2764-12-T

2764-12-T = RL-2080

A THEORY OF PHOTON TRANSPORT IN DISPERSIVE MEDIA

by

Edward H. Klevans

May 1962

This report is identical to a dissertation submitted
in partial fulfillment of the requirements for the
degree of Doctor of Philosophy in the University of Michigan, 1962

ARPA Order Nr. 120-61, Project Code Nr. 7400
Contract Nr. DA 36-039 SC-75041
Department of the Army Project Nr. 3A99-23-001-01

The work described in this report was partially supported by the
ADVANCED RESEARCH PROJECTS AGENCY
ARPA Order Nr. 120-61, Project Code Nr. 7400

Prepared For

The Advanced Research Projects Agency
and the
U. S. Army Signal Research and Development Laboratory
Ft. Monmouth, New Jersey

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Professor R. K. Osborn for his invaluable guidance and assistance throughout the course of this investigation. He also wishes to acknowledge the stimulating discussions of the problem with his colleague, Ercument Ozizmir, and the criticism of the presentation given by Professor William Kerr.

In addition, he wishes to acknowledge both the moral support and typing assistance given by his wife.

Part of this work was supported by the Advanced Research Projects Agency and the U. S. Army Signal Research and Development Laboratories under ARPA Order No. 120-61, Project Code No. 7400, Contract DA 36-039 SC-75041 and carried out at the Radiation Laboratory, Electrical Engineering Department, The University of Michigan. Financial support for the completion of the investigation was provided by Conductron Corporation, Ann Arbor, Michigan. The author is especially grateful to both of these organizations.

TABLE OF CONTENTS

TABLE OF CONTENTS
(Cont'd.)

LIST OF ILLUSTRATIONS

A THEORY OF PHOTON TRANSPORT IN DISPERSIVE MEDIA

Edward Harris Klevans

ABSTRACT

A systematic, self-contained development of an equation describing photon transport in a dispersive, nonrelativistic medium is given. The postulate-deduction procedure has been developed along lines similar to those used by Ono in his quantum-statistical theory of neutral gas transport phenomena. The photon balance equation is exhibited as a time rate of change of a coarse-grained photon distribution function. The photons described have an energy-momentum relation $\hbar\omega = \frac{\hbar c|\underline{k}|}{\mu}$ where μ is the index of refraction. Transport of these photons is characterized by a photon speed which is different from the speed of light. Transition probabilities for scattering, emission and absorption processes are computed to the lowest nonvanishing order.

The development of the transport equation is self-contained in the sense that the transverse dispersion relations for different systems of interest are determined within the context of the analysis. With respect to these calculations, a method employed by Mead in his discussion of the neutral gas has been employed. Whenever comparison is possible the dispersion relations are found to be in agreement with those computed by other methods. Except for propagation parallel to the magnetic field, the method is not applicable to the plasma in an external magnetic field.

Radiation from a plane plasma layer is studied and reasonably good agreement with experiment is obtained, although the radiative transfer equation for the dispersive medium is not found to be in agreement with the more conventional equation which is developed by phenomenological balance considerations.

Emission and absorption coefficients are modified as a result of the dispersive medium. For bremsstrahlung in a fully ionized gas, for instance, as the photon frequency approaches the plasma frequency the emission is reduced and absorption enhanced. The possibility of Cerenkov radiation for a plasma in an external magnetic field is also indicated.

Photon transport through an Einstein model crystal is considered. The scattering cross section developed in x-ray scattering theory is obtained. In the limit of the infinite homogeneous crystal, the Bragg scattering condition emerges.

The transport equation developed here is expected to be valid when (a) the medium is isotropic; (b) spatial variation of particle and photon distributions is slow over regions characterized by a length many times larger than the maximum wavelength of photons under consideration; (c) the frequency should be sufficiently above resonance regions (or the plasma frequency for the fully ionized gas) that radiation damping effects can be ignored. These conditions are only intended to be sufficient for the validity of the equation, but may not be necessary.

I. INTRODUCTION

The problem of radiant energy transport is of fundamental importance in the study of astrophysics, in the shielding of a nuclear reactor, and in attempting to maintain thermonuclear reactions in a controlled nuclear fusion device. Gamma ray transport associated with fission reactors is discussed by Goldstein (1). The photons under consideration are very high energy, (>0.5 Mev) and the most important interaction processes are pair production, Compton scattering and the photo-electric ionization. Furthermore, the photons are far from equilibrium.

For radiation transport in stars and laboratory plasmas, we are usually concerned with low energy systems. Pair creation and annihilation are negligible and scattering is characterized by the Thomson cross section rather than the Compton cross section. For stellar systems the Thomson scattering may be important, but for laboratory plasmas it is usually negligible because of the small cross section. Depending upon the mean energy of the system, bremsstrahlung (and its inverse), ionization and recombination, and excitation and de-excitation can be important radiation processes. If a magnetic field is present, cyclotron radiation may also be important.

We will be concerned in this thesis with a description of low energy photon systems. The equation of radiative transfer is usually developed by phenomenological consideration. In a medium in which the index refraction is unity the equation is written as (2, 3)

$$\frac{dI_\lambda}{ds} = S_\lambda - \alpha_\lambda I \tag{I-1}$$

where the intensity of radiation $I_\lambda(\underline{x}, \omega, \underline{\Omega}, t)$ is defined so that $I_\lambda dS d\Omega dt$ is the energy of polarization λ passing through a surface dS located at the position $\underline{x}$ going in the direction $\underline{\Omega}$ in solid angle $d\Omega$ and with $\underline{\Omega}$ oriented normal to the surface dS of frequency ω in $d\omega$ in the time interval dt. The absorption coefficient $\alpha_\lambda(\underline{x}, \omega, \underline{\Omega})$ is the probability per unit path traveled that energy of polarization λ radiated with frequency ω and going in direction $\underline{\Omega}$ is absorbed, and the source term S_λ is defined so that $S_\lambda(\underline{x}, \omega, \underline{\Omega})\, d^3x\, d\omega\, d\Omega\, dt$ is the energy of polarization λ radiated in time dt in the volume d^3x about $\underline{x}$ with frequency ω in $d\omega$ going in direction $\underline{\Omega}$ in $d\Omega$. The operation $\frac{d}{ds}$ means differentiation along the path, i.e.

$$\frac{d}{ds} = \frac{dx_j}{ds} \frac{\partial}{\partial x_j} = \Omega_j \frac{\partial}{\partial x_j} . \tag{I-2}$$

When the medium is characterized by a spatially varying index of refraction $\mu(\underline{x})$, equation (I-1) is modified. According to the derivation of Woolley (4), one obtains

$$\frac{d}{ds}\left(\frac{I_\omega}{\mu_\omega^2}\right) = \frac{S_\omega}{\mu_\omega^2} - \alpha_\omega \frac{I_\omega}{\mu_\omega^2} . \tag{I-3}$$

As seen from equation (I-3), we do not have the simple conservation equation given by equation (I-1). The intensity is changing from point to point, not only from creation and destruction, but from a convergence or divergence of the pencils of rays resulting from a change in the properties of the medium.

Turning again to equation (I-1), we observe that in equilibrium, $S_\omega = \alpha_\omega B_\omega(T)$, where $B_\omega(T)$ is the intensity of black body radiation of frequency ω of temperature T. It will be shown later that $B_\omega(T)$ is the Planck black body distribution function. The relation

$$\frac{S_\omega}{\alpha_\omega} = B_\omega(T)$$

is known as Kirchoff's law. When $\mu_\omega \neq 1$, we obtain

$$\frac{S_\omega}{\alpha_\omega} = \mu_\omega^2 B_\omega . \qquad \text{(I-4)}$$

We can obtain greater insight into the emission and absorption terms in equation (I-1) by considering the microscopic processes. There are two types of source terms, spontaneous emission and induced emission, the latter being proportional to the intensity of the radiation present.

Consider a simple two level system, with state 1 as the normal state and state 2 as an excited state (2, 3). The radiative transfer equation can be derived by use of the Einstein coefficients B_{12}, A_{21} and B_{21}, where $B_{12} I_\nu$ is the probability per unit time that an atom exposed to isotropic radiation of intensity $I_\nu \, d\nu$ will absorb the quantum $h\nu$ making a transition to the state 2; A_{21} is the probability per unit time that an atom will decay by spontaneous emission from the state 2 to the state 1 and emit a photon of energy $h\nu$; and $B_{21} I_\nu$ is the probability per unit time that an atom, exposed to isotropic radiation I_ν will make a transition from the state 2 to the state 1. When q_1 and q_2 are the statistical weights of the states 1 and 2, the Einstein coefficients are related by

$$\frac{A_{21}}{B_{12}} = \frac{2h\nu^3}{c^2} \frac{q_1}{q_2} \; ; \; \frac{B_{21}}{B_{12}} = \frac{q_1}{q_2} \tag{I-5}$$

where c is the speed of light. The coefficient B_{12} is related to the atomic absorption coefficient. Let $n_1(\nu)\, d\nu$ be the number of atoms in the state n_1 capable of absorbing radiation of frequency ν in $d\nu$. The number of such absorptions per second and per unit volume is $n_1(\nu)\, d\nu\, B_{12}\, I_\nu$. The incident isotropic radiation is given by $4\pi\, I_\nu\, d\nu$. Thus, the absorption coefficient per atom $\sigma(\nu)$ is

$$\sigma(\nu) = \frac{n_1(\nu)\, d\nu\, B_{12}\, I_\nu\, h\nu}{4\pi\, I_\nu\, d\nu\, n_1} = \frac{B_{12}\, h\nu\, n_1(\nu)}{4\pi\, n_1} .$$

Recalling that $\int n_1(\nu)\, d\nu = n_1$, and assuming $n_1(\nu)$ is small except in the vicinity of ν_o,

$$\int \sigma(\nu)\, d\nu \simeq \sigma(\nu_o)\, \Delta\nu_o = \frac{B_{12}\, h\nu_o}{4\pi} . \tag{I-6}$$

