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KINETIC EQUATIONS FOR PLASMAS

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ABSTRACT

This paper presents a unified development of kinetic equations describing particle and photon transport in plasmas subjected to non-constant and non-uniform external fields. It is found that the equations for the particle distributions are a generalization to the plasma of the equations postulated by Uehling and Uhlenbeck+ for neutral quantum gases. As the equations describing photon transport have been developed and discussed previously, only a brief discussion is incorporated here for completeness. It is then shown that the present description of the plasma is sufficiently complete and consistently developed that an H-theorem is demonstrable.

+ E. A. Uehling and G. E. Uhlenbeck, Phys. Rev. 43, 552(1933).

INTRODUCTION

The purpose of this report is to present - from one particular point of view - a summary of some preliminary investigations of particle and photon transport in fully ionized gases, [1, 2, 3]. The emphasis on a particular point of view is not intended to suggest that it is necessarily the best vantage point from which to inspect the subject, but rather that it is a seemingly simplifying and clarifying - and yet so far somewhat unexploited - vantage point. Furthermore, mention of this emphasis serves to warn that no attempt shall be made herein to review the many interesting and different approaches to this problem that have been developed in the past few years, [4].

Because recourse to experiment to test semi-intuitive models of the plasma is not often feasible, it seems necessary at the present time to investigate the validity (or range of approximate validity) of such models from strictly theoretical considerations. The accomplishment of such an objective requires firstly a comprehensive axiomatic statement of the problem (the axioms being reasonably widely agreed upon, of course), followed secondly by a deduction of descriptions of the plasma to which the various models purportedly correspond. Needless to say, no such ambitious program has yet been achieved.

The present discussion is restricted to the delineation of an approach to the problem of determining the validity of Boltzmann type equations for the description of particle and photon balance in the fully ionized plasma. It is admitted at the outset that this approach essentially fails with respect to both of the main points indicated above. In the first place the selection of axioms is hardly universally agreed upon and in the second place the deduction of consequences from the chosen axioms is far less rigorous than is desirable. Nevertheless the results seem suggestive and represent somewhat of a generalization of those usually discussed in the context of the present problem. Furthermore, though the deductions herein proceed via many approximations (none of which have been investigated in detail), the steps required for their testing are usually discernible.

The discussion will be divided into several sections. Section I will incorporate a statement of the axioms and some discussion thereof. Section II will be devoted to an approximate deduction of a balance relation for the particles in the plasma and some consideration of Maxwell's equations. Section III will present a similar development of a transport equation for photons. In Section IV some of the implications of these balance relations for the thermodynamic state of the plasma will be examined. In particular, an H-theorem for the particle photon system will be sketched.

I
THE AXIOMS

The axioms required for the description of systems of the type presently under consideration are usually considered to be of two kinds. The first of these is for the purpose of specifying the dynamics of the interactions between the particles that comprise the plasma, whereas the second is for the purpose of introducing statistical concepts into a description of a system characterized by a huge number of degrees of freedom. The dynamical axiom is conveniently expressed in terms of a Hamiltonian for the system; from which, according to the canonical equations whether classically or quantum mechanically interpreted, all information may be deduced. Since we are here concerned with electrodynamics, we may expect that the dynamical axiom will be reasonably firm and non controversial; at least within certain self evident limitations such as, for example, non-relativistic treatment of the particles.

The statistical axiom is usually introduced via the concept of ensembles of systems in terms of which the probability of finding the given system in a given state at a given instant can be meaningfully formulated. Though usually considered necessary (whether the system be dealt with in classical or quantum terms), we shall avoid the explicit introduction of such concepts into the present discussion. It is for this reason that our axiomatization of the system may be considered controversial to say the least. Instead we shall treat the system quantum mechanically and apparently rely solely upon the statistical concepts inherent in such a treatment. The equivocation is a recognition of the possibility that justification of some of the approximations to be invoked subsequently may require the ensemble concept - but such a necessity is not evident at the moment. We will note that all of the results of the conventional statistical treatments of systems similar to the one considered here are forthcoming from the present analysis.

The dynamical axiom will be stated in the form of an energy density for fields of interacting charged particles and photons, and the Schroedinger equation for the wave function which characterizes the states of such a system. The field theoretic formalism is dictated by the desire to deal with photon transport on the same footing as one deals with particle transport, and so far there has been no indication that this is feasible in the classical, or semi-classical context [5]. It has the slight, further, formal advantage that the singlet densities whose equations we seek can be defined in terms of expectation values. In non-degenerate plasmas it is not expected that quantum effects will play a significant role in the description of particle transport. In view of these remarks we have [6]

$$H\Psi = i\hbar \frac{\partial \Psi}{\partial t}, \qquad (1)$$

where

$$H = \int_{\underline{x}} \mathcal{H}(\underline{x})\, d^3x, \qquad (2)$$

and

$$\mathcal{H}(\underline{x}) = -\sum_{\sigma} \frac{1}{2m_{\sigma}} \left[(i\hbar\underline{\nabla} - \frac{e_{\sigma}}{c}\underline{A} - \frac{e_{\sigma}}{c}\underline{A}^{e})_j \psi_{\sigma}^{+} \right]$$

$$\times \left[(i\hbar\underline{\nabla} + \frac{e_{\sigma}}{c}\underline{A} + \frac{e_{\sigma}}{c}\underline{A}^{e})_j \psi_{\sigma} \right] + \left[2\pi c^2 \underline{P}^2 \right.$$

$$\left. + \frac{1}{8\pi} (\underline{\nabla} \times \underline{A})^2 \right] + \sum_{\sigma} e_{\sigma} \phi \psi_{\sigma}^{+} \psi_{\sigma} \tag{3}$$

$$+ \sum_{\sigma\sigma'} \frac{e_{\sigma} e_{\sigma'}}{1 + \delta_{\sigma\sigma'}} \int d^3x' \frac{\psi_{\sigma}^{+}(\underline{x}) \psi_{\sigma'}^{+}(\underline{x}') \psi_{\sigma}(\underline{x}) \psi_{\sigma'}(\underline{x}')}{|\underline{x} - \underline{x}'|} .$$

In equation (3), ψ_{σ} is a wave operator for a field of particles of the σth kind, $\underline{A}^e$ and ϕ are the vector and scalar potentials of the "external fields", and $\underline{A}$ and $\underline{P}$ are the "transverse" magnetic and electric operators for the photon field. The external fields are presumed known and hence are unquantized, whereas the "internal" fields are described by operators which satisfy the commutation relations,

$$\left[\psi_{\sigma}(\underline{x}), \psi_{\sigma'}^{+}(\underline{x}') \right]_{\pm} = \psi_{\sigma}(\underline{x}) \psi_{\sigma'}^{+}(\underline{x}')$$

$$\pm \psi_{\sigma'}^{+}(\underline{x}') \psi_{\sigma}(\underline{x}) = \delta_{\sigma\sigma'} \delta(\underline{x}-\underline{x}') ,$$

and

$$\left[A_j(\underline{x})\,,\; P_\ell(\underline{x}')\right]_- = A_j(\underline{x})\, P_\ell(\underline{x}') - P_\ell(\underline{x}')\, A_j(\underline{x}) = i\hbar\, \delta_{j\ell}\, \delta(\underline{x}-\underline{x}') - i\hbar\, \nabla_j \nabla_\ell' \left(\frac{1}{4\pi\, |\underline{x}-\underline{x}'|}\right), \tag{4}$$

all other quantities commuting. The particles satisfy anti-commutation or commutation rules depending upon whether they are fermions or bosons. This distinction is of no importance for the non-degenerate plasma, but will be maintained throughout the early stages of the analysis, for the sake of generality. The assertion of the transversality of the field operators is rendered formally precise by the statements

$$(\underline{\nabla} \cdot \underline{A}) = (\underline{\nabla} \cdot \underline{P}) = 0. \tag{5}$$

The energy density, equation (3), may be rewritten variously by regrouping terms or by transforming coordinates. Expression of the energy density in various ways is desirable since some calculations proceed most naturally from one form of equation (3) whereas others require other ways of writing it. For this reason we expend a little effort here upon the rewriting of (3) in two different ways - one most suitable for the discussion of particle transport while the other simplifies the treatment of photon transport. Both ways of exhibiting (3) are, of course, equivalent. For the purpose of dealing with particle transport it is convenient first of all to decompose the vector potential, $\underline{A}$, into two parts, i.e.,

$$A = \underline{A}^s + \underline{A}^f , \tag{6}$$

where $\underline{A}^s$ refers to the "slowly varying" part of $\underline{A}$, while $\underline{A}^f$ denotes the "rapidly varying" part of $\underline{A}$. The distinction between "slow" and "rapid" variation will be determined only in the context of a given situation and hence will not be discussed further at this point. We then introduce the notation,

$$\underline{R} = \underline{A}^e + \underline{A}^s , \tag{7}$$

i.e., $\underline{R}$ represents the superposition of the known, externally applied field and the "slowly varying" part of the internal field. Given this decomposition of the internally induced electromagnetic field, it is convenient to rewrite equation (3) as,

$$\mathcal{H}(\underline{x}) = -\sum_\sigma \frac{1}{2m_\sigma}\left[\left(i\hbar\underline{\nabla} - \frac{e_\sigma}{c}\underline{R}\right)_j \psi_\sigma^+\right]\left[\left(i\hbar\underline{\nabla} + \frac{e_\sigma}{c}\underline{R}\right)_j \psi\right]$$

$$-\sum_\sigma \frac{e_\sigma}{2m_\sigma c}\left\{\left[\left(i\hbar\underline{\nabla} - \frac{e_\sigma}{c}\underline{R}\right)_j \psi_\sigma^+\right] A_j^f \psi_\sigma\right.$$

$$\left. - A_j^f \psi_\sigma^+ \left[\left(i\hbar\underline{\nabla} + \frac{e_\sigma}{c}\underline{R}\right)_j \psi_\sigma\right]\right\} + \sum_\sigma \frac{e_\sigma^2}{2m_\sigma c^2}\psi_\sigma^+\psi_\sigma\left(\underline{A}^f\right)^2$$

$$+\left[2\pi c^2 \underline{P}^2 + \frac{1}{8\pi}\left(\underline{\nabla}\times\underline{A}\right)^2\right] + \sum_\sigma e_\sigma \phi \psi_\sigma^+ \psi_\sigma$$

(continued)

$$+\sum_{\sigma\sigma'} \frac{e_\sigma e_{\sigma'}}{1+\delta_{\sigma\sigma'}} \int d^3x' \frac{\psi_\sigma^+(\underline{x})\,\psi_{\sigma'}^+(\underline{x}')\,\psi_\sigma(\underline{x})\,\psi_{\sigma'}(\underline{x}')}{|\underline{x}-\underline{x}'|} \tag{8}$$

For convenient treatment of photon transport, it is useful to employ a different regrouping of the terms in equation (3). Introducing the operator,

$$\underline{\Pi}^\sigma = -i\hbar\underline{\nabla} - \frac{e_\sigma}{c} \underline{A}^e , \tag{9}$$

we express the energy density as,

$$\mathcal{H}(\underline{x}) = \left[2\pi c^2 \underline{P}^2 + \frac{1}{8\pi}(\underline{\nabla}\times\underline{A})^2\right]$$

$$+\sum_\sigma \frac{1}{2m_\sigma} (\underline{\Pi}^{\sigma *}\psi_\sigma^+)\cdot(\underline{\Pi}^\sigma\,\psi_\sigma)$$

$$-\sum_\sigma \frac{e_\sigma}{2m_\sigma c} (\underline{\Pi}^{\sigma *}\psi_\sigma^+)\cdot\underline{A}\,\psi_\sigma + \underline{A}\,\psi_\sigma^+\cdot(\underline{\Pi}^\sigma\psi_\sigma)$$

$$+\sum_{\sigma\sigma'} \frac{e_\sigma e_{\sigma'}}{1+\delta_{\sigma\sigma'}} \int d^3x' \frac{\psi_\sigma(\underline{x})\,\psi_{\sigma'}^+(\underline{x}')\,\psi_\sigma(\underline{x})\,\psi_{\sigma'}(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$+\sum_\sigma \frac{e_\sigma^2}{2m_\sigma c^2}\psi_\sigma^+\psi_\sigma A^2 + \sum_\sigma e_\sigma \phi\,\psi_\sigma^+\psi_\sigma \;. \tag{10}$$

For some calculational purposes it is convenient to transform to momentum space. Accordingly we introduce the fourier analyses,

$$\underline{A} = \sqrt{\frac{2\pi\hbar c}{V}} \sum_{\underline{k}\lambda} \frac{e^{-i\underline{k}\cdot\underline{x}}}{\sqrt{k}} \underline{\zeta}_{\lambda}^{+}(\underline{k}),$$

$$\underline{P} = i\sqrt{\frac{\hbar}{8\pi c V}} \sum_{\underline{k}\lambda} \sqrt{k}\, e^{-i\underline{k}\cdot\underline{x}} \zeta_{\lambda}^{-}(\underline{k}) \tag{11}$$

$$\psi_{\sigma} = \frac{1}{\sqrt{V}} \sum_{\underline{K}} a_{\sigma}(\underline{K})\, e^{i\underline{K}\cdot\underline{x}} .$$

