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GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS FOR PLASMA SYSTEMS WITH LARGE CURRENTS

by

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ABSTRACT

The differential equations for number densities, mean velocities, temperatures, stresses, and heat fluxes of the electrons and ions, obtained by taking velocity moments of the Boltzmann equations, are closed out by utilizing Grad 13-moment approximations for the distribution functions. These equations, coupled with Maxwell's equations, constitute a closed set. The approximate distribution function, for a given type particle, is such that the first term is Maxwellian in a coordinate system moving with the mean velocity of those particles; the higher order terms are nonsymmetric in velocity space and have for coefficients the non-hydrostatic (traceless) stress tensor and the heat flux vector defined relative to the same mean velocity. The range of validity of the resulting magnetohydrodynamic equations is anticipated to extend to systems in which the difference in flow velocity of plasma constituents is not negligible relative to the thermal velocity of the electrons. By limiting attention to systems with slowly varying flows, it is found possible to exhibit transport properties to third order in the ratio of the difference in constituent flow velocity to the electron thermal velocity. Transport properties may also be exhibited for systems in which the above ratio is large.

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I. INTRODUCTION

Accepting the Boltzmann equation as providing an adequate description for the behavior of gas constituents, continuum equations for macroscopic properties of gases have been developed by many authors. The Chapman-Enskog method has been developed in detail in the classical treatise by Chapman and Cowling, "The Mathematical Theory of Non-Uniform Gases"[1]. A different method, which utilizes a Hermite polynomial expansion for the distribution function, has been developed for a one constituent gas by Grad[2,3], and generalized to include gas mixtures by Kolodner[4]. These methods have been found successful in providing descriptions for systems of neutral particle gases close to equilibrium.

With the increased interest, in recent years, in plasma systems, attempts have been made to generalize these developments to include systems in which the gas constituents are ions and electrons [5,6,7,8]. Although these developments provide closed sets of equations for macroscopic properties for systems close to equilibrium, their value is questionable for systems away from equilibrium. Of particular interest are systems in which the difference in flow velocity of the constituents is appreciable relative to the random (thermal) velocity of electrons. (See, for example, comments by Tuck, Allis, Burgers, and others [9]).

In the Marshall development, which utilizes the Chapman-Enskog method, attention is restricted to systems in which collision effects and external magnetic field effects dominate the behavior of the plasma, giving rise to a first order velocity space solution, for each constituent which is Maxwellian relative to the plasma flow velocity. To this order, the difference in flow velocity of the constituents is zero. With the second order velocity space solution, a first approximation for this difference can be obtained, however, it is not anticipated to be adequate for systems in which this difference is appreciable[10].

With the realization that a Maxwellian distribution, relative to the plasma flow velocity, is not an adequate first approximation for systems in which the difference in constituent flow velocity is appreciable, an attempt was made to modify the Chapman-Enskog approach in such a way that the resulting distribution functions would be appropriate for this wider class of systems. In this attempt, only like-particle collisions and magnetic field effects were taken to be of first order, giving rise to a first order solution, for each constituent, Maxwellian relative to the constituent flow velocity. Since like-particle collisions are more important than unlike-particle collisions for transfer of ion momentum and energy, and electron energy, the unlike-particle collisions were taken to be of second order. The second order velocity space solutions were obtained, the moment equations closed out, and transport properties exhibited. However, one shortcoming was in evidence. With regard to collision transfer of momentum, the important coupling with heat flux did not enter until third order; and unfortunately the third order solution is very difficult to obtain as a result of mathematical complexities. The above difficulty is a result of the fact that, for momentum transfer of electrons, unlike-particle collisions are of the same order of importance as like-particle collisions. As a result of this difficulty, and for reasons of lesser importance, attention was diverted to an adaptation of the Grad 13-moment method.

The Grad 13-moment method has been utilized for plasma systems by Kolodner[6] and by Burgers[7]. In these developments, the distribution functions assumed are based on Hermite polynomial velocity space expansions. As in the Marshall distributions, the first term, for each constituent, is Maxwellian relative to the plasma flow velocity. The non-symmetric higher order terms have for coefficients the constituent flow velocity, stresses, and heat flux. Closed sets of equations for macroscopic properties have been exhibited, as well as transport coefficients to first order in the ratio of the difference in constituent flow velocity to the random electron velocity.

In the method developed herein, the Grad method is modified such that the results are expected to have validity over the extended range of systems in which the above ratio is not small. The Hermite polynomial velocity space expansion is utilized; however, the independent variable is taken to be the particle velocity relative to the flow velocity of the constituent. The first term of the expansion is then Maxwellian relative to the constituent flow velocity, and the non-symmetric higher order terms have for coefficients the stresses and heat flux defined relative to the constituent flow velocity.

With regard to the Boltzmann equation, which is assumed to provide an adequate description for plasma systems, the point of view is taken that the binary collision model is appropriate for collisions with impact parameter less than the Debye shielding distance, and that distant encounters (those with impact parameter greater than the Debye shielding distance) are accounted for by the use of the inhomogeneous Maxwell's equations[4,5].

In the next section the basis of the general theory is outlined and the method of approach presented. The method of computing the collision transfer terms is given in section III, followed in section IV by a presentation of the resulting set of magnetohydrodynamic equations, which coupled with Maxwell's field equations, constitutes a closed set. In section V, attention is restricted to systems with slowly varying flows. The equations are simplified and transport parameters are exhibited to third order in the ratio of the difference in component flow velocity to the random electron velocity. Comparison is made with the results obtained from the Kolodner method, as exhibited by Herdan and Liley[8]. Attention is also given to systems in which the ratio of the difference in component flow velocity to the random electron velocity is very large. Here again, the equations are reduced and transport coefficients exhibited. Finally in section VI, a general discussion is given on the method of development and the results obtained.

Gaussian units are used throughout, and all symbols are defined where they first appear.

II. GENERAL THEORY AND METHOD OF APPROACH

The Boltzmann equation for the distribution function of particles of type α may be written,

$$\frac{\partial f^\alpha}{\partial t} + v_i \frac{\partial f^\alpha}{\partial r_i} + \frac{e^\alpha}{m^\alpha} (E_i + \frac{1}{c} \varepsilon_{ijk} v_j H_k) \frac{\partial f^\alpha}{\partial v_i} = \sum_\beta{}' I(f^\alpha f^\beta) \tag{II-1}$$

where e^α and m^α are respectively the charge and mass of the α-type particle. $I(f^\alpha f^\beta)$ expresses the change in f^α per unit time due to binary collisions of the α-type particles with β-type particles. These collision terms are considered in detail in section III. The electric and magnetic fields are governed by Maxwell's field equations,

$$\text{(a)} \quad \varepsilon_{ijk} \frac{\partial H_k}{\partial r_j} = \frac{1}{c} \frac{\partial E_i}{\partial t} + \frac{4\pi}{c} J_i$$

$$\text{(b)} \quad \varepsilon_{ijk} \frac{\partial E_k}{\partial r_j} = -\frac{1}{c} \frac{\partial H_i}{\partial t}$$

(II-2)

$$\text{(c)} \quad \frac{\partial H_i}{\partial r_i} = 0$$

$$\text{(d)} \quad \frac{\partial E_i}{\partial r_i} = 4\pi \; q \, ,$$

with appropriate boundary conditions to account for externally applied fields. The current density and charge density, J_k and q, are defined by

$$\text{(a)} \quad J_k = n^i Z e w_k^i - n^e e w_k^e$$

(II-3)

$$\text{(b)} \quad q = n^i Z e - n^e e.$$

The number densities n^{α} and the constituent flow velocities w_k^{α} are velocity moments of the distribution function f^{α}, that is,

$$
\begin{aligned}
&\text{(a)} \quad n^{\alpha} = \int f^{\alpha} d^3v \\
&\text{(b)} \quad w_k^{\alpha} = \frac{1}{n^{\alpha}} \int f^{\alpha} v_k \, d^3v.
\end{aligned}
\tag{II-4}
$$

Transport equations for these and higher order moments of the distribution functions (temperatures, stresses, heat fluxes, etc.) are obtained by taking velocity moments of Boltzmann's equations (II-1). For this purpose it is convenient to transform independent variables in equation (II-1) from the actual particle velocity v_i to the random velocity $u_i = v_i - w_i^{\alpha}$. With these transformations, equations (II-1) become,

$$
\frac{D_{\alpha} f^{\alpha}}{Dt} + u_i \frac{\partial f^{\alpha}}{\partial r_i} + A_i^{\alpha} \frac{\partial f^{\alpha}}{\partial u_i} + \frac{e^{\alpha}}{m^{\alpha} c} \mathcal{E}_{ijk} u_j H_k \frac{\partial f^{\alpha}}{\partial u_i} - \frac{\partial f^{\alpha}}{\partial u_i} u_j \frac{\partial w_j^{\alpha}}{\partial r_i} = \sum_{\beta} I(f^{\alpha} \bar{f}^{\beta})
\tag{II-5}
$$

where

$$
\begin{aligned}
&\text{(a)} \quad \frac{D_{\alpha}}{Dt} = \frac{\partial}{\partial t} + w_i^{\alpha} \frac{\partial}{\partial r_i} \\
&\text{(b)} \quad A_i^{\alpha} = \left(\frac{e^{\alpha}}{m^{\alpha}} E_i + \frac{e^{\alpha}}{m^{\alpha} c} \mathcal{E}_{ijk} w_j^{\alpha} H_k - \frac{D_{\alpha} w_i^{\alpha}}{Dt} \right)
\end{aligned}
\tag{II-6}
$$

On multiplication of equation (II-5) by an arbitrary function $\psi(u_k)$, and integrating over the infinite space of u_k, we obtain,

$$\frac{D_\alpha\left\{n^\alpha\left\langle\psi\right\rangle\right\}}{Dt}+\frac{\partial\left\{n^\alpha\left\langle u_i\psi\right\rangle\right\}}{\partial r_i}-A_i^\alpha\, n^\alpha\left\langle\frac{\partial\psi}{\partial u_i}\right\rangle$$

$$-\frac{e^\alpha n^\alpha}{m^\alpha c}\ \varepsilon_{ijk}\, H_k\left\langle\frac{\partial\left\{u_j\,\psi\right\}}{\partial u_i}\right\rangle+n^\alpha\left\langle\frac{\partial u_j}{\partial u_i}\ \psi\right\rangle\frac{\partial w_i^\alpha}{\partial r_j} \tag{II-7}$$

$$=\sum_\beta\int I(f^\alpha\,\bar{f}^\beta)\ \psi\ d^3u$$

where,

$$\left\langle\psi\right\rangle=\frac{1}{n^\alpha}\int f^\alpha\ \psi\ d^3u\ . \tag{II-8}$$

Taking ψ equal to 1, $m^\alpha u_k$, $\frac{1}{2}\ m^\alpha u^2$, $m^\alpha \overset{o}{u_k u_\ell}$, and $\frac{1}{2}\ m^\alpha u^2 u_k$; transport equations are obtained for the number densities, flow velocities, temperatures, non-hydrostatic (traceless) stress tensors, and heat fluxes of the plasma constituents.

(a) $\dfrac{D_\alpha n^\alpha}{Dt}+n^\alpha\dfrac{\partial w_i^\alpha}{\partial r_i}=0$

(b) $\dfrac{D_\alpha w_i^\alpha}{Dt}=\dfrac{e^\alpha}{m^\alpha}\left(E_i+\dfrac{1}{c}\ \varepsilon_{ijk}\ w_j^\alpha H_k\right)-\dfrac{1}{\rho^\alpha}\ \dfrac{\partial P_{ij}^\alpha}{\partial r_j}+\dfrac{1}{\rho^\alpha}\ S_i^\alpha$

(II-9)

(c) $\dfrac{D_\alpha T^\alpha}{Dt}=\dfrac{2}{3n^\alpha k}\left\{-\dfrac{\partial Q_i^\alpha}{\partial r_i}-P_{ij}^\alpha\ \dfrac{\partial w_j^\alpha}{\partial r_i}+R^\alpha\right\}$

(d) $\dfrac{D_\alpha \overset{o}{P}{}_{rs}^\alpha}{Dt}=-\dfrac{\partial}{\partial r_i}\ \rho^\alpha\left\langle u_i\ \overset{o}{u_r\, u_s}\right\rangle+\dfrac{2e^\alpha}{m^\alpha c}\ \varepsilon_{rjk}\ \overset{o}{P}{}_{is}^\alpha\, H_k-\overset{o}{P}{}_{rs}^\alpha\ \dfrac{\partial w_i^\alpha}{\partial r_i}$

$\qquad\qquad -\ 2\ \overline{\overline{P_{jr}^\alpha\ \dfrac{\partial w_s^\alpha}{\partial r_j}}}^{\,o}+\overset{o}{L}{}_{rs}^\alpha$

$$\text{(e)}\quad \frac{D_\alpha Q^\alpha_r}{Dt} = -\frac{\partial}{\partial r_i}\frac{\rho^\alpha}{2}\left\langle u^2 u_i u_r\right\rangle + \frac{5}{2}p^\alpha A^\alpha_r + \overset{o}{P}{}^\alpha_{ir} A^\alpha_i + \frac{e^\alpha}{m^\alpha c}$$

(II-9) cont.

$$\mathcal{E}_{rjk} Q^\alpha_j H_k - Q^\alpha_r \frac{\partial w^\alpha_i}{\partial r_i} - Q^\alpha_j \frac{\partial w^\alpha_r}{\partial r_j} - \rho^\alpha \left\langle u_i u_j u_r\right\rangle \frac{\partial w^\alpha_i}{\partial r_j} + M^\alpha_r$$

ρ^α is the mass density, S^α_i, R^α, $\overset{o}{L}{}^\alpha_{rs}$ and M^α_r are the collision transfer of momentum, energy, traceless stress, and heat flux respectively; that is,

$$\text{(a)}\quad S^\alpha_r = \sum_\beta \int (m^\alpha u_r)\, I(f^\alpha \bar{f}^\beta)\, d^3u$$

$$\text{(b)}\quad R^\alpha = \sum_\beta \int (\tfrac{1}{2} m^\alpha u^2)\, I(f^\alpha \bar{f}^\beta)\, d^3u \qquad \text{(II-10)}$$

$$\text{(c)}\quad \overset{o}{L}{}^\alpha_{rs} = \sum_\beta \int (m^\alpha \overset{o}{u_r u_s})\, I(f^\alpha \bar{f}^\beta)\, d^3u$$

$$\text{(d)}\quad M^\alpha_r = \sum_\beta \int (\tfrac{1}{2} m^\alpha u^2 u_r)\, I(f^\alpha \bar{f}^\beta)\, d^3u.$$

A circle above a second rank tensor is used to designate a traceless tensor, and a double bar is used to designate a symmetrized tensor. That is,

$$\text{(a)}\quad \overset{o}{T}_{rs} = T_{rs} - \frac{1}{3}\delta_{rs} T_{ii}$$

(II-11)

$$\text{(b)}\quad \overline{\overline{T}}_{rs} = \frac{1}{2}(T_{rs} + T_{sr}) \ .$$

The temperature, stress, and heat flow of a given constituent are defined relative to the flow velocity of that constituent:

$$
\begin{aligned}
&(a) \quad T^{\alpha} = \frac{2}{3k} \left\langle \frac{m^{\alpha}}{2} u^2 \right\rangle \\
&(b) \quad P^{\alpha}_{rs} = n^{\alpha} \left\langle m^{\alpha} u_r u_s \right\rangle \\
&(c) \quad Q^{\alpha}_{r} = n^{\alpha} \left\langle \frac{m^{\alpha}}{2} u^2 u_r \right\rangle \quad .
\end{aligned}
\tag{II-12}
$$

The temperature and pressure tensor are related by the expression,

$$
n^{\alpha} k \, T^{\alpha} = \frac{1}{3} P^{\alpha}_{ii} \quad , \tag{II-13}
$$

as may be seen from (II-12 , a and b.) Also

$$
p^{\alpha} = n^{\alpha} k \, T^{\alpha} = \frac{1}{3} P^{\alpha}_{ii} \tag{II-14}
$$

defines the hydrostatic pressure of the α constituent relative to the constituent flow velocity.

The set of equations (II-9) coupled with Maxwell's equations may be closed out if a representation of the distribution functions f^{α} can be found in terms of the macroscopic properties, n^{α}, w^{α}_{k}, T^{α}, $\overset{o}{P}{}^{\alpha}_{rs}$, and Q^{α}_{r} ; permitting the determination of $\left\langle u_i \overset{o}{u_r u_s} \right\rangle$, $\left\langle u^2 u_i u_r \right\rangle$, S^{α}_{r} , R^{α}, $\overset{o}{L}{}^{\alpha}_{rs}$, and M^{α}_{r} in terms of these properties.

A representation of f^{α} meeting these requirements is the Grad 13-moment approximation[2, 3] , obtained by expanding the distribution functions in three dimensional Hermite polynomials and retaining terms up to and including a contraction of the third rank polynomial.