We now consider $\frac{I_\nu \Delta\nu\, d\Omega\, dS}{h\nu}$ photons impinging upon a surface dS of a slab of thickness ds. Then, counting gains and losses

$$\frac{dI_\nu}{ds} \Delta\nu = \frac{h\nu}{4\pi} \left[n_2 (A_{21} + I_\nu B_{21}) - n_1 B_{12} I_\nu \right] \tag{I-7}$$

for isotropic emission and absorption. From the relations between the Einstein coefficients, we find

$$\frac{dI_\nu}{ds} = \frac{q_1}{q_2} n_2 \frac{2h\nu^3}{c^2} \sigma_\nu - (n_1 - \frac{q_1}{q_2} n_2) \sigma_\nu I_\nu . \quad \text{(I-8)}$$

Clearly, $S = \frac{q_1}{q_2} n_2 \frac{2h\nu^3}{c^2} \sigma_\nu$ accounts for spontaneous emission and $\alpha_\nu = (n_1 - \frac{q_1}{q_2} n_2) \sigma_\nu$ is the effective absorption coefficient (absorption minus induced emission). This result can be rewritten as

$$\frac{dI_\nu}{ds} = - \alpha_\nu \left[I_\nu - B_\nu \right] \quad \text{(I-9)}$$

where

$$B_\nu = \frac{\frac{q_1}{q_2} n_2 \frac{2h\nu^2}{c^2}}{n_1 - \frac{q_1}{q_2} n_2} . \quad \text{(I-10)}$$

If the particles are in equilibrium, $\frac{n_2}{n_1} = \frac{q_2}{q_1} e^{-\frac{h\nu}{kT}}$ and equation (I-10) reduces to

$$\bar{B}_\nu (T) = 2 \frac{h\nu^3}{c^2} / (e^{h\nu/kT} - 1) \quad \text{(I-11)}$$

which is the Planck distribution. We see then that in kinetic equilibrium, where the particles are in equilibrium although the radiation isn't,

$$S_\nu = \alpha_\nu \bar{B}_\nu \tag{I-12}$$

and

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + \alpha_\nu \bar{B}_\nu \, . \tag{I-13}$$

This is the equation for local thermodynamic equilibrium used in astrophysics. When we are no longer in local thermodynamic equilibrium it is seen from equation (I-10) that we need to know the distribution of n_1 and n_2.

With this derivation we have progressed to a higher level of sophistication and insight than was achieved previously. Nonetheless, there are still some very important questions which this analysis does not answer. First, there is no obvious way to generalize this procedure to include an index of refraction, and second, the derivation does not exhibit in a clear cut manner, the approximations and limitations involved.

To pursue the implications of the last comment, consider the analogous equation in neutral particle gas transport, the Boltzmann equation. This equation was obtained by Boltzmann by phenomenological balance arguments similar to those used to obtain equation (I-1). For instance, the Boltzmann equation does not reveal the relationship of the one-particle distribution to the two-particle or three-particle distribution functions. It does not indicate what, if any, modifications to the collision cross section might be necessary for different types of systems, e.g. the plasma. Also, we note that the Boltzmann equation is irreversible, i.e. it is not variant under the transformation $t \longrightarrow -t$, $\underline{v} \longrightarrow -\underline{v}$ whereas the

Liouville equation, the fundamental equation of statistical mechanics, is invariant under this transformation. The relationship of these equations is not revealed by Boltzmann's phenomenological derivation. A considerable effort (5, 6, 7, 8) has been expended in the past twenty years in an attempt to achieve greater insight into the Boltzmann equation and to understand some of its limitations.

A derivation of a radiation transport equation from fundamental considerations was made recently by Osborn and Klevans (9), who introduced a photon distribution function in analogy with the quantum mechanical Wigner distribution function for particles. A photon transport equation was developed for nonrelativistic systems and this equation was reduced to the conventional radiative transfer equation (I-1). A set of conditions was specified within which this equation would be expected to be valid. Further discussion of these points will be given in Chapter II.

The extension of these fundamental considerations to dispersive media was begun in Section V of reference (9), where it was shown that certain aspects of collective particle behavior could be introduced quite naturally into the description of photon transport in a homogeneous, fully ionized gas. We present in this thesis a further extension and reformulation of reference (9) so that greater detail of collective particle phenomena can be incorporated into the analysis from the beginning. A general framework is thereby provided within which we can discuss modifications of emission and absorption coefficients, as well as of transport of photons in dispersive media.

In reference (9), the photon balance equation was developed for photons with an energy-momentum relation $\hbar\omega = \hbar ck$. This is equivalent to saying that the dispersion relation for transverse electromagnetic waves is given by $\omega(k) = ck$ or that the index of refraction of the medium μ is given by $\mu \equiv \frac{ck}{\omega(k)} = 1$. For the dispersive medium we can write $\omega = \frac{ck}{\mu_\omega}$. In Chapter II we discuss photons which have an energy-momentum relation $\hbar \omega_\lambda = \frac{\hbar c k}{\mu_\omega}$ where the subscript λ specifies the polarization. These photons are referred to (10, 11) as "dressed" photons, and they contain the effect of the collective particle phenomena. They propagate with a speed v^γ which is different from c. A systematic, self-contained development of a transport equation for these photons is presented in Chapter II. The procedure is deductive, i.e. it is based upon clearly stated postulates. Nonetheless, it will be necessary to introduce many approximations in order to obtain the photon balance equation. These approximations will be made without exploring the corrections although it would be possible, in principle, to do so. Particular attention is given to the transport term. It is shown that, for a medium which is inhomogeneous, i.e. one which has spatial variation in the particle density or temperature, the procedure used in this chapter leads to an ambiguous result. The various interaction processes - emission, absorption and scattering - are computed to a convenient low order.

In Chapter III, the dispersion relations for various systems are computed by the method suggested in Chapter II. For the fully ionized gas with no external fields, we find the transverse dispersion relation previously obtained by Bohm and Pines (12). For the fully ionized plasma in a constant, uniform magnetic field, we obtain, for propagation along the magnetic field, two circularly polarized waves, the ordinary and extraordinary waves. The zero temperature limit is in agreement with Spitzer (13), and our first order temperature correction has been given previously by Pradhan (14). For propagation perpendicular to the magnetic field, we have a wave polarized parallel to the magnetic field and one polarized perpendicular to the magnetic field. For the former wave, there is no coupling with the longitudinal mode and our results are in agreement with those obtained by the procedure employed by Bernstein (15). For the other wave, however, there is coupling between the longitudinal and transverse modes and our analysis is applicable only at frequencies sufficiently high that decoupling results. Lastly, we apply our technique to a neutral gas of one species and obtain the Kramers-Heisenberg dispersion formula (10).

Because of the ambiguities which arose in the transport term in Chapter II, a new derivation of the photon balance equation is given in Chapter IV. The transport term is developed unambiguously, although it involves certain approximations. The photon speed is found to be different than the photon speed obtained in Chapter II. However, for the frequency region where the equation is expected to be valid, the two photon speeds are essentially the same. Additional terms which did not appear in the earlier derivation are present. They account for processes for which both emission and transport are involved. A condition is specified which enables us to ignore these terms.

In Chapter V a brief discussion of the thermodynamics of particles and dressed photons is presented. An H-theorem is indicated, and a black body energy density spectrum in the dispersive medium is obtained. This spectrum, which differs from the Planck distribution by a factor $\mu_\lambda^2 \, c/v^\gamma$ where $v^\gamma = \frac{\partial\omega}{\partial k}$ is in agreement with the result presented by Landau and Lifshitz (16).

In Chapter VI, the photon balance relation is reduced to a radiative transfer equation which is a balance equation for the radiation intensity. This equation differs from equation (I-3), although it is difficult to evaluate the differences directly from the equations. In order to compare them, the radiation from a plane slab of plasma is computed from both equations. After the solution of equation (I-3) is modified to account for internal reflection, the solution of our equation is in qualitative agreement with that found from the phenomenological equation. Both solutions give reasonably good agreement with the experimental results of Bekefi and Brown (17).

In Chapter VII, modifications to absorption and emission terms as a result of dispersion are indicated. In the fully ionized gas, for instance, emission of bremsstrahlung is strongly reduced as we approach the plasma frequency, whereas absorption is strongly enhanced.

It is also shown in Chapter VII that Cerenkov radiation is possible for the plasma in a magnetic field. The calculation is severely limited, however, because the anisotropic nature of the medium is not considered.

The problem of photon transport in an Einstein model crystal is discussed in Chapter VIII. The cross section for x-ray scattering is obtained and the Debye-Waller factor emerges naturally from the analysis. In the limit of the infinite, homogeneous crystal, the Bragg scattering condition will be found. An analogy with the low energy neutron scattering in a moderator is suggested.

Finally, in Chapter IX, we present the conclusions of our analysis.

It should be emphasized that a primary goal of this thesis is the delineation of difficulties inherent in the development of a momentum-configuration space transport equation for photons in a nonrelativistic, dispersive medium. Questions will be raised which will not be answered and approximations will be invoked without investigation of corrections. But by pinpointing the difficult and obscure aspects of the development of such a theory, we hope to focus attention upon areas in need of further investigation.

II. DEVELOPMENT OF THE PHOTON BALANCE RELATION

The deductive development of the photon transport equation presented in this chapter is based upon two types of fundamental postulates, the dynamical postulate and the statistical postulate.

The dynamical postulate characterizes the dynamics of the interactions between the particles and between particles and photons. It consists of specification of the hamiltonian of the system, and the Schroedinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H \Psi \qquad \text{(II-1)}$$

for the wave function Ψ which specifies the state of the system. For the non-relativistic systems of interest this postulate can be considered to be on a reasonably firm foundation.

The statistical postulate is introduced for the purpose of describing the behavior of a system with a large number of degrees of freedom. It is presented in a formal manner and consists of specifying a distribution function for photons. This specification is not unique and we do not wish to claim that the particular distribution function chosen in this chapter provides one with the best method of development. It does appear to be quite adequate for systems homogeneous in the particle distribution, although for inhomogeneous systems, an ambiguity arises in the study of the transport term. For this reason an alternative derivation of the transport equation based upon a different photon distribution function will be presented in Chapter IV.