In the relations (11), V represents the volume of quantization. It is the volume with respect to which the quantities $\underline{A}$, $\underline{P}$, and ψ_{σ} obey periodic boundary conditions. The operators, $\zeta_{\lambda}^{\pm}(\underline{k})$, are given by

$$\zeta_{\lambda}^{\pm}(\underline{k}) = \alpha_{\lambda}^{+}(\underline{k})\, \underline{\epsilon}_{\lambda}(\underline{k}) \pm \alpha_{\lambda}(-\underline{k})\, \underline{\epsilon}_{\lambda}(-\underline{k}), \tag{12}$$

where the $\underline{\epsilon}_{\lambda}(\underline{k})$ are the unit polarization vectors of the photon field. The indices λ take on two values corresponding to the two states of polarization for the photons. The sums over $\underline{k}$ and $\underline{K}$ are the usual sums over the integers permitted by the requirement that $\underline{A}$, $\underline{P}$, and ψ_{σ} be periodic on the boundaries of the volume V. The quantities $\alpha_\lambda^+(\underline{k})$ and $\alpha_\lambda(\underline{k})$ are the creation and destruction operators for photons of momentum $\hbar\,\underline{k}$ and polarization λ while $a_\sigma^+(\underline{K})$ and $a_\sigma(\underline{K})$ are the creation and destruction operators for particles of kind σ and momentum $\hbar\,\underline{K}$. The commutation rules governing the creation and destruction operators are,

$$\left[\alpha_\lambda(\underline{k}),\ \alpha_{\lambda'}^+(\underline{k}')\right]_- = \delta_{\lambda\lambda'}\,\delta(\underline{k} - \underline{k}'),$$

and

$$\left[a_\sigma(\underline{K}),\ a_{\sigma'}^+(\underline{K}')\right]_+ = \delta_{\sigma\sigma'}\,\delta(\underline{K} - \underline{K}'). \tag{13}$$

Note that in equation (13) the functions $\delta(\underline{k} - \underline{k}')$ and $\delta(\underline{K} - \underline{K}')$ represent Kronecker delta's since the arguments take on discrete values, whereas the function $\delta(\underline{x} - \underline{x}')$ appearing in equation (4) is a Dirac delta function. We shall continue to use the same notation for the two kinds of delta functions, letting the context reveal which interpretation of the symbol is appropriate in a given case.

II

BALANCE RELATIONS FOR THE PARTICLES

Our prime concern in this section shall be for the deduction of a Boltzmann-type equation for a singlet particle density. Thus our initial task must be the identification of a quantity which, in some sense, may be interpreted as the expected number of particles to be found in a small element of volume in phase space. Since we have formulated the problem in quantum mechanical terms, it is evident that some difficulty will be encountered here, as it is impossible to localize particles with arbitrary precision in phase space. A way out (and the one chosen here as it has been many times elsewhere) is simply to give up the notion of "fine grained meaningfulness" of the singlet density. Alternatively (and equivalently) we may solely require of the singlet density that it be a proper weight function for the calculation of observable averages.

With these remarks in mind, we define a singlet density for particles of kind σ by

$$f_\sigma(\underline{x}, \underline{K}, t) = \frac{8}{V}\int d^3z e^{-2i\underline{K}\cdot\underline{z}} \left(\Psi, \psi_\sigma^+(\underline{x}-\underline{z})\, \psi_\sigma(\underline{x}+\underline{z})\Psi\right)$$

$$= \frac{8}{V}\sum_{Q} e^{-2i\underline{x}\cdot\underline{Q}}\left(\Psi, a_\sigma^+(\underline{K}+\underline{Q})a_\sigma(\underline{K}-\underline{Q})\Psi\right). \tag{14}$$

The density defined by (14) is not everywhere positive but assumes negative values as well because of the impossibility of simultaneously specifying the ($\underline{x}$, $\underline{K}$) coordinates of a particle within arbitrarily small ranges. Nevertheless, it is convenient (when meaningful) to interpret f_σ as the expected number of particles of kind σ to be found at the point ($\underline{x}$, $\underline{K}$) per unit volume in configuration space+. Force is given to this interpretation by the observations that

$$f_\sigma(\underline{x}, t) = \sum_{\underline{K}} f_\sigma(\underline{x}, \underline{K}, t) = (\Psi, \psi_\sigma^+(\underline{x}) \; \psi_\sigma(\underline{x}) \Psi), \tag{15}$$

and

$$f_\sigma(\underline{K}, t) = \int d^3x \, f_\sigma(\underline{x}, \underline{K}, t) = (\Psi, a_\sigma^+(\underline{K}) \, a_\sigma(\underline{K}) \Psi), \tag{16}$$

i.e., that $f_\sigma(\underline{x}, t)$ and $f_\sigma(\underline{K}, t)$ are indeed the expected particle densities or particle numbers in either configuration space or momentum space separately. Further reinforcement follows from the observation to be made subsequently that $f_\sigma(\underline{x}, \underline{K}, t)$ is persumably truly interpretable as a particle density in the classical limit. It should be noted in passing that the density defined by equation (14) is simply the field theoretic equivalent of the Wigner distribution function [7, 8] , and has been employed for purposes similar to those that concern us here many times previously.

+ Note that f_σ is not a density in $\underline{K}$-space (momentum space). This is a direct consequence of having defined the Fourier transforms (11) one of which connects the two expressions for f_σ given in equation (14) in a finite volume in configuration space. However, in spite of the fact that the variable, $\underline{K}$ (and $\underline{k}$ for photons), is discretely distributed, we shall assume that the operation of differentiation of functions of $\underline{K}$ with respect to $\underline{K}$ is meaningful.

Some more notation will prove useful. Define the operators

$$\rho_\sigma(\underline{x}, \underline{z}) = \psi^+_\sigma(\underline{x} - \underline{z})\, \psi_\sigma(\underline{x} + \underline{z}) \quad ,$$

and

$$\rho_\sigma(\underline{K}, \underline{Q}) = a^+_\sigma(\underline{K} + \underline{Q})\, a_\sigma(\underline{K} - \underline{Q}) \quad . \tag{17}$$

Then equation (14) may also be written as

$$f_\sigma(\underline{x}, \underline{K}, t) = \frac{8}{V} \int d^3 z\, e^{-2i\underline{K} \cdot \underline{z}} \ \mathrm{Tr}\ \rho_\sigma(\underline{x}, \underline{z})\, D$$

$$= \frac{8}{V} \sum_{\underline{Q}} e^{-2i\underline{x} \cdot \underline{Q}} \ \mathrm{Tr}\ \rho_\sigma(\underline{K}, \underline{Q})\, D \quad , \tag{18}$$

where Tr AB means take the trace of the product of the matrices A and B, and D is the density matrix for the system in the Schroedinger representation. Because of the invariance of traces to unitary transformations, the forms (18) for $f_\sigma(\underline{x}, \underline{K}, t)$ will prove most useful for calculational purposes.

Now recall the expression for the energy density given in equation (18) and write the total energy as

$$H = \int_{\underline{x}} d^3 x\, \mathcal{H}(\underline{x}) = H^o + H^{p\gamma} + H^{pp} \quad , \tag{19}$$

where

$$H^{o} = -\sum_{\sigma} \frac{1}{2m_{\sigma}} \int d^{3}x \left[\left(i\hbar \underline{\nabla} - \frac{e_{\sigma}}{c} \underline{R} \right)_{j} \psi^{+}_{\sigma} \right] \left[\left(i\hbar \underline{\nabla} + \frac{e_{\sigma}}{c} \underline{R} \right)_{j} \psi_{\sigma} \right]$$

$$+ \sum_{\sigma} e_{\sigma} \int d^{3}x \, \phi \, \psi^{+}_{\sigma} \psi_{\sigma}$$

$$+ \sum_{\sigma, \sigma'} \frac{e_{\sigma} e_{\sigma'}}{1 + \delta_{\sigma\sigma'}} \int d^{3}x \, d^{3}x' \, \psi^{+}_{\sigma}(\underline{x}) \, \psi^{+}_{\sigma'}(\underline{x}') \, \psi_{\sigma}(\underline{x}) \, \psi_{\sigma'}(\underline{x}') \, u(\underline{x}, \underline{x}') \tag{19}$$

$$+ \int d^{3}x \left[2\pi c^{2} \underline{P}^{2} + \frac{1}{8\pi} (\underline{\nabla} \times \underline{A})^{2} \right] , \tag{20a}$$

$$H^{p\gamma} = \sum_{\sigma} \frac{e_{\sigma}}{2m_{\sigma} c} \int d^{3}x \left\{ A^{F}_{j} \psi^{+}_{\sigma} \left[\left(i\hbar \underline{\nabla} + \frac{e_{\sigma}}{c} \underline{R} \right)_{j} \psi_{\sigma} \right] - \left[\left(i\hbar \underline{\nabla} - \frac{e_{\sigma}}{c} \underline{R} \right)_{j} \psi^{+}_{\sigma} \right] A^{F}_{j} \psi_{\sigma} \right\} , \tag{20b}$$

$$H^{pp} = \sum_{\sigma\sigma'} \frac{e_{\sigma} e_{\sigma'}}{1 + \delta_{\sigma\sigma'}} \int d^{3}x \, d^{3}x' \, \psi^{+}_{\sigma}(\underline{x}) \, \psi^{+}_{\sigma'}(\underline{x}') \, \psi_{\sigma}(\underline{x}) \, \psi_{\sigma'}(\underline{x}') \times S(\underline{x}) \, S(\underline{x}') \, \frac{1}{|\underline{x} - \underline{x}'|} , \tag{20c}$$

where

$$u(\underline{x}, \underline{x}') = \left[S(\underline{x})\, s(\underline{x}') + S(\underline{x}')\, s(\underline{x}) + s(\underline{x})\, s(\underline{x}') \right] \frac{1}{|\underline{x} - \underline{x}'|} \quad . \tag{21}$$

The step functions S and s require definition, and the breakup of the coulomb energy according to the parts appearing in equations (20a) and (20c) requires some explanation. We are envisaging the selection within the system of a sub-volume, V, whose size, shape, and location will remain largely unspecified for the moment. We may hope that under some practical circumstances V can be chosen small compared to regions over which macroscopic spatial variations are significant. We have then defined step functions according to

$$S(\underline{x}) = 1, \quad \underline{x} \in V,$$
$$= 0, \quad \text{otherwise}$$

and

$$s(\underline{x}) = 0, \quad \underline{x} \in V,$$
$$= 1, \quad \text{otherwise} \quad , \tag{22}$$

and decomposed the coulomb energy of the system into two parts. The part incorporated into H^o, equation (20a), corresponds to the interaction energy between all particles not both in V, whereas the other part, H^{pp}, is the remainder, i.e., the energy of interaction between particles both in V. The utility and validity of this breakup of the coulomb energy will be manifested and investigated as we go along.

We now transform to an interaction representation by

$$\Psi = U\Phi, \quad U^{+} = U^{-1}, \tag{23}$$

where U satisfies the equation

$$i\hbar \frac{\partial U}{\partial t} = H^{o}\, U(t) \quad , \tag{24}$$

so that Φ satisfies the equation,

$$(\bar{H}^{p} + H^{pl})\Phi = i\hbar \frac{\partial \Phi}{\partial t} \quad . \tag{25}$$

Because U is a unitary transformation, we may rewrite the expressions for the particle singlet density, equation (18), as

$$f_{\sigma} = \frac{8}{V}\int d^{3}z e^{-2i\underline{K}\cdot\underline{z}}\, \mathrm{Tr}\, \bar{\rho}_{\sigma}(\underline{x}, \underline{z}, t)\bar{D} = \frac{8}{V}\sum_{\underline{Q}} e^{-2i\underline{x}\cdot\underline{Q}}\, \mathrm{Tr}\, \bar{\rho}_{\sigma}(\underline{K}, \underline{Q}, t)\bar{D}, \tag{26}$$

where

$$\bar{D} = U^{+} D U \text{ and } \bar{\rho} = U^{+} \rho U \quad . \tag{27}$$

It then follows that $\bar{\rho}$ and $\bar{D}$ satisfy the equations

$$\frac{\partial \bar{\rho}_{\sigma}}{\partial t} = \frac{i}{\hbar}\left[\bar{H}^{o}, \bar{\rho}_{\sigma}\right] , \tag{28a}$$

and

$$\frac{\partial \bar{D}}{\partial t} = \frac{i}{\hbar}\left[\bar{D}, \bar{H}^{p\gamma} + \bar{H}^{pp}\right] \quad . \tag{28b}$$

Thus by means of the transformation (23) we have invested ρ_{σ} with an explicit time dependence resulting from the interaction of particles of the σ'th kind with the externally applied electromagnetic fields, ($\underline{A}^{e}$ and ϕ), with the "slowly varying" part of the transverse internal field, $(\underline{A}^{s})$, and with the coulomb fields of other particles outside of a volume V surrounding the point $\underline{x}$ at which f_σ is to be evaluated. Conversely, the same transformation implies that the time dependence of the density matrix $\bar{D}$ results from the interactions of particles with the "rapidly varying" part of the internal transverse field and between particles in the sub-volume V. We may anticipate that it is essentially the time dependence of $\bar{\rho}_\sigma$ that describes the influence of external and self-consistent internal fields upon the temporal variation of f_σ, whereas the time dependence of $\bar{D}$ will contribute the effects of coulomb "collisions" and particle-photon collisions to the temporal variation of f_σ.