The three dimensional Hermite polynomials are defined[3],

$$\text{(a)} \quad \mathcal{H}^{(n)}_{i_1 i_2 \ldots i_n} = \frac{(-1)^n}{\omega} \frac{\partial^n \omega}{\partial c_{i_1} \partial c_{i_2} \ldots \partial c_{i_n}}$$

where (II-15)

$$\text{(b)} \quad \omega = \frac{1}{(2\pi)^{3/2}} e^{-c^2/2}$$

The first few polynomials are,

$$\text{(a)} \quad \mathcal{H}^{(o)} = 1$$

$$\text{(b)} \quad \mathcal{H}^{(1)}_{i} = c_i$$

$$\text{(c)} \quad \mathcal{H}^{(2)}_{ij} = c_i c_j - \delta_{ij} \qquad \text{(II-16)}$$

$$\text{(d)} \quad \mathcal{H}^{(3)}_{ijk} = c_i c_j c_k - (c_i \delta_{jk} + c_j \delta_{ik} + c_k \delta_{ij}).$$

An arbitrary scalar function $g(\vec{c})$ may be represented by the expansion,

$$g(\vec{c}) = \omega^{1/2} \sum_{n=0}^{\infty} a^{(n)}_{i_1 i_2 \ldots i_n} \mathcal{H}^{(n)}_{i_1 i_2 \ldots i_n} , \qquad \text{(II-17)}$$

and the coefficients determined by multiplying both sides of (II-17) by $\omega^{1/2} \mathcal{H}^{(n)}_{i_1 i_2 \ldots i_n}$ and integrating over the infinite space of c_k. That is

$$a^{(n)}_{i_1 i_2 \ldots i_n} = \frac{1}{n!} \int \omega^{1/2} \mathcal{H}^{(n)}_{i_1 i_2 \ldots i_n} g(c_k)\, d^3 c \qquad \text{(II-18)}$$

Letting $c_k = \sqrt{m^\alpha / kT^\alpha}\, u_k$, the distribution functions may be represented by the expansions

$$\omega^{-1/2} f^\alpha = \omega^{1/2} \sum_{n=0}^{\infty} a^{(n)}_{i_1 i_2 \ldots i_n} \mathcal{H}^{(n)}_{i_1 i_2 \ldots i_n} \qquad \text{(II-19)}$$

Terminating the expansions with a contraction of $\mathcal{H}^{(3)}_{ijk}$ (i.e. $\delta_{ij}\mathcal{H}^{(3)}_{ijk} = c^2 c_k - 5c_k$), and evaluating coefficients, we obtain,

$$f^\alpha = n^\alpha \left(\beta^\alpha/2\pi\right)^{3/2} e^{-\frac{\beta^\alpha}{2}u^2} \left\{ 1 + \frac{\beta^{\alpha 2}}{2\rho^\alpha} \overset{o}{P}{}^\alpha_{ij} u_i u_j + \frac{\beta^{\alpha 3}}{5\rho^\alpha} Q^\alpha_i \left(u^2 u_i - \frac{5}{\beta^\alpha} u_i\right) \right\} , \quad \text{(II-20)}$$

where $\beta^\alpha = \dfrac{m^\alpha}{kT^\alpha}$.

It may be noted that, for each constituent, the coefficient of the polynomial of rank n involves the nth velocity moment of the distribution function; hence, the use of these solutions implies a restriction to systems in which moments of the distribution functions higher than those which define the heat fluxes are small relative to the traceless pressure tensors and heat fluxes. To my knowledge, no velocity moments beyond the heat flux have been found to be of physical importance in hydrodynamics or in magnetohydrodynamics, hence it is anticipated that the solutions have validity over a wide range of systems. More detailed arguments in behalf of these solutions are given by Grad[(2)].

As pointed out previously, the approximate velocity space solutions (II-20) make it possible to determine $\left\langle u_i \overset{o}{u_r} u_s \right\rangle$, $\left\langle u^2 u_i u_r \right\rangle$, S^α_r, R^α, $\overset{o}{L}{}^\alpha_{rs}$ and M^α_r in terms of n^α , w^α_k , T^α, $\overset{o}{P}{}^\alpha_{rs}$, and Q^α_r . Then the transport equations (II-20) coupled with Maxwell's equations (II-2) constitute a closed set.

The next section is devoted to the determination of the collision transfer of momentum, energy, stress, and heat flux. In section IV, the expressions for $\left\langle u_i \overset{o}{u_r} u_s \right\rangle$ and $\left\langle u^2 u_i u_r \right\rangle$ are given, and the resulting closed set of equations exhibited explicitly.

III. COLLISION TRANSFER

The collision transfer terms may be written in the conventional form, (see, for example, chapter 3 of reference 1),

$$\int \psi^{\alpha}\, I(f^{\alpha} f^{\gamma}) d^3u^{\alpha} = -\int f^{\alpha} f^{\gamma} (\psi^{\alpha} - \psi^{\alpha'}) v_r\, b\, db\, d\phi\; d^3u^{\gamma} d^3u^{\alpha} \quad . \tag{III-1}$$

The prime is used to designate post collision variables. The range of integration on the azimuthal angle ϕ is from 0 to 2π, and on the impact parameter b, from 0 to the Debye shielding distance h. Also, $v_r = |\vec{v}^{\alpha} - \vec{v}^{\gamma}|$, the relative velocity between interacting particles. The Greek symbols α and γ can correspond to either electrons or ions. For example, if α corresponds to electrons, γ to ions, and ψ^{α} to $\psi^{e} = \frac{m^e}{2} u^{e^2} u^e_j$, expression (III-1) is the transfer of electron heat flow resulting from electron-ion collisions. Similarly, if α corresponds to ions, γ to ions, and ψ^{α} to $\psi^{i} = m^i u^i_r u^i_s$, expression (III-1) is the transfer of ion stress resulting from ion-ion collisions

For the Coulomb interactions involved in a fully ionized plasma, the collision cross section is given by the well-known Rutherford formula,

$$\sigma(\chi, \phi) = \frac{1}{4} \left(\frac{e^{\alpha} e^{\gamma}}{m^{\alpha\gamma}} \right)^2 \frac{1}{v_r^4 \sin^4 \frac{\chi}{2}} \quad , \tag{III-2}$$

where χ is the angle through which v_r is deflected, and

$$m^{\alpha\gamma} = \frac{m^{\alpha} m^{\gamma}}{m^{\alpha} + m^{\gamma}} \quad . \tag{III-3}$$

Hence,

$$b\,db\,d\phi = \sigma d\vec{\Omega} = \sigma \sin\chi\, d\chi\, d\phi = \left(\frac{e^{\alpha} e^{\gamma}}{m^{\alpha\gamma}}\right)^2 \frac{1}{v_r^4} \frac{d(\sin\frac{\chi}{2})\, d\phi}{\sin^3 \frac{\chi}{2}} \quad . \tag{III-4}$$

The limits on $\sin \frac{\chi}{2}$ are from 1 to $1/\sqrt{1+\left(\frac{h v_r^2 m^{\alpha\gamma}}{e^{\alpha} e^{\gamma}}\right)^2}$ which correspond to the limits on the impact parameter b, from 0 to the Debye distance h.

To facilitate the integration over scattering angles, $(\psi^{\alpha}-\psi^{\alpha'})$ are expressed in terms of a collision invariant vector c_k^{α} and the relative velocity between interacting particles which rotates through the angle χ on collision.

To simplify the notation we let

(a) $$M^{\alpha} = \frac{m^{\alpha}}{m^{\alpha} + m^{\gamma}}$$

(b) $$M^{\gamma} = \frac{m^{\gamma}}{m^{\alpha} + m^{\gamma}}$$

(c) $$\beta^{\alpha} = \frac{m^{\alpha}}{k\,T^{\alpha}} \tag{III-5}$$

(d) $$\beta = \beta^{\alpha} + \beta^{\gamma}$$

(e) $$\beta' = \frac{\beta^{\alpha} \beta^{\gamma}}{\beta^{\alpha} + \beta^{\gamma}}$$

(f) $$\beta^{o} = \frac{\beta^{e} \beta^{i}}{\beta^{e} + \beta^{i}}$$

Utilizing the relations for conservation of energy and momentum, it is easily shown that,

$$\text{(a)} \qquad u_k^\alpha = c_k^\alpha + M^\gamma v_{r_k}$$

$$\text{(b)} \qquad u_k^{\alpha'} = c_k^\alpha + M^\gamma v'_{r_k} \quad . \tag{III-6}$$

The vector c_k^α is equal to the velocity of the center of mass minus the mean flow velocity for particles of type α. Then, for $\psi^\alpha = m^\alpha u_k^\alpha$; $m^\alpha u_k^\alpha u_\ell^\alpha$; $\frac{m^\alpha}{2} u^{\alpha^2} u_k^\alpha$; the expressions for $(\psi^\alpha - \psi^{\alpha'})$ may be written

$$\text{(a)} \qquad m^\alpha (u_k^\alpha - u_k^{\alpha'}) = m^{\alpha\gamma} (v_{r_k} - v'_{r_k})$$

$$\text{(b)} \qquad m^\alpha (u_k^\alpha u_\ell^\alpha - u_k^{\alpha'} u_\ell^{\alpha'}) = 2m^{\alpha\gamma} \overline{\overline{c_k^\alpha (v_{r_\ell} - v'_{r_\ell})}} + m^{\alpha\gamma} M^\gamma (v_{r_k} v_{r_\ell} - v'_{r_k} v'_{r_\ell}) \tag{III-7}$$

$$\text{(c)} \qquad \frac{m^\alpha}{2} (u^{\alpha^2} u_k^\alpha - u^{\alpha'^2} u_k^{\alpha'}) = \frac{m^{\alpha\gamma}}{2} c^{\alpha^2} (v_{r_k} - v'_{r_k}) + \frac{m^{\alpha\gamma}}{2} M^{\gamma^2} v_r^2 (v_{r_k} - v'_{r_k})$$

$$+ m^{\alpha\gamma} c_k^\alpha c_\ell^\alpha (v_{r_\ell} - v'_{r_\ell}) + m^{\alpha\gamma} M^\gamma c_\ell^\alpha (v_{r_k} v_{r_\ell} - v'_{r_k} v'_{r_\ell}) \quad .$$

Perhaps it should be pointed out that the collision exhange term in which $\psi^\alpha = m^\alpha u_k^\alpha u_\ell^\alpha$ contains the information required for both the energy balance equation (II-9, c) and the traceless stress equation (II-9, d) since,

$$\text{(a)} \qquad \frac{m^{\alpha}}{2} u^{\alpha 2} = \frac{\delta_{k\ell}}{2} m^{\alpha} u_k^{\alpha} u_\ell^{\alpha}$$

$$\text{(b)} \qquad m^{\alpha} \overset{o}{u_k^{\alpha} u_\ell^{\alpha}} = m^{\alpha} (u_k^{\alpha} u_\ell^{\alpha} - \frac{\delta_{k\ell}}{3} u^{\alpha 2}) \quad . \qquad \text{(III-8)}$$

It is now possible to carry out the integrations over angles of scatter. On substitution of expressions (III-7) and (III-4) into (III-1), we see that only two types of integrals are involved. These have been carried out in Appendix A, where it is found that,

$$\text{(a)} \int (v_{r_k} - v'_{r_k}) \frac{d(\sin \frac{\chi}{2})}{\sin^3 \frac{\chi}{2}} d\phi = 4\pi \, v_{r_k} \Omega^{(1)}_{\alpha\gamma}$$

$$\text{(b)} \int (v_{r_k} v_{r_\ell} - v'_{r_k} v'_{r_\ell}) \frac{d(\sin \frac{\chi}{2})}{\sin^3 \frac{\chi}{2}} d\phi = 12\pi \, \overset{o}{v_{r_k} v_{r_\ell}} \Omega^{(2)}_{\alpha\gamma} \quad , \qquad \text{(III-9)}$$

where,

$$\text{(a)} \Omega^{(1)}_{\alpha\gamma} \equiv \frac{1}{2} \ln \left\{ 1 + \left(\frac{m^{\alpha\gamma} h}{e^{\alpha} e^{\gamma}} \right)^2 v_r^4 \right\}$$

$$\text{(b)} \Omega^{(2)}_{\alpha\gamma} \equiv \frac{1}{2} \ln \left\{ 1 + \left(\frac{m^{\alpha\gamma} h}{e^{\alpha} e^{\gamma}} \right)^2 v_r^4 \right\} - \frac{1}{2} \left[\frac{\left(\frac{m^{\alpha\gamma} h}{e^{\alpha} e^{\gamma}} \right)^2 v_r^4}{1 + \left(\frac{m^{\alpha\gamma} h}{e^{\alpha} e^{\gamma}} \right)^2 v_r^4} \right] \quad . \qquad \text{(III-10)}$$

Inasmuch as the value that should be used for the upper limit of the impact parameter is not fixed precisely, and the results are not very sensitive to the relative velocity, we replace v_r^2 by an approximate average obtained by assuming the distribution functions of the interacting particles to be given by the Maxwellian parts only of distributions (II-20). This approximate average is found to be,

$$\overline{v_r^2} = \frac{1}{\beta'} (3 + \varepsilon^2), \text{ where } \varepsilon \equiv \beta^{o^{1/2}} \left| w_k^{\alpha} - w_k^{\gamma} \right|. \tag{III-11}$$

Also, $\frac{m^{\alpha\gamma} h}{e^{\alpha} e^{\gamma}} \overline{v_r^2} \gg 1$, so we take,

$$\text{(a)}\ \Omega^{(1)}_{\alpha\gamma} = \ln \left\{ (3+\varepsilon^2) \frac{m^{\alpha\gamma}}{\beta'} \frac{h}{e^{\alpha} e^{\gamma}} \right\}$$

$$\text{(b)}\ \Omega^{(2)}_{\alpha\gamma} = \ln \left\{ (3+\varepsilon^2) \frac{m^{\alpha\gamma}}{\beta'} \frac{h}{e^{\alpha} e^{\gamma}} \right\} - \frac{1}{2} \quad . \tag{III-12}$$

The collision integrals (III-1) may now be written in the form,

$$- \frac{4\pi (e^{\alpha} e^{\gamma})^2 \Omega^{(1)}_{\alpha\gamma}}{m^{\alpha\gamma}} \int f^{\alpha} f^{\gamma} \frac{\Psi^{\alpha}}{v_r^3} d^3 u_{\alpha} d^3 u_{\gamma} \quad , \tag{III-13}$$

where, for momentum transfer, stress transfer, and heat flux transfer, Ψ^{α} is equal respectively to,

$$\text{(a)} \qquad v_{r_k}$$

$$\text{(b)} \qquad 2 \overline{\overline{c_k^{\alpha} v_{r_\ell}}} + 3M^{\gamma} \frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}} \overset{o}{v_{r_k} v_{r_\ell}} \tag{III-14}$$

$$\text{(c)} \qquad \frac{1}{2} c^{\alpha^2} v_{r_k} + \frac{1}{2} M^{\gamma^2} v_r^2 v_{r_k} + c_k^{\alpha} c_\ell^{\alpha} v_{r_\ell} + 3M^{\gamma} \frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}} c_\ell^{\alpha} \overset{o}{v_{r_k} v_{r_\ell}} \quad .$$

To facilitate the integrations over velocity spaces, we define new dimensionless variables, g_k and G_k, by the relations

$$\text{(a)} \quad \beta^{\alpha\,1/2}\, u_k^{\alpha} \equiv b^{\alpha}\left\{ a^{\alpha} G_k + g_k - \varepsilon_k^{\alpha} \right\}$$

$$\text{(b)} \quad \beta^{\gamma\,1/2}\, u_k^{\gamma} \equiv b^{\gamma}\left\{ a^{\gamma} G_k + g_k - \varepsilon_k^{\alpha} \right\}, \tag{III-15}$$

where

$$\text{(a)} \quad a^{\alpha} \equiv \left(\frac{\beta^{\alpha}}{\beta^{\gamma}}\right)^{1/2}$$

$$\text{(b)} \quad b^{\alpha} \equiv \left(\frac{\beta^{\gamma}}{\beta}\right)^{1/2}$$

$$\text{(c)} \quad a^{\gamma} \equiv -\left(\frac{\beta^{\gamma}}{\beta^{\alpha}}\right)^{1/2} \tag{III-16}$$

$$\text{(d)} \quad b^{\gamma} \equiv -\left(\frac{\beta^{\alpha}}{\beta}\right)^{1/2}$$

and

$$\varepsilon_k^{\alpha} \equiv \beta^{o\,1/2}\,(w_k^{\alpha} - w_k^{\gamma}) \quad . \tag{III-17}$$

