain

$$\sum_{n=1}^{M}\left[y_{2n-1}\psi_n(ka) - y_{2n}\psi_n'(ka)\right]\left[\frac{P_n^1(\cos\theta)}{\sin\theta} - \frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right] = 0, \qquad \text{(B.9)}$$

$$\theta_o < \theta < \pi ,$$

$$\sum_{n=1}^{M}\left[y_{2n-1}\psi_n(ka) - y_{2n}\psi_n'(ka)\right]\left[\frac{P_n^1(\cos\theta)}{\sin\theta} - \frac{\partial}{\partial\theta}P_n^1(\cos\theta)\right] = 0, \qquad \text{(B.10)}$$

$$0 < \theta < \theta_o ,$$

from which we must have

$$y_{2n-1}\psi_n(ka) - y_{2n}\psi'_n(ka) = 0 \quad , \qquad \text{(B.11)}$$

$$y_{2n-1}\frac{1}{\zeta_n(ka)} - y_{2n}\frac{1}{\zeta_n'(ka)} = 0 \quad , \qquad \text{(B.12)}$$

for $n = 1, 2, \ldots, M$.

It is clear from Eqs. (B.8) and (B.12) that $y_1 = y_2 = \ldots = y_{2M} = 0$ is the only solution for any finite value of ka. However, it should be noted that when $\theta_o = 0$, Eqs. (B.3) and (B.4) are deleted, and hence we have only Eqs. (B.7) and (B.11). Obviously the solution of these two equations is not unique when $\psi_n(ka) = 0$ or $\psi_n'(ka) = 0$, and there now exists a non-zero solution.

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13. ABSTRACT

A preliminary study of the scattering behavior of a spherical shell with a circular aperture is presented. The shell is assumed to be perfectly conducting and infinitesimally thin, and is illuminated by a plane electromagnetic wave symmetrically incident upon the aperture. Several different approaches to the solution of this problem are discussed, and the most promising one, the method of least square error, is described in detail. A numerical approach based on this scheme is devised, and values for the surface and back scattered far fields are given. In effect, the aperture and cavity provide a reactive load which modifies the scattering behavior of the complete shell (or sphere), and an example shows the cross section reduction achievable in this manner. Experimental confirmation was obtained, and the study is continuing.

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