To realize the content of these remarks, consider

$$\frac{\partial f_\sigma}{\partial t} = \frac{8}{V}\int d^3z\, e^{-2i\underline{K}\cdot\underline{z}}\, \mathrm{Tr}\, \frac{\partial\bar{\rho}_\sigma}{\partial t}\, \bar{D} + \frac{8}{V}\sum_{\underline{Q}} e^{-2i\underline{x}\cdot\underline{Q}}\, \mathrm{Tr}\, \bar{\rho}_\sigma \frac{\partial\bar{D}}{\partial t}\ , \tag{29}$$

where the different contributions to the time derivative of f_σ have been expressed in different coordinate systems for calculational convenience. Noting that

$$\mathrm{Tr}\, \frac{\partial\bar{\rho}_\sigma}{\partial t}\, \bar{D} = \mathrm{Tr}\, \frac{i}{\hbar}\left[\bar{H}^o, \bar{\rho}_\sigma\right]\bar{D} = \frac{i}{\hbar}\, \mathrm{Tr}\left[H^o, \rho_\sigma\right] D\ , \tag{30}$$

one finds after tedious but wholly straightforward manipulations, including the transition to the continuum in $\underline{K}$-space, that equation (29) may be written as

$$
\frac{\partial f_\sigma}{\partial t} + \frac{\hbar K_j}{m_\sigma}\frac{\partial f_\sigma}{\partial x_j} - \frac{e_\sigma}{m_\sigma c}\,\mathrm{Tr}\,R_j \cos\left(\frac{\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_K}{2}\right)\frac{\partial}{\partial x_j}\Gamma_{Kz}\,\rho_\sigma D
$$

$$
- \frac{2}{\hbar} e_\sigma \phi \sin\left(\frac{\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_K}{2}\right) f_\sigma - \sum_{\sigma'} \frac{2 e_\sigma e_{\sigma'}}{\pi^3 \hbar}\int d^3x'\, d^3z
$$

$$
\times\, e^{-2i\underline{K}\cdot\underline{z}}\left(\Psi\, \psi^+_\sigma(\underline{x}-\underline{z})\, \psi^+_{\sigma'}(\underline{x}')\, \psi_\sigma(\underline{x}+\underline{z})\, \psi_{\sigma'}(\underline{x}')\,\Psi\right) \sin\left(\frac{\overleftarrow{\nabla}_K \cdot \overrightarrow{\nabla}_x}{2}\right) u(\underline{x},\underline{x}')
$$

$$
+ \frac{2e_\sigma}{m_\sigma c} K_j\,\mathrm{Tr}\,R_j \sin\left(\frac{\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_K}{2}\right)\Gamma_{Kz}\,\rho_\sigma D - \frac{e_\sigma^2}{\hbar m_\sigma c^2}\,\mathrm{Tr}\,R_j^2 \sin\left(\frac{\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_K}{2}\right)\Gamma_{Kz}\,\rho_\sigma D
$$

$$
= \frac{V}{(2\pi)^3}\,\frac{8}{V}\sum_{\underline{Q}} e^{-2i\underline{x}\cdot\underline{Q}}\,\mathrm{Tr}\,\rho_\sigma(\underline{K},\underline{Q},t)\,\frac{\partial \overline{D}}{\partial t}\,, \tag{31}
$$

where, for further compression of notation, we have introduced the symbol Γ_{Kz} for the Fourier operator, i.e.,

$$
\Gamma_{Kz} = \frac{8}{V}\int d^3z\, e^{-2i\underline{K}\cdot\underline{z}} \,. \tag{32}
$$

Transcendental operators are defined by their power series representations.

Equation (31) is exact, but almost contentless as it stands, as it is no more than a relation between the time derivative of f_σ and diverse functionals not obviously related to f_σ. To reduce (31) to recognizable and usable form, numerous approximations must be invoked. Our attitude at this point will be to introduce the approximations with operational precision, but to expend little effort in attempts to evaluate or justify them We first note that $\hbar \underline{K}$ is particle momentum (and $\hbar K_j/m_\sigma = v_j$, the j'th component of the velocity of the σ'th particle) and hence that the trigonometric functions of the operator $\underline{\nabla}_K$ have a natural representation as power series in $\hbar$. Since the left-hand-side of equation (31) describes the influence on f_σ of explicit time variation, transport, and the interactions of the particles represented by f_σ with external and "slowly varying" internal electromagnetic fields, it is not expected that specifically quantum effects will be significant. Thus we retain only the explicit, lowest order dependence upon $\hbar$ in these terms. We are then led immediately to

$$
\frac{\partial f_\sigma}{\partial t} + \frac{p_j}{m_\sigma}\frac{\partial f_\sigma}{\partial x_j} - \frac{e_\sigma}{m_\sigma c}\,\mathrm{Tr}\,R_j \frac{\partial}{\partial x_j}\,\Gamma_{Kz}\,\rho_\sigma D - e_\sigma \frac{\partial \phi}{\partial x_j}\frac{\partial f_\sigma}{\partial p_j}
$$

$$
- \sum_{\sigma'} \frac{e_\sigma e_{\sigma'}}{\hbar \pi^3}\int d^3x'\, d^3z\, e^{-2i\underline{K}\cdot\underline{z}} \left(\Psi, \psi^+_\sigma(\underline{x}-\underline{z})\, \psi^+_{\sigma'}(\underline{x}')\, \psi_\sigma(\underline{x}+\underline{z})\, \psi_{\sigma'}(\underline{x}')\, \Psi\right)
$$

$$
\times \left(\overleftarrow{\underline{\nabla}}_K \cdot \overrightarrow{\underline{\nabla}}_x\right) U(\underline{x}, \underline{x}') + \frac{e_\sigma}{m_\sigma c} K_j\, \mathrm{Tr}\, R_j \left(\overleftarrow{\underline{\nabla}}_x \cdot \overrightarrow{\underline{\nabla}}_K\right) \Gamma_{Kz}\, \rho_\sigma D
$$

$$
- \frac{e_\sigma^2}{2\hbar m_\sigma c^2}\, \mathrm{Tr}\, R_j^2 \left(\overleftarrow{\underline{\nabla}}_x\, \overrightarrow{\underline{\nabla}}_K\right) \Gamma_{Kz}\, \rho_\sigma D = \frac{V}{(2\pi\hbar)^3}\, \frac{8}{V} \sum_{\underline{Q}} e^{-2i\underline{x}\cdot\underline{Q}}\, \mathrm{Tr}\, \bar{\rho}_\sigma(\underline{K}, \underline{Q}, t)\, \frac{\partial \bar{D}}{\partial t} \,. \tag{33}
$$

To simplify further, the left-hand-side of (33) requires a more serious and subtle sort of approximation. First consider the terms of the form

$$\mathrm{Tr}\ \zeta(\underline{R})\ \Gamma_{Kz}\ \rho_\sigma\ D. \tag{34}$$

where $\zeta(\underline{R})$ stands for appropriate operator functionals of the portion of the electromagnetic field represented by $\underline{R}$. Note that if $\underline{R}$ stood solely for the external (given) field, (34) could be written as

$$\zeta(\underline{R})\mathrm{Tr}\ \Gamma_{Kz}\rho_\sigma\ D = \zeta(\underline{R})\ f_\sigma\ , \tag{35}$$

i.e., all such terms as (34) could be written as explicit functionals of f_σ - the singlet density for which we are attempting to deduce a balance relation. In actuality, of course, $\underline{R}$ is dependent in part upon a portion of the internal vector potential $(\underline{A}^s)$, and consequently the replacement of (34) by (35) involves the approximation of replacing averages of products by the product of averages, i.e., instead of (35) we have

$$\mathrm{Tr}\,\zeta(\underline{R})\,\Gamma_{Kz}\ \rho_\sigma D \simeq \left[\mathrm{Tr}\ \zeta(\underline{R})D\right]\ \left[\mathrm{Tr}\ \Gamma_{Kz}\ \rho_\sigma\ D\right] = \left[\mathrm{Tr}\ \zeta(\underline{R})D\right]\ f_\sigma\ . \tag{36}$$

Introducing the notation $\langle\underline{R}\rangle = \mathrm{Tr}\ \underline{R}\ D$, and making the replacements (36) in (33) as well as the replacement[+]

$$(\Psi, \psi^+_\sigma(\underline{x}-\underline{z})\, \psi^+_{\sigma'}(\underline{x}')\, \psi_\sigma(\underline{x}+\underline{z})\, \psi_{\sigma'}(\underline{x}')\, \Psi)$$

$$\simeq (\Psi, \psi^+_\sigma(\underline{x}-\underline{z})\, \psi_\sigma(\underline{x}+\underline{z})\, \Psi)\, (\Psi, \psi^+_{\sigma'}(\underline{x}')\, \psi_{\sigma'}(\underline{x}')\, \Psi) \quad , \tag{37}$$

which is facilitated by the nature of the potential $u(\underline{x}, \underline{x}')$, i.e., the replacement of doublet densities by products of singlet densities is presumably the more justified the farther apart the space points upon which they depend, we obtain

$$\frac{\partial f_\sigma}{\partial t} + \left(\frac{p_j}{m_\sigma} - \frac{e_\sigma}{m_\sigma c}\left\langle R_j\right\rangle\right)\frac{\partial f_\sigma}{\partial x_j} - e_\sigma \frac{\partial \phi}{\partial x_j}\frac{\partial f_\sigma}{\partial p_j} - e_\sigma \frac{\partial f_\sigma}{\partial p_j}\frac{\partial}{\partial x_j}\sum_{\sigma'} e_{\sigma'} \int_{\underline{x}'} d^3x' u(\underline{x}, \underline{x})f_{\sigma'}(\underline{x}; t)$$

$$+ \frac{e_\sigma}{m_\sigma c}\, p_j \frac{\partial\langle \dot{R}_j\rangle}{\partial x_\ell}\frac{\partial f_\sigma}{\partial p_\ell} - \frac{e^2_\sigma}{2m_\sigma c^2}\frac{\partial\langle R_j\rangle^2}{\partial x_\ell}\frac{\partial f_\sigma}{\partial p_\ell}$$

$$= \frac{V}{(2\pi\hbar)^3}\, \frac{8}{V}\sum_{\underline{Q}} e^{-2i\underline{x}\cdot \underline{Q}}\, \mathrm{Tr}\, \overline{\rho}_\sigma(\underline{K}, \underline{Q}, t)\, \frac{\partial}{\partial t} \quad , \tag{38}$$

where we have also approximated

$$\left\langle R_j^2\right\rangle = \left\langle R_j\right\rangle^2 \quad , \tag{39}$$

and employed the notation of equation (15). Now define a total, longitudinal electric

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[+] For consistency with later analysis which explicitly incorporates exchange effects in the representation of collisions, the exchange contributions to the replacements (37) should be retained here. For a discussion of this matter, see Oldwig von Roos, Phys. Rev., 119, 1174 (1960).

field by

$$E_j^L = -\frac{\partial \phi}{\partial x_j} - \frac{\partial}{\partial x_j}\sum_{\sigma'} e_{\sigma'} \int_{\underline{x}'} d^3 x' u(\underline{x}, \underline{x}') f_{\sigma'}(\underline{x}', t)$$

$$= -\frac{\partial \phi}{\partial x_j} - \frac{\partial}{\partial x_j}\sum_{\sigma'} \int_{\underline{x}'} d^3 x' \, s(\underline{x}') \frac{e_{\sigma'} f_{\sigma'}(\underline{x}', t)}{|\underline{x} - \underline{x}'|} , \qquad (40)$$

where we make explicit use of the nature of the function $u(\underline{x}, \underline{x}')$, equation (21), and of the fact that the point $\underline{x}$ is in V. Evidently the second term in (40) is the coulomb field at the point $\underline{x}$ due to all charged particles in the system outside of the volume V about $\underline{x}$. We then may write equation (38) as

$$\frac{\partial f_\sigma}{\partial t} + \left(\frac{p_j}{m_\sigma} - \frac{e_\sigma}{m_\sigma c}\langle R_j\rangle\right)\frac{\partial f_\sigma}{\partial x_j} + \left[e_\sigma E_j^L + \frac{e_\sigma}{m_\sigma c} p_\ell \frac{\partial \langle R_\ell\rangle}{\partial x_j} - \frac{e_\sigma^2}{2 m_\sigma c^2}\frac{\partial \langle R_\ell\rangle^2}{\partial x_j}\right]\frac{\partial f_\sigma}{\partial p_j}$$

$$= \frac{V}{(2\pi \hbar)^3}\frac{8}{V}\sum_{\underline{Q}} e^{-2i\underline{x}\cdot\underline{Q}} \, \mathrm{Tr}\, \bar{\rho}_\sigma(\underline{K}, \underline{Q}, t) \frac{\partial \bar{D}}{\partial t} . \qquad (41)$$

The left-hand-side of equation (41) has now assumed the conventional form of the rate equation for a singlet density in configuration and momentum (instead of velocity) space without collisions and with self-consistent fields. The fact that the fields, $\underline{E}^L$ and $\langle\underline{R}\rangle$, satisfy appropriate Maxwell's equations with sources in the plasma is not necessarily obvious and will be discussed in some detail later. At the moment we turn our attention to the right-hand-side of (41) which should, in some sense, describe how close encounters between particles and the interaction of particles with the "high frequency" part of the electromagnetic field influences the temporal variation of f_σ.