With this change of variables, it may be shown that:

$$\text{(a)} \quad v_{r_k} = \frac{1}{\beta'^{1/2}}\, g_k$$

$$\text{(b)} \quad c_k^{\alpha} = \frac{1}{\beta'^{1/2}}\left\{ \left(\frac{\beta'}{\beta}\right)^{1/2} G_k + \chi^{\alpha}\, g_k - \frac{\beta^{\gamma}}{\beta}\,\varepsilon_k^{\alpha} \right\} \quad , \tag{III-18}$$

so that the expressions (III-14) for $\underline{\Psi}^{\alpha}$ may be written, in terms of the new variables,

(a) $\frac{1}{\beta'^{1/2}} \quad g_k$

(b) $\frac{1}{\beta'} \left\{ d^{(1)}_{\alpha\gamma} \; \overline{\overline{\varepsilon^{\alpha}_k g_\ell}} \; + d_{k\ell rs} g_r g_s + d^{(4)}_{\alpha\gamma} \; \overline{\overline{G_k g_\ell}} \right\}$ (III-19)

(c) $\frac{1}{\beta'^{3/2}} \left\{ h^{(1)}_{kr} \, g_r + h^{(2)}_{krs} \, g_r g_s + e^{(4)}_{\alpha\gamma} \, g^2 g_k + h^{(3)}_{krst} \, g_r G_s G_t \right.$

$\left. + h^{(4)}_{krs} \, G_r g_s + h^{(5)}_{krst} G_r g_s g_t \right\}$,

where

(a) $d_{k\ell rs} = d^{(2)}_{\alpha\gamma} \delta_{kr} \; \delta_{\ell s} + d^{(3)}_{\alpha\gamma} \; \delta_{k\ell} \delta_{rs}$

(b) $h^{(1)}_{kr} = e^{(1)}_{\alpha\gamma} \left(\frac{\varepsilon^{\alpha^2}}{2} \delta_{kr} + \varepsilon^{\alpha}_k \; \varepsilon^{\alpha}_r \right)$

(c) $h^{(2)}_{krs} = e^{(2)}_{\alpha\gamma} \; \delta_{ks} \; \varepsilon^{\alpha}_r + e^{(3)}_{\alpha\gamma} \delta_{rs} \; \varepsilon^{\alpha}_k$

(d) $h^{(3)}_{krst} = e^{(5)}_{\alpha\gamma} \left(\frac{1}{2} \delta_{kr} \, \delta_{st} + \delta_{ks} \; \delta_{rt} \right)$

(e) $h^{(4)}_{krs} = e^{(6)}_{\alpha\gamma} \left(\delta_{kr} \, \varepsilon^{\alpha}_s + \delta_{ks} \, \varepsilon^{\alpha}_r + \delta_{rs} \, \varepsilon^{\alpha}_k \right)$

(f) $h^{(5)}_{krst} = e^{(7)}_{\alpha\gamma} \; \delta_{rs} \; \delta_{kt} + e^{(8)}_{\alpha\gamma} \; \delta_{st} \; \delta_{kr}$

(g) $d^{(1)}_{\alpha\gamma} = -2 \; \frac{\beta^{\gamma}}{\beta}$

(h) $d^{(2)}_{\alpha\gamma} = 3M^{\gamma} \frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}} + 2\,\kappa^{\alpha}$ (III-20)

(i) $$d^{(3)}_{\alpha\gamma} = -M^{\gamma}\frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}}$$ (III-20) (cont'd)

(j) $$d^{(4)}_{\alpha\gamma} = 2\left(\frac{\beta'}{\beta}\right)^{1/2}$$

(k) $$e^{(1)}_{\alpha\gamma} = \left(\frac{\beta_\gamma}{\beta}\right)^2$$

(l) $$e^{(2)}_{\alpha\gamma} = -\ \frac{\beta_\gamma}{\beta}\left(2\,\kappa^{\alpha}+3M^{\gamma}\frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}}\right)$$

(m) $$e^{(3)}_{\alpha\gamma} = -\ \frac{\beta_\gamma}{\beta}\left(\kappa^{\alpha} - M^{\gamma}\frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}}\right)$$

(n) $$e^{(4)}_{\alpha\gamma} = \frac{3}{2}\,\kappa^{\alpha^2} + \frac{1}{2}\ M^{\gamma^2} + 2\,\kappa^{\alpha}\,M^{\gamma}\,\frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}}$$

(o) $$e^{(5)}_{\alpha\gamma} = \frac{\beta'}{\beta}$$

(p) $$e^{(6)}_{\alpha\gamma} = -\left(\frac{\beta'}{\beta}\right)^{1/2}\ \frac{\beta_\gamma}{\beta}$$

(q) $$e^{(7)}_{\alpha\gamma} = \left(\frac{\beta'}{\beta}\right)^{1/2}\left(2\,\kappa^{\alpha}+3M^{\gamma}\frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}}\right)$$

(r) $$e^{(8)}_{\alpha\gamma} = \left(\frac{\beta'}{\beta}\right)^{1/2}\left(\kappa^{\alpha} - M^{\gamma}\frac{\Omega^{(2)}_{\alpha\gamma}}{\Omega^{(1)}_{\alpha\gamma}}\right)$$

$$\text{(s)} \qquad \varkappa^{\alpha} = \frac{\beta^{o}\, k}{(m^{\alpha}+m^{\gamma})}\,(T^{\alpha}-T^{\gamma}) \quad . \qquad \text{(III-20) (cont'd)}$$

Making the change of variables (III-15) in the approximate distribution functions (II-20), it may be shown after considerable manipulation that

$$f^{\alpha} f^{\gamma}\, d^3u^{\alpha} d^3u^{\gamma} = \frac{n^{\alpha} n^{\gamma}}{(2\pi)^3}\, e^{-\left[\frac{G^2}{2}+\frac{(\vec{g}-\vec{\varepsilon})^2}{2}\right]} \left[1+\sum_{\nu=\alpha,\gamma} P'^{\nu}_{ij}(A^{\nu}_{ij}+B^{\nu}_{ij}) + \sum_{\nu=\alpha,\gamma} Q^{\nu}_{i}(C^{\nu}_{i}+D^{\nu}_{i})\right] d^3g\, d^3G \quad , \qquad \text{(III-21)}$$

where

$$\text{(a)} \qquad A^{\nu}_{ij} = a^{\nu^2}\, \overset{\circ}{G_i G_j} + \overset{\circ}{g_i g_j} + \overset{\circ}{\varepsilon^{\alpha}_i \varepsilon^{\alpha}_j} - 2\, \overset{\circ}{g_i \varepsilon^{\alpha}_j} \qquad \text{(III-22)}$$

$$\text{(b)} \qquad B^{\nu}_{ij} = 2\, a^{\nu}\,(\overset{\circ}{G_i g_j} - \overset{\circ}{G_i \varepsilon^{\alpha}_j})$$

$$\text{(c)} \qquad C^{\nu}_{i} = a^2_{\nu}\,(G^2 g_i + 2G_iG_jg_j - G^2\varepsilon^{\alpha}_i - 2G_iG_j\varepsilon^{\alpha}_j - 5g_i + 5\varepsilon^{\alpha}_i) + (g^2 g_i - g^2\varepsilon^{\alpha}_i + \varepsilon^{\alpha^2} g_i - \varepsilon^{\alpha^2}\varepsilon_i$$
$$-2\varepsilon^{\alpha}_j g_j g_i + 2 g_j \varepsilon^{\alpha}_j \varepsilon^{\alpha}_i - 5 g_i + 5\varepsilon^{\alpha}_i)$$

$$\text{(d)} \qquad D^{\nu}_{i} = a^3_{\nu}\,(G^2G_i - 5G_i) + a_{\nu}(g^2G_i + 2G_jg_jg_i - 2 g_j \varepsilon^{\alpha}_j G_i - 2G_jg_j\varepsilon^{\alpha}_i - 2G_j\varepsilon^{\alpha}_j g_i$$
$$+\varepsilon^{\alpha^2} G_i + 2G_j\varepsilon^{\alpha}_j\varepsilon^{\alpha}_i - 5G_i)$$

(e) $$P'^{\nu}_{ij} = \frac{\overset{o}{P}{}^{\nu}_{ij}\, b^{\nu^2}}{2\, p^{\nu}}$$ (III-22) (cont'd)

(f) $$Q'^{\nu} = \frac{\beta^{\nu 3/2}\, Q^{\nu}_{i}\, b^{\nu^3}}{5\, \rho^{\nu}} \quad .$$

In taking the product of the approximate distribution functions, terms involving the products of pressure tensors and heat fluxes have been assumed negligible. (The integrals involving the larger of these product terms were carried out and found to be less than five percent of the terms retained.)

As may be seen by substituting expressions (III-18, a; III-19; III-21, and III-22) into (III-13), hundreds of terms are involved in the collision transfer expressions. However, all the velocity space integrals are of the form,

$$\frac{1}{4\pi(2\pi)^{3/2}} \int e^{-\left[\frac{G^2}{2} + \frac{(\vec{g} - \vec{\varepsilon})^2}{2}\right]} \frac{G_{i_1} G_{i_2} \ldots G_{i_n} g_{j_1} g_{j_2} \ldots g_{j_m}}{g^3}\, d^3G\, d^3g, \tag{III-23}$$

which may be carried out exactly. The method used in carrying out these integrations is outlined and the results tabulated in Appendix B.

Finally, after considerable manipulation, the collision transfer of momentum, energy, traceless stress, and heat flux may be expressed as,

$$\text{(a)}\quad S^{\alpha}_{k} = \mathcal{S}^{(1)}\varepsilon^{\alpha}_{k} + \mathcal{S}^{(2)}\sum_{\nu=\alpha,\gamma} P^{\nu'}_{kj}\,\varepsilon^{\alpha}_{j} + \mathcal{S}^{(3)}\sum_{\nu=\alpha,\gamma} P^{\nu'}_{ij}\,\varepsilon^{\alpha}_{i}\,\overset{o}{\varepsilon}{}^{\alpha}_{j}\,\varepsilon^{\alpha}_{k} \qquad \text{(III-24)}$$

$$+\mathcal{S}^{(4)}\sum_{\nu=\alpha,\gamma} Q^{\nu'}_{k} + \mathcal{S}^{(5)}\sum_{\nu=\alpha,\gamma} Q^{\nu'}_{i}\,\varepsilon^{\alpha}_{i}\,\varepsilon^{\alpha}_{k}$$

$$\text{(b)}\quad R^{\alpha} = \mathcal{R}^{(1)}_{\alpha\gamma} + \mathcal{R}^{(2)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} P^{\nu'}_{ij}\,\varepsilon^{\alpha}_{i}\,\varepsilon^{\alpha}_{j} + \mathcal{R}^{(3)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} a_{\nu}\,P^{\nu'}_{ij}\,\varepsilon^{\alpha}_{i}\,\varepsilon^{\alpha}_{j}$$

$$+\mathcal{R}^{(4)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} Q^{\nu'}_{i}\,\varepsilon^{\alpha}_{i} + \mathcal{R}^{(5)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} a_{\nu}\,Q^{\nu'}_{i}\,\varepsilon^{\alpha}_{i}$$

$$\text{(c)}\quad \overset{o}{L}{}^{\alpha}_{k\ell} = \mathcal{L}^{(1)}_{\alpha\gamma}\,\varepsilon^{\alpha}_{k}\,\overset{o}{\varepsilon}{}^{\alpha}_{\ell} + \mathcal{L}^{(2)}_{\alpha\gamma}\,P^{\nu'}_{k\ell} + \mathcal{L}^{(3)}_{\alpha\gamma}\,P^{\gamma'}_{k\ell} + \mathcal{L}^{(4)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} a_{\nu}\,P^{\nu'}_{k\ell}$$

$$+\mathcal{L}^{(5)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} \overset{o}{\overline{\overline{P^{\nu'}_{kj}\,\varepsilon^{\alpha}_{j}\,\varepsilon^{\alpha}_{\ell}}}} + \mathcal{L}^{(6)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} a_{\nu}\,\overset{o}{\overline{\overline{P^{\nu'}_{kj}\,\varepsilon^{\alpha}_{j}\,\varepsilon^{\alpha}_{\ell}}}}$$

$$+\mathcal{L}^{(7)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} P^{\nu'}_{ij}\,\varepsilon^{\alpha}_{i}\,\varepsilon^{\alpha}_{j}\,\varepsilon^{\alpha}_{k}\,\overset{o}{\varepsilon}{}^{\alpha}_{\ell} + \mathcal{L}^{(8)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} \overset{o}{\overline{\overline{Q^{\nu'}_{k}\,\varepsilon^{\alpha}_{\ell}}}}$$

$$+\mathcal{L}^{(9)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} a_{\nu}\,\overset{o}{\overline{\overline{Q^{\nu'}_{k}\,\varepsilon^{\alpha}_{\ell}}}} + \mathcal{L}^{(10)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} Q^{\nu'}_{i}\,\varepsilon^{\alpha}_{i}\,\overset{o}{\varepsilon}{}^{\alpha}_{k}\,\varepsilon^{\alpha}_{\ell}$$

$$+\mathcal{L}^{(11)}_{\alpha\gamma}\sum_{\nu=\alpha,\gamma} a_{\nu}\,Q^{\nu'}_{i}\,\varepsilon^{\alpha}_{i}\,\varepsilon^{\alpha}_{k}\,\overset{o}{\varepsilon}{}^{\alpha}_{\ell}$$

$$\text{(d)}\quad M_k^\alpha = \mathcal{M}_{\alpha\gamma}^{(1)} \varepsilon_k^\alpha + \mathcal{M}_{\alpha\gamma}^{(2)} \sum_{\nu=\alpha,\gamma} P_{kj}^{\nu'} \varepsilon_j^\alpha + \mathcal{M}_{\alpha\gamma}^{(3)} \sum_{\nu=\alpha,\gamma} a_\nu P_{kj}^{\nu'} \varepsilon_j^\alpha \tag{III-24 (cont'd)}$$

$$+\mathcal{M}_{\alpha\gamma}^{(4)} \sum_{\nu=\alpha,\gamma} a_\nu^2 P_{kj}^{\nu'} \varepsilon_j^\alpha + \mathcal{M}_{\alpha\gamma}^{(5)} \sum_{\nu=\alpha,\gamma} P_{ij}^{\nu'} \varepsilon_i^\alpha \varepsilon_j^\alpha \varepsilon_k^\alpha$$

$$+\mathcal{M}_{\alpha\gamma}^{(6)} \sum_{\nu=\alpha,\gamma} a_\nu P_{ij}^{\nu'} \varepsilon_i^\alpha \varepsilon_j^\alpha \varepsilon_k^\alpha + \mathcal{M}_{\alpha\gamma}^{(7)} Q_k^{\alpha'} + \mathcal{M}_{\alpha\gamma}^{(8)} Q_k^{\gamma'}$$

$$+\mathcal{M}_{\alpha\gamma}^{(9)} \sum_{\nu=\alpha,\gamma} a_\nu Q_k^{\nu'} + \mathcal{M}_{\alpha\gamma}^{(10)} \sum_{\nu=\alpha,\gamma} a_\nu^2 Q_k^{\nu'} + \mathcal{M}_{\alpha\gamma}^{(11)} \sum_{\nu=\alpha,\gamma} Q_i^{\nu'} \varepsilon_i^\alpha \varepsilon_k^\alpha$$

$$+\mathcal{M}_{\alpha\gamma}^{(12)} \sum_{\nu=\alpha,\gamma} a_\nu Q_i^{\nu'} \varepsilon_i^\alpha \varepsilon_k^\alpha + \mathcal{M}_{\alpha\gamma}^{(13)} \sum_{\nu=\alpha,\gamma} a_\nu^2 Q_i^{\nu'} \varepsilon_i^\alpha \varepsilon_j^\alpha \quad .$$

Whereas in the notation used for the individual collision transfer integrals, the Greek symbols α and γ could independently correspond to either electrons or ions; in the above expressions (III-24) the symbols α and γ are to correspond to particles of different type, that is, if α corresponds to electrons, then γ corresponds to ions and vice versa. The coefficients in (III-24) are given by;

$$\text{(a)}\quad \mathcal{J}^{(1)} = \frac{\Lambda_{ei}}{\beta^{1/2}} \mathcal{F} \tag{III-25}$$

$$\text{(b)}\quad \mathcal{J}^{(2)} = \frac{\Lambda_{ei}}{\beta^{1/2}} \left(\frac{2}{\varepsilon^2}\mathcal{G} - \frac{6}{\varepsilon^2}\mathcal{F}\right)$$

(III-25) (cont'd)

(c) $\mathcal{S}^{(3)} = \dfrac{\Lambda_{ei}}{\beta_o^{1/2}} \left(- \dfrac{1}{\varepsilon^2} \mathcal{J} - \dfrac{5}{\varepsilon^4} \mathcal{J} + \dfrac{15}{\varepsilon^4} \mathcal{F} \right)$

(d) $\mathcal{S}^{(4)} = \dfrac{\Lambda_{ei}}{\beta_o^{1/2}} \left(- \mathcal{J} \right)$

(e) $\mathcal{S}^{(5)} = \dfrac{\Lambda_{ei}}{\beta_o^{1/2}} \left(\mathcal{J} \right)$.