Both derivations are being presented because we feel that the development of a photon balance equation for a dispersive medium from a postulate-deduction approach is quite complicated. At this stage of the development it seems desirable to look at it from various viewpoints. Each derivation presented here illuminates different aspects of the problem and each has different shortcomings.

The Hamiltonian for a system of nonrelativistic charged particles interacting with a radiation field can be written as[18]

$$H = T^{\gamma} + T^{P} + H^{Pe} + V + H^{P\gamma 1} + H^{P\gamma e} + H^{P\gamma 2} \qquad \text{(II-2)}$$

where

$$T^{\gamma} = \int d^3x \left[2\pi c^2 P^2 + \frac{1}{8\pi} (\underline{\nabla} \times \underline{A})^2 \right] \qquad \text{(II-3a)}$$

$$T^{P} = \sum_{\sigma} \frac{p^{\sigma 2}}{2\, m_{\sigma}} = - \sum_{\sigma} \frac{\hbar^2}{2\, m_{\sigma}} \nabla_{\sigma}^2 \qquad \text{(II-3b)}$$

$$V = \sum_{\sigma, \sigma'} \frac{e_{\sigma} e_{\sigma'}}{\left| \underline{x}^{\sigma} - \underline{x}^{\sigma'} \right|} \qquad \text{(II-3c)}$$

$$H^{Pe} = \sum_{\sigma} \left[\frac{i\hbar e_{\sigma}}{m_{\sigma} c} \underline{A}^{e} \cdot \underline{\nabla}_{\sigma} + \frac{e_{\sigma}^2}{2 m_{\sigma} c^2} \underline{A}^{e\,2} \right] \qquad \text{(II-3d)}$$

$$H^{P\gamma 1} = \sum_{\sigma} \left[\frac{i\hbar e_{\sigma}}{m_{\sigma} c} \underline{A}\,(\underline{x}^{\sigma}) \cdot \underline{\nabla}_{\sigma} \right] \qquad \text{(II-3e)}$$

$$H^{P\gamma e} = \sum_{\sigma} \frac{e_{\sigma}^2}{2 m_{\sigma} c^2} \underline{A}^{e} \cdot \underline{A}\,(\underline{x}^{\sigma}) \qquad \text{(II-3f)}$$

$$H^{P\gamma 2} = \sum_{\sigma} \frac{e_\sigma^2}{2m_\sigma c^2} \underline{A}^2 . \qquad \text{(II-3g)}$$

In equations (II-3), $\underline{A}^e$ is the divergenceless vector potential for the external magnetic field and $\underline{A}$ and $\underline{P}$ are canonically conjugate transverse wave operators for the photon field. They satisfy the commutation relations

$$\left[A_j(\underline{x}), P_\ell(\underline{x}')\right] = A_j(\underline{x}) P_\ell(\underline{x}') - P_\ell(\underline{x}') A_j(\underline{x})$$

$$= i\hbar\, \delta_{j\ell}\, \delta(\underline{x} - \underline{x}') - i\hbar \frac{\partial}{\partial x_j} \frac{\partial}{\partial x'_\ell} \left(\frac{1}{4\pi |\underline{x} - \underline{x}'|} \right) . \qquad \text{(II-4)}$$

The index σ refers to the σth particle, and the sum over σ and σ' is to be interpreted in the following sense: If σ and σ' are the same kind of particle, take $\sigma' < \sigma$; if they are different kinds of particles, the sums are performed without restrictions.

The various terms in equation (II-2) can be interpreted as follows: T^γ represents the energy of the photon (or electromagnetic) field; T^P specifies the kinetic energy of the particles; H^{Pe} is the interaction energy for particles with the external magnetic field; V is the Coulomb interaction between particles; $H^{P\gamma 1}$ and $H^{P\gamma 2}$ are particle-photon interactions, with $H^{P\gamma 1}$ representing one photon processes - emission and absorption - and $H^{P\gamma 2}$ being responsible for photon scattering from particles; and $H^{P\gamma e}$ is the interaction energy for particles, photons and the external magnetic field.

The particles in the medium generate fields covering a wide spectrum of possible frequencies. Following Osborn[19], we separate the vector potential into two parts A^S and A^F, where the former is associated with the "slowly varying" part of the field and the latter describes a "rapidly varying" part. The "slowly varying" field is of importance in developing a particle transport equation, and Maxwell's equations are used in the discussion of this part of the field (19). The "rapidly varying" field is employed in the study of photons.

It is convenient for the development of a photon balance relation to regroup the terms in the Hamiltonian. This is accomplished by selecting from the system a cubical subvolume $V(\underline{x})$, located about the point $\underline{x}$ with respect to some coordinate system, whose size will be unspecified for the moment. Then

$$H = H_c + H_R + H_{cR} \qquad \text{(II-5)}$$

where H_c represents the energy of the particles and photons in the cell, H_R represents the energy of the system outside the cell, and H_{cR} is the interaction energy.

The "rapidly varying" field in the cell constitutes the photons within the cell. The "slowly varying" transverse field within the cell is treated as an additional external field within which the particles move. Interactions between particles in different cells via long range Coulomb forces constitute a small effect upon photon-particle interactions within the cell and we may expect that neglect of this effect produces negligible error. Coulomb interactions within the cell are retained, however, and will be of importance in the discussion of bremsstrahlung.

Within the cell we expand the transverse wave operators into a fourier series with coefficients which are creation and destruction operators, i.e.

$$\underline{A}^F = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_{\lambda \underline{k}} \frac{e^{-i\underline{k}\cdot\underline{x}}}{\sqrt{\omega_{\lambda k}}} \underline{\xi}_\lambda^+ (\underline{k}) \qquad \text{(II-6a)}$$

$$\underline{P}(\underline{x}) = i\sqrt{\frac{\hbar}{8\pi c^2 V}} \sum_{\lambda \underline{k}} \sqrt{\omega_{\lambda k}}\, e^{-i\underline{k}\cdot\underline{x}} \underline{\xi}_\lambda^- (\underline{k}) \qquad \text{(II-6b)}$$

where

$$\underline{\xi}_\lambda^{\pm}(\underline{k}) = \underline{\mathcal{E}}_\lambda(\underline{k})\, \alpha_\lambda^+(\underline{k}) \pm \underline{\mathcal{E}}_\lambda(-\underline{k})\, \alpha_\lambda(-\underline{k}) .$$

The operators $\alpha_\lambda^+(\underline{k})$ and $\alpha_\lambda(\underline{k})$ are creation and destruction operators for photons of momentum $\hbar\underline{k}$ and polarization λ $(\lambda = 1, 2)$, and $\underline{\mathcal{E}}_\lambda(\underline{k})$ is a unit polarization vector of the photon field. The creation and destruction operators satisfy the commutation relations

$$\left[\alpha_\lambda(\underline{k}), \alpha_{\lambda'}^+(\underline{k}')\right] = \delta_{\lambda\lambda'}\, \delta(\underline{k} - \underline{k}') . \qquad \text{(II-7)}$$

(We will not be particularly careful to make a distinction between Dirac and Kronecker deltas, letting the context reveal which is appropriate.) The volume V in equations (II-6) is the quantization volume and is presumed the same as the cell volume.

The use of an arbitrary oscillation frequency in the expansions of $\underline{A}$ and $\underline{P}$ is somewhat unconventional. It was first used by Bohm and Pines[12] in their discussion of collective transverse oscillations in an electron gas. The condition for specifying $\omega_{\lambda k}$ as a function of k will appear later and will yield the desired dispersion relations. The creation and destruction operators create and destroy photons of frequency $\omega_{\lambda k}$. They are different from the free space operators, although they become the same for $\omega = ck$. If $\bar{\alpha}_\lambda^+$ and $\bar{\alpha}_\lambda$ are the creation and destruction operators in free space, Mead[10] showed that

$$\alpha_\lambda(\underline{k}) = \frac{1}{2}\left(\frac{\omega_\lambda}{ck}\right)^{1/2}\left[\left(1+\frac{ck}{\omega_\lambda}\right)\bar{\alpha}_\lambda(\underline{k}) + \left(1-\frac{ck}{\omega_\lambda}\right)\bar{\alpha}_\lambda^+(-\underline{k})\right] \quad \text{(II-8a)}$$

$$\alpha_\lambda^+(\underline{k}) = \frac{1}{2}\left(\frac{\omega_\lambda}{ck}\right)^{1/2}\left[\left(1-\frac{ck}{\omega_\lambda}\right)\bar{\alpha}_\lambda(-\underline{k}) + \left(1+\frac{ck}{\omega_\lambda}\right)\bar{\alpha}_\lambda^+(\underline{k})\right] \quad \text{(II-8b)}$$

$$\bar{\alpha}_\lambda(\underline{k}) = \frac{1}{2(\omega_\lambda ck)^{1/2}}\left[(ck+\omega_\lambda)^{1/2}\alpha_\lambda(\underline{k}) + (ck-\omega_\lambda)^{1/2}\alpha_\lambda^+(-\underline{k})\right] \quad \text{(II-8c)}$$

$$\bar{\alpha}_\lambda^+(\underline{k}) = \frac{1}{2(\omega_\lambda ck)^{1/2}}\left[(ck+\omega_\lambda)^{1/2}\alpha_\lambda^+(\underline{k}) + (ck-\omega_\lambda)^{1/2}\alpha_\lambda(-\underline{k})\right]. \quad \text{(II-8d)}$$