We assume that we may so choose the volume V about $\underline{x}$ that the left-hand-side of (41) does not vary appreciably within it. We then integrate (41) over the volume V obtaining (employing the notation $L^\sigma f_\sigma$ for the left-hand-side),

$$\int_{\underline{x} \in V} d^3x \, L^\sigma f_\sigma \cong V L^\sigma f_\sigma = \frac{V}{(2\pi\hbar)^3} \operatorname{Tr} \bar{\rho}_\sigma(\underline{K}, 0, t) \frac{\partial \bar{D}}{\partial t} . \tag{42}$$

This integrating, or averaging, is often referred to as spatial course-graining, and is a procedure resorted to elsewhere in similar contexts [9] . Apparently the bulk of the task remaining to us is the estimation of the trace in (42) as some functional of f_σ. This we shall accomplish by a cavalier recourse to approximations which, though reasonably well-defined, we shall make no attempt to justify herein.

Recalling equations (17) and (27), we write (42) as

$$L^\sigma f_\sigma = (2\pi\hbar)^{-3} \operatorname{Tr} U^+ \rho_\sigma(\underline{K}) \, U \frac{\partial \bar{D}}{\partial t} = (2\pi\hbar)^{-3} \operatorname{Tr} \rho_\sigma(\underline{K}) \, U \frac{\partial \bar{D}}{\partial t} U^+ . \tag{43}$$

We now approximate

$$U \frac{\partial \bar{D}}{\partial t} U^+ \simeq U(t) \left[\frac{\bar{D}(t+s) - \bar{D}(t)}{s} \right] U^+(t) = \frac{1}{s} \left[U(t) U^+(t+s) D(t+s) U(t+s) U^+(t) - D(t) \right], \tag{44}$$

where s is some small but finite time interval which is long compared to collision time but short compared to intervals over which macroscopic quantities ~~vary~~

appreciably, e.g., the external fields. By virtue of the latter of these restrictions on s, we have

$$U(t+s) \simeq e^{-iH^{o}s/\hbar}\, U(t) \; : \tag{45}$$

so that (44) may be further approximated as

$$U \frac{\partial \bar{D}}{\partial t} U^{+} \simeq \frac{1}{s}\left[e^{iH^{o}s/\hbar} D(t+s) e^{-iH^{o}s/\hbar} - D(t) \right] \; . \tag{46}$$

The "coarsening" of the time derivative explicit in (44) is also a procedure that has been employed before [10] and has essentially the same effect as the usual temporal coarse-graining [11] . Inserting (46) into (43) leads us to

$$L^{\sigma} f_{\sigma} \cong (2\pi\hbar)^{-3} s^{-1} \left[\mathrm{Tr}\, \rho_{\sigma}(\underline{K}) e^{iH^{o}s/\hbar} D(t+s) e^{-iH^{o}s/\hbar} - \mathrm{Tr}\, \rho_{\sigma}(\underline{K}) D(t) \right] . \tag{47}$$

Recall that H^{o} has been so defined as to include the kinetic energies of the particles and photons inside the box of quantization (T), the kinetic energies of particles and photons outside the box plus their interaction energies with other particles outside the box and with the external fields (H^{e}) , and the energy of interaction between the particles in the box and the external and "self-consistent" internal fields (H^{ie}) -- the latter generated by particles outside the box. Throughout the remainder of the discussion in this section, we shall neglect H^{ie} compared to T and H^{e} . This approximation in the present context corresponds to the assumption that when two particles are sufficiently closely associated that their interaction may be described in collision terms they may be regarded as effectively decoupled from their environment. Then noting that both T and H^e commute with $\rho_\sigma(\underline{K})$ (the number operator for particles in the box), we find that equation (47) becomes

$$L^\sigma f_\sigma \cong (2\pi\hbar)^{-3} s^{-1}\left[\mathrm{Tr}\,\rho_\sigma(\underline{K})D(t+s) - \mathrm{Tr}\,\rho_\sigma(\underline{K})D(t)\right]$$

$$\cong (2\pi\hbar)^{-3} s^{-1}\left[\mathrm{Tr}\, e^{iHs/\hbar}\rho_\sigma(\underline{K})e^{-iHs/\hbar}D(t) - \mathrm{Tr}\,\rho_\sigma(\underline{K})D(t)\right]. \tag{48}$$

again for appropriately small s. According to the above remarks, the Hamiltonian may be written as

$$H = T + H^e + H^{ei} + H^{pp} + H^{p\gamma} = T + H^{e'} + H^{ei} + H^{pp} + H^{p\gamma'} , \tag{49}$$

where we have redefined H^e and $H^{p\gamma}$ so that the portion of $H^{p\gamma}$ describing the interactions of particles and photons outside of V has been added to H^e yielding $H^{e'}$, the remainder of $H^{p\gamma}$ being designated $H^{p\gamma'}$. As previously agreed, we shall ignore the coupling between particles inside and outside of V implied by H^{ei} and will assume the effective commutivity of $(T+H^{pp}+H^{p\gamma'})$ with $H^{e'}$. Then since $H^{e'}$ also commutes with $\rho_\sigma(\underline{K})$ (the interactions between particles outside of V cannot change the number of particles in a given state within V), we find for equation (48),

$$L^\sigma f_\sigma \cong (2\pi\hbar)^{-3} s^{-1}\Big[-\mathrm{Tr}\,\rho_\sigma(\underline{K})D(t)$$

$$+ \mathrm{Tr}\,\rho_\sigma(\underline{K})e^{-i(T+H^{pp}+H^{p\gamma'})s/\hbar}D(t)e^{i(T+H^{pp}+H^{p\gamma'})s/\hbar}\Big]. \tag{50}$$

To continue from here it is useful to employ a specific representation for the explicit calculation of matrix elements. It will be convenient to choose a representation for the system which is in part the number representation for both particles and photons in the volume V and which has the following properties among others:

$$a^{+}_{\sigma}(\underline{K})a_{\sigma}(\underline{K})\,|n\eta\alpha\rangle = n_{\sigma\underline{K}}\,|n\eta\alpha\rangle \;.$$

$$\alpha^{+}_{\lambda}(\underline{k})\alpha_{\lambda}(\underline{k})\,|n\eta\alpha\rangle = \eta_{\lambda\underline{k}}\,|n\eta\alpha\rangle, \tag{51}$$

and

$$\langle n'\eta'\alpha'|n\eta\alpha\rangle = \delta_{nn'}\,\delta_{\eta\eta'}\,\delta_{\alpha\alpha'} \;. \tag{52}$$

The eigenlabels $(n\eta)$ are the usual sets of numbers required for the specification of occupancy of momentum states by particles and photons in V. The labels α are a sufficient set to complete the specification of the states of the whole system. The density matrix has, of course, been defined for the whole system and must therefore depend upon the labels α. Nevertheless we shall suppress the dependence of all quantities upon these labels, as both $\rho_{\sigma}(\underline{K})$ and $e\left[-i(T+H^{pp}+H^{p\gamma'})\,s/\hbar\right]$ are diagonal with respect to this part of our representation. In view of these remarks we find that

$$\left(e^{-i(T+H^{pp}+H^{p\gamma'})s/\hbar}\,D(t)\,e^{i(T+H^{pp}+H^{p\gamma'})s/\hbar}\right)_{n\eta n\eta}$$

$$= \sum_{n'\eta'}\left|\left(e^{-i(T+H^{pp}+H^{p\gamma'})s/\hbar}\right)_{n\eta n'\eta'}\right|^{2} D_{n'\eta' n'\eta'}(t)$$

$$+\text{(terms proportional to off-diagonal elements of D).} \tag{53}$$

We will ignore the contribution of the off-diagonal elements of D to the relation (53).

It is convenient to introduce the notation

$$W_{n\eta n'\eta'} = (2\pi\hbar)^{-3} s^{-1} \left| \left(e^{-i(T+H^{pp}+H^{p\gamma'})s/\hbar} \right)_{n\eta n'\eta'} \right|^2 \qquad (54)$$

so that equation (50) becomes

$$L^{\sigma} f_{\sigma} \cong \sum_{n\eta n'\eta'} n_{\sigma K} W_{n\eta n'\eta'} D_{n'\eta' n'\eta'}(t) - (2\pi\hbar)^{-3} s^{-1} \sum_{n\eta} n_{\sigma \underline{K}} D_{n\eta n\eta}(t)$$

$$= \sum_{n\eta n'\eta'}{}' \left[n'_{\sigma \underline{K}} - n_{\sigma \underline{K}} \right] W_{n\eta n'\eta'} D_{n\eta n\eta}(t) \ , \qquad (55)$$

where the prime on the summation means that the term in the sum for which n=n' and $\eta = \eta'$ is not to be included. Some manipulation is required in the development of equation (55), hinging primarily on the symmetry of W and the fact that

$$\sum_{n'\eta'} W_{n\eta n'\eta'} = (2\pi\hbar)^{-3} s^{-1} \ . \qquad (56)$$

The explicit evaluation of the transition probabilities, W, is facilitated (at least perturbation-wise) by re-expressing the exponential operators in equation (54) as

$$e^{-i(T+H^{pp}+H^{p\gamma})s/\hbar} = e^{-iTs/\hbar} (I+Q) \ , \qquad (57)$$

where I is the identity operator. Recalling equation (54), we see that now

$$W_{n\eta n'\eta'} = (2\pi\hbar)^{-3} s^{-1} \left| \left(e^{-iTs/\hbar} \left\{ I + Q \right\} \right)_{n\eta n'\eta'} \right|^2 = (2\pi\hbar)^{-3} s^{-1} \left| Q_{n\eta n'\eta'} \right|^2 , \tag{58}$$

because the kinetic energy operator T is diagonal in the number representation. The operator Q may be computed to various orders of approximation from the formula,

$$Q(s) = \sum_{j=1}^{\infty} \left(-\frac{i}{\hbar} \right)^j \int_{s_1=0}^{s} \dots \int_{s_j=0}^{s_{j-1}} ds_1 \dots ds_j \left(\bar{H}^{pp} + \bar{H}^{p\gamma} \right)_{s_1} \dots \left(\bar{H}^{pp} + \bar{H}^{p\gamma} \right)_{s_j} , \tag{59}$$

where

$$\left(\bar{H}^{pp} + \bar{H}^{p\gamma} \right)_s = e^{iTs/\hbar} \left(H^{pp} + H^{p\gamma} \right) e^{-iTs/\hbar} . \tag{60}$$

A straightforward but tedious calculation reveals that through terms of fourth degree in the interactions, QQ^+ may be expressed as

$$Q_{n\eta n'\eta'} Q^+_{n'\eta' n\eta} \cong \frac{4}{\hbar^2} \frac{\sin^2 \frac{\omega_{n\eta} - \omega_{n'\eta'}}{2} s}{(\omega_{n\eta} - \omega_{n'\eta'})^2} \left[\left| \left(H^{pp} + H^{p} \right)_{n\eta n'\eta'} \right|^2 + \left| \sum_{m\alpha} \frac{\left(H^{pp} + H^{p\gamma} \right)_{n\eta m\alpha} \left(H^{pp} + H^{p\gamma} \right)_{m\alpha n'\eta'}}{\hbar \left(\omega_{n\eta} - \omega_{m\alpha} \right)} \right|^2 \right] , \tag{61}$$

where we have introduced $\hbar\omega_{n\eta} = E_{n\eta}$. The complete disregard of the interactions in H^o while calculating the effects of interactions between particles (or between particles and photons) in the volume V is equivalent to the assertion that the particles in V are decoupled from the environment created by other particles outside of V.

A little reflection reveals that

$$\left|(H^{pp}+H^{p\gamma})_{n\eta n'\eta'}\right|^2 = \left|H^{pp}_{n\eta n'\eta'}\right|^2 + \left|H^{p\gamma}_{n\eta n'\eta'}\right|^2 , \qquad (62)$$

since the matrix elements of H^{pp} are non-zero only if the photon number does not change while those of $H^{p\gamma}$ are non-vanishing only if the photon number changes. In fact, as we shall see, the matrix elements of H^{pp} describe particle-particle scattering, whereas the elements of $H^{p\gamma}$ --containing terms both linear and bilinear in the creation and destruction operators for photons -- describe either radiative particle transitions or particle-photon scattering. Since we are not in a position here to discuss particle transport in systems in which particle transitions to, from, or between bound states are of much importance; and since free particles cannot emit or absorb a single photon, it follows that only photon scattering will be considered in the evaluation of $\left|H^{p\gamma}_{n\eta n'\eta'}\right|^2$.