(III-26)

(a) $\mathcal{R}^{(1)}_{\alpha\gamma} = \dfrac{3}{2} \dfrac{\Lambda_{ei}}{\beta^{o}} \left\{ d^{(1)}_{\alpha\gamma} \left(\dfrac{1}{3} \varepsilon^2 \mathcal{F} \right) + d^{(2)}_{\alpha\gamma} \left(\dfrac{1}{3} \mathcal{J} + \dfrac{1}{3} \varepsilon^2 \mathcal{F} \right) + d^{(3)}_{\alpha\gamma} \left(\mathcal{J} + \varepsilon^2 \mathcal{F} \right) \right\}$

(b) $\mathcal{R}^{(2)}_{\alpha\gamma} = \dfrac{3}{2} \dfrac{\Lambda_{ei}}{\beta^{o}} \left\{ d^{(1)}_{\alpha\gamma} \left(- \dfrac{1}{3} \mathcal{J} - \dfrac{1}{\varepsilon^2} \mathcal{J} + \dfrac{3}{\varepsilon^2} \mathcal{F} \right) + d^{(2)}_{\alpha\gamma} \left(- \dfrac{1}{3\varepsilon^2} \mathcal{J} + \dfrac{1}{\varepsilon^2} \mathcal{F} \right) + d^{(3)}_{\alpha\gamma} \left(- \dfrac{1}{\varepsilon^2} \mathcal{J} + \dfrac{3}{\varepsilon^2} \mathcal{F} \right) \right\}$

(c) $\mathcal{R}^{(3)}_{\alpha\gamma} = \dfrac{3}{2} \dfrac{\Lambda_{ei}}{\beta^{o}} \left\{ d^{(4)}_{\alpha\gamma} \left(\dfrac{2}{3\varepsilon^2} \mathcal{J} - \dfrac{2}{\varepsilon^2} \mathcal{F} \right) \right\}$

(d) $\mathcal{R}^{(4)}_{\alpha\gamma} = \dfrac{3}{2} \dfrac{\Lambda_{ei}}{\beta^{o}} \left\{ d^{(1)}_{\alpha\gamma} \left(- \dfrac{\mathcal{L}}{3} + \dfrac{\varepsilon^2}{3} \mathcal{J} \right) + d^{(2)}_{\alpha\gamma} \left(\dfrac{\mathcal{L}}{3} \right) + d^{(3)}_{\alpha\gamma} \left(\mathcal{J} \right) \right\}$

(e) $\mathcal{R}^{(5)}_{\alpha\gamma} = \dfrac{3}{2} \dfrac{\Lambda_{ei}}{\beta^{o}} \left\{ d^{(1)}_{\alpha\gamma} \left(- \mathcal{J} \right) \right\}$.

$$\text{(a)}\quad \mathcal{L}^{(1)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(1)}_{\alpha\gamma}(\mathcal{I}) + d^{(2)}_{\alpha\gamma}\left(\frac{1}{\varepsilon^2}\mathcal{J} + \mathcal{I} - \frac{3}{\varepsilon^2}\mathcal{I}\right)\right\} \tag{III-27}$$

$$\text{(b)}\quad \mathcal{L}^{(2)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(2)}_{\alpha\gamma}\left(\frac{2}{\varepsilon^2}\mathcal{J} + 2\mathcal{I} - \frac{6}{\varepsilon^2}\mathcal{I}\right) + \frac{\Lambda_{\alpha\gamma}}{\Lambda_{ei}}\, d^{(2)}_{\alpha\alpha}\left(\frac{8}{15}\right)\right\}$$

$$\text{(c)}\quad \mathcal{L}^{(3)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(2)}_{\alpha\gamma}\left(\frac{2}{\varepsilon^2}\mathcal{J} + 2\mathcal{I} - \frac{6}{\varepsilon^2}\mathcal{I}\right)\right\}$$

$$\text{(d)}\quad \mathcal{L}^{(4)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(4)}_{\alpha\gamma}\quad (2\mathcal{I})\right\}$$

$$\text{(e)}\quad \mathcal{L}^{(5)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(1)}_{\alpha\gamma}\left(\frac{2}{\varepsilon^2}\mathcal{J} - \frac{6}{\varepsilon^2}\mathcal{I}\right) + d^{(2)}_{\alpha\gamma}\left(-\frac{20}{\varepsilon^4}\mathcal{J} - \frac{12}{\varepsilon^2}\mathcal{I} + \frac{60}{\varepsilon^4}\mathcal{I}\right)\right\}$$

$$\text{(f)}\quad \mathcal{L}^{(6)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(4)}_{\alpha\gamma}\left(\frac{2}{\varepsilon^2}\mathcal{J} - \frac{6}{\varepsilon^2}\mathcal{I}\right)\right\}$$

$$\text{(g)}\quad \mathcal{L}^{(7)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(1)}_{\alpha\gamma}\left(-\frac{1}{\varepsilon^2}\mathcal{J} - \frac{5}{\varepsilon^4}\mathcal{J} + \frac{15}{\varepsilon^4}\mathcal{I}\right) + d^{(2)}_{\alpha\gamma}\left(\frac{2}{\varepsilon^4}\mathcal{J} + \frac{35}{\varepsilon^6}\mathcal{J} + \frac{15}{\varepsilon^4}\mathcal{I} - \frac{105}{\varepsilon^6}\mathcal{I}\right)\right\}$$

$$\text{(h)}\quad \mathcal{L}^{(8)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(1)}_{\alpha\gamma}(-\mathcal{J}) + d^{(2)}_{\alpha\gamma}\left(\frac{4}{\varepsilon^2}\mathcal{J} - \frac{12}{\varepsilon^2}\mathcal{I}\right)\right\}$$

$$\text{(i)}\quad \mathcal{L}^{(9)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(4)}_{\alpha\gamma}\left(-\mathcal{J} + \frac{4}{\varepsilon^2}\mathcal{J} - \frac{12}{\varepsilon^2}\mathcal{I}\right)\right\}$$

$$\text{(j)}\quad \mathcal{L}^{(10)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(1)}_{\alpha\gamma}(\mathcal{J}) + d^{(2)}_{\alpha\gamma}\left(-\frac{2}{\varepsilon^2}\mathcal{J} - \frac{10}{\varepsilon^4}\mathcal{J} + \frac{30}{\varepsilon^4}\mathcal{I}\right)\right\}$$

$$\text{(k)}\quad \mathcal{L}^{(11)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta^{o}}\left\{ d^{(4)}_{\alpha\gamma}\left(-\frac{2}{\varepsilon^2}\mathcal{J} - \frac{10}{\varepsilon^4}\mathcal{J} + \frac{30}{\varepsilon^4}\mathcal{I}\right)\right\}$$

(III-28)

$$\text{(a)}\quad \mathcal{M}^{(1)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_0^{3/2}} \Big\{ e^{(1)}_{\alpha\gamma}\Big(\frac{3\varepsilon^2}{2}\mathcal{F}\Big) + e^{(2)}_{\alpha\gamma}\big(\mathcal{J} - 2\mathcal{F} + \varepsilon^2\mathcal{F}\big) + e^{(3)}_{\alpha\gamma}\big(\mathcal{J} + \varepsilon^2\mathcal{F}\big) + e^{(4)}_{\alpha\gamma}\big(\mathcal{J} - \mathcal{F} + \varepsilon^2\mathcal{F}\big) + e^{(5)}_{\alpha\gamma}\Big(\frac{5}{2}\mathcal{F}\Big) \Big\}$$

$$\text{(b)}\quad \mathcal{M}^{(2)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_0^{3/2}} \Big\{ e^{(1)}_{\alpha\gamma}\big(\mathcal{J} - 3\mathcal{F}\big) + e^{(5)}_{\alpha\gamma}\Big(\frac{5}{\varepsilon^2}\mathcal{J} - \frac{15}{\varepsilon^2}\mathcal{F}\Big) + e^{(2)}_{\alpha\gamma}\Big(-\frac{8}{\varepsilon^2}\mathcal{J} - 4\mathcal{F} + \frac{24}{\varepsilon^2}\mathcal{F}\Big) + e^{(4)}_{\alpha\gamma}\Big(-\frac{2}{\varepsilon^2}\mathcal{J} - 2\mathcal{F} + \frac{6}{\varepsilon^2}\mathcal{F}\Big) \Big\}$$

$$\text{(c)}\quad \mathcal{M}^{(3)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_0^{3/2}} \Big\{ e^{(6)}_{\alpha\gamma}\big(2\mathcal{J} - 2\mathcal{F}\big) + e^{(7)}_{\alpha\gamma}\Big(\frac{4}{\varepsilon^2}\mathcal{J} + 2\mathcal{F} - \frac{12}{\varepsilon^2}\mathcal{F}\Big) + e^{(8)}_{\alpha\gamma}\big(-2\mathcal{F}\big) \Big\}$$

$$\text{(d)}\quad \mathcal{M}^{(4)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_0^{3/2}} \Big\{ e^{(5)}_{\alpha\gamma}\big(2\mathcal{F}\big) \Big\}$$

$$\text{(e)}\quad \mathcal{M}^{(5)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_0^{3/2}} \Big\{ e^{(1)}_{\alpha\gamma}\Big(-\frac{3}{2}\mathcal{J} - \frac{11}{2\varepsilon^2}\mathcal{J} + \frac{33}{2\varepsilon^2}\mathcal{F}\Big) + e^{(5)}_{\alpha\gamma}\Big(-\frac{5}{2\varepsilon^2}\mathcal{J} - \frac{25}{2\varepsilon^4}\mathcal{J} + \frac{75}{2\varepsilon^4}\mathcal{F}\Big) + e^{(2)}_{\alpha\gamma}\Big(\frac{1}{\varepsilon^2}\mathcal{J} + \frac{20}{\varepsilon^4}\mathcal{J} + \frac{9}{\varepsilon^2}\mathcal{F} - \frac{60}{\varepsilon^4}\mathcal{F}\Big) + e^{(3)}_{\alpha\gamma}\Big(-\frac{1}{\varepsilon^2}\mathcal{J} + \frac{3}{\varepsilon^2}\mathcal{F}\Big) + e^{(4)}_{\alpha\gamma}\Big(\frac{5}{\varepsilon^4}\mathcal{J} + \frac{3}{\varepsilon^2}\mathcal{F} - \frac{15}{\varepsilon^4}\mathcal{F}\Big) \Big\}$$

$$\text{(f)}\quad \mathcal{M}^{(6)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_0^{3/2}} \Big\{ e^{(6)}_{\alpha\gamma}\Big(\frac{4}{\varepsilon^2}\mathcal{J} - \frac{12}{\varepsilon^2}\mathcal{F}\Big) + e^{(7)}_{\alpha\gamma}\Big(-\frac{10}{\varepsilon^4}\mathcal{J} - \frac{6}{\varepsilon^2}\mathcal{F} + \frac{30}{\varepsilon^4}\mathcal{F}\Big) \Big\}$$

$$\text{(g)}\quad \mathcal{M}^{(7)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_0^{3/2}} \Big\{ e^{(1)}_{\alpha\gamma}\Big(-\frac{\varepsilon^2}{2}\mathcal{J}\Big) + e^{(5)}_{\alpha\gamma}\Big(-\frac{5}{2}\mathcal{J}\Big) + e^{(2)}_{\alpha\gamma}\big(2\mathcal{J} - 6\mathcal{F}\big) + e^{(4)}_{\alpha\gamma}\big(-2\mathcal{F}\big) + 2\Big(\frac{1}{3}e^{(7)}_{\alpha\alpha} - \frac{5}{3}e^{(8)}_{\alpha\alpha}\Big)\frac{\Lambda_{\alpha\alpha}}{\Lambda_{ei}} \Big\}$$

(III-28)
(cont'd)

$$\text{(h)}\quad \mathcal{M}^{(8)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_o^{3/2}} \left\{ e^{(1)}_{\alpha}\left(-\frac{\varepsilon^2}{2}\mathcal{G}\right) + e^{(5)}_{\alpha\gamma}\left(-\frac{5}{2}\mathcal{G}\right) + e^{(2)}_{\alpha\gamma}(2\mathcal{G} - 6\mathcal{F}) + e^{(4)}_{\alpha\gamma}(-2\mathcal{F}) \right\}$$

$$\text{(i)}\quad \mathcal{M}^{(9)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_o^{3/2}} \left\{ e^{(6)}_{\alpha\gamma}(-\varepsilon^2\mathcal{G} + 4\mathcal{G} - 12\mathcal{F}) + e^{(7)}_{\alpha\gamma}(-\mathcal{G} + 4\mathcal{F}) + e^{(8)}_{\alpha\gamma}(-\mathcal{G} - 2\mathcal{F}) \right\}$$

$$\text{(j)}\quad \mathcal{M}^{(10)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_o^{3/2}} \left\{ e^{(5)}_{\alpha\gamma}(2\mathcal{G} + 9\mathcal{F}) \right\}$$

$$\text{(k)}\quad \mathcal{M}^{(11)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_o^{3/2}} \left\{ e^{(1)}_{\alpha\gamma}\left(-\mathcal{G} + \frac{3}{2}\varepsilon^2\mathcal{G}\right) + e^{(5)}_{\alpha\gamma}\left(\frac{5}{2}\mathcal{G}\right) + e^{(2)}_{\alpha\gamma}\left(-\mathcal{G} - \frac{6}{\varepsilon^2}\mathcal{G} + \frac{18}{\varepsilon^2}\mathcal{F}\right) \right.$$
$$\left. + e^{(3)}_{\alpha\gamma}(\mathcal{G}) + e^{(4)}_{\alpha\gamma}\left(-\frac{2}{\varepsilon^2}\mathcal{G} + \frac{6}{\varepsilon^2}\mathcal{F}\right) \right\}$$

$$\text{(l)}\quad \mathcal{M}^{(12)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_o^{3/2}} \left\{ e^{(6)}_{\alpha\gamma}\left(-8\mathcal{G} - \frac{12}{\varepsilon^2}\mathcal{G} + \frac{36}{\varepsilon^2}\mathcal{F}\right) + e^{(7)}_{\alpha\gamma}\left(\frac{4}{\varepsilon^2}\mathcal{G} - \frac{12}{\varepsilon^2}\mathcal{F}\right) + e^{(8)}_{\alpha\gamma}\left(-\frac{2}{\varepsilon^2}\mathcal{G} + \frac{6}{\varepsilon^2}\mathcal{F}\right) \right\}$$

$$\text{(m)}\quad \mathcal{M}^{(13)}_{\alpha\gamma} = \frac{\Lambda_{ei}}{\beta_o^{3/2}} \left\{ e^{(5)}_{\alpha\gamma}\left(\frac{9}{\varepsilon^2}\mathcal{G} - \frac{27}{\varepsilon^2}\mathcal{F}\right) \right\},$$

where

(III-29)

$$\text{(a)}\quad \Lambda_{ei} = -4\sqrt{2\pi}\; n^e n^i \; \frac{(Ze^2)^2}{m^{ei}} \; \beta_o^{3/2} \; \Omega^{(1)}_{ei}$$

$$\text{(b)}\quad \Lambda_{\alpha\alpha} = -4\sqrt{\pi}\; \frac{(n^{\alpha} e^{\alpha 2})^2}{m^{\alpha}} \; \beta^{\alpha\, 3/2} \; \Omega^{(1)}_{\alpha\alpha}$$

$$\text{(c)}\quad \mathcal{G} = e^{-\varepsilon^2/2}$$

(III-29)
(cont'd)

$$\text{(d)} \qquad \mathcal{J} = \frac{1}{\varepsilon^3} \int_0^{\varepsilon} e^{-x^2/2} \; x^2 \, dx \qquad .$$

The above collision transfer expressions are obviously extremely complicated. Considerable simplification has, however, been found possible for systems in which $\varepsilon < 1$ (terms of order ε^4 negligible), and for systems in which $\varepsilon^2 \gg 1$. These reduced expressions are exhibited in section V .