The Hamiltonian within the cell can be written as

$$H_c = H^o + H^{(1)} + H^{(2)} = H^o + H^I \quad \text{(II-9a)}$$

where

$$H^o = T^{\gamma o} + H^M \quad \text{(II-9b)}$$

$$H^{M} = (T^{P} + H^{Pe}) + V \qquad \text{(II-9c)}$$

$$H^{(1)} = H^{P\gamma 1} + H^{P\gamma e} \qquad \text{(II-9d)}$$

$$H^{(2)} = (H_{o}^{P\gamma 2} + T^{\gamma 1}) + (H_{1}^{P\gamma 2} + T^{\gamma 2}) + H_{2}^{P\gamma 2} \qquad \text{(II-9e)}$$

$$T^{\gamma o} = \sum_{\lambda \underline{k}} \hbar \omega_{\lambda k} \, \alpha_{\lambda}^{+}(\underline{k}) \, \alpha_{\lambda}(\underline{k}) \qquad \text{(II-10a)}$$

$$H^{M} = \sum_{\sigma} \frac{1}{2m_{\sigma}} \underline{\Pi}^{\sigma} \cdot \underline{\Pi}^{\sigma} + \sum_{\sigma, \sigma'} \frac{e_{\sigma} e_{\sigma'}}{\left| \underline{x}^{\sigma} - \underline{x}^{\sigma'} \right|} S(\underline{x}^{\sigma}, \underline{x}^{\sigma'}) \qquad \text{(II-10b)}$$

$$H^{(1)} = \sum_{\sigma, \lambda \underline{k}} \frac{e_{\sigma}}{m_{\sigma}} \sqrt{\frac{2\pi\hbar}{V}} \frac{e^{-i\underline{k} \cdot \underline{x}^{\sigma}}}{\sqrt{\omega_{\lambda k}}} \, \underline{\xi}_{\lambda}^{+}(\underline{k}) \cdot \underline{\Pi}^{\sigma} \qquad \text{(II-10c)}$$

$$H_{o}^{P\gamma 2} + T^{\gamma 1} = \frac{\hbar}{2} \sum_{\lambda \underline{k}} \left[\sum_{\sigma} \frac{4\pi e_{\sigma}^{2}}{m_{\sigma} \omega_{\lambda k} V} + \frac{c^{2} k^{2} - \omega_{\lambda k}^{2}}{\omega_{\lambda k}} \right] \alpha_{\lambda}^{+}(\underline{k}) \, \alpha_{\lambda}(\underline{k}) \qquad \text{(II-10d)}$$

$$H_{1}^{P\gamma 2} + T^{\gamma 2} = \frac{\hbar}{4} \sum_{\lambda \underline{k}} \left[\sum_{\sigma} \frac{4\pi e_{\sigma}^{2}}{m_{\sigma} \omega_{\lambda k} V} + \frac{c^{2} k^{2} - \omega_{\lambda k}^{2}}{\omega_{\lambda k}} \right] \left[\alpha_{\lambda}^{+}(\underline{k}) \, \alpha_{\lambda}^{+}(-\underline{k}) + \alpha_{\lambda}(\underline{k}) \, \alpha_{\lambda}(-\underline{k}) \right] \qquad \text{(II-10e)}$$

$$H^{P\gamma 2} = \sum_{\substack{\sigma\lambda\underline{k} \\ \lambda'\underline{k}'}}{}' \frac{\pi\hbar e_\sigma^2}{m_\sigma V} \frac{e^{-i(\underline{k}-\underline{k}')\cdot\underline{x}^\sigma}}{(\omega_{\lambda k}\,\omega_{\lambda' k'})^{1/2}}$$

$$(x)\quad \underline{\zeta}_\lambda^+(\underline{k}) \cdot \underline{\zeta}_{\lambda'}^+(-\underline{k}') \qquad \text{(II-10f)}$$

where $\underline{\Pi}^\sigma = -i\hbar\,\underline{\nabla}_\sigma - \frac{e_\sigma}{c}\underline{A}^e - \frac{e_\sigma}{c}\underline{A}^S$

and $S(\underline{x}^\sigma, \underline{x}^{\sigma'}) = 1$ if both $\underline{x}^\sigma$ and $\underline{x}^{\sigma'}$ are in the box V, and $S(\underline{x}^\sigma, \underline{x}^{\sigma'}) = 0$ if they are not. The sum over σ is now restricted to particles within the box, but the conditions on the double sum over σ and σ' still apply.

We introduce the statistical axiom by defining the singlet photon distribution function

$$G_\lambda(\underline{x}, \underline{k}, t) = \frac{8}{V}\sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}}\,(\Psi, \rho_\lambda(\underline{k}, \underline{q})\Psi) \qquad \text{(II-11)}$$

where

$$\rho_\lambda(\underline{k}, \underline{q}) = \alpha_\lambda^+(\underline{k}+\underline{q})\,\alpha_\lambda(\underline{k}-\underline{q}) .$$

The function $G_\lambda(\underline{x}, \underline{k}, t)$, known as the fine-grained distribution function, can be interpreted, at least partially, through two of its properties. First,

$$\int_{V(\underline{x})} d^3x'\, G_\lambda(\underline{x}', \underline{k}, t) = (\Psi, \alpha_\lambda^+(\underline{k})\,\alpha_\lambda(\underline{k})\Psi) \qquad \text{(II-12)}$$

represents the expected number of photons of momentum $\hbar \underline{k}$ and polarization λ in the cell; and second,

$$\int_{V(\underline{x})} d^3x' \sum_{\lambda \underline{k}} G(\underline{x}', \underline{k}, t)\, \hbar \omega_{\lambda k}$$

$$= \sum_{\lambda \underline{k}} \hbar \omega_{\lambda k} \left(\Psi, \alpha_\lambda^+(\underline{k})\, \alpha_\lambda(\underline{k}) \Psi\right) \qquad \text{(II-13)}$$

can be interpreted as the total photon energy in the cell, assuming $\hbar \omega_{\lambda k}$ is the energy of a photon in the medium. The sense in which this assumption is valid is shown later in this section.

A quantity of more direct physical interpretation is the coarse-grained distribution function

$$F_\lambda(\underline{x}, \underline{k}, t) = \frac{1}{V} \int_{V(\underline{x})} d^3x' \; G(\underline{x}', \underline{k}, t) \qquad \text{(II-14)}$$

which is the expected number of photons per unit volume of momentum $\hbar \underline{k}$ and polarization λ located in the volume $V(\underline{x})$ about the point $\underline{x}$.

The equation of motion for $G_\lambda(\underline{x}, \underline{k}, t)$ can be written

$$\dot{G}_\lambda(\underline{x}, \underline{k}, t) - \frac{8}{V} \sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}} \left(\frac{i}{\hbar}\right) \left(\Psi, \left[H^0, \rho_\lambda(\underline{k}, \underline{q})\right] \Psi\right)$$

$$= \frac{8}{V} \sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}} \left(\frac{i}{\hbar}\right) \left(\Psi, \left[H^I, \rho_\lambda(\underline{k}, \underline{q})\right] \Psi\right) . \qquad \text{(II-15)}$$

The second term on the left hand side represents photon transport and will be designated $\dot{G}^T$. By straightforward manipulation, we find

$$\dot{G}_\lambda^T = \frac{8}{V} \sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}} \; \underline{\Omega} \cdot {}_{\underline{x}}\overleftarrow{\nabla} \left(\Psi, \frac{\partial\omega_{\lambda k}}{\partial k} \rho_\lambda(\underline{k}, \underline{q})\Psi\right)$$

$$+ \left\{ \text{Terms of } \mathcal{O}(\hbar^2) \right\}. \qquad \text{(II-16)}$$

Assuming that terms of $\mathcal{O}(\hbar^2)$ give negligible contribution to transport, and writing

$$\left(\Psi, \frac{\partial\omega_{\lambda k}}{\partial k} \rho_\lambda(\underline{k}, \underline{q})\Psi\right)$$

$$\simeq \left[\frac{\partial}{\partial k} (\Psi, \omega_{\lambda k}\Psi)\right](\Psi, \rho_\lambda(\underline{k}, \underline{q})\Psi)$$

$$= v^\gamma (\Psi, \rho_\lambda(\underline{k}, \underline{q})\Psi),$$

equation (II-16) becomes

$$\dot{G}_\lambda^T \simeq v^\gamma \underline{\Omega} \cdot \underline{\nabla}\, G_\lambda \qquad \text{(II-17)}$$

where $v^\gamma = \frac{\partial \bar{\omega}_{\lambda k}}{\partial k}$. From equation (II-16) it is seen that v^γ in equation (II-17) does not appear to the right of the divergence. However, from the expansions of $\underline{A}$ and $\underline{P}$ - equations (II-6) - it is clear that $\omega_{\lambda k}$ cannot be a function of $\underline{x}$ within the cell. Since the development of this term was carried out within the cell, it is equally valid to write

$$\dot{G}_\lambda^T = \underline{\Omega} \cdot \underline{\nabla} v^\gamma G_\lambda .$$

Of course, this does not prohibit either ω or v^γ from varying from cell to cell.

Integrating $\dot{G}_\lambda^T$ over the cell, we find, for the coarse-grained transport term

$$\dot{F}_\lambda^T = \frac{1}{V} \int_{V(\underline{x})} d^3x' \; \dot{G}_\lambda^T (\underline{x}')$$

$$= v^\gamma \underline{\Omega} \cdot \frac{1}{V} \int d^3x' \; \underline{\nabla} G_\lambda (\underline{x}') \qquad \text{(II-18)}$$

where we have again observed that within the cell the photon speed is constant.

The difficulty we face is the interpretation of equation (II-18) in terms of the quantity $\underline{\nabla} F$ which is defined, in the one dimensional model shown in Fig. 1 as

$$\frac{d F (X_1)}{dx} \equiv \frac{F (X_2) - F (X_1)}{L} . \qquad \text{(II-19)}$$

Observe that for the inhomogeneous medium, i.e. a medium in which v^γ and ω change from cell to cell, the definition (II-19) clearly distinguishes between the quantities $\frac{d v^\gamma F}{dx}$ and $v^\gamma \frac{dF}{dx}$. This distinction is not possible from equation (II-18), which is defined within the cell. It is because of this that the relation of equation (II-18) to a transport term expressed in terms of F is ambiguous in the inhomogeneous medium.