As it is our intention to consider the influence of bremsstrahlung and inverse-bremsstrahlung on photon balance, we should here consider first order (one photon) radiative particle-particle scattering. Such processes are accounted for by the terms

$$\left|\sum_{m\alpha} \frac{H^{pp}_{n\eta m\alpha}\,H^{p\gamma}_{m\alpha n'\eta'} + H^{p\gamma}_{n\eta m\alpha}\,H^{pp}_{m\alpha n'\eta'}}{\hbar\,(\omega_{n\eta} - \omega_{m\alpha})}\right|^2 , \qquad (63)$$

if we employ plane wave states for the description of the radiating particles, or by the terms

$$\left| H^{p\gamma}_{n\eta\, n'\eta'} \right|^2 , \tag{64}$$

if we employ positive energy coulomb states. It is, in fact, the latter that we employ in the discussion of thermodynamics later.

Utilizing equation (58) we may rewrite (55) as

$$L^{\sigma} f_{\sigma} = (2\pi\hbar)^{-3} s^{-1} \sum_{n\eta n'\eta'}{}' \left[n'_{k} - n_{\sigma K} \right] \left| Q_{n\eta n'\eta'} \right|^2 D_{n\eta n\eta} . \tag{65}$$

Then the calculation of the quantities $|Q|^2$ proceeds straightforwardly from the formula (61) and our knowledge of the Hamiltonian for the system. One finds after considerable manipulation that equation (65) becomes

$$L^{\sigma} f_{\sigma} = \sum_{\sigma' K_1 K_2 K_3} C^{\sigma K,\, \sigma' K_3}_{\sigma K_1,\, \sigma' K_2} \sum_{n\eta} \Big[n_{\sigma K_1} n_{\sigma' K_2} (1 \pm n_{\sigma K})(1 \pm n_{\sigma' K_3})$$

$$- n_{\sigma K} n_{\sigma' K_3} (1 \pm n_{\sigma K_1})(1 \pm n_{\sigma' K_2}) \Big] D_{n\eta n\eta} + \sum_{K_1 \lambda k \lambda' k'} S^{K,\lambda k}_{K_1,\lambda' k'} \sum_{n\eta} \Big[n_{\sigma K_1} \eta_{\lambda' k'} (1 \pm n_{\sigma K})(1 + \eta_{\lambda k})$$

$$- n_{\sigma K} \eta_{\lambda k} (1 \pm n_{\sigma K_1})(1 + \eta_{\lambda' k'}) \Big] D_{n\eta n\eta} , \tag{66}$$

where the quantities C and S are essentially the transition probabilities per unit time for elastic charged particle scattering and for the scattering of particles by photons respectively. Here we are ignoring the contributions to the particle balance relation due to radiative coulomb scattering. We return to this question when we consider the thermodynamics of the system in Section IV. The transition probabilities represented by C and S are symmetric, e.g.,

$$C^{\sigma K, \sigma' K_3}_{\sigma K_1, \sigma' K_2} = C^{\sigma K_1, \sigma' K_2}_{\sigma K, \sigma' K_3} . \tag{67}$$

The choice of sign in the factors $(1 \pm n)$ depends upon whether the particles whose number is represented by n are bosons or fermions. These factors appear in our balance relation because of the dependence of reaction rates upon the densities of particles or photons in the final states. In particular, if the particles are fermions (e.g., electrons) so that the factors are of the form $(1-n)$ and the only allowed values of n are 0 and 1, we see that transitions to occupied states are forbidden as must be the case because of the exclusion principle. However, as indicated earlier, we should not expect this dependence of reaction rates upon the density of particles in final states to be significant in the non-degenerate plasma (a system in which the number of available states greatly exceeds the number of particles).

The direct, formal evaluation of the occupation number sums appearing in (66) leads to the introduction of higher order densities (higher order than the singlet density, e.g., doublet, triplet, and quartet densities) into the balance relation. To circumvent this complication at this point, we resort again to the approximation of replacing averages of products by products of averages -- bearing in mind that the average of an $n_{\sigma K}$ (or an $\eta_{\lambda k}$) with respect to the density matrix D is just the singlet density for particles of kind σ having momentum $\hbar \underline{K}$

(or for photons of polarization λ and momentum $\hbar$ k) multiplied by the volume of quantization, V. Thus we find that (66) may be expressed as

$$L^{\sigma} f_{\sigma} = \sum_{\sigma'} \int_{\underline{p}_1, \underline{p}_2, \underline{p}} d^3 p_1 \, d^3 p_2 \, d^3 p \; C(\sigma \underline{p}_1, \sigma' \underline{p}_2; \sigma \underline{p}, \sigma' \underline{p}_3) \Big[f_{\sigma}(\underline{p}_1) f_{\sigma'}(\underline{p}_2) \Big\{ (2\pi\hbar)^{-3}$$

$$\pm f_{\sigma}(\underline{p}) \Big\} \Big\{ (2\pi\hbar)^{-3} \pm f_{\sigma'}(\underline{p}_3) \Big\} - f_{\sigma}(\underline{p}) f_{\sigma}(\underline{p}_3) \Big\{ (2\pi\hbar)^{-3} \pm f_{\sigma}(\underline{p}_1) \Big\} \Big\{ (2\pi\hbar)^{-3} \pm f_{\sigma'}(\underline{p}_2) \Big\} \Big]$$

$$+ \sum_{\lambda\lambda'} \int_{\underline{p}_1, \underline{k}, \underline{k}'} d^3 p_1 \, dk \, d\Omega \, dk' \, d\Omega' \, S_{\sigma}(\underline{p}_1 \lambda' \underline{k}'; \underline{p} \lambda \underline{k}) \Big[f_{\sigma}(\underline{p}_1) F_{\lambda'}(\underline{k}') \Big\{ (2\pi\hbar)^{-3}$$

$$\pm f_{\sigma}(\underline{p}) \Big\} \Big\{ k^2 (2\pi)^{-3} + F_{\lambda}(\underline{k}) \Big\} - f_{\sigma}(\underline{p}) F_{\lambda}(\underline{k}) \Big\{ (2\pi\hbar)^{-3} \pm f_{\sigma}(\underline{p}_1) \Big\} \Big\{ k'^2 (2\pi)^{-3} + F_{\lambda'}(\underline{k}') \Big\} \Big] .$$

(68)

The F_{λ}'s represent the photon singlet densities -- to be discussed in Part III. The transition probabilities are exhibitable as

$$C(\sigma \underline{p}_1, \sigma' \underline{p}_2; \sigma \underline{p}, \sigma' \underline{p}_3) = \frac{(2\pi\hbar)^6}{\mu^2_{\sigma\sigma'}} \sigma(E_C, \Theta) \delta(E_C - E_C') \delta(\underline{p} + \underline{p}_3 - \underline{p}_1 - \underline{p}_2), \qquad (69a)$$

and

$$S_{\sigma}(\underline{p}_1\lambda'\underline{k}';\underline{p}\lambda\underline{k})=(2\pi)^6\hbar^4\left(\frac{e_{\sigma}^2}{m_{\sigma}c}\right)^2\frac{c^2}{kk'}\left|\underline{\epsilon}_{\lambda}(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}')\right|^2\delta(E_{pk}-E_{p_1k'})$$

$$\times\,\delta(\underline{p}_1+\hbar\underline{k}'-\underline{p}-\hbar\underline{k})\,, \tag{69b}$$

where $\mu_{\sigma\sigma'}$ is the reduced mass of the scattering pair, and E_C and Θ are the energy of the scattering pair and the scattering angle in the center of mass coordinate system. We have introduced the symbol $\sigma(E_C, \Theta)$ to represent the coulomb cross section, i.e.,

$$\sigma(E_C, \Theta)=\left(\frac{e_{\sigma}e_{\sigma'}}{4E_C}\right)^2\sin^{-4}\frac{\Theta}{2}\left[1+\delta_{\sigma\sigma'}\frac{1-\cos\Theta}{1+\cos\Theta}\right]^2 . \tag{70}$$

The factor modifying the usual formula for coulomb scattering which differs from unity only when the scattering particles are identical arises because of exchange, i.e., because of the indistinguishability of targets from projectiles. Analogously equation (69b) contains the Thomson scattering cross section (as would be expected since we are dealing solely with non-relativistic particles) as is readily demonstrated by averaging over photon polarization states.

Recalling the definition of the operator L^{σ}, equation (41), we re-express our densities in position and velocity space rather than position and momentum space, obtaining

$$\frac{\partial f_\sigma}{\partial t} + v_j \frac{\partial f_\sigma}{\partial x_j} + \frac{e_\sigma}{m_\sigma} E_j \frac{\partial f_\sigma}{\partial v_j} + \frac{e_\sigma}{m_\sigma c} (\underline{v} \times \underline{H})_j \frac{\partial f_\sigma}{\partial v_j} = \sum_{\sigma'} \int d^3 v_1 \, d^3 v_2 \, d^3 v_3 \, C(\sigma \underline{p}_1, \sigma' \underline{p}_2; \sigma \underline{p}, \sigma' \underline{p}_3)$$

$$\times \left[f_\sigma(v_1) f_{\sigma'}(v_2) \left\{ \left(\frac{m_\sigma}{2\pi\hbar} \right)^3 \pm f_\sigma(v) \right\} \left\{ \left(\frac{m_{\sigma'}}{2\pi\hbar} \right)^3 \pm f_{\sigma'}(v_3) \right\} - f_\sigma(v) f_{\sigma'}(v_3) \left\{ \left(\frac{m_\sigma}{2\pi\hbar} \right)^3 \pm f_\sigma(\underline{v}_1) \right\} \right.$$ [redacted]

$$\left. \times \left\{ \left(\frac{m_{\sigma'}}{2\pi\hbar} \right)^3 \pm f_{\sigma'}(\underline{v}_2) \right\} \right] + \sum_{\lambda\lambda'} \int d^3 v_1 \, dk \, d\Omega \, dk' \, d\Omega' \, S_\sigma(\underline{p}_1 \lambda' \underline{k}'; \underline{p} \lambda \underline{k}) \left[f_\sigma(v_1) F_{\lambda'}(\underline{k}') \left\{ \left(\frac{m_\sigma}{2\pi\hbar} \right)^3 \right. \right.$$

$$\left. \left. \pm f_\sigma(\underline{v}) \right\} \left\{ \frac{k^2}{(2\pi)^3} + F_\lambda(\underline{k}) \right\} - f_\sigma(\underline{v}) F_\lambda(\underline{k}) \left\{ \left(\frac{m_\sigma}{2\pi\hbar} \right)^3 \pm f_\sigma(\underline{v}_1) \right\} \left\{ \frac{k'^2}{(2\pi)^3} + F_{\lambda'}(\underline{k}') \right\} \right] . \qquad (71)$$

The velocity variable was here introduced according to the identification,

$$\underline{v} = (\underline{p} - \frac{e_\sigma}{c} \langle \underline{R} \rangle)/m_\sigma \; ;$$

and we have employed the notation

$$\underline{E} = \underline{E}^L - \frac{1}{c} \frac{\partial}{\partial t} \langle \underline{R} \rangle \; , \qquad (72a)$$

and

$$\underline{H} = \underline{\nabla} \times \langle \underline{R} \rangle \; . \qquad (72b)$$

Recalling equations (7) and (40), we see that $\underline{E}$ and $\underline{H}$ are interpretable as the superposition of the externally applied and a portion of the internally generated electric and magnetic fields. Thus if we ignore the scattering of particles by photons and take the classical limit of the terms describing the scattering of particles by particles, we find that equation (71) is just the conventional Boltzmann equation with self-consistent fields; except for the implication of some restriction upon the interpretation of the self-consistent fields as well as upon the strength and variation of the permissible applied fields.

The nature of these restrictions deserves a little attention at this point. In the present context they stem at least partially from the seeming necessity for the operations of spatial and temporal coarse-graining. The spatial coarse-graining required that the distribution functions for the particles and the fields $\underline{E}$ and $\underline{H}$ be essentially constant over an appropriately chosen volume V. Thus given V, this puts an obvious limitation on the space rates of change of the fields, whereas given the inhomogeneities of the fields a lower bound on the dimensions of V is immediately indicated. Furthermore the distribution functions are expected to represent the average number of particles in the volume V -- hence the particles must be presumed to be localized within V -- thus the least linear dimension of this volume must be large compared to the DeBroglie wavelength of the majority of particles of interest. It should be noted that the collision description of the interaction of closely associated particles also requires that their DeBroglie wavelengths (actually their relative DeBroglie wavelengths) be small compared to any linear measure of V. Thus rapid field variations require a fine-grained average, whereas systems of low-mean-energy-particles require a coarse-grained average and in some systems these opposing demands may not be met. This is unlikely however in the fully ionized plasma. A serious complication is introduced by the presence of strong magnetic fields, even if homogeneous. Evidently, in such a circumstance, it is required that the dimensions of the quantization volume must be small compared to the radius of gyration of the lightest particle of interest in the system. Otherwise, the employment of the plane wave representation for the particles in V would be unsuitable. Finally, the assumption of the localizability of photons in the volume of quantization implies that its least linear dimension be large compared to the wavelengths of such photons.