IV. GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS

Utilizing the approximate distribution (II-20), it may be shown that,

$$
\begin{aligned}
&\text{(a)} \quad \left\langle u_i u_r u_s \right\rangle = \frac{2}{5}\,\frac{Q_k^\alpha}{\rho^\alpha}\left(\delta_{kr}\delta_{is} + \delta_{ki}\delta_{rs} + \delta_{ks}\delta_{ri}\right) \\
&\text{(b)} \quad \left\langle u^2 u_r u_s \right\rangle = \frac{5}{\beta^{\alpha 2}}\,\delta_{rs} + \frac{7}{\beta^\alpha \rho^\alpha}\,\mathring{P}^\alpha_{rs}
\end{aligned}
\qquad \text{(IV-1)}
$$

On substitution of these expressions into the transport equations (II-9), we obtain

$$
\begin{aligned}
&\text{(a)} \quad \frac{D_\alpha n^\alpha}{Dt} + n^\alpha \frac{\partial w_i^\alpha}{\partial r_i} = 0 \\
&\text{(b)} \quad \frac{D_\alpha w_i^\alpha}{Dt} = \frac{e^\alpha}{m^\alpha}\left(E_i + \frac{1}{c}\,\varepsilon_{ijk}\, w_j^\alpha H_k\right) - \frac{1}{\rho^\alpha}\,\frac{\partial P_{ij}^\alpha}{\partial r_j} + \frac{1}{\rho^\alpha}\, S_i^\alpha \\
&\text{(c)} \quad \frac{D_\alpha T^\alpha}{Dt} = \frac{2}{3 n^\alpha k}\left\{ - \frac{\partial Q_i^\alpha}{\partial r_i} - P_{ij}^\alpha \frac{\partial w_j^\alpha}{\partial r_i} + R^\alpha \right\} \\
&\text{(d)} \quad \frac{D_\alpha \mathring{P}^\alpha_{rs}}{Dt} = -\frac{4}{5}\,\overset{o}{\overline{\overline{\frac{\partial Q_s^\alpha}{\partial r_r}}}} + 2\,\frac{e^\alpha}{m^\alpha c}\,\varepsilon_{rjk}\,\mathring{P}^\alpha_{js} H_k - \mathring{P}^\alpha_{js}\,\frac{\partial w_i^\alpha}{\partial r_i} \\
&\qquad\qquad - 2 P_{jr}^\alpha\,\overset{o}{\overline{\overline{\frac{\partial w_s^\alpha}{\partial r_j}}}} + \mathring{L}^\alpha_{rs} \\
&\text{(e)} \quad \frac{D_\alpha Q_r^\alpha}{Dt} = -\frac{7}{5}\, Q_r^\alpha \frac{\partial w_i^\alpha}{\partial r_i} - Q_j^\alpha \frac{\partial w_r^\alpha}{\partial r_j} - \frac{4}{5}\, Q_s^\alpha\,\overline{\overline{\frac{\partial w_s^\alpha}{\partial r_r}}} + \frac{\mathring{P}^\alpha_{ir}}{\rho^\alpha}\,\frac{\partial P_{ij}^\alpha}{\partial r_j} - \frac{kT^\alpha}{m^\alpha}\,\frac{\partial \mathring{P}^\alpha_{rs}}{\partial r_s} \\
&\qquad\qquad - \frac{7}{2}\,\frac{k}{m^\alpha}\,\mathring{P}^\alpha_{rs}\,\frac{\partial T^\alpha}{\partial r_s} - \frac{5}{2}\,\frac{kp^\alpha}{m^\alpha}\,\frac{\partial T^\alpha}{\partial r_r} + \frac{e^\alpha}{m^\alpha c}\,\varepsilon_{rjk}\, Q_j^\alpha H_k - \left(\frac{5}{2}\, p^\alpha \delta_{ir} + \mathring{P}^\alpha_{ir}\right) \\
&\qquad\qquad \times \frac{1}{\rho^\alpha}\, S_i^\alpha + M_r^\alpha
\end{aligned}
\qquad \text{(IV-2)}
$$

Alternate forms of the equations for conservation of particles (IV-2, a), and conservation of momentum (IV-2, b), are of interest. These alternate forms express conservation of the properties of the mixture, as opposed to conservation of properties of the constituents, that is, equations for total mass density ρ, total charge density q, total momentum density $\rho\, w_k^o$, and a generalized Ohm's law for the conduction current density j_k. The latter macroscopic properties of the plasma are defined by:

$$
\begin{aligned}
&(a) \quad \rho = \rho^i + \rho^e = n^i m^i + n^e m^e \\
&(b) \quad q = q^i + q^e = n^i Z e - n^e e \\
&(c) \quad \rho\, w_k^o = \rho^i\, w_k^i + \rho^e\, w_k^e \\
&(d) \quad j_k = (q^i\, w_k^i - q^e\, w_k^e) - q\, w_k^o = J - q\, w_k^o
\end{aligned}
\tag{IV-3}
$$

On multiplying (IV-2, a) by m^α and adding the ion equation to that of electrons, the total mass conservation equation is obtained, namely,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r_i}\, \rho\, w_i^o = 0 \quad . \tag{IV-4}$$

Multiplication by e^α and addition of the constituent equations leads to the charge conservation equation:

$$\frac{\partial q}{\partial t} + \frac{\partial J_i}{\partial r_i} = 0 \ . \tag{IV-5}$$

Similarly, the equation for conservation of total momentum and the generalized Ohm's law equation are obtained using the multiplicative factors ρ^α and m^α/e^α.

These equations may be written,

$$(a)\quad \rho \frac{D_o w_j^o}{Dt} = - \frac{\partial}{\partial r_i}\left(P_{ij}^e + P_{ij}^i + \frac{m^e}{n^e e^2} j_i j_j\right) + q\left(E_j + \frac{1}{c}\,\mathcal{E}_{jk\ell}\, w_k^o H_\ell\right) + \frac{1}{c}\,\mathcal{E}_{jk\ell}\, j_k H_\ell$$

$$(b)\quad \frac{m^e}{n^e e^2}\left(\frac{D_o j_j}{Dt} + j_j \frac{\partial w_i^o}{\partial r_i} + j_i \frac{\partial w_j^o}{\partial r_i}\right) = \frac{1}{n^e e}\,\frac{\partial}{\partial r_i}\left(P_{ij}^e + \frac{m^e}{n^e e^2} j_i j_j\right) \qquad \text{(IV-6)}$$

$$+ \left(E_j + \frac{1}{c}\,\mathcal{E}_{jk\ell}\, w_k^o H_\ell\right) - \frac{1}{n^e ec}\,\mathcal{E}_{jk\ell}\, j_k H_\ell - \frac{1}{n^e e}\, S_j^e \,,$$

where

$$\frac{D_o}{Dt} = \frac{\partial}{\partial t} + w_k^o \frac{\partial}{\partial r_k} \quad . \qquad \text{(IV-7)}$$

In obtaining the equations for total momentum and current, use has been made of the relations,

$$(a)\qquad m^i \gg Z\, m^e$$

$$(b)\qquad n^i m^i \gg n^e m^e$$

$$(c)\qquad n^e m^i \gg Z^2 n^i m^e \,,$$

which are, of course, valid for most systems of interest.

Equations (IV-4), (IV-5), (IV-6), (IV-2, c), (IV-2, d), and (IV-2, e); with expressions (III-24) for the collision transfer terms, S_i^α, R^α, $\overset{o}{L}{}_{rs}^{\alpha}$, and M_r^α; constitute a generalized set of magnetohydrodynamic equations. The electric and magnetic fields which appear are governed by Maxwell's equations (II-2).

Since the magnetohydrodynamic equations exhibited by Spitzer, "Physics of Fully Ionized Gases"(11) have received wide attention; it is worthy of note that, making allowance for the difference in notation, the Spitzer equations for particle conservation (2-14) and (2-15), total momentum conservation (2-11) and the current equation (2-12) with definition (2-13), are the same as the set of equations consisting of; the ion conservation equation (IV-2, a) above with α corresponding to ions; the equation for electron number density obtained by eliminating the ion flow velocity from equations (IV-3, c) and (IV-3, d) and substituting the resulting expression for electron flow velocity into (IV-2, a) with α corresponding to electrons; and the reduced forms of (IV-6, a, b) obtained by using the approximations specified by Spitzer on page 20 of "Physics of Fully Ionized Gases."

The need herein for the equations for traceless stress and heat flux is a result of the dependence of the momentum transfer on these properties [expression (III-24, a)]. Under certain limiting conditions, the momentum transfer may be expressed independent of heat flux and traceless stress as indicated by Spitzer in section 5.4 of "Physics of Fully Ionized Gases".

V. REDUCED MAGNETOHYDRODYNAMIC EQUATIONS AND TRANSPORT PROPERTIES.

The magnetohydrodynamic equations in their general form are seen to be extremely complicated. Fortunately, however, considerable simplification is possible. For systems with slowly varying flows in which $\mathcal{E} < 1$, the collision transfer terms (III-24) are reduced to a manageable form by expanding the $\mathcal{E}$-dependent functions $\mathcal{G}$ and $\mathcal{F}$ in powers of $\mathcal{E}$ and retaining terms to order $\mathcal{E}^3$. (Recall from section III that

$$\mathcal{E}_k^\alpha = \beta^{o^{1/2}} (w_k^\alpha - w_k^\gamma) ; \qquad \beta^o = \frac{\beta^e \beta^i}{\beta^e + \beta^i} ; \qquad \beta^\alpha = \frac{m^\alpha}{kT^\alpha} ;$$

$$\mathcal{E} = |\mathcal{E}_k^\alpha| ; \qquad \mathcal{G} = e^{-\mathcal{E}^2/2} ; \text{ and } \mathcal{F} = \frac{1}{\mathcal{E}^3} \int_0^{\mathcal{E}} e^{-x^2/2} x^2 dx.)$$

On the other hand, for slowly varying systems in which $\mathcal{E}^2 >> 1$, the expansion is made in inverse powers of $\mathcal{E}$ and only the highest order terms are retained. Further, the slowly varying flow restriction is such that the differential equations for pressure tensors and heat fluxes reduce to algebraic transport relations for these properties.

A. Plasma Systems in which $\mathcal{E} < 1$.

To obtain reduced forms of the collision transfer terms, use is made of the assumptions,

$$\text{(a)} \quad \beta^e << \beta^i \tag{V-1}$$

$$\text{(b)} \quad K^e = \frac{\beta^o k}{(m^e + m^i)} (T^e - T^i) \approx \frac{m^e}{m^i} \frac{(T^e - T^i)}{T^e} << 1$$

which are anticipated to be valid for most systems. Further, for order of magnitude purposes, it is assumed that,

$$
\begin{aligned}
&\text{(a)}\quad Q'^{e}_{k} = \frac{\beta^{o^{3/2}}\, Q^{e}_{k}}{5\,\rho^{e}} \approx \varepsilon_{k} \\
&\text{(b)}\quad P^{e'}_{ij} = \frac{\overset{o}{P}{}^{e}_{ij}}{2p^{e}} \approx \varepsilon_{i}\,\varepsilon_{j} \\
&\text{(c)}\quad Q'^{i}_{k} = \frac{\beta^{o^{3/2}}\, Q^{i}_{k}}{5\,\rho^{i}} \ll Q'^{e}_{k} \\
&\text{(d)}\quad P'^{i}_{ij} = \frac{\beta^{e}}{\beta^{i}}\,\frac{\overset{o}{P}{}^{i}_{ij}}{2p^{i}} \ll P'^{e}_{ij}
\end{aligned}
\qquad \text{(V-2)}
$$

The validity of these latter assumptions may be verified in retrospect. Representing the collision transfer terms in powers of ε, ignoring terms of higher order than ε^{3}, and utilizing the above assumptions; it may be shown that the collision transfer terms may be put in the form

$$
\text{(a)}\quad S^{e}_{k} = -S^{i}_{k} = -\frac{\rho^{e}}{\beta^{e}\lambda^{e}}\left\{\left(\frac{1}{3}-\frac{\varepsilon^{2}}{10}\right)\varepsilon^{e}_{k} - \frac{2}{5}P^{e'}_{kj}\,\varepsilon^{e}_{j} + \left(-1+\frac{\varepsilon^{2}}{2}\right)Q'^{e}_{k} + Q'^{e}_{j}\,\varepsilon^{e}_{j}\,\varepsilon^{e}_{k}\right\}
$$

$$
\text{(b)}\quad R^{e} = -\frac{\rho^{e}}{\beta^{e^{3/2}}\lambda^{e}}\left\{K^{e}\left(1-\frac{\varepsilon^{2}}{6}+Q'^{e}_{i}\,\varepsilon^{e}_{i}\right) - \frac{\varepsilon^{2}}{3} + Q'^{e}_{i}\,\varepsilon^{e}_{i}\right\}
\qquad \text{(V-3)}
$$

$$
\text{(c)}\quad R^{i} = -\frac{\rho^{e}}{\beta^{e^{3/2}}\lambda^{e}}\left\{-K^{e}\left(1-\frac{\varepsilon^{2}}{6}+Q'^{e}_{i}\,\varepsilon^{e}_{i}\right) - \frac{\beta^{e}}{\beta^{i}}\,\frac{\varepsilon^{2}}{3} + \frac{\beta^{e}}{\beta^{i}}\,Q'^{e}_{i}\,\varepsilon^{e}_{i}\right\}
$$

(d) $$\overset{o}{L}{}^{e}_{k\ell} = -\frac{\rho^{e}}{\beta^{e\,3/2}\lambda^{e}}\Bigg\{\left(-\frac{2}{3}+\frac{2}{5}\,\frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}\right)\overset{o}{\mathcal{E}^{e}_{k}\mathcal{E}^{e}_{\ell}}$$

$$+\frac{4}{5}\left(\frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}+\frac{n^{e}}{\sqrt{2}\;Z^{2}\,n^{i}}\,\frac{\Omega^{(2)}_{ee}}{\Omega^{(1)}_{ei}}\right)P'^{e}_{k\ell}+\left(2-\frac{12}{5}\,\frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}\right)\overset{o}{\overline{\overline{Q'^{e}_{k}\;\mathcal{E}^{e}_{\ell}}}}\Bigg\}$$

(e) $$\overset{o}{L}{}^{i}_{k\ell} = -\frac{\rho^{e}}{\beta^{e\,3/2}\lambda^{e}}\Bigg\{\left(-\frac{2}{3}\,\frac{\beta^{e}}{\beta^{i}}+\frac{2}{5}\,\frac{m^{e}}{m^{i}}\,\frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}-\frac{4}{15}\,K^{e}\right)\overset{o}{\mathcal{E}^{e}_{k}\;\mathcal{E}^{e}_{\ell}}$$

$$+\frac{4}{15}\left(3\,\frac{m^{e}}{m^{i}}\,\frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}-2\,K^{e}-20\,\frac{\beta^{e}}{\beta^{i}}\right)P'^{e}_{k\ell}$$

$$+\frac{4}{5}\left(3\,\frac{m^{e}}{m^{i}}\,\frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}-2\,K^{e}-7\,\frac{\beta^{e}}{\beta^{i}}\right)\overset{o}{\overline{\overline{Q'^{e}_{k}\;\mathcal{E}^{e}_{\ell}}}}$$

$$+\frac{4}{3}\left(1+\frac{3}{5\sqrt{2}}\,\frac{Z^{2}n^{i}}{n^{e}}\,\frac{m^{e}}{m^{i}}\,\frac{\beta^{i\,3/2}}{\beta^{e\,3/2}}\,\frac{\Omega^{(2)}_{ii}}{\Omega^{(1)}_{ei}}\right)P'^{i}_{k\ell}\Bigg\}$$

$$\text{(f)}\quad M_k^e = -\frac{\rho^e}{\beta^{e^2}\lambda^e}\left\{\left[\frac{1}{3}+\left(\frac{7}{15}-\frac{4}{15}\frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right)\mathcal{E}^2\right]\mathcal{E}_k^e\right.$$

$$+\left(-\frac{4}{5}\frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}-\frac{2}{15}\right)P'^e_{kj}\,\mathcal{E}_j^e+\left[\left(-\frac{1}{3}+\frac{4}{3\sqrt{2}}\frac{n^e}{Z^2n^i}\frac{\Omega_{ee}^{(2)}}{\Omega_{ei}^{(1)}}\right)\right.$$

$$\left.\left.+\left(-\frac{2}{5}+\frac{6}{5}\frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right)\mathcal{E}^2\right]Q_k^{e'}+\left(-\frac{4}{5}+\frac{2}{5}\frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right)Q'^e_i\,\mathcal{E}_i^e\,\mathcal{E}_k^e\right\} \qquad \text{(V-3) cont.}$$

$$\text{(g)}\quad M_k^i = -\frac{\rho^e}{\beta^{e^2}\lambda^e}\left\{\left(5+\frac{4}{3\sqrt{2}}\frac{Z^2n^i}{n^e}\frac{m^e}{m^i}\frac{\beta^{i^{3/2}}}{\beta^{e^{3/2}}}\frac{\Omega_{ii}^{(2)}}{\Omega_{ei}^{(1)}}\right)Q'^i_k\right\}$$

In the above relations and those to follow, it has been found convenient to introduce the electron mean free path λ^e, and collision time $\mathcal{T}^e$, defined by

$$\text{(a)}\quad \lambda^e = \frac{(k\,T^e)^2}{4\sqrt{2\pi}\,n^iZ^2\,e^4\,\Omega_{ei}^{(1)}}$$

$$\text{(b)}\quad \mathcal{T}^e = \lambda^e\,\beta^{e^{1/2}}. \qquad \text{(V-4)}$$

With the exception of ion heat flux transfer, terms have been retained to order $\mathcal{E}^3$. The ion heat flux is sufficiently small relative to electron heat flux that inclusion of terms of order $\mathcal{E}^3$ is unnecessary. To within the limits imposed by (IV-8, a, b) and (V-1, a), $\mathcal{E}_k^e = - \frac{\beta^{e^{1/2}}}{n^e e} j_k$; hence, the collision transfer terms could equally as well be expressed in terms of the conduction current instead of $\mathcal{E}_k^e$.