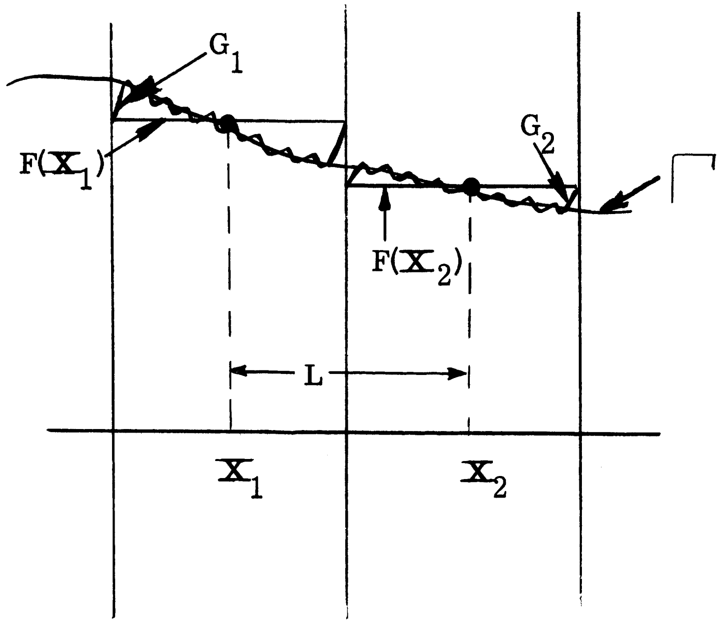

Figure 1. Two Cells of a One-Dimensional Model of The System

To pursue this point, let us again consider the one-dimensional model of of our system. (See Fig. 1). The quantity Γ represents the fine-grained distribution function for the whole system and G_1 and G_2 are the fine-grained distribution functions of the different cells. They are Fourier representations of Γ within each cell and are expected to represent Γ quite well, except near the boundaries of the cells. The bars across the cells indicate the histogram approximation which characterizes the coarse-grained distribution function.

From equation (II-18) we would write

$$\dot{F}_{\lambda}^{T} = v^{\gamma} \frac{1}{L} \int_{X-\frac{L}{2}}^{X+\frac{L}{2}} dx' \frac{dG_{\lambda}}{dx'} .$$

Because of periodic boundary conditions, $G\,(X+\frac{L}{2}) = G\,(X-\frac{L}{2})$ so that evaluation of the integral at the end points would not give a meaningful result. However, as noted in Fig. 1, if we move a small distance ϵ away from the boundary, we expect G to approximately represent Γ, so that

$$\frac{1}{L} \int_{X-\frac{L}{2}+\epsilon}^{X+\frac{L}{2}-\epsilon} dx' \frac{dG}{dx'} \simeq \frac{\Gamma(X+\frac{L}{2}) - \Gamma(X-\frac{L}{2})}{L} .$$

For Γ slowly varying we might expect that this expression is a reasonably good approximation of equation (II-19). Thus, for the homogeneous medium, we are able to relate equation (II-18) in an approximate, but unambiguous manner to obtain

$$\dot{F}_{\lambda}^{T} \simeq v^{\gamma} \, \underline{\Omega} \cdot \nabla F_{\lambda} .$$

However, for the inhomogeneous medium it is simply not clear whether we should have

$$\dot{F}_\lambda^T = \underline{\Omega} \cdot \underline{\nabla} v^\gamma F_\lambda \qquad \text{or} \qquad \dot{F}_\lambda^T = v^\gamma \underline{\Omega} \cdot \underline{\nabla} F_\lambda .$$

In Chapter IV we will present an alternative derivation of the transport equation in which the cell procedure will be treated more carefully. It is found in that derivation that the transport term should appear as

$$\dot{F}_\lambda^T = \underline{\Omega} \cdot \underline{\nabla} v^\gamma F_\lambda . \qquad \text{(II-20)}$$

The photon speed v^γ is slightly different in Chapter IV, but, as will be shown, it is approximately the same as the one presented here for systems for which our transport equation is expected to be valid.

The right hand side of equation (II-15) represents the rate of change of G through interactions. Integrating over equation (II-15) we find, with the aid of equation (II-20), that

$$\dot{F}_\lambda (\underline{x}, \underline{k}, t) + \underline{\Omega} \cdot \underline{\nabla} v^\gamma F_\lambda$$

$$= \frac{1}{V} \left(\frac{i}{\hbar}\right) \left(\Psi, \left[H^I, \rho_\lambda (\underline{k}, 0)\right] \Psi\right) . \qquad \text{(II-21)}$$

We have not, as yet, discussed the limitations on the size of the cell. The left hand side of equation (II-21) provides an upper limit for the cell size, for if we wish to associate the behavior of the photons within the cell with the point $\underline{x}$, then $G_\lambda (\underline{x}, \underline{k}, t)$ must be slowly varying in space within the cell.

A lower limit of the cell size is partially indicated by the interaction term. An emitting particle localized within the box has an uncertainty in its change of momentum, dictated by $\Delta K_{f,i} > \frac{1}{L}$. This uncertainty appears as a similar uncertainty in the photon momentum, i.e. $\Delta K_{f,i} \simeq \Delta k > \frac{1}{L}$. If we wish $\frac{\Delta k}{k} \ll 1$ we must certainly require that $\frac{\lambda_{max}}{L} << 1$ where λ_{max} is the maximum wavelength of the radiation to be considered.

We now assume the existence of a complete and orthonormal set of eigenfunctions $\{ | n\eta R> \}$ such that

$$H_R \, | n \eta R > \; = \; E_R | n\eta R > \tag{II-22a}$$

$$H_c^M + T_c^{\gamma 0} | n\eta R > \; = \; (E_n + \epsilon_\eta) \, | n\eta R > . \tag{II-22b}$$

The eigenlabel n specifies the eigenstate for particles within the cell, while the label η refers to the set of numbers needed for specification of the occupancy of photon momentum states within the cell. The label R is attached to the eigenstate used to complete the specification of a state of the whole system.

Expanding

$$\Psi = \sum_{n\eta R} b_{n\eta R} \, | n\eta R >$$

and defining

$$D_{n\eta R n'\eta' R'} = b^*_{n'\eta' R'} \, b_{n\eta R} \, ,$$

the interaction term can be written

$$\frac{1}{V}\left(\frac{i}{\hbar}\right)\left(\Psi, \left[H^I, \rho_\lambda(\underline{k}, 0)\right]\Psi\right)$$

$$= \frac{1}{V} \sum_{n\eta R} \eta_{\lambda k} \dot{D}_{n\eta R\, n\eta R}(t) \,, \qquad \text{(II-23)}$$

where we have noted that

$$\rho_{\lambda\, n'\eta' R'\, n\eta R}(\underline{k}, 0) = \eta_{\lambda \underline{k}} \; \delta_{n'\eta' R'\, n\eta R}$$

and

$$\left[D, H^0\right]_{n\eta R,\, n\eta R} = 0 \,.$$

Substituting equation (II-23) into (II-21), we obtain

$$\dot{F}_\lambda + \underline{\Omega} \cdot \underline{\nabla}_{v^\gamma} F_\lambda = \frac{1}{V} \sum_{n\eta} \eta_{\lambda \underline{k}} \; \dot{D}_{n\eta R n\eta R} \,. \qquad \text{(II-24)}$$

The succeeding steps in the study of $\dot{D}_{n\eta R n\eta R}$ are similar to those presented by Van Hove[20] in deriving the Pauli equation. We begin by applying a temporal coarse-graining[21] whereby

$$\dot{D}_{n\eta R n\eta R}(t) \simeq \frac{D_{n\eta R n\eta R}(t+s) - D_{n\eta R n\eta R}(t)}{s} \qquad \text{(II-25)}$$

in which s is a time interval short compared to the lifetime τ of a state, but long compared to the period of oscillation of the emitted, absorbed, or scattered photon. (These conditions are to be considered as sufficient for our purposes, although they may not be necessary.)

From the Schroedinger equation

$$\Psi(t+s) = e^{-iHs/\hbar}\,\Psi(t)$$

$$= e^{-i(H^R + H^c)s/\hbar}\;\Psi(t)$$

$$= e^{-iH^R s/\hbar}\; U(s)\;\Psi(t)\;. \qquad \text{(II-26)}$$

where $U(s) = e^{-iH^c s/\hbar}$ and where we have assumed the commutivity of H^R and H^c. Noting that $U(s)$ is diagonal with respect to the R part of the representation

$$D_{n\eta Rn\eta R}(t+s) = \left|< n\eta R \,\middle|\, \Psi(t+s) >\right|^2$$

$$= \sum_{\substack{n'\eta'R' \\ n''\eta''R''}} D_{n'\eta'R'n''\eta''R''}(t)\;\left(e^{-iH^R s/\hbar}\;U(s)\right)^*_{n\eta Rn''\eta''R''}$$

$$(x)\quad \left(e^{-iH^R s/\hbar}\;U(s)\right)_{n\eta Rn'\eta'R'}$$

$$= \sum_{\substack{n'\eta' \\ n''\eta''}} D_{n'\eta'Rn''\eta''R}(t)\;U^*_{n\eta Rn''\eta''R}(s)\,U_{n\eta Rn'\eta'R}(s)\;. \qquad \text{(II-27)}$$

For the remainder of the discussion, we will suppress the label R.