The equations governing the behavior of $\underline{E}$ and $\underline{H}$ remain to be developed in the present context. As they represent superpositions of internally generated and externally applied field, and as the external fields are presumed known, it suffices to consider only the portions of $\underline{E}$ and $\underline{H}$ which arise from charges and currents within the plasma. If we designate these portions by $\underline{\mathcal{E}}$ and $\underline{\mathcal{H}}$ respectively, then by equation (40) we have

$$\mathcal{E}_j^L(\underline{x}, t) = -\frac{\partial}{\partial x_j}\sum_\sigma e_\sigma \int_{\underline{x}'} d^3x'\, s(\underline{x}')\, \frac{f_\sigma(\underline{x}', t)}{|\underline{x} - \underline{x}'|} , \tag{73}$$

for the longitudinal part of the self-consistent electric field. For the transverse part of this field we have

$$\mathcal{E}_j^T(\underline{x}, t) = -\frac{1}{c}\frac{\partial}{\partial t} \text{Tr } A_j^s(\underline{x}) \, D(t) \; . \qquad (74)$$

whereas for the self-consistent magnetic field we have

$$\mathcal{H}_j(\underline{x}, t) = \text{Tr}\left\{ \underline{\nabla} \times \underline{A}^s(\underline{x}) \right\}_j D(t) \; . \qquad (75)$$

It is immediately evident that $\underline{\mathcal{H}}$ is divergenceless, for

$$\underline{\nabla} \cdot \underline{\mathcal{H}} = \underline{\nabla} \cdot \text{Tr}\left\{ \underline{\nabla} \times \underline{A}^s \right\} D = \text{Tr}\left\{ \underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^s \right\} D = 0 \; . \qquad (76)$$

It is perhaps also equally obvious that $\underline{\mathcal{E}}^L$ is the solution to Maxwell's equation

$$\underline{\nabla} \cdot \underline{\mathcal{E}}^L = 4\pi \sum_\sigma e_\sigma f_\sigma(\underline{x}, t) \quad , \qquad (77)$$

giving the longitudinal part of the electric field at $\underline{x}$ due to charges outside of the volume V about $\underline{x}$. Recalling equation (74), it is seen that

$$\underline{\nabla} \times \underline{\mathcal{E}}^T = -\frac{1}{c} \text{Tr}\left\{ \underline{\nabla} \times \underline{A}^s \right\} \dot{D} \quad , \qquad (78)$$

which, according to equation (75), leads to

$$\underline{\nabla} \times \underline{\mathcal{E}}^T = -\frac{1}{c}\frac{\partial}{\partial t} \underline{\mathcal{H}} \; . \qquad (79)$$

Thus three of Maxwell's equations are established almost trivially as descriptive of the "slowly varying" part of the internally generated electric and magnetic fields. But not so, apparently, for the fourth equation relating the fields to the currents in the system. The difficulty here seems to stem from the necessity of calculating explicitly operators representing the time derivatives of $\underline{A}^s$, i.e., of calculating commutators of $\underline{A}^s$ with the Hamiltonian. But such calculations have here been complicated by the fact that $\underline{A}^S$ represents only the "low frequency" part of the vector potential generated by charges and currents in the plasma. A semi intuitive circumvention of this difficulty is accomplished by considering an equation satisfied by the exact fields in the plasma, i.e., the fields which have not been decomposed into parts of the "fast" and "slow" variation. Labeling these "complete" fields by $\mathcal{E}'$ and $\mathcal{H}'$ one finds (as shown elsewhere[1]) that they satisfy the equation:

$$\underline{\nabla} \times \underline{\mathcal{H}}' - \frac{1}{c}\frac{\partial}{\partial t}\underline{\mathcal{E}}' = \frac{4\pi}{c} \mathrm{Tr}\, \underline{J}_{op} D, \tag{80}$$

where

$$\underline{J}_{op} = \sum_{\sigma} \left[\frac{i\hbar e_{\sigma}}{2m_{\sigma}} \left\{ (\underline{\nabla}\,\psi_{\sigma}^{+})\,\psi_{\sigma} - \psi_{\sigma}^{+}\,\underline{\nabla}\,\psi_{\sigma} \right\} - \frac{e_{\sigma}^{2}}{m_{\sigma}c}\,\underline{A}\,\psi_{\sigma}^{+}\,\psi_{\sigma} \right]. \tag{81}$$

This is the anticipated relation between the exact fields and the exact currents in the plasma. However, it is desirable to translate the above description of the current to the conventional macroscopic description of plasma currents, i.e.,

$$\underline{J}_{macro} = \sum_{\sigma} e_{\sigma} \int \underline{v}\, f_{\sigma}\, d^3v \quad . \tag{82}$$

This is readily accomplished, for

$$\sum_\sigma e_\sigma \int \underline{v} f_\sigma d^3 v = \sum_\sigma \frac{e_\sigma}{m_\sigma} \int (\underline{p} - \frac{e_\sigma}{c} \langle \underline{A} \rangle f_\sigma d^3 p$$

$$= \sum_\sigma \frac{e_\sigma \hbar}{m_\sigma} \int \underline{K} f_\sigma d^3 K - \sum_\sigma \frac{e^2_\sigma}{m_\sigma c} \langle \underline{A} \rangle \langle \psi^+_\sigma \psi_\sigma \rangle , \tag{83}$$

where here of course, $\underline{A}$ is interpreted as the total vector potential in the plasma. A little calculation reveals that the first term on the right-hand-side of equation (83) is the same as the average of the first term on the right-hand-side of equation (81); so that

$$\text{Tr}\, \underline{J}_{op} D \equiv \langle \underline{J}_{op} \rangle = \underline{J}_{macro}$$

$$+ \sum_\sigma \frac{e^2_\sigma}{m_\sigma c} \left[\langle \underline{A} \rangle \langle \psi^+_\sigma \psi_\sigma \rangle - \langle \underline{A} \psi^+_\sigma \psi_\sigma \rangle \right] . \tag{84}$$

Entering this relation into equation (80), and decomposing $\underline{\mathcal{E}}'$ and $\underline{\mathcal{H}}'$ into parts of "fast" and "slow" variation we obtain

$$\underline{\nabla} \times \underline{\mathcal{H}} - \frac{1}{c} \frac{\partial}{\partial t} \underline{\mathcal{E}} - \frac{4\pi}{c} \underline{J}_{macro} + \underline{\nabla} \times \underline{\mathcal{H}}^F - \frac{1}{c} \frac{\partial}{\partial t} \underline{\mathcal{E}}^F$$

$$- \frac{4\pi}{c} \sum_\sigma \frac{e^2_\sigma}{m_\sigma c} \left[\langle \underline{A}^F \rangle \langle \psi^+_\sigma \psi_\sigma \rangle - \langle \underline{A}^F \psi^+_\sigma \psi_\sigma \rangle \right] = 0 . \tag{85}$$

Note that $\underline{A}^F$ only enters into the "correlation term" in (85), since we have already employed assumptions equivalent to the statement that

$$\langle \underline{A}^s \rangle \langle \psi_\sigma^+ \psi_\sigma \rangle - \langle \underline{A}^s \psi_\sigma^+ \psi_\sigma \rangle = 0 . \tag{86}$$

It is at least intuitively reasonable to argue at this point that the macroscopic current $\underline{J}_{macro}$, contributes only to the slowly varying fields $\underline{\mathcal{E}}$ and $\underline{\mathcal{H}}$, whereas the "correlation current" contributes only to the rapidly varying component of the internal fields. The latter however, have already presumably been adequately accounted for in terms of photon distributions, hence the only relevant part of equation (85) is

$$\underline{\nabla} \times \underline{\mathcal{H}} - \frac{1}{c} \frac{\partial}{\partial t} \underline{\mathcal{E}} = \frac{4\pi}{c} \underline{J}_{macro} . \tag{87}$$

We now assert that equations (76), (77), (79), and (87) provide an appropriate description of the self-consistent fields appearing in the particle transport equations (71).

III

THE BALANCE RELATION FOR THE PHOTON DISTRIBUTIONS

In order to complete our description of the plasma, we require the equations governing the photon distributions, F_λ. The deduction of a set of such equations has been described in considerable detail elsewhere [2], hence only the highlights of that deduction will be sketched here.

We define a photon singlet density in close analogy to the definition (14) of the particle singlet density, i.e.,

$$F_\lambda(\underline{x}, \underline{k}, t) = \frac{8}{V} \sum_{\underline{q}} e^{-2i\,\underline{x}\cdot\underline{q}} (\Psi, \alpha_\lambda^+(\underline{k}+\underline{q})\,\alpha_\lambda(\underline{k}-\underline{q})\,\Psi)$$

$$= \Gamma_{xq} \operatorname{Tr} \rho_\lambda(\underline{k}, \underline{q})\, D, \qquad (88)$$

where we have introduced the notation

$$\Gamma_{xq} = \frac{8}{V} \sum_{\underline{q}} e^{-2i\,\underline{x}\cdot\underline{q}} \qquad (89)$$

for the fourier sum operator, and

$$\rho_\lambda(\underline{k}, \underline{q}) = \alpha_\lambda^+(\underline{k}+\underline{q})\,\alpha_\lambda(\underline{k}-\underline{q}) \qquad (90)$$

for the photon singlet density operator. To illuminate the sense in which (88) defines an appropriate photon singlet density, we first integrate F_λ over the volume, V; obtaining,

$$F_\lambda(\underline{k}, t) = \int_V d^3x \; F_\lambda(\underline{x}, \underline{k}, t)$$

$$= \mathrm{Tr}\, \alpha_\lambda^+(\underline{k})\; \alpha_\lambda(\underline{k})\; D. \tag{91}$$

Thus $F_\lambda(\underline{k}, t)$ is just the expected number of photons of polarization λ having momentum $\hbar k$ to be found in the volume V at time t - suggesting the interpretation of $F_\lambda(\underline{x}, \underline{k}, t)\, d^3x$ as the corresponding expected number in d^3x. Next we observe that

$$\int_V d^3x \sum_{\lambda \underline{k}} (\hbar c k)\, F_\lambda(\underline{x}, \underline{k}, t)$$

$$= \int_V d^3x \left(\psi, \frac{(\underline{E}^T)^2 + (\underline{H})^2}{8\pi}\, \psi\right) - \frac{1}{2} \sum_{\lambda \underline{k}} \hbar c k, \tag{92}$$

where we have made use of the relation $\underline{E}^T = -4\pi c \underline{P}$. The second term on the right-hand-side of (92) is just the zero-point energy implicit in the first term on the right-hand side; hence the difference on the right represents the energy of the actual photons in the volume V. This suggests that, in some sense, we might identify

$$\sum_{\underline{k}\lambda} \hbar c k \; F_\lambda(\underline{x}, \underline{k}, t)$$

$$\sim \left(\Psi, \frac{(\underline{E}^T)^2 + (\underline{H})^2}{8\pi} \Psi \right) - \frac{1}{2} \sum_{\underline{k}\lambda} \frac{\hbar ck}{V}, \tag{93}$$

and hence - since $\hbar c k$ is the energy per photon of momentum $\hbar \underline{k}$ - further interpret $\sum_{\lambda \underline{k}} F_\lambda(\underline{x}, \underline{k}, t)$ as the expected number of photons per unit volume at the point $\underline{x}$. Of course, the identification (93) is not quantitative, since the functions identified actually differ by quantities whose integral over V vanish Evidently, the concept of the spatial localization of photons is somewhat more obscure than is the case for particles.