With regard to the momentum transfer term S_k^e, it is of interest to note that, in addition to the usual heat flux coupling $(-Q_k^{e'})$, (see, for example, reference 6), there is higher order heat flux coupling $(\frac{\mathcal{E}^2}{2} Q_k^{e'} + Q_j^{'e} \mathcal{E}_j^e \mathcal{E}_k^e)$, and pressure tensor coupling $(-\frac{2}{5} P_{kj}^{e'} \mathcal{E}_j^e)$. Attention will be given to the influence of this coupling on conductivity later in this section.

To obtain transport relations for the traceless stress and heat flux of electrons, attention is restricted to systems in which space and time variations are small. Explicitly, it is required that

$$\text{(a)} \quad \frac{\tau^e}{\bar{t}} \ll \mathcal{E}^2 < 1; \qquad \frac{\tau^e w^e}{\lambda^e} \ll 1; \qquad \frac{\tau^e w^o}{\lambda^e} \ll 1; \tag{V-5}$$

$$\text{(b)} \quad \frac{\lambda^e}{\bar{r}} \gtrsim \mathcal{E}^3;$$

where τ^e and λ^e are the electron collision time and mean free path respectively; and $\bar{t}$ and $\bar{r}$ are respectively the characteristic time and characteristic distance for macroscopic changes in the system.

For electron densities in the range 10^{12} to 10^{15} particles/cm^3 and a temperature of 10^5 ^{o}K (i.e. 8.6 ev), the mean free path ranges from 8×10^3 to 8 cm., and the electron collision time ranges from 6×10^{-5} to 6×10^{-8} seconds. The limitation on characteristic distance is seen to be too severe for many laboratory plasmas.

With these restrictions, the differential equations for electron traceless stress and heat flux (IV-2, d, e) reduce to coupled algebraic equations. These equations may be decoupled, however, since the traceless stress to second order requires the heat flux to first order only in $\mathcal{E}$, and the latter is independent of the stress. Hence, the procedure is to solve for the heat flux to first order in $\mathcal{E}$. The first order heat flux is then substituted into the traceless stress equation and the traceless stress determined to second order in $\mathcal{E}$. Finally, the second order stress is substituted into the heat flux equation, and the heat flux determined to third order in $\mathcal{E}$. The relations which result are exhibited below.

The heat flux, to first order in $\mathcal{E}$, is given by

$$Q_r^e = \frac{\rho^e}{\beta^{e\,3/2}} \xi^e \alpha_{ri}^e \mathcal{E}_i^e \qquad \text{(V-6)}$$

where

$$\xi^e = \frac{15}{13} \frac{1}{\left(1 + \frac{8}{13\sqrt{2}} \frac{n^e}{Z^2 n^i} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right)} \quad . \qquad \text{(V-7)}$$

With the choice of a local coordinate system in which the magnetic field is in the z-direction,

$$\text{(a)} \quad \alpha_{ri}^e = \begin{pmatrix} \chi^{(1)} & \chi^{(2)} & 0 \\ -\chi^{(2)} & \chi^{(1)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(V-8)}$$

$$\text{(b)} \quad \chi^{(1)} = \frac{1}{1 + \omega^{e\,2} \mathcal{T}_{(1)}^{e\,2}} \quad .$$

$$\text{(c)} \quad \chi^{(2)} = \frac{\omega^e \mathcal{T}^e_{(1)}}{1 + \omega^{e^2} \mathcal{T}^{e^2}_{(1)}} \quad . \tag{V-8 cont.}$$

$\mathcal{T}^e_{(1)}$ is the collision time for electron heat flux transfer, and $|\omega^e|$ is the cyclotron frequency for electrons:

$$\text{(a)} \quad \mathcal{T}^e_{(1)} = 2 \, \xi^e \mathcal{T}^e$$

$$\text{(b)} \quad \omega^e = - \frac{e H}{m^e c} \quad . \tag{V-9}$$

Representation of the above expression for heat flux in the local coordinate system specified does not imply that the direction of the magnetic field is known.

The traceless stress, to second order in $\mathcal{E}$, is given by

$$\overset{o}{P}{}^{e}_{rs} - 2 \, \omega^e \mathcal{T}^e_{(2)} \, \varepsilon_{rmn} \, P^e_{ms} \, \hat{H}_n = p^e \, \phi^e \, T_{rs} \tag{V-10}$$

where $\hat{H}_n$ is a unit vector in the direction of the magnetic field, and $\mathcal{T}^e_{(2)}$ is the collision time for electron stress transfer given by,

$$\mathcal{T}^e_{(2)} = \frac{5}{2} \, \phi^e \mathcal{T}^e \tag{V-11}$$

The dimensionless quantities ϕ^e and T_{rs} are given by,

$$\text{(a)} \quad \phi^e = \frac{1}{\left(\dfrac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}} + \dfrac{n^e}{\sqrt{2} \, Z^2 n^i} \; \dfrac{\Omega^{(2)}_{ee}}{\Omega^{(1)}_{ei}} \right)} \tag{V-12}$$

$$\text{(b)}\quad T_{rs} = \overline{\left(a^{(1)}\,\delta_{ri} + a^{(2)}\,\alpha^{e}_{ri}\right)\,\mathcal{E}^{e}_{i}\,\mathcal{E}^{e}_{s}}^{\,o} \qquad \text{(V-12) cont.}$$

where

$$\text{(a)}\quad a^{(1)} = \left(\frac{5}{3} - \frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}\right) \qquad \text{(V-13)}$$

$$\text{(b)}\quad a^{(2)} = \left(-1 + \frac{6}{5}\,\frac{\Omega^{(2)}_{ei}}{\Omega^{(1)}_{ei}}\right)\xi^{e}$$

With the choice of a local coordinate system in which the magnetic field is in the z-direction and $\mathcal{E}_k$ in the x-z plane, the traceless stress components may be written,

$$\text{(a)}\quad \overset{o}{P}{}^{e}_{zz} = p^{e}\,\phi^{e}\left\{b^{(1)}\,\mathcal{E}^{e}_{z}\,\mathcal{E}^{e}_{z} - b^{(2)}\,\mathcal{E}^{e}_{x}\,\mathcal{E}^{e}_{x}\right\}$$

$$\text{(b)}\quad \overset{o}{P}{}^{e}_{xx} = \frac{p^{e}\,\phi^{e}}{1 + 4\,\omega^{e^{2}}\,\mathcal{T}^{e^{2}}_{(2)}}\left\{b^{(3)}\,\mathcal{E}^{e}_{x}\,\mathcal{E}^{e}_{x} - b^{(4)}\,\mathcal{E}^{e}_{z}\,\mathcal{E}^{e}_{z}\right\}$$

$$\text{(c)}\quad \overset{o}{P}{}^{e}_{yy} = -\frac{p^{e}\,\phi^{e}}{1 + 4\,\omega^{e^{2}}\,\mathcal{T}^{e^{2}}_{(2)}}\left\{b^{(5)}\,\mathcal{E}^{e}_{x}\,\mathcal{E}^{e}_{x} + b^{(4)}\,\mathcal{E}^{e}_{z}\,\mathcal{E}^{e}_{z}\right\} \qquad \text{(V-14)}$$

$$\text{(d)}\quad \overset{o}{P}{}^{e}_{xy} = \overset{o}{P}{}^{e}_{yx} = -\frac{p^{e}\,\phi^{e}}{1 + 4\,\omega^{e^{2}}\,\mathcal{T}^{e^{2}}_{(2)}}\left\{b^{(6)}\,\mathcal{E}^{e}_{x}\,\mathcal{E}^{e}_{x}\right\}$$

$$\text{(e)}\quad \overset{o}{P}{}^{e}_{xz} = \overset{o}{P}{}^{e}_{zx} = \frac{p^e \phi^e}{1 + \omega^{e^2} \mathcal{T}^{e^2}_{(2)}} \left\{ b^{(7)} \mathcal{E}^e_x \mathcal{E}^e_z \right\}$$

$$\text{(f)}\quad \overset{o}{P}{}^{e}_{zy} = \overset{o}{P}{}^{e}_{yz} = - \frac{p^e \phi^e}{1 + \omega^{e^2} \mathcal{T}^{e^2}_{(2)}} \left\{ b^{(8)} \mathcal{E}^e_x \mathcal{E}^e_z \right\} \qquad \text{(V-14) cont.}$$

where,

$$\text{(a)}\quad b^{(1)} = \frac{2}{3} (a^{(1)} + a^{(2)})$$

$$\text{(b)}\quad b^{(2)} = \frac{1}{3} (a^{(1)} + a^{(2)} \chi^{(1)})$$

$$\text{(c)}\quad b^{(3)} = \frac{2}{3} (a^{(1)} + a^{(2)} \chi^{(1)}) (1 + \omega^{e^2} \mathcal{T}^{e^2}_{(2)}) - a^{(2)} \chi^{(2)} \omega^e \mathcal{T}^e_{(2)}$$

$$\text{(d)}\quad b^{(4)} = \frac{1}{3} (a^{(1)} + a^{(2)}) (1 + 4\, \omega^{e^2} \mathcal{T}^{e^2}_{(2)}) \qquad \text{(V-15)}$$

$$\text{(e)}\quad b^{(5)} = \frac{1}{3} (a^{(1)} + a^{(2)} \chi^{(1)})(1 - 2\, \omega^{e^2} \mathcal{T}^{e^2}_{(2)}) - a^{(2)} \chi^{(2)} \omega^e \mathcal{T}^e_{(2)}$$

$$\text{(f)}\quad b^{(6)} = (a^{(1)} + a^{(2)} \chi^{(1)})\, \omega^e \mathcal{T}^e_{(2)} + \frac{1}{2}\, a^{(2)} \chi^{(2)}$$

$$\text{(g)}\quad b^{(7)} = \left[a^{(1)} + \frac{1}{2}\, a^{(2)} (\chi^{(1)} + 1) \right] - \frac{1}{2}\, a^{(2)} \chi^{(2)}\, \omega^e \mathcal{T}^e_{(2)}$$

$$\text{(h)}\quad b^{(8)} = \left[a^{(1)} + \frac{1}{2}\, a^{(2)} (\chi^{(1)} + 1) \right] \omega^e \mathcal{T}^e_{(2)} + \frac{1}{2}\, a^{(2)} \chi^{(2)}$$

For the class of systems considered here, there does not appear to be a reliable expression for the electron stress tensor with which this result could be compared. With regard to the result exhibited by Herdan and Liley, the restrictions are in one respect less severe in that the influence of space variations of plasma flow velocity is retained and the stress tensor is then a function of this variation. On the other hand, the influence of the space variation of heat flux has been ignored, which implies that the difference between electron flow velocity and the ion flow velocity is much less than the plasma flow velocity. (The latter assertion may be verified by substituting the relation which they obtain for electron heat flux into their equations for electron stress). In view of the difference in the classes of systems considered, a comparison of results is not meaningful.

If, in the reduction of the transport equations, we were to relieve the restriction on space variations, that is, take $\lambda^e/\bar{r}$ to be of the order of $\mathcal{E}$; an expression for the electron stress could be exhibited which would reflect the influence of both the space variation of electron flow velocity and the space variation of electron heat flux. However, this relation would be very complex by virtue of the tensorial coefficient in the expression for heat flux.

The heat flux, to third order in $\mathcal{E}$, is given by

$$Q_r^e = \frac{\rho^e}{\beta^{e\,3/2}} \alpha_{ri}^e \left\{ \xi^e \mathcal{E}_i^e - \nu^e \frac{\partial T^e}{\partial r_i} + \xi^{e'} \mathcal{E}_i^e + \Gamma_{ij}^e \mathcal{E}_j^e \right\} \qquad \text{(V-16)}$$

where

$$\text{(a)} \quad \nu^e = 5\, \xi^e \frac{\lambda^e}{T^e}$$

$$\text{(b)} \quad \xi^{e'} = \xi^e \left[\left(-\frac{43}{30} + \frac{8}{15} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right) \mathcal{E}^2 + \left(\frac{33}{25} - \frac{4}{25} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right) \xi^e \alpha_{ij}^e \mathcal{E}_i^e \mathcal{E}_j^e \right]$$

$$\text{(V-17)}$$

$$\text{(c)} \quad \Gamma_{ij}^e = \xi^e \left[\left(-\frac{1}{5} + \frac{4}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right) \frac{\overset{o}{P}{}_{ij}^e}{p^e} - \frac{2}{5} \xi^e \frac{\overset{o}{P}{}_{ik}^e}{p^e} \alpha_{kj}^e + \left(\frac{33}{50} - \frac{12}{25} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}}\right) \xi^e \mathcal{E}^2 \alpha_{ij}^e \right]$$

Expressed in the notation used here, Herdan and Liley obtain,

$$Q^e_r = \frac{\rho^e}{{\beta^e}^{3/2}} \alpha^e_{ri} \left\{ \xi^e \varepsilon^e_i - \nu^e \frac{\partial T^e}{\partial r_i} \right\} . \tag{V-18}$$

Hence, the latter two terms in (V-16) constitute a correction to the Herdan and Liley result. It may be noted that these terms are of order ε^3. The thermal conductivity is given by,

$$\nu^{e'} = 5\, \xi^e \frac{k\, n^e \lambda^e}{{\beta^e}^{1/2}} \tag{V-19}$$

Attention is now directed to the current equation (generalized Ohm's law). To third order in ε, the equation may be written,

$$\frac{1}{n^e e} \frac{\partial p^e}{\partial r_k} + (E_k + \frac{1}{c} \varepsilon_{k\ell m} w^o_\ell H_m) - \frac{1}{n^e e c} \varepsilon_{k\ell m} j_\ell H_m$$

$$= - \frac{\xi^e}{e} \alpha^e_{ki} \frac{\partial\, kT^e}{\partial r_i} + \eta^o (\delta_{k\ell} - \frac{3}{5} \xi^e \alpha_{k\ell}) j_\ell \tag{V-20}$$

$$+ \left[\eta^o \left\{ (-\frac{3}{10} \varepsilon^2 + \frac{3}{5} \xi^e \alpha^e_{ij} \varepsilon^e_i \varepsilon^e_j) \delta_{k\ell} + (- \frac{3}{5} \xi^{e'} + \frac{3}{10} \xi^e \varepsilon^2) \alpha^e_{k\ell} \right.\right.$$

$$\left.\left. - \frac{3}{5} \frac{P^{oe}_{k\ell}}{p^e} - \frac{3}{5} \alpha^e_{km} \Gamma^e_{m\ell} \right\} j_\ell \right]$$

where

$$\eta^o = \frac{m^e}{3 \tau^e n^e e^2} , \tag{V-21}$$

The use of both ε_k^e and j_k is a matter of convenience. As pointed out previously, $\varepsilon_k^e \approx -\frac{\beta^{e^{1/2}}}{n^e e} j_k$, and $\varepsilon = |\varepsilon_k^e|$.

With the exception of the terms inside the bracket this relation is the same as that exhibited by Herdan and Liley. The terms inside the bracket constitute correction terms of third order in ε.

To first order in ε, the resistivity is, of course, the same as that given by Herdan and Liley. To exhibit an explicit relationship for the resistivity, to third order in ε, in the presence of a magnetic field, is a formidable task as a result of the complex coupling with the electron stress tensor. However, in the absence of a magnetic field, considerable simplification results. For this special case, the current equation reduces to,

$$\frac{1}{n^e e}\frac{\partial p^e}{\partial r_i} + \frac{\xi^e}{e}\frac{\partial k T^e}{\partial r_i} + E_i$$

$$= \eta^o \Big[(1 - \frac{3}{5}\xi^e) + (-\frac{3}{10} + \frac{36}{25}\xi^e - \frac{201}{250}\xi^{e^2} - \frac{4}{15}\phi^e - \frac{6}{25}\xi^e\phi^e + \frac{22}{375}\xi^{e^2}\phi^e + \frac{4}{125}\xi^{e^3}\phi^e)\varepsilon^2\Big] j_i . \tag{V-22}$$

It may be noted that the coefficient of the current vector is a function of the current. The resistivity (assuming this nomenclature appropriate) is given by

$$\eta = \eta^o \Big[(1 - \frac{3}{5}\xi^e) + (-\frac{3}{10} + \frac{36}{25}\xi^e - \frac{201}{250}\xi^{e^2} - \frac{4}{15}\phi^e - \frac{6}{25}\xi^e\phi^e + \frac{22}{375}\xi^{e^2}\phi^e + \frac{4}{125}\xi^{e^3}\phi^e)\varepsilon^2\Big] \tag{V-23}$$

By comparison, Herdan and Liley obtain,

$$\eta = \eta^{o} \left(1 - \frac{3}{5}\ \xi^{e}\right) \tag{V-24}$$

whereas Chen[11], who has approximated the distribution functions by using only the symmetric parts of the distribution functions used herein (i.e. the Maxwellian portion), obtains

$$\eta = \eta^{o}\left(1 - \frac{3}{10}\varepsilon^{2}\right). \tag{V-25}$$

For a numerical comparison for the cases where Z = 1 and where Z is large we find,

	Herdan and Liley	Chen	Results Herein
Z = 1	$\eta^{o}\,(.53)$	$\eta^{o}(1 - \frac{3}{10}\,\varepsilon^{2})$	$\eta^{o}(.53 - .44\,\varepsilon^{2})$
Z (large)	$\eta^{o}\,(.31)$	$\eta^{o}(1 - \frac{3}{10}\,\varepsilon^{2})$	$\eta^{o}(.31 - .12\,\varepsilon^{2})$

(V-26)

The Chen result differs from the result herein by a factor of from two to three (depending on Z), even for systems in which $\varepsilon^{2} \ll 1$. The source of this difference is the use of the Maxwellian part only of the distribution functions. As may be seen herein, the heat flux coupling in the momentum transfer, which gives rise to the term $(-\frac{3}{5}\ \xi^{e}\eta^{o}\ j_{k})$ in (V-22), is very significant even to first order in ε.