If an average is performed over the phases of the expansion coefficients $b_{n\eta}(t)$ at the time t, and we invoke the postulate of a random a priori phases, we find from equation (II-27)

$$D_{n\eta n\eta}(t+s) - D_{n\eta n\eta}(t) = \sum_{n'\eta'} \Big[D_{n'\eta' n'\eta'}(t) \left| U_{n\eta n'\eta'}(s) \right|^2$$

$$- D_{n\eta n\eta}(t) \left| U_{n'\eta' n\eta}(s) \right|^2 \Big] . \tag{II-28}$$

The method of radiation damping theory[22, 23, 24] is employed to calculate the matrix elements of $U(s)$. Restricting ourselves to those transition and scattering processes which are accounted for by the lowest power of H^I, we find[1]

$$U^{(1)}_{n'\eta' n\eta}(s) = (-i)\, H^I_{n'\eta' n\eta}$$

$$(x)\ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dz \frac{e^{zs/\hbar}}{(z + i\epsilon_{n\eta} + \frac{i\hbar}{2} \Gamma_{n\eta n\eta}(z))(z + i\epsilon_{n'\eta'} + \frac{i\hbar}{2}\Gamma_{n'\eta' n'\eta'}(z))}$$

(II-29)

where, to second order in the interaction strength e,

$$\frac{1}{2} i\hbar \Gamma_{n\eta n\eta}(z) = i H^I_{n\eta n\eta} - (i)^2 \sum_{n'\eta' \neq n\eta} \frac{H^I_{n\eta n'\eta'} H^I_{n'\eta' n\eta}}{z + i\epsilon_{n'\eta'}} . \tag{II-30}$$

The integral in equation (II-29) can be performed when the poles of the integrand are known. We wish to show that all of these poles are on the imaginary axis. Suppressing the index η, we define

1. See Appendix A.

$$K_n(z) = \frac{1}{z + i\epsilon_n + \frac{i\hbar}{2}\Gamma_{nn}(z)}$$

$$W_n(z) = \frac{1}{2}\, i\hbar\, \Gamma_{n\eta}(z) - i\, H^I_{n\eta}\,.$$

The poles of $K_n(z)$ occur at the zeros of $K_n^{-1}(z)$, and

$$K_n^{-1}(z) = z + i\epsilon_n + i\, H^I_{nn} + W_n(z)\,.$$

Let $z = x + iy$. Then

$$W(x + iy) = \sum_{n' \neq n} \frac{H^I_{nn'}\, H^I_{n'n}}{x + i(y + \epsilon_{n'})}$$

$$= x \sum_{n' \neq n} \frac{H^I_{nn'}\, H^I_{n'n}}{x^2 + (y + \epsilon_{n'})^2} - i \sum_{n' \neq n} \frac{(y + \epsilon_{n'})}{x^2 + (y + \epsilon_{n'})^2}\, H^I_{nn'}\, H^I_{n'n}\,.$$

Then

$$K_n^{-1}(x + iy) = x \left[1 + \sum_{n' \neq n} \frac{H^I_{nn'}\, H^I_{n'n}}{x^2 + (y + \epsilon_{n'})^2} \right]$$

$$+ i \left[(y + \epsilon_n + H^I_{nn}) - \sum_{n' \neq n} \frac{y + \epsilon_{n'}}{x^2 + (y + \epsilon_{n'})^2}\, H^I_{nn'}\, H^I_{n'n} \right]\,.$$

The first term in brackets is positive definite, so that K_n^{-1} cannot vanish unless $x = 0$. We see then that

$$\lim_{x \to 0} K_n^{-1}(x + i\,y) = 0 \quad \text{if and only if}$$

$$y = -\epsilon_n - H^I_{nn} + \sum_{n' \neq n} \frac{H^I_{nn'} \, H^I_{n'n}}{y + \epsilon_{n'}} \,. \tag{II-31}$$

Assuming the last two terms are small compared to ϵ_n, we can write

$$y \simeq -\epsilon_n - H^I_{nn} + \sum_{n' \neq n} \frac{H^I_{nn'} \, H^I_{n'n}}{-\epsilon_n + \epsilon_{n'}} \,. \tag{II-32}$$

The choice of y given by equation (II-32) is equivalent to evaluating the integral in equation (II-29) using

$$\Gamma_{n\eta n\eta}(z) = \lim_{x \to 0^+} \Gamma_{n\eta n\eta}(x - i\,\epsilon_{n\eta}) \,.$$

Corrections to this are discussed elsewhere[24], but are not of importance for the processes considered here.

Define $B_{n\eta}$ and $S_{n\eta}$ by

$$B_{n\eta} \equiv \mathrm{Re} \lim_{x \to 0^+} i \, \Gamma_{n\eta n\eta}(x - i\,\epsilon_{n\eta}) \tag{II-33a}$$

and $$S_{n\eta} \equiv \mathrm{Im} \lim_{x \to 0^+} \frac{i\hbar}{2} \, \Gamma_{n\eta n\eta}(x - i\,\epsilon_{n\eta}) \,. \tag{II-33b}$$

where Re and Im mean the real and imaginary parts. From equation (II-30) we find

$$B_{n\eta} = \frac{2\pi}{\hbar^2} \sum_{n'\eta' \neq n\eta} \left| H^I_{n'\eta' n\eta} \right|^2 \delta(\omega_{n\eta} - \omega_{n'\eta'}) \tag{II-34a}$$

$$S_{n\eta} = \left[H^I_{n\eta n\eta} + \sum_{n'\eta'} P \frac{H^I_{n\eta n'\eta'} \; H^I_{n'\eta' n\eta}}{\epsilon_{n\eta} - \epsilon_{n'\eta'}} \right] \tag{II-34b}$$

where P indicates that a principle value should be taken.

The quantity $B_{n\eta}$ is referred to as the level broadening, while $S_{n\eta}$ is called the level shift for bound state problems, although it also exists for continuum states. A physical interpretation of these quantities is obtained from equation (II-29) by assuming $B_{n\eta} \simeq B_{n'\eta'}$. Then writing $\bar{\epsilon}_{n\eta} = \epsilon_{n\eta} + S_{n\eta}$, equation (II-29) becomes

$$\left| U^{(1)}_{n'\eta' n\eta}(s) \right|^2 = \frac{4}{\hbar^2} \left| H^I_{n'\eta' n\eta} \right|^2 e^{-B_{n\eta} s} \frac{\sin^2\left(\frac{\bar{\epsilon}_{n\eta} - \bar{\epsilon}_{n'\eta'}}{2\hbar} s \right)}{(\bar{\epsilon}_{n\eta} - \bar{\epsilon}_{n'\eta'})^2/\hbar^2} . \tag{II-35}$$

Recall that $\left| U^{(1)}_{n'\eta' n\eta}(s) \right|^2$ is the probability that the system will make a transition from the state $|n\eta>$ to the state $|n'\eta'>$ during the time interval s. Of course, the probability of being in the state $|n\eta>$ is decreasing with time as the result of transitions out of the state. The factor $e^{-B_{n\eta} s}$ accounts for the depopulation of the state $|n\eta>$, and suggests that $B_{n\eta}$ is interpretable as the total transition probability per unit time for transitions out of the state $|n\eta>$, and that $B_{n\eta} \simeq \frac{1}{\tau}$ where τ is the lifetime of the state. If we limit s such that

$s/\tau << 1$, we see that $e^{-B_{n\eta} s} \simeq 1$. This is equivalent to taking $\lim B_{n\eta} \to 0$ in the integrand of equation (II-29) before performing the integral.

The quantity $S_{n\eta}$ is associated with a shift of the energy level $\epsilon_{n\eta}$. Note from equation (II-34b) that when no photons are present, S_{n0} is nonvanishing and represents the self energy of the medium. It will be shown in Chapter III that

$$S_{n\eta} - S_{n0} = \sum_{\lambda \underline{k}} \eta_{\lambda \underline{k}} \hbar \omega_{\lambda k} g(\omega_{\lambda k})$$

where $g(\omega_{\lambda k})$ is a function of $\omega_{\lambda k}$ and includes sums over particle momenta. Recalling equations (II-9b), (II-10a), and (II-22b), we write

$$\bar{\epsilon}_{n\eta} = \epsilon_{n\eta} + S_{n\eta} = E_n + \bar{\mathcal{E}}_\eta$$

where

$$\bar{\mathcal{E}}_\eta = \sum_{\lambda \underline{k}} \eta_{\lambda \underline{k}} \hbar \omega_{\lambda k} \left[1 + g(\omega_{\lambda k})\right]. \qquad \text{(II-36)}$$

We have discarded the infinite self energy which could have been removed at an earlier stage by a mass renormalization. It is not of any particular interest to us.

In a representation in which $\omega = ck$, the additional term would give a finite energy level shift from the unperturbed eigenvalues. This additional energy results from the virtual interactions of the photons with the medium. Tidman[11] refers to this energy as the polarization energy of the medium, while Van Hove[25], in discussing systems with continuous eigenvalues, refers to this as the persistent effect of the medium.

It is now possible to appreciate the advantage of the arbitrary $\omega_{\lambda k}$. By choosing $\omega_{\lambda k}$ so that $g(\omega_{\lambda k}) = 0$, we will be working in a representation in which the polarization energy of the medium is included in the photons from the beginning.[2] These photons, which have a dispersion relation different from $\omega = ck$, are the "dressed" photons mentioned in Chapter I. The unperturbed photon eigenvalues are given by

$$\bar{\varepsilon}_{\eta} = \sum_{\lambda \underline{k}} \eta_{\lambda \underline{k}} \hbar \omega_{\lambda k}$$

with $\omega_{\lambda k}$ chosen so that $g(\omega_{\lambda k}) = 0$ or, equivalently, $S_{n\eta} - S_{n0} = 0$. For simplicity we drop the bar in the eigenvalue notation.

Equation (II-25) can now be expressed in the form of the Pauli equation

$$\dot{D}_{n\eta n\eta}(t) = \sum_{m\alpha} W_{n\eta m\alpha} \left[D_{m\alpha m\alpha} - D_{n\eta n\eta} \right] \tag{II-37}$$

where $W^{(1)}_{n\eta m\alpha}$, the transition probability per unit time for a transition from the state $| m\alpha >$ to the state $| n\eta >$, is related to equation (II-35) by

$$W^{(1)}_{n\eta m\alpha} \equiv \lim_{s \to \infty} \frac{\left| U^{(1)}_{n\eta m\alpha} \right|^2}{s} = \frac{2\pi}{\hbar^2} \left| H^{I}_{n\eta m\alpha} \right|^2 \delta(\omega_{n\eta} - \omega_{m\alpha}) \tag{II-38}$$

2. The idea of incorporating the level shift into the photons was borrowed from Mead.[10] By modifying the radiation damping theory of Heitler[23] and others[22, 24] we have adapted Mead's calculation to the photon transport problem.

with $\omega_{n\eta} = \epsilon_{n\eta}/\hbar$. The limit $s \rightarrow \infty$ reflects the condition $s\,\omega >> 1$ discussed earlier. The transition probability $W_{n\eta m\alpha}$ has the symmetry property $W_{n\eta m\alpha} = W_{m\alpha n\eta}$.

To proceed further, we note that

$$\left| H^{I}_{n\eta m\alpha} \right|^2 = \left[\left| H^{(1)}_{n\eta m\alpha} \right|^2 + \left| (H_2^{P\gamma 2})_{n\eta m\alpha} \right|^2 \right].$$

No cross terms appear because $H^{(1)}$ is linear in creation or destruction operators and $H_2^{P\gamma 2}$ is bilinear; thus, a matrix element of both terms cannot be simultaneously nonvanishing.

The transitions described by $H_2^{P\gamma 2}$ are photon-particle scattering, while those accomplished by $H^{(1)}_{n\eta m\alpha}$ include: (a) electron transitions between two magnetic states, resulting in emission or absorption of cyclotron radiation; (b) free-bound and bound-free electron transitions, producing recombination radiation or photoelectric absorption, respectively; (c) transitions giving excitation absorption or de-excitation emission; and (d) free-free electron transitions, where the electron is under the influence of a Coulomb potential. These latter transitions account for bremsstrahlung and inverse bremsstrahlung. It will also be shown in Chapter VII, in connection with cyclotron radiation calculations, that Cerenkov radiation is possible for a plasma in a magnetic field.

Because of the inverse mass dependence of the matrix elements, only electrons need to be considered for both scattering and radiative transitions. We will specify the particle state according to the process and medium under consideration. The subject of x-ray scattering in a crystal will receive special attention in Chapter VIII.

For other cases discussed here, we will treat the system as a collection of spinless particles. It will be assumed that the Hamiltonian of the medium H^M can be written as a sum of terms, each representing either an electron (for the fully ionized gas) or a molecule (for the neutral gas). This allows us to employ appropriate product wave functions for the system under consideration. For example, for the fully ionized gas, product wave functions for the electrons will be chosen. Relevant aspects of completeness of the particle states have been discussed in reference (9).

For specific calculations, the following approximations will be employed:

(a) Emission and absorption of cyclotron radiation - Ignore interactions between particles and take the particle states as electron magnetic states. Cerenkov radiation will appear as transitions in which the component of the particle momenta along the magnetic field changes.

(b) Recombination radiation and photoelectric absorption - The bound state is chosen as the state of an appropriate central potential, e.g. the hydrogen atom state of a Coulomb potential. Plane waves are used for the free particle state.

(c) De-excitation radiation and excitation absorption - Use Coulomb wave functions without any external fields for both bound states.

(d) Bremsstrahlung and inverse bremsstrahlung - Use positive energy Coulomb wave functions for both initial and final particle state. Expand the wave functions in a perturbation series[26] and keep only the first nonvanishing matrix element.

From equations (II-25) and (II-37) we write (with a little manipulation)

$$\dot{F}^{I} = \sum_{n\eta n'\eta'} \frac{W^{(1)}_{n\eta n'\eta'}}{V} \quad D_{n\eta n\eta} \left[\eta_{\lambda k} - \eta'_{\lambda k} \right] . \qquad \text{(II-39)}$$

In order to simplify the remaining calculations in this section, we will neglect photon scattering. Then, from equation (II-38)

$$\frac{1}{V} \, W^{(1)}_{n\eta n'\eta'} = \frac{2\pi}{V\hbar^2} \left| H^{(1)}_{n\eta n'\eta'} \right|^2 \delta \left(\omega_{n\eta} - \omega_{n'\eta'} \right) .$$

The expression for $H^{(1)}_{n\eta n'\eta'}$ is found from equation (II-10c) to be

$$H^{(1)}_{n\eta n'\eta'} = - \sum_{\sigma, \underline{k}, \lambda} \frac{e_\sigma}{m_\sigma} \sqrt{\frac{2\pi\hbar}{V\omega_{\lambda k}}} \left[< \eta \left| \alpha^{+}_{\lambda} (\underline{k}) \right| \eta' > \right.$$

$$(x) \int d\,\tau \, \psi^{*}_{n} \, e^{-i \underline{k} \cdot \underline{x}^{\sigma}} \, \mathcal{E}_{\lambda} (\underline{k}) \cdot \underline{\Pi}^{\sigma} \psi_{n'}$$

$$\left. + < \eta \left| \alpha_{\lambda} (\underline{k}) \right| \eta' > \int d\tau \, \psi^{*}_{n} \, e^{i \underline{k} \cdot \underline{x}^{\sigma}} \, \mathcal{E}_{\lambda} (\underline{k}) \cdot \underline{\Pi}^{\sigma} \psi_{n'} \right] \qquad \text{(II-40)}$$

where $d\tau = \prod_{i=1}^{N} dx_i$.

The relation between the decomposition of $\psi_n (\underline{x}_1 \underline{x}_2, \ldots \underline{x}_N)$ into appropriately symmetrized product wave and the second quantized formalism used by Osborn and Klevans[9] is given by Landau and Lifshitz[27] for both bosons and fermions. We obtain

$$\sum_{\sigma} \int d\tau \, \psi_n^* \, e^{-i\underline{k}\cdot\underline{x}^{\sigma}} \, \mathcal{E}_{\lambda}(\underline{k}) \cdot \underline{\Pi}^{\sigma} \psi_{n'}$$

$$= \sqrt{(1 \pm n_{K_1}) \, n_K} \int d^3x \, \psi_{\underline{K}_1}(\underline{x}_1) \, e^{-i\underline{k}\cdot\underline{x}} \, \mathcal{E}_{\lambda}(\underline{k}) \cdot \underline{\Pi} \, \psi_{\underline{K}}(\underline{x}) \qquad \text{(II-41)}$$

where the positive sign indicates bosons and the negative sign fermions. The labels $\underline{K}$ and $\underline{K}_1$ appended to the one particle wave functions represent a sufficient set of labels to specify the one particle state.

Then we have

$$\frac{1}{V} \, W^{(1)}_{n\eta n'\eta'} = \sum_{\beta, K, K_1} T^{K_1}_{\beta K}(\lambda\underline{k}) \; n_K (1 \pm n_{K_1})$$

$$(x) \quad \left| < \eta \right| \alpha_{\lambda}^{+}(\underline{k}) \left| \eta' > \right|^2 \qquad \text{(II-42a)}$$

$$\text{or} \quad = \sum_{\beta, K, K_1} T^{aK_1}_{\beta_K}(\lambda\underline{k}) \; n_K (1 \pm n_{K_1}) \left| <\eta \right| \alpha_{\lambda}(\underline{k}) \left| \eta' > \right|^2 \qquad \text{(II-42b)}$$

depending upon whether we have emission or absorption. The quantities $T^{K_1}_{\beta_K}$ and $T^{aK_1}_{\beta_K}$ are given by

$$T^{K_1}_{\beta_K}(\lambda\, \underline{k}) = \left(\frac{4\pi^2 c^2}{\hbar V^2}\right)\left(\frac{e}{mc}\right)^2 \frac{1}{\omega_{\lambda k}} \quad \delta(\omega_{K_1 k} - \omega_K)$$

$$(x) \left| < \underline{K_1} \right| e^{-i\underline{k}\cdot\underline{x}} \; \mathcal{E}_\lambda(\underline{k}) \cdot \underline{\Pi} \left| \underline{K} > \right|^2 \qquad \text{(II-43a)}$$

$$T^{aK_1}_{\beta_K}(\lambda\, \underline{k}) = \left(\frac{4\pi^2 c^2}{\hbar c^2}\right)\left(\frac{e}{mc}\right)^2 \frac{1}{\omega_{\lambda k}} \; \delta(\omega_{K_1 k} - \omega_K)$$

$$(x) \left| < \underline{K}_1 \right| e^{i\underline{k}\cdot\underline{x}} \; \mathcal{E}_\lambda(\underline{k}) \cdot \underline{\Pi} \left| \underline{K} > \right|^2 . \qquad \text{(II-43b)}$$

The index β refers to the various emission and absorption processes. For the emission term $\eta'_{\lambda k} = \eta_{\lambda k} - 1$ and for the absorption term $\eta'_{\lambda k} = \eta_{\lambda k} + 1$. Equation (II-37) can therefore be written

$$\dot{F}^I = \sum_{\beta n \eta K K_1} D_{n\eta n\eta} \left[T^{K_1}_{\beta_K}(\lambda \underline{k})\, n_K (1 \pm n_{K_1})(\eta_{\lambda_k} + 1) \right.$$

$$\left. - T^{aK_1}_{\beta_K}(\lambda\, \underline{k})\, n_K (1 \pm n_{K_1})\, \eta_{\lambda k} \right] . \qquad \text{(II-44)}$$

It will be seen in Chapter III that $\omega_{\lambda k}$ is dependent upon the particle occupation numbers so that T is dependent upon particle occupation numbers. Replacing the averages of products of particle and photon occupation numbers by products of averages, and employing the symmetry property

$$T^{K_1}_{\beta_K}(\lambda \underline{k}) = T^{aK}_{\beta_{K_1}}(\lambda \underline{k})$$

we obtain

$$\dot{F}^{I} = \sum_{\beta K K_1} \bar{T}^{K_1}_{\beta_K}(\lambda \underline{k}) \left[\left\{ V F_\lambda(\underline{k}) + 1 \right\} V f_\sigma(\underline{K}) \left\{ 1 \pm V f_\sigma(\underline{K}_1) \right\} \right.$$

$$\left. - V F_\lambda(\underline{k}) V f_\sigma(\underline{K}_1) \left\{ 1 \pm V f_\sigma(\underline{K}) \right\} \right] \qquad \text{(II-45)}$$

where

$$\bar{T}^{K_1}_{\beta_K}(\lambda \underline{k}) = \sum_{n\eta} D_{n\eta} T^{K_1}_{\beta_K}(\lambda \underline{k})$$

$$\simeq \frac{4\pi^2 c^2}{\hbar V^2} \left(\frac{e}{mc}\right)^2 \frac{1}{\bar{\omega}_{\lambda k}} \delta(\bar{\omega}_{K_1 k} - \omega_K)$$

$$(x) \quad \left| < \underline{K}_1 \right| e^{-i\underline{k}\cdot\underline{x}} \, \underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{\Pi} \left| \underline{K} > \right|^2 . \qquad \text{(II-46)}$$

We have used the approximation in equation (II-46) (and it will be used throughout) that

$$\sum_{n\eta} D_{n\eta n\eta} A(\omega) = A(\bar{\omega})$$

where $A(\bar{\omega})$ is some function of ω, and where

$$\bar{\omega} = \sum_{n\eta} D_{n\eta n\eta} \omega_{\lambda k} .$$

Finally, proceeding along entirely similar lines, the contribution of photon scattering can be computed. We obtain for the photon balance relation

$$\dot{F}_\lambda + \Omega \cdot \underline{\nabla}_v{}^\gamma F_\lambda = \sum_{\beta K K_1} \bar{T}^{K_1}_{\beta_K} \left[\left\{ V F_\lambda (\underline{k}) + 1 \right\} V f (\underline{K}) \right.$$

$$(x) \left\{ 1 \pm V f (\underline{K}_1) \right\} - V F_\lambda (\underline{k}) V f (\underline{K}_1) \left. \left\{ 1 \pm V f (\underline{K}) \right\} \right]$$

$$+ \sum_{K K_1 \lambda' k'} \bar{S}^{K\lambda k}_{K_1 \lambda' k'} \left[V F_{\lambda'} (\underline{k}') \left\{ V F_\lambda (\underline{k}) + 1 \right\} V f (\underline{K}_1) \left\{ 1 \pm V f (\underline{K}) \right\} \right.$$

$$\left. - V F_\lambda (\underline{k}) \left\{ V F_{\lambda'} (\underline{k}') + 1 \right\} V f (\underline{K}) \left\{ 1 \pm V f (\underline{K}_1) \right\} \right] \qquad \text{(II-47)}$$

where

$$\bar{S}^{K\lambda k}_{K_1, \lambda' k'} = \left(\frac{e^2}{mc^2} \right)^2 \frac{8\pi^3}{V^3} \frac{c^4}{\bar{\omega}_{\lambda k} \bar{\omega}_{\lambda' k'}} \left| \underline{\mathcal{E}}_\lambda (\underline{k}) \cdot \underline{\mathcal{E}}_{\lambda'} (\underline{k}') \right|^2$$

$$(x) \quad \delta (\underline{k}' + \underline{K}' - \underline{k} - \underline{K}) \; \delta (\bar{\omega}_{Kk} - \bar{\omega}_{K'k'}) \; . \qquad \text{(II-48)}$$

III. DISPERSION RELATIONS

A. General Considerations

In Chapter II, it was pointed out that the unperturbed photon eigenvalues are given by

$$\mathcal{E}_{\eta} = \sum_{\lambda \underline{k}} \eta_{\lambda \underline{k}} \hbar \omega_{\lambda k}$$

where $\omega_{\lambda k}$ is chosen such that $S_{n\eta} - S_{no} = 0$. In this chapter, we will investigate this condition.

The quantity $S_{n\eta}$ was given in Chapter II as

$$S_{n\eta} = \left[H^{I}_{n\eta n\eta} + \sum_{r\sigma} P \frac{H^{I}_{n\eta r\sigma} \; H^{I}_{r\sigma n\eta}}{\epsilon_{n\eta} - \epsilon_{r\sigma}} \right] . \qquad \text{(II-34b)}$$

It is readily established that

$$H^{I}_{n\eta n\eta} = (H_{o}^{p\gamma 2} + T^{\gamma 1})_{n\eta n\eta}$$

$$= \frac{1}{2} \sum_{\lambda \underline{k}} \eta_{\lambda \underline{k}} \hbar \omega_{\lambda k} \left[\frac{c^2 k^2 + \frac{4\pi \; e^2}{m V} \sum_{K} n_K - \omega_{\lambda k}^{\;2}}{\omega^2_{\lambda k}} \right]$$

(III-1a)

and

$$\sum_{r\sigma} P \frac{H^{I}_{n\eta r\sigma} \; H^{I}_{r\sigma n\eta}}{\epsilon_{n\eta} - \epsilon_{r\sigma}} = \sum_{\lambda k K K_1}{}' \hbar \omega_{\lambda k} \left(\frac{e}{m}\right)^2 \frac{2\pi}{V} \frac{(1 \pm n_K) n_{K1}}{\omega^2_{\lambda k}}$$

$$(x) \left[\frac{\eta_{\lambda k} \left| I_{\lambda k}(\underline{K}_1 | \underline{K}) \right|^2}{E_{K_1} - E_K + \hbar \omega_{\lambda k}} + \frac{(\eta_{\lambda k} + 1) \left| I_{\lambda k}(\underline{K} | \underline{K}_1) \right|^2}{E_{K_1} - E_K - \hbar \omega_{\lambda k}} \right] \qquad \text{(III-1b)}$$

where

$$I_{\lambda k}(\underline{K}_1 \,|\, \underline{K}) \equiv \langle \underline{K}_1 | \, e^{-i\underline{k}\cdot\underline{x}} \, \underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{\Pi} \, | \underline{K} \rangle \qquad \text{(III-2)}$$

and where we now employ a prime to indicate the exclusion from the sum terms for which a denominator vanishes.

By straightforward manipulations we find (except for the self energy of the medium)

$$S_{n\eta} = \sum_{\lambda k} \eta_{\lambda k} \hbar \, \omega_{\lambda k} \, g(\omega_{\lambda k}) \qquad \text{(III-3)}$$

where

$$g(\omega_{\lambda k}) = \left(\frac{1}{\omega_{\lambda k}}\right)^2 \frac{1}{2} \left[\left(\frac{4\pi e^2}{m V}\right) \sum_{\underline{K}} n_K + c^2 k^2 - \omega_{\lambda k}{}^2 \right.$$

$$\left. + \left(\frac{4\pi e^2}{V m^2}\right) \sum_{K K_1} \frac{|I_{\lambda k}(\underline{K}_1 | \underline{K})|^2}{E_K - E_{K_1} - \hbar \omega_{\lambda k}} \left\{ n_K - n_{K_1} \right\} \right] . \qquad \text{(III-4)}$$

As discussed in Chapter II, we choose $\omega_{\lambda k}$ so that $g(\omega_{\lambda k})$ vanishes. The obvious non-trivial choice is

$$\omega^2_{\lambda k} = c^2 k^2 + \frac{4\pi e^2}{m V} \left[\sum_K n_K \right.$$

$$\left. - \frac{1}{m} \sum_{K K_1} \frac{|I_{\lambda k}(\underline{K}_1 | \underline{K})|^2}{E_K - E_{K_1} - \hbar \omega_{\lambda k}} (n_K - n_{K_1}) \right] . \qquad \text{(III-5)}$$

In accordance with remarks in the last chapter, averages of functions of ω will be replaced by functions of the average of ω.

Throughout our discussion we will find it convenient to apply various approximations with respect to the averaging of $\omega_{\lambda k}$. In particular, we will use

$$\overline{\omega^2_{\lambda k}} \simeq (\bar{\omega}_{\lambda k})^2$$

$$\overline{\left(\frac{1}{\omega_{\lambda k}}\right)} \simeq \frac{1}{\bar{\omega}_{\lambda k}}$$

$$\overline{\left(\frac{1}{\omega_{\lambda k}\omega_{\lambda' k'}}\right)} \simeq \frac{1}{\bar{\omega}_{\lambda k}\ \bar{\omega}_{\lambda' k'}} \ .$$

Although these approximations are employed in a rather cavalier manner, it should be remembered that, at least in principle, it would be possible to investigate the errors involved.

Applying an average to equation (III-5), we now find

$$\overline{\omega^2_{\lambda k}} \cong c^2 k^2 + \omega_p^2 \left[1 - \rho_{\lambda k}\right] \qquad \text{(III-6)}$$

where $\omega_p^2 = 4\pi n e^2/m$

$$\rho_{\lambda k} = \frac{1}{m} \sum_{KK_1} \frac{|I_{\lambda k}(\underline{K}_1|\underline{K})|^2}{E_K - E_{K_1} - \hbar\bar{\omega}_{\lambda k}} \left[\bar{f}_1(\underline{K}_1) - \bar{f}(\underline{K})\right]$$

$$= \frac{1}{m} \sum_{KK_1} \bar{f}(\underline{K}_1) \left[\frac{|I_{\lambda k}(\underline{K}_1|\underline{K})|^2}{E_K - E_{K_1} - \hbar\bar{\omega}_{\lambda k}} + \frac{|I_{\lambda k}(\underline{K}|\underline{K}_1)|^2}{E_K - E_{K_1} + \hbar\bar{\omega}_{\lambda k}}\right]$$

(III-7)

and $\bar{f}(\underline{K}) \equiv \frac{1}{n} f(\underline{K})$. From equation (III-6), the index of refraction μ_λ can be written

$$\bar{\mu}_\lambda^2 = \left(\frac{ck}{\bar{\omega}_{\lambda k}}\right)^2 = 1 - \frac{\omega_p^2}{\bar{\omega}_{\lambda k}^2}\left[1 - \rho_{\lambda k}\right]. \qquad \text{(III-8)}$$

Because we will only be interested in $\bar{\omega}$ and $\bar{\mu}$ throughout the remainder of the thesis, we will drop the bar and use ω and μ as the averaged quantities.

In order to compute $I_{\lambda k}(\underline{K}_1 | \underline{K})$, it is necessary to specify the particle eigenfunctions. Three special systems will be considered, and the eigenfunctions will be chosen accordingly. They include: the fully ionized plasma with no external magnetic field; the fully ionized plasma in a constant homogeneous magnetic field; and the neutral gas of one species ,

B. Fully Ionized Gas With No External Fields

For the fully ionized gas without external fields, we use plane wave states for the particles. After going to the continuum in velocity space, we obtain

$$\omega_{\lambda k}^2 = c^2 k^2 + \omega_p^2 \left[1 + P \int d^3 v \,\bar{f}(v) \, \frac{(