In this instance also it is convenient to select an appropriate interaction representation in which to calculate the time rate of change of F_λ. To identify the transformation leading to a useful interaction representation, it is desirable first to regroup terms in the Hamiltonian (10) after introducing the fourier analysis (11) of the vector potential $\underline{A}$. One finds that

$$\cdot\ H = \sum_{\underline{k}\lambda} (\hbar c k)/2 + \sum_{\underline{k}\lambda} (\hbar c k)\, \alpha^{+}_{\lambda}(\underline{k})\, \alpha_{\lambda}(\underline{k})$$

$$\times \left\{ 1 + \frac{1}{c^2 k^2} \sum_{\sigma} \frac{4\pi e_\sigma^2}{2m_\sigma} \frac{1}{V} \int d^3x \psi^{+}_{\sigma} \psi_\sigma \right\} + \sum_{\sigma} \frac{1}{2m_\sigma} \int d^3x (\underline{\Pi}^{\sigma *} \psi^{+}_{\sigma}) \cdot (\underline{\Pi}^{\sigma} \psi_\sigma)$$

$$- \sum_{\sigma \underline{k} \lambda} \frac{e_\sigma}{2m_\sigma c} \sqrt{\frac{2\pi \hbar c}{kV}} \int d^3x\, e^{-i\underline{k}\cdot\underline{x}} \left[(\underline{\Pi}^{\sigma *} \psi^{+}_{\sigma}) \cdot \underline{\zeta}^{+}_{\lambda}(\underline{k})\, \psi_\sigma + \underline{\zeta}^{+}_{\lambda}(\underline{k}) \psi^{+}_{\sigma} \cdot (\underline{\Pi}^{\sigma} \psi_\sigma) \right]$$

$$+ \sum_{\sigma\sigma'} \frac{e_\sigma e_{\sigma'}}{1 + \delta_{\sigma\sigma'}} \int d^3x\, d^3x' \frac{\psi^{+}_{\sigma}(\underline{x})\, \psi^{+}_{\sigma'}(\underline{x}') \psi_\sigma(\underline{x})\, \psi_{\sigma'}(\underline{x}')}{|\underline{x} - \underline{x}'|}$$

$$+ \sum_{\sigma} e_\sigma \int d^3x\, \phi \psi^{+}_{\sigma} \psi_\sigma + \sum_{\sigma k \lambda} \frac{\pi \hbar e_\sigma^2}{m_\sigma c V k} \left\{ \underline{\epsilon}_{\lambda}(k)\, \underline{\epsilon}_{\lambda}(-k) \left[\alpha^{+}_{\lambda}(k)\, \alpha^{+}_{\lambda}(-\underline{k}) \right.\right.$$

$$\left.\left. + \alpha_\lambda(-\underline{k})\, \alpha_\lambda(\underline{k}) \right] + 1 \right\} \int d^3x\, \psi^{+}_{\sigma} \psi_\sigma$$

$$+ \sum_{\sigma k \lambda k' \lambda'}{}' \frac{\pi \hbar e^2}{m_\sigma c V} \frac{\underline{\zeta}^{+}_{\lambda}(\underline{k}) \cdot \underline{\zeta}^{+}_{\lambda'}(-\underline{k})}{\sqrt{kk'}} \int d^3x\, e^{-i\underline{x}\cdot(\underline{k}-\underline{k}')} \psi^{+}_{\sigma} \psi_\sigma . \qquad (94)$$

The prime on the last sum implies that the terms for which $\lambda = \lambda'$ and $\underline{k} = \underline{k}'$ are not to be included in the sum. They appear explicitly in the terms immediately preceding and in the second set of terms which are proportional to the photon number operator, $\alpha^{+}_{\lambda}(\underline{k})\,\alpha_{\lambda}(\underline{k})$ The first sum on the right-hand-side of (94) is simply the zero-point energy of the radiation field. Now define a "free photon" Hamiltonian by

$$H^{0} = \sum_{\underline{k}\,\lambda} (\hbar c k)\, \alpha^{+}_{\lambda}(\underline{k})\, \alpha_{\lambda}(\underline{k})$$

$$\times \left\{ 1 + \frac{1}{(ck)^{2}} \sum_{\sigma} \frac{4\pi e_{\sigma}^{2}}{2m_{\sigma}} \frac{1}{V} \int d^{3}x\, \psi^{+}_{\sigma}\, \psi_{\sigma} \right\} \tag{95}$$

and an "interaction" Hamiltonian H^{I} by

$$H = H^{0} + H^{I}. \tag{96}$$

The transformation to the desired interaction representation is now accomplished according to

$$\Psi = U\, \Phi\,, \tag{97}$$

where

$$U = e(-i\, H^{0} t/\hbar\ ). \tag{98}$$

In the new representation the expression for the photon density becomes

$$F_\lambda(\underline{x}, \underline{k}, t) = \Gamma_{xq} \operatorname{Tr} \bar{\rho}_\lambda(k, q, t) \bar{D}, \tag{99}$$

where

$$\frac{\partial \bar{\rho}_\lambda}{\partial t} = \frac{i}{\hbar} \left[\bar{H}^0, \bar{\rho}_\lambda\right], \tag{100a}$$

and

$$\frac{\partial \bar{D}}{\partial t} = \frac{i}{\hbar} \left[\bar{D}, \bar{H}^I\right]. \tag{100b}$$

From here on the calculation of the time rate of change of F_λ proceeds in a manner formally identical to the corresponding calculations for the particle densities described in Section II. (It should be noted in passing that a more elaborate and more subtle treatment of photon transport in dispersive media is readily accessible here. However, as such investigations are only in progress and not complete, discussion of them will be deferred). Thus we shall dispense with further discussion of deductive detail and go immediately to the final results which are embodied in the equation

$$\frac{\partial F_\lambda}{\partial t} + v_g \underline{\Omega} \cdot \underline{\nabla} F_\lambda$$

$$= \sum_{\sigma\lambda'} \int d^3v d^3v_1 \, dk' \, d\Omega' \; S_\sigma(\underline{p}_1 \lambda' \underline{k}'; \underline{p} \lambda \underline{k})$$

(continued)

$$\times \left[f_\sigma(\underline{v}_1) F_{\lambda'}(\underline{k}') \left\{ \left(\frac{m_\sigma}{2\pi h} \right)^3 \pm f_\sigma(\underline{v}) \right\} \left\{ \frac{k^2}{(2\pi)^3} + F_\lambda(\underline{k}) \right\} \right.$$

$$\left. - f_\sigma(\underline{v}) F_\lambda(\underline{k}) \left\{ \left[\left(\frac{m_\sigma}{2\pi h} \right)^3 \pm f_\sigma(\underline{v}_1) \right\} \left\{ \frac{k'^2}{(2\pi)^3} + F_{\lambda'}(\underline{k}') \right\} \right]$$

$$+ \epsilon^\lambda(\underline{k}) \left[F_\lambda(\underline{k}) + \frac{k^2}{(2\pi)^3} \right] - \alpha^\lambda(\underline{k}) F_\lambda(\underline{k}) \, . \tag{101}$$

We have here introduced the symbol v_g to represent the group speed of the photons in the medium, and have neglected higher order space derivatives of F_λ than the first. To the extent that we may replace averages of products by products of averages, we find for v_g in this case (because of our choice of H^o, equation (95), which is the prime consideration in the determination of the manner in which photons propagate between interactions with the particles of the plasma),

$$v_g \cong \frac{\partial}{\partial k} \left[ck + \sum_\sigma \frac{4\pi N_\sigma e_\sigma^2}{2ck\, m_\sigma} \right] , \tag{102}$$

where N_σ is the average density of particles of kind σ in the volume V. Since only the electrons will contribute appreciably to the sum in (102), we note that

$$v_g \simeq c - \omega_e^2/2ck^2, \tag{103}$$

where $\omega_e^2 = 4\pi N_e^2 e^2/m_e$. Thus photons of momentum $\hbar \underline{k}$ such that $\omega_e^2/2c^2k^2 > 1$ do not propagate in the medium as is expected. Evidently the energy of the "free" photons in the medium is given by

$$\mathcal{E}_k = \hbar\,\omega(k) = \hbar\, c\, k(1 + \omega_e^2/2c^2k^2). \tag{104}$$

The vector $\underline{\Omega}$ is just the unit vector in the direction of propagation of the photons, i.e., $\underline{\Omega} = \underline{k}/k$. The scattering frequency, S_σ, is the same as the one encountered earlier (69b) in the equations governing particle transport. The quantities $\epsilon^\lambda(\underline{k})$ and $\alpha^\lambda(\underline{k})$ are the transition probabilities per unit time for the emission or absorption of photons of polarization λ and propagation vector $\underline{k}$. Of course, these transition probabilities are space and time dependent as are the distributions F_λ. They may be explicitly evaluated in the sense of first order perturbation theory from the formulas,

$$\epsilon^\lambda(\underline{k}) = \sum_{\underline{K}\underline{K}'\sigma} T_{\sigma K}^{K'}(\lambda \underline{k})\, \bar{f}_\sigma(\underline{K}) \left\{1 \pm V \bar{f}_\sigma(\underline{K}')\right\}, \tag{105a}$$

$$\alpha^\lambda(\underline{k}) = \sum_{\underline{K}\underline{K}'\sigma} T_{\sigma K}^{K'}(\lambda\, \underline{k})\, \bar{f}_\sigma(\underline{K}') \left\{1 \pm V \bar{f}_\sigma(\underline{K})\right\}, \tag{105b}$$

where

$$T_{\sigma K}^{K'}(\lambda\, \underline{k}) = \left| \left\langle \underline{K}' \right| \frac{2\pi}{\sqrt{ck}}\, e^{-i\underline{k}\cdot\underline{x}}\, \underline{\epsilon}_\lambda(k)\cdot e_\sigma \underline{v}^\sigma \left| \underline{K} \right\rangle \right|^2 \delta(E_{K'k} - E_K). \tag{106}$$

Because the particle transition accompanying photon emission or absorption may be between, to, or from discrete states, it is not generally feasible to convert the sums in (105a, b) into integrals. Thus the $\underline{K}$'s in these formulas are simply to be interpreted as a sufficient set of labels for the complete specification of particle eigenstates. The velocity operator appearing in equation (106) is to be represented as $\underline{v}^{\sigma} = (\underline{p} - \frac{e_{\sigma}}{c}\underline{A}^{e})/m_{\sigma}$. The details of some calculations of ϵ^{λ} and α^{λ} for specific mechanisms of emission and absorption have been presented elsewhere [2], and therefore will not be entered into here.

IV

THERMODYNAMICS OF THE FULLY IONIZED PLASMA

The present description of the plasma is complete and irreversible (in the sense of the many and varied approximations that have been introduced into its development). It is complete in the sense that we have as many equations as we have unknown functions described by them, and these equations involve functional parameters (scattering frequencies, emission and absorption transition probabilities etc.) whose analytical representations have been specified. It is irreversible in the sense that the whole set is not invariant under the transformation

$$(\underline{v},\ \underline{k},\ \underline{H},\ \underline{t}) \longrightarrow (-\underline{v},\ -\underline{k},\ -\underline{H},\ -\underline{t}).$$

Thus this description of the plasma should contain the important implication that, under certain circumstances at least, the system progresses irreversibly to a unique state referred to as the thermodynamic state; and that in that state the system variables assume specified forms. Thus it is of some interest to explore these implications here, as this has not hitherto been accomplished from the present point of view for the plasma.

However, in order that such an implication be realizable it is also necessary that our description be consistent. That is, the equations must explicitly describe the full effect of any given interaction upon all distributions concerned to the same level of approximation. But this has not yet been done here, for although due account has been taken of the effect of emission and absorption processes on the photon distributions (as is obviously essential), the effect of these same phenomena upon the time rate of change of the particle distributions has so far been neglected. It was pointed out earlier that for the sake of consistency inelastic corrections to scattering should indeed be retained in the equations for the particle distributions, (71), but that their retention implied only a small correction to the scattering cross section and hence would be ignored. Thus their quantitative importance to the description of the fully ionized plasma is most probably negligible. But to achieve a proper qualitative appreciation of the approach to equilibrium, the influence of these mechanisms of interaction between particles and photons on the particle distributions cannot be ignored. Since the description of the effects that concern us here (to the present level of approximation, i.e., first order perturbation theory) would simply enter additively into equation (71), we merely sketch at this point the calculation of these requisite extra terms.

According to equation (65), we should compute

$$\left.\frac{\delta f_\sigma}{\delta t}\right)_{ea} = \sum_{n\eta n'\eta'} (n'_{\sigma K} - n_{\sigma K})\, W^{ea}_{n\eta n'\eta'}\, D_{n\eta n\eta} \qquad (107)$$

where $W^{ea}_{n\eta n'\eta'}$ is appropriately chosen to represent particle interactions involving single photon emission or absorption. Since the transition probabilities for these processes are inversely proportional to the masses of the emitting and absorbing particles, it is evident that we need to evaluate (107) for the electrons in the plasma only. Furthermore, for the discussion of the particle distributions at least, we are restricted to fully ionized systems. Also, as we are solely concerned with the approach to the thermodynamic state in this section, we shall presume that our system is not exposed to external fields of any kind. Thus the only events that concern us here are the electronic "free-free" transitions executed in ionic coulomb fields. In order to keep the manipulations to a minimum and the bring the results smoothly into line with the formalism of equations (101), (105a, b) and (106) for the photons, we shall calculate the desired transition probabilities by first order perturbation theory assuming positive energy coulomb wave-functions for the representation of the electronic states. Accordingly, the quantity to be calculated is

$$W^{ea}_{n\eta n'\eta'} = \frac{4}{\hbar^2}\left(\frac{m}{2\pi\hbar}\right)^3 \frac{\sin^2 \frac{\omega_{n\eta} - \omega_{n'\eta'}}{2} s}{s(\omega_{n\eta} - \omega_{n'\eta'})^2}$$

$$\times \left| \left\langle n\eta \left| \frac{e}{c}\int d^3x\, \psi_e^+ (\underline{A} \cdot \underline{p}/m)\psi_e \right| n'\eta' \right\rangle \right|^2 . \qquad (108)$$

Again, after tedious but straight forward manipulations, we find for (107),

$$\left.\frac{\delta f_e}{\delta t}\right)_{ea} = \sum_{\lambda} \int d^3 v' \, dk \, d\Omega \; V \, T_K^{K'}(\lambda \, \underline{k})$$

$$\times \left[\left\{ \left(\frac{m_e}{2 \pi \hbar} \right)^3 - f_e(\underline{v}) \right\} f_e(\underline{v}') \, F_\lambda(\underline{k}) \right.$$

$$- f_e(\underline{v}) \left\{ \left(\frac{m_e}{2 \pi \hbar} \right)^3 - f_e(\underline{v}') \right\} \left\{ \frac{k^2}{(2\pi)^3} + F_\lambda(\underline{k}) \right\}$$

$$+ \sum_{\lambda} \int d^3 v' \, dk \, d\Omega \; V \, T_{K'}^{K}(\lambda \, \underline{k})$$

$$\times \left[\left\{ \left(\frac{m_e}{2 \pi \hbar} \right)^3 - f_e(\underline{v}) \right\} f_e(\underline{v}') \left\{ \frac{k^2}{(2\pi)^3} + F_\lambda(\underline{k}) \right\} \right.$$

$$\left. - f_e(\underline{v}) \left\{ \left(\frac{m_e}{2 \pi \hbar} \right)^3 - f_e(\underline{v}') \right\} F_\lambda(\underline{k}) \right] . \qquad (109)$$

In this expression, $T_K^{K'}(\lambda \, k)$ is the same as is given in equation (106).