As pointed out previously, to third order in ε, additional heat flux coupling and stress coupling becomes evident. This coupling accounts for the corrections of order ε^{3} to the results exhibited by Herdan and Liley. If for example, $\varepsilon^{2} = 1/3$, the correction ranges from 13 per cent for large Z to 27 per cent for Z = 1. The correction is such that the resistance is reduced.

B. Plasma Systems in which $\mathcal{E}^2 >> 1$.

To reduce the general magnetohydrodynamic equations of Section IV to a manageable form for systems in which $\mathcal{E}^2 >> 1$, we again restrict attention to systems with slowly varying flows. Explicitly, it is required that

$$\text{(a)} \left\{ \left(\frac{m^i}{m^e}\right)^2 \left(\frac{\beta^e}{\beta^i}\right)^{3/2} \frac{1}{Z^2} \right\} \frac{\mathcal{T}^e}{\bar{t}} << 1;$$

$$\left\{ \left(\frac{m^i}{m^e}\right)^2 \left(\frac{\beta^e}{\beta^i}\right)^{3/2} \frac{1}{Z^2} \right\} \frac{\mathcal{T}^e w^i}{\bar{r}} << 1; \quad Z \frac{\mathcal{T}^e w^e}{\bar{r}} << 1; \qquad \text{(V-27)}$$

$$\text{(b)} \left\{ \left(\frac{m^i}{m^e}\right)^2 \left(\frac{\beta^e}{\beta^i}\right)^{3/2} \frac{1}{Z^2} \right\} \frac{\mathcal{T}^e}{\bar{r}\beta^i w^i} << 1; \quad Z \frac{\mathcal{T}^e}{\bar{r}\ \beta^e w^e} << 1.$$

[Since relativistic effects have not been accounted for, a limitation is imposed on the difference in constituent flow velocity. If, for example, we require that, $| w_k^e - w_k^i | \lesssim c/5$, where c is the velocity of light; then for $kT^e = 100$ ev., $\mathcal{E} = \beta^{e^{1/2}} |w_k^e - w_k^i| \lesssim \beta^{e^{1/2}} \frac{c}{5} \approx 15.$]

The $\mathcal{E}$ dependent functions in the collision transfer expressions (III-24) are expanded in inverse powers of $\mathcal{E}$, and only the highest order terms in the collision transfer expressions are retained.

After considerable manipulation the traceless stress equation for particles of type α reduces to the form,

$$\overset{\circ\alpha}{P}_{rs} - 2\,\omega^\alpha \mathcal{T}^\alpha_{(3)}\ \mathcal{E}_{rmn} \overset{\circ\alpha}{P}_{ms} \hat{H}_n = -2\,\mu^\alpha \underline{\underline{\frac{\partial w_s^\alpha}{\partial r_r}}}^{\circ} \qquad \text{(V-29)}$$

where $\mathcal{T}^{\alpha}_{(3)}$ is the collision time for traceless stress transfer, for particles of type α given by,

$$\mathcal{T}^{\alpha}_{(3)} = \frac{5}{8\sqrt{\pi}} \quad \frac{m^{\alpha^2}}{n^{\alpha} e^{\alpha^4} \beta^{\alpha^{3/2}} \Omega^{(2)}_{\alpha\alpha}} , \tag{V-29}$$

and μ^{α} is the viscosity coefficient for α type particles, given by,

$$\mu^{\alpha} = p^{\alpha} \mathcal{T}^{\alpha}_{(3)} \tag{V-30}$$

Again choosing a coordinate system in which the magnetic field is in the Z-direction, the components of $\overset{o}{P}^{\alpha}_{rs}$ may be written,

$$\text{(a)} \quad \overset{o}{P}^{\alpha}_{zz} = - 2 \mu^{\alpha} U_{zz}$$

$$\text{(b)} \quad \overset{o}{P}^{\alpha}_{xx} = - \frac{2\mu^{\alpha}}{1 + 4 \omega^{\alpha^2} \mathcal{T}^{\alpha^2}_{(3)}} \left\{ U_{xx} + \frac{1}{2} (U_{xx} + U_{yy}) \, 4\omega^{\alpha^2} \mathcal{T}^{\alpha^2}_{(3)} + U_{xy} \, 2 \, \omega^{\alpha} \mathcal{T}^{\alpha}_{(3)} \right\}$$

$$\text{(c)} \quad \overset{o}{P}^{\alpha}_{yy} = - \frac{2\mu^{\alpha}}{1 + 4 \omega^{\alpha^2} \mathcal{T}^{\alpha^2}_{(3)}} \left\{ U_{yy} + \frac{1}{2} (U_{xx} + U_{yy}) \, 4 \, \omega^{\alpha^2} \mathcal{T}^{\alpha^2}_{(3)} - U_{xy} \, 2\omega^{\alpha} \mathcal{T}^{\alpha}_{(3)} \right\}$$

$$\text{(d)} \quad \overset{o}{P}^{\alpha}_{xy} = \overset{o}{P}^{\alpha}_{yx} = - \frac{2\mu^{\alpha}}{1 + 4 \omega^{\alpha^2} \mathcal{T}^{\alpha^2}_{(3)}} \left\{ U_{xy} + \frac{1}{2} (U_{yy} - U_{xx}) \, 2 \, \omega^{\alpha} \mathcal{T}^{\alpha}_{(3)} \right\} \tag{V-31}$$

$$\text{(e)} \quad \overset{o}{P}^{\alpha}_{xz} = \overset{o}{P}^{\alpha}_{zx} = - \frac{2\mu^{\alpha}}{1 + \omega^{\alpha^2} \mathcal{T}^{\alpha^2}_{(3)}} \left\{ U_{zx} + \omega^{\alpha} \mathcal{T}^{\alpha}_{(3)} U_{zy} \right\}$$

$$\text{(f)}\quad \overset{o}{P}{}^{\alpha}_{zy} = \overset{o}{P}{}^{\alpha}_{yz} = -\frac{2\mu^{\alpha}}{1+\omega^{\alpha^2}\mathcal{T}^{\alpha^2}_{(3)}}\left\{U_{zy} - \omega^{\alpha}\mathcal{T}^{\alpha}_{(3)}\,U_{zx}\right\} \tag{V-31 cont.}$$

where,

$$U_{ij} = \overset{o}{\overline{\overline{\frac{\partial w_i^{\alpha}}{\partial r_i}}}} \tag{V-32}$$

The transport relation for heat flux for particles of type α reduces to the form,

$$Q_r^{\alpha} = \alpha_{ri}^{\alpha}\left\{-\nu^{\alpha}\,\frac{\partial T^{\alpha}}{\partial r_i}\right\} \tag{V-33}$$

where

$$\alpha_{ri}^{\alpha} = \begin{pmatrix} \dfrac{1}{1+\omega^{\alpha^2}\mathcal{T}^{\alpha^2}_{(4)}} & \dfrac{\omega^{\alpha}\mathcal{T}^{\alpha}_{(4)}}{1+\omega^{\alpha^2}\mathcal{T}^{\alpha^2}_{(4)}} & 1 \\ -\dfrac{\omega^{\alpha}\mathcal{T}^{\alpha}_{(4)}}{1+\omega^{\alpha^2}\mathcal{T}^{\alpha^2}_{(4)}} & \dfrac{1}{1+\omega^{\alpha^2}\mathcal{T}^{\alpha^2}_{(4)}} & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

in a local coordinate system in which the z direction is taken to be along the magnetic field H_k. $\mathcal{T}^{\alpha}_{(4)}$ is the collision time for heat transfer of the α type particles, and ν^{α} is the thermal conductivity:

$$\text{(a)}\ \mathcal{T}^{\alpha}_{(4)} = \frac{m^{\alpha^2}}{16\sqrt{\pi}} \frac{m^{\alpha^2}}{n^{\alpha} e^{\alpha^4} \beta^{\alpha^{3/2}} \Omega^{(2)}_{\alpha\alpha}} \tag{V-35}$$

$$\text{(b)}\ \nu^{\alpha} = \frac{5}{2} \frac{kp^{\alpha}}{m^{\alpha}} \mathcal{T}^{\alpha}_{(4)} \ .$$

The generalized Ohm's law equation for conduction current reduces to,

$$\frac{m^e}{n^e e^2}\left\{\frac{D_o j_k}{Dt} + j_k \frac{\partial w^o_i}{\partial r_i} + j_i \frac{\partial w^o_k}{\partial r_i}\right\} = \frac{1}{n^e e} \frac{\partial}{\partial r_i}\left\{P^e_{ik} + \frac{m^e}{n^e e^2} j_i j_k\right\}$$

$$+ (E_k + \frac{1}{c} \varepsilon_{k\ell m} w^o_{\ell} H_m) - \frac{1}{n^e ec} \varepsilon_{k\ell m} j_{\ell} H_m - \eta^{(2)} j_k \tag{V-36}$$

where the resistivity $\eta^{(2)}$ is given by,

$$\eta^{(2)} = (4\pi \frac{n^i}{n^e} \frac{Ze^2}{m^e} \beta^{e^{3/2}} \Omega^{(1)}_{ei}) \frac{1}{\varepsilon^3} \ . \tag{V-37}$$

It may be noted that the expressions for stress and heat flux have the same form as would be obtained for a one constituent gas. (See, for example, sections 18.43 and 18.44 of Reference 1). That is, for systems in which $\varepsilon^2 >> 1$, the unlike particle collisions are ineffective relative to like particle collisions in changing either the stress or the heat flux. This decoupling of the plasma constituents results from the decrease in the unlike-particle interaction cross section as the difference in constituent velocity, and hence the relative velocity of interacting particles, becomes large.

With regard to momentum transfer, the like particle collisions have no influence. Hence, although small , the influence of unlike particle collisions must be retained. The resistivity which reflects the influence of unlike particle collisions is seen to be proportional to $1/\varepsilon^3$.

Although the above expressions for stress and heat flux constitute new results; the expression for resistivity has been previously obtained by Driecer[13] and by Burgers[7].

VI. SUMMARY

Grad 13-moment approximation distribution functions (expressions II-12) have been assumed to provide adequate velocity space solutions for the Boltzmann equations of the plasma constituents. These solutions are anticipated to have validity over a wide class of systems which includes those in which the difference in constituent flow velocity is appreciable relative to the electron thermal velocity, in addition to those close to equilibrium.

With the use of these distribution functions it has been found possible to derive generalized magnetohydrodynamic equations, which coupled with Maxwell's equations, constitute a closed set [Equations (IV-4, 5, 6, 2c, 2d, 2e) and (II-2)].

In their general form, these equations are so extremely complicated that their usefulness is obscure. Hence, to demonstrate their usefulness, transport relations [(V-14) and (V-16)] to third order in ε (ratio of the difference in constituent flow velocity to random electron velocity) have been exhibited for the electron heat flux and stress, for systems with slowly varying flows in which ε is less than one, [restrictions (V-5)]. For this class of systems, the generalized Ohm's law is also exhibited [equation (V-20)] and the resistivity is determined to third order in ε for systems in which the magnetic field is zero [relation (V-23)]. With the exception of the relation for electron stress, the results are found to be in agreement with those of Herdan and Liley[6] to order ε, giving corrections to order ε^3. With regard to electron stress, there does not appear to be a reliable expression for the class of systems under consideration, with which to compare. The numerical comparison of the resistivity for the case where the magnetic field is zero [(V-26)] indicates that, for systems in which $\mathcal{E}^2 \approx 1/3$, the resistivity is less than that exhibited by Herdan and Liley by from 13 per cent to 27 per cent, depending on Z. Further, the inadequacy of the Maxwellian part only of distribution (II-12), which has been used by some authors (see, for example, references (12) and (13)), has been made evident.

Simplified transport relations [(V-31), (V-33), and (V-36)] have also been exhibited for systems with slowly varying flows in which $\mathcal{E}^2 \gg 1$ [restrictions (V-27)]. The expressions for heat flux and stress are found to have the same form as those for a one constituent gas. The resistivity [relation (V-37)] is seen to vary inversely with the current cubed as previously noted by Driecer[(12)] and Burgers[(7)].

Hence, integral (A-1, a) becomes,

$$4\pi\, v_{r_k} \int \frac{d\,(\sin\frac{x}{2})}{\sin\frac{x}{2}} = 2\pi\, v_{r_k}\, \ln\left\{ 1 + \left(\frac{m^{\alpha\gamma} h}{e^{\alpha} e^{\gamma}}\right)^2 v_r^4 \right\} \qquad \text{(A-4)}$$

The tensor $(v_{r_k} v_{r_\ell} - v'_{r_k} v'_{r_\ell})$ may be represented by,

$$\begin{bmatrix} -v_r^2 \sin^2 x \cos^2\phi & -v_r^2 \sin^2 x \cos\phi \sin\phi & -v_r^2 \cos x \sin x \cos\phi \\ -v_r^2 \sin^2 x \cos\phi \sin\phi & -v_r^2 \sin^2 x \sin^2\phi & -v_r^2 \cos x \sin x \sin\phi \\ -v_r^2 \cos x \sin x \cos\phi & -v_r^2 \cos x \sin x \sin\phi & v_r^2 (1 - \cos^2 x) \end{bmatrix} \qquad \text{(A-5)}$$

which on integration over $d\,\phi$ becomes,

$$\pi \begin{bmatrix} -v_r^2 \sin^2 x & 0 & 0 \\ 0 & -v_r^2 \sin^2 x & 0 \\ 0 & 0 & 3\, v_r^2 \sin^2 x - v_r^2 \sin^2 x \end{bmatrix} . \qquad \text{(A-6)}$$

In index notation, with the direction of v_r unspecified, (A-6) may be written

$$3\pi \sin^2 x \left(v_{r_k} v_{r_\ell} - \frac{\delta_{k\ell}}{3} v_r^2\right) = 12\pi \sin^2\frac{x}{2} \cos^2\frac{x}{2} \left(v_{r_k} v_{r_\ell} - \frac{\delta_{k\ell}}{3} v_r^2\right) . \qquad \text{(A-7)}$$

Hence, expression (A-1, b) may be written,

$$12\pi \left(v_{r_k} v_{r_\ell} - \frac{\delta_{k\ell}}{3} v_r^2\right) \int \frac{\cos^2\frac{x}{2}}{\sin\frac{x}{2}}\, d\,(\sin\frac{x}{2}) . \qquad \text{(A-8)}$$

Finally, on integration over the range from 1 to $\dfrac{1}{\sqrt{1+\left(\dfrac{m^{\alpha\gamma}h}{e^{\alpha}e^{\gamma}}\right)^{2} v_r^4}}$ on $\sin\frac{x}{2}$ we obtain,

$$6\pi\left(v_{r_k} v_{r_\ell} - \frac{\delta_{k\ell}}{3} v_r^2\right)\left[\ln\left\{1+\left(\frac{m^{\alpha\gamma}h}{e^{\alpha}e^{\gamma}}\right)^{2} v_r^4\right\} - \frac{\left(\dfrac{m^{\alpha\gamma}h}{e^{\alpha}e^{\gamma}}\right)^{2} v_r^4}{1+\left(\dfrac{m^{\alpha\gamma}h}{e^{\alpha}e^{\gamma}}\right) v_r^4}\right] \qquad \text{(A-9)}$$

APPENDIX A

INTEGRATIONS OVER ANGLES OF SCATTER.

The integrals over angles of scatter were found in section II to be of the form,

$$\text{(a)} \quad \int (v_{r_k} - v'_{r_k}) \frac{d(\sin\frac{x}{2})}{\sin^3\frac{x}{2}} \, d\phi \tag{A-1}$$

and

$$\text{(b)} \quad \int (v_{r_k} v_{r_\ell} - v'_{r_k} v'_{r_\ell}) \frac{d(\sin\frac{x}{2})}{\sin^3\frac{x}{2}} \, d\phi \quad .$$

The limits of integration are from 0 to 2π on ϕ ; and 1 to $\frac{1}{\sqrt{1 + (\frac{m^{\alpha\gamma} h}{e^\alpha e^\gamma})^2 v_r^4}}$ on $\sin\frac{x}{2}$.