We now undertake the task of proving an H-theorem for systems that are not exposed to external fields, that have already become spatially homogeneous, that exhibit no internal electric currents, but that are still temporally varying. By virtue of these assumptions we may presume the absence of self consistent fields and may take for our system equations:

$$\frac{\partial f_\sigma}{\partial t} = \sum_{\sigma'} \int d^3v_1\, d^3v_2\, d^3v_3\, C\,(\sigma\, p_1\; \sigma'\, p_2\, ;\, \sigma\, \underline{p}\, ,\, \sigma'\, \underline{p_3})$$

$$\times \left[\rho_\sigma \pm f_\sigma(\underline{v})\right]\left[\rho_{\sigma'} \pm f_{\sigma'}(\underline{v_3})\right]\left[\rho_\sigma \pm f_\sigma(\underline{v_1})\right]\left[\rho_{\sigma'} \pm f_{\sigma'}(\underline{v_2})\right]$$

$$\times \left[\frac{f_\sigma(\underline{v_1})}{\rho_\sigma \pm f_\sigma(\underline{v_1})}\; \frac{f_{\sigma'}(\underline{v_2})}{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v_2})} - \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})}\; \frac{f_{\sigma'}(\underline{v_3})}{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v_3})}\right]$$

$$+ \sum_{\lambda\lambda'} \int d^3v_1\, dk\, d\Omega\, dk'\, d\Omega'\, S_\sigma\,(p_1\; \lambda'\, \underline{k'}\, ;\, \underline{p}\, \lambda\, \underline{k})$$

$$\times \left[\rho_\sigma + f_\sigma(\underline{v})\right]\left[\rho_k + F_\lambda(\underline{k})\right]\left[\rho_\sigma + f_\sigma(\underline{v_1})\right]\left[\rho_{k'} + F_{\lambda'}(\underline{k'})\right]$$

$$\times \left[\frac{f_\sigma(\underline{v_1})}{\rho_\sigma \pm f_\sigma(\underline{v_1})}\; \frac{F_{\lambda'}(\underline{k'})}{\rho_{k'} + F_{\lambda'}(\underline{k'})} - \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})}\; \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})}\right] + \left.\frac{\delta f_e}{\delta t}\right)_{ea}, \qquad (110)$$

and

$$\frac{\partial F_\lambda}{\partial t} = \sum_{\sigma\lambda'} \int d^3v\, d^3v_1\, dk'\, d\Omega'\, S_\sigma(\underline{p}_1\, \lambda'\, \underline{k}' ; \underline{p}\, \lambda\, \underline{k})$$

$$\times \left[\rho_\sigma \pm f_\sigma(\underline{v}_1)\right] \left[\rho_{k'} + F_{\lambda'}(\underline{k}')\right] \left[\rho_\sigma \pm f_\sigma(\underline{v})\right] \left[\rho_k + F_\lambda(\underline{k})\right]$$

$$\times \left[\frac{f_\sigma(\underline{v}_1)}{\rho_\sigma \pm f_\sigma(\underline{v}_1)} \frac{F_{\lambda'}(\underline{k}')}{\rho_{k'} + F_{\lambda'}(\underline{k}')} - \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})} \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})}\right]$$

$$+ \int d^3v\, d^3v'\, VT_K^{K'}(\lambda\, k) \left[\rho_k + F_\lambda(\underline{k})\right] \left[\rho_e - f_e(\underline{v})\right] \left[\rho_e - f_e(v')\right]$$

$$\times \left[\frac{f_e(\underline{v})}{\rho_e - f_e(v)} - \frac{f_e(\underline{v}')}{\rho_e - f(\underline{v}')} \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})}\right]. \qquad (111)$$

These equations follow directly from (71) and (101), with due regard for (105 a, b) and (106), by an obvious regrouping of terms, the introduction of the symbols $\rho_\sigma = (m_\sigma/2\pi\hbar)^3$ and $\rho_k = k^2/(2\pi)^3$, and by a conversion of the sums in (105a, b) to integrals on the assumption that the particle states between which emission and absorption transitions occur are sufficiently densely distributed. We now define a function S according to

$$S = -\kappa \sum_{\sigma} \int d^3v \left[f_\sigma(\underline{v}) \ln f_\sigma(\underline{v}) \pm \rho_\sigma \ln \rho_\sigma \right.$$

$$\left. + \left\{ \rho_\sigma + f_\sigma(\underline{v}) \right\} \ln \left\{ \rho_\sigma \pm f_\sigma(\underline{v}) \right\} \right]$$

$$-\kappa \sum_{\lambda} \int dk\, d\Omega \left[F_\lambda(\underline{k}) \ln F_\lambda(\underline{k}) + \rho_k \ln \rho_k \right.$$

$$\left. - \left\{ \rho_k^\circ + F_\lambda(\underline{k}) \right\} \ln \left\{ \rho_k + F_\lambda(\underline{k}) \right\} \right] . \tag{112}$$

It is then readily shown that

$$\frac{dS}{dt} = -\kappa \sum_{\sigma} \int d^3v \ln \left\{ \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})} \right\} \dot{f}_\sigma$$

$$-\kappa \sum_{\lambda} \int dk\, d\Omega \ln \left\{ \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})} \right\} \dot{F}_\lambda . \tag{113}$$

Substituting (110) and (111) for the derivatives of the distributions in (113) and performing the usual manipulations on the variables of integration while noting the symmetries explicit in the scattering frequencies C and S_σ, one finds that

$$\frac{dS}{dt} = -K \sum_{\sigma\sigma'} \int d^3v\, d^3v_1\, d^3v_2\, d^3v_3 \; C(\sigma \underline{p}_1, \sigma' \underline{p}_2 ; \sigma \underline{p}, \sigma' \underline{p}_3)$$

$$\times \left[\rho_\sigma \pm f_\sigma(\underline{v})\right] \left[\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_3)\right] \left[\rho_\sigma \pm f_\sigma(\underline{v}_1)\right] \left[\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_2)\right]$$

$$\times \ln \left[\frac{f_\sigma(\underline{v})\, f_{\sigma'}(\underline{v}_3) \left\{\rho_\sigma \pm f_\sigma(\underline{v}_1)\right\} \left\{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_2)\right\}}{f_\sigma(\underline{v}_1)\, f_{\sigma'}(\underline{v}_2) \left\{\rho_\sigma \pm f_\sigma(\underline{v})\right\} \left\{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_3)\right\}} \right]$$

$$\times \left[\frac{f_\sigma(\underline{v}_1)}{\rho_\sigma \pm f_\sigma(\underline{v}_1)} \quad \frac{f_{\sigma'}(\underline{v}_2)}{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_2)} - \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})} \quad \frac{f_{\sigma'}(\underline{v}_3)}{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_3)} \right]$$

$$-K \sum_{\sigma\lambda\lambda'} \int d^3v\, d^3v_1\, dk\, d\Omega\, dk'\, d\Omega' \; S_\sigma(\underline{p}_1\, \lambda'\, \underline{k}' ; \underline{p}\, \lambda\, \underline{k})$$

(continued)

$$\times \left[\rho_\sigma \pm f_\sigma(\underline{v})\right] \left[\rho_k + F_\lambda(\underline{k})\right] \left[\rho_\sigma \pm f_\sigma(\underline{v}_1)\right] \left[\rho_{k'} + F_{\lambda'}(\underline{k}')\right]$$

$$\times \ln \left[\frac{f_\sigma(\underline{v})\, F_\lambda(\underline{k}) \left\{\rho_\sigma \pm f_\sigma(\underline{v}_1)\right\} \left\{\rho_{k'} + F_{\lambda'}(\underline{k}')\right\}}{f_\sigma(\underline{v}_1)\, F_{\lambda'}(\underline{k}') \left\{\rho_\sigma \pm f_\sigma(\underline{v})\right\} \left\{\rho_k + F_\lambda(\underline{k})\right\}} \right]$$

$$\times \left[\frac{f_\sigma(\underline{v}_1)}{\rho_\sigma \pm f_\sigma(\underline{v}_1)} \, \frac{F_{\lambda'}(\underline{k}')}{\rho_{k'} + F_{\lambda'}(\underline{k}')} - \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})} \, \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})} \right]$$

$$- K \sum_\lambda \int d^3v\, d^3v'\, dk\, d\Omega\, V\, T_K^{K'}(\lambda\, \underline{k})$$

$$\times \left[\rho_e - f_e(\underline{v})\right] \left[\rho_e - f_e(\underline{v}')\right] \left[\rho_k + F_\lambda(\underline{k})\right]$$

$$\times \ln \left[\frac{f_e(\underline{v}')\, F_\lambda(\underline{k}) \left\{\rho_e - f_e(\underline{v})\right\}}{\left\{\rho_e - f_e(\underline{v}')\right\} f_e(\underline{v}) \left\{\rho_k + F_\lambda(\underline{k})\right\}} \right]$$

$$\times \left[\frac{f_e(\underline{v})}{\rho_e - f_e(\underline{v})} - \frac{f_e(\underline{v}')}{\rho_e - f_e(\underline{v}')} \, \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})} \right] . \qquad (114)$$

Since $f_\sigma/\rho_\sigma \leq 1$ for all fermions (because the number of fermions in V must never exceed the number of available states) it is seen that all of the integrands in equation (114) are everywhere negative or zero. Thus it follows that

$$\frac{dS}{dt} \geqslant 0, \qquad (115)$$

and vanishes only when

$$\frac{f_\sigma(\underline{v}_1)}{\rho_\sigma \pm f_\sigma(\underline{v}_1)} \; \frac{f_{\sigma'}(\underline{v}_2)}{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_2)} = \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})} \; \frac{f_{\sigma'}(\underline{v}_3)}{\rho_{\sigma'} \pm f_{\sigma'}(\underline{v}_3)},$$

$$\frac{f_\sigma(\underline{v}_1)}{\rho_\sigma \pm f_\sigma(\underline{v}_1)} \; \frac{F_{\lambda'}(\underline{k}')}{\rho_{k'} + F_{\lambda'}(\underline{k}')} = \frac{f_\sigma(\underline{v})}{\rho_\sigma \pm f_\sigma(\underline{v})} \; \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})},$$

$$\frac{f_e(\underline{v})}{\rho_e - f_e(\underline{v})} = \frac{f_e(\underline{v}')}{\rho_e - f_e(\underline{v}')} \; \frac{F_\lambda(\underline{k})}{\rho_k + F_\lambda(\underline{k})}. \qquad (116)$$

By virtue of the conservation of energy implied in C, S, and T, it follows that all of the relations (116) imply that

$$f_\sigma(\underline{v}) = \frac{(m_\sigma/2\pi\hbar)^3}{B_\sigma e^{E(v)/\theta} \mp 1}, \qquad F_\lambda(\underline{k}) = \frac{k^2/(2\pi)^3}{e^{\mathcal{E}(k)/\theta} - 1} \qquad (117)$$

where $E(v) = mv^2/2$, $\mathcal{E}(k) = \hbar c k\left[1 + \omega_e^2/2c^2k^2\right]$ and B_σ is a normalization constant for the particle distributions to be determined by requirements of conservation of particles. The particle distributions are recognized to be those appropriate for Bosons or Fermions as the case may be, while that for the photons is a modified Planck distribution - the modification disappearing in the vacuum since there ω_e vanishes. The fact that S is an always increasing function of the time as the system changes state with time suggests its identification as the system entropy; and, hence, the further identification of the distributions (117) as those appropriate to the thermodynamic state. Of course, these identifications are not complete until the so far arbitrary constants $\mathcal{K}$ (equation (112)) and θ (equation (117)) have been specified. The fact that θ is the only parameter common to all of the distributions, and the fact that temperature is defined to be the property of systems in equilibrium which is the same for all systems, suggests that θ is a function of temperature. Appeal to experiment is required for the explicit identification, $\theta = \mathcal{K}\,T$.

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