Choosing a coordinate system in which v_{r_k} is in the z-direction, the vector $(v_{r_k} - v'_{r_k})$ may be represented by

$$\begin{pmatrix} -v_r \sin x \cos\phi \\ -v_r \sin x \sin\phi \\ v_r (1 - \cos x) \end{pmatrix} , \tag{A-2}$$

which on integration over $d\phi$ becomes,

$$2\pi v_{r_k} (1 - \cos x) = 4\pi v_{r_k} \sin^2\frac{x}{2} \quad . \tag{A-3}$$

APPENDIX B

INTEGRATION OVER VELOCITY SPACES

All terms in the collision transport expressions were found in section II to involve integrals of the form,

$$\frac{1}{4\pi(2\pi)^{3/2}} \int e^{-\left[\frac{G^2}{2} + \frac{(\vec{g}-\vec{\varepsilon})^2}{2}\right]} \frac{G_{i_1}G_{i_2}\cdots G_{i_n}g_{j_1}g_{j_2}\cdots g_{j_m}}{g^3} d^3G d^3g. \tag{B-1}$$

Before proceeding to evaluate the required integrals, it is found convenient to introduce a special notation. The third order tensor $\{\delta\varepsilon\}_{ijk}$ is formed by taking the sum of all distinct products of the Kronecker delta and the vector ε_r which arise on permutation of the subscripts, each term occuring once. That is,

$$\{\delta\varepsilon\}_{ijk} = \delta_{ij}\varepsilon_k + \delta_{ik}\varepsilon_j + \delta_{jk}\varepsilon_i \quad . \tag{B-2}$$

Similarly,

$$\{\delta\varepsilon\varepsilon\}_{ijk\ell} = \delta_{ij}\varepsilon_k\varepsilon_\ell + \delta_{ik}\varepsilon_j\varepsilon_\ell + \delta_{i\ell}\varepsilon_j\varepsilon_k + \delta_{jk}\varepsilon_i\varepsilon_\ell + \delta_{j\ell}\varepsilon_i\varepsilon_k + \delta_{k\ell}\varepsilon_i\varepsilon_j \quad . \tag{B-3}$$

Other tensorial expressions of this type, $\{\delta\delta\}_{ijk\ell}$, $\{\delta\varepsilon\varepsilon\varepsilon\}_{ijk\ell m}$, etc., are formed in an analogous manner.

Turning attention to integrals (B-1), the G-space integrations have been carried out by Grad[2,3]. Those required are tabulated below.

(B-3)

(a) $$\frac{1}{(2\pi)^{3/2}} \int e^{-G^2/2} d^3G = 1$$

(b) $$\frac{1}{(2\pi)^{3/2}} \int e^{-G^2/2} G_i G_j d^3G = \delta_{ij}$$

(c) $$\frac{1}{(2\pi)^{3/2}} \int e^{-G^2/2} G^2 d^3G = 3$$

(d) $$\frac{1}{(2\pi)^{3/2}} \int e^{-G^2/2} G_i G_j G_k G \quad d^3G = \{\delta\delta\}_{ijk\ell}$$

(e) $$\frac{1}{(2\pi)^{3/2}} \int e^{-G^2/2} G^2 G_i G_j d^3G = 5\delta_{ij}$$

(f) $$\frac{1}{(2\pi)^{3/2}} \int e^{-G^2/2} G^4 d^3 G = 15 \quad .$$

To facilitate the g-space integrations, we define,

(B-4)

(a) $$\left\langle g_{i_1} g_{i_2} \cdots g_{i_n} \right\rangle \equiv \frac{1}{4\pi} \int e^{-\frac{(\vec{g}-\vec{\varepsilon})^2}{2}} \frac{g_{i_1} g_{i_2} \cdots g_{i_n} d^3 g}{g^3}$$

(b) $$I_k \equiv \frac{1}{4\pi} \int e^{-\frac{g^2}{2} + \vec{g}\cdot\vec{\varepsilon}} \frac{g_k}{g^3} d^3 g \quad ,$$

and observe that,

$$\left\langle g_{i_1} g_{i_2} \cdots g_{i_n} \right\rangle = e^{-\varepsilon^2/2} \frac{\partial^n I_{i_n}}{\partial \varepsilon_{i_1} \partial \varepsilon_{i_2} \cdots \partial \varepsilon_{i_{n-1}}} \quad : \tag{B-5}$$

That is, all of the g-space integrals of interest may be expressed in terms of derivatives of the integral I_k with respect to the vector ε_ℓ .

To evaluate I_k, a change of variables is made from g_k to $x_k = g_k - \varepsilon_k$, which transforms I_k to the form

$$I_k = \frac{1}{4\pi} e^{\varepsilon^2/2} \int e^{-x^2/2} \frac{(x_k + \varepsilon_k)}{|\vec{x} + \vec{\varepsilon}|^3} d^3x \tag{B-6}$$

Choosing a coordinate system in which the vector ε_k is in the z-direction, and noting that the components perpendicular to ε_k are zero, we write,

$$I_k = \frac{1}{4\pi} e^{\varepsilon^2/2} \int e^{-x^2/2} \frac{(x \cos\theta + \varepsilon) x^2 \sin\theta \, d\nu \, d\theta \, dx}{(x^2 + 2x\,\varepsilon \cos\theta + \varepsilon^2)^{3/2}} \left(\frac{\varepsilon_k}{\varepsilon}\right) \quad , \tag{B-7}$$

where (x, θ, ν) are spherical coordinate variables. Integrating over ν from 0 to 2π, and letting $\mu = -\cos\theta$, we obtain,

$$\begin{aligned} I_k &= \frac{1}{2} e^{\varepsilon^2/2} \int e^{-x^2/2} x^2 \frac{(-x\mu + \varepsilon)}{(x^2 - 2x\varepsilon\mu + \varepsilon^2)^{3/2}} d\mu \, dx \left(\frac{\varepsilon_k}{\varepsilon}\right) \\ &= \frac{\varepsilon_k}{2\varepsilon} e^{\varepsilon^2/2} \int_0^\infty dx \, e^{x^2/2} x^2 \left[- \frac{\partial}{\partial\varepsilon} \int_{-1}^{1} \frac{1}{(x^2 - 2x\varepsilon\mu + \varepsilon^2)^{1/2}} d\mu \right] \end{aligned} \tag{B-8}$$

Expressing the integrand in terms of associated Legendre polynomials, the μ integration is easily carried out as follows,

$$\int_{-1}^{1} \frac{1}{(x^2-2x\varepsilon\mu+\varepsilon^2)^{1/2}}\, d\mu = \begin{cases} \displaystyle\int_{-1}^{1} \sum_{\ell} \mathcal{P}_{\ell}(\mu) \frac{x^{\ell}}{\varepsilon^{\ell+1}}\, d\mu = \frac{2}{\varepsilon}\ ; & x < \varepsilon \\ \displaystyle\int_{-1}^{1} \sum_{\ell} \mathcal{P}_{\ell}(\mu) \frac{\varepsilon^{\ell}}{x^{\ell+1}}\, d\mu = \frac{2}{x}\ ; & x > \varepsilon\ . \end{cases} \tag{B-9}$$

The integral I_k may now be written,

$$I_k = \frac{\varepsilon_k}{\varepsilon^3} e^{-\varepsilon^2/2} \int_0^{\varepsilon} x^2 e^{-x^2/2}\, dx \quad . \tag{B-10}$$

As expressed by relation (B-5), the required g-space integrations may be obtained by differentiations of I_k with respect to the vector ε_{ℓ} . The operations are straightforward, though lengthy, hence it will suffice to simply tabulate the required expressions. Defining,

(B-11)

(a) $\mathcal{F} \equiv \frac{1}{\varepsilon^3} \int_0^{\varepsilon} e^{-x^2/2} x^2\, dx$

(b) $\mathcal{J} \equiv e^{-\varepsilon^2/2}$

(c) $\mathcal{G}^{(1)} \equiv \frac{1}{\varepsilon^2}\mathcal{J} + (1 - \frac{3}{\varepsilon^2})\mathcal{F}$

(d) $\mathcal{G}^{(2)} \equiv \frac{1}{\varepsilon^2}(1 - \frac{5}{\varepsilon^2})\mathcal{J} + (1 - \frac{6}{\varepsilon^2} + \frac{15}{\varepsilon^4})\mathcal{F}$

(B-11)
(cont'd)

(e) $$\mathcal{G}^{(3)} \equiv \frac{1}{\varepsilon^2}\left(1-\frac{8}{\varepsilon^2}+\frac{35}{\varepsilon^4}\right)\mathcal{J} + \left(1-\frac{9}{\varepsilon^2}+\frac{45}{\varepsilon^4}-\frac{105}{\varepsilon^6}\right)\mathcal{F}$$

(f) $$\mathcal{G}^{(4)} \equiv \frac{1}{\varepsilon^2}\left(1-\frac{11}{\varepsilon^2}+\frac{77}{\varepsilon^4}-\frac{315}{\varepsilon^6}\right)\mathcal{J} + \left(1-\frac{12}{\varepsilon^2}+\frac{90}{\varepsilon^4}-\frac{420}{\varepsilon^6}+\frac{945}{\varepsilon^8}\right)\mathcal{F}$$

(g) $$\mathcal{G}^{(5)} \equiv (3+\varepsilon^2)\mathcal{J} + (-1+2\varepsilon^2+\varepsilon^4)\mathcal{F}$$

(h) $$\mathcal{G}^{(6)} \equiv \left(-\frac{1}{\varepsilon^2}+4+\varepsilon^2\right)\mathcal{J} + \left(\frac{3}{\varepsilon^2}-3+3\varepsilon^2+\varepsilon^4\right)\mathcal{F} ,$$

the g-space integrals required are given by,

(B-12)

(a) $$\langle g_k \rangle = \mathcal{F}\varepsilon_k$$

(b) $$\langle g_i g_j \rangle = \mathcal{F}\delta_{ij} + \mathcal{G}^{(1)}\varepsilon_i\varepsilon_j$$

(c) $$\langle g_i g_j g_k \rangle = \mathcal{G}^{(1)}\{\delta\varepsilon\}_{ijk} + \mathcal{G}^{(2)}\varepsilon_i\varepsilon_j\varepsilon_k$$

(d) $$\langle g^2 g_i \rangle = (5\mathcal{G}^{(1)} + \mathcal{G}^{(2)}\varepsilon^2)\varepsilon_i$$

(e) $$\langle g_i g_j g_k g_\ell \rangle = \mathcal{G}^{(1)}\{\delta\delta\}_{ijk\ell} + \mathcal{G}^{(2)}\{\delta\varepsilon\varepsilon\}_{ijk\ell} + \mathcal{G}^{(3)}\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_\ell$$

(f) $$\langle g^2 g_k g_\ell \rangle = (5\mathcal{G}^{(1)} + \mathcal{G}^{(2)}\varepsilon^2)\delta_{k\ell} + (7\mathcal{G}^{(2)} + \mathcal{G}^{(3)}\varepsilon^2)\varepsilon_k\varepsilon_\ell$$

(B-12)
(cont'd)

(g) $$\langle g_i g_j g_k g_\ell g_m \rangle = \mathcal{U}^{(2)} \{\delta\delta\varepsilon\}_{ijk\ell m} + \mathcal{U}^{(3)} \{\delta\varepsilon\varepsilon\varepsilon\}_{ijk\ell m}$$

$$+ \mathcal{U}^{(4)} \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_\ell \varepsilon_m$$

(h) $$\langle g^2 g_i g_j g_k \rangle = (7\mathcal{U}^{(2)} + \mathcal{U}^{(3)} \varepsilon^2) \{\delta\varepsilon\}_{ijk} + (9\mathcal{U}^{(3)} + \mathcal{U}^{(4)} \varepsilon^2) \varepsilon_i \varepsilon_j \varepsilon_k$$

(i) $$\langle g^4 g_i \rangle = (35\mathcal{U}^{(2)} + 14\varepsilon^2 \mathcal{U}^{(3)} + \varepsilon^4 \mathcal{U}^{(4)}) \varepsilon_i$$

(j) $$\langle g^4 g_i g_k \rangle = \mathcal{U}^{(5)} \delta_{ik} + \mathcal{U}^{(6)} \varepsilon_i \varepsilon_k \quad .$$

APPENDIX C

COMMENTS WITH REGARD TO DEFINITIONS OF TEMPERATURE, STRESSES, AND HEAT FLUXES IN PLASMA SYSTEMS.

Definitions of constituent temperature, stress, and heat flux relative to the plasma flow velocity, and definitions relative to the constituent flow velocity, are both in common usage. Hence, some comments with regard to the relationships between these definitions seem appropriate.

We let,

$$\text{(a)} \quad v_k = w_k^o + c_k = w_k^o + \bar{c}_k^\alpha + u_k = w_k^\alpha + u_k \quad \text{(particle velocity in laboratory coordinate system)}$$

$$\text{(b)} \quad w_k^o = \sum_{\alpha = e, i} \frac{\rho^\alpha}{\rho} w_k^\alpha \quad \text{(plasma flow velocity)}$$

$$\text{(c)} \quad w_k^\alpha = \frac{1}{n^\alpha} \int f^\alpha v_k \, d^3 v \quad \text{(constituent flow velocity)} \tag{C-1}$$

$$\text{(d)} \quad \bar{c}_k^\alpha = \frac{1}{n^\alpha} \int f^\alpha c_k \, d^3 c \quad \text{(constituent flow velocity relative to plasma flow velocity).}$$

In the development herein, the constituent temperature, stress, and heat flux are defined relative to the constituent flow velocity. That is,

$$\text{(a)} \quad \frac{3}{2} n^\alpha k T^\alpha = \frac{1}{2} m^\alpha \int f^\alpha u^2 \, d^3 u$$

$$\text{(b)} \quad P_{ij}^\alpha = m^\alpha \int f^\alpha u_i u_j \, d^3 u \tag{C-2}$$

$$\text{(c)} \quad Q_r^\alpha = \frac{1}{2} m^\alpha \int f^\alpha u^2 u_r \, d^3 u$$

These definitions, brought forward from section I, are consistent with those of Spitzer[11].

By contrast, Chapman and Cowling[1] and Herdan and Liley[8] define these properties relative to the plasma flow velocity. That is,

$$\text{(a)}\quad \frac{3}{2}\, n^{\alpha}\, k\, T^{\alpha'} = \frac{1}{2}\, m^{\alpha} \int f^{\alpha}\, c^{2}\, d^{3}c$$

$$\text{(b)}\quad P^{\alpha'}_{ij} = m^{\alpha} \int f^{\alpha}\, c_{i} c_{j}\, d^{3}c \qquad \text{(C-3)}$$

$$\text{(c)}\quad Q^{\prime\alpha}_{r} = \frac{1}{2}\, m^{\alpha} \int f^{\alpha}\, c^{2}\, c_{r}\, d^{3}c$$

It may easily be shown that definitions (C-5) and definitions (C-6) are related by

$$\text{(a)}\quad \frac{3}{2}\, n^{\alpha}\, kT^{\alpha'} = \frac{3}{2}\, n^{\alpha}\, kT^{\alpha} + \frac{1}{2}\, n^{\alpha}\, m^{\alpha}\, \bar{c}^{\alpha 2}$$

$$\text{(b)}\quad P^{\alpha'}_{ij} = P^{\alpha}_{ij} + n^{\alpha}\, m^{\alpha}\, \bar{c}^{\alpha}_{i}\, \bar{c}^{\alpha}_{j} \qquad \text{(C-4)}$$

$$\text{(c)}\quad Q^{\alpha'}_{r} = Q^{\alpha}_{r} + \frac{1}{2}\, n^{\alpha}\, m^{\alpha}\, \bar{c}^{\alpha 2}\, \bar{c}^{\alpha}_{r} + \left(\frac{3}{2}\, n^{\alpha}\, k\, T^{\alpha}\right) \bar{c}^{\alpha}_{r} + P^{\alpha}_{rs}\, \bar{c}^{\alpha}_{s}$$

For systems in which $\varepsilon^{2} = \beta^{e}\,(w^{e}_{k} - w^{i}_{k})(w^{e}_{k} - w^{i}_{k}) \ll 1$, these relations reduce to

$$\text{(a)}\quad T^{\alpha'} = T^{\alpha}$$

$$\text{(b)}\quad P^{\alpha'}_{ij} = P^{\alpha}_{ij} \qquad \text{(C-5)}$$

$$\text{(c)}\quad Q^{\alpha'}_{r} = Q^{\alpha}_{r} + \frac{5}{2}\, p^{\alpha}\, \bar{c}^{\alpha}_{r} \quad .$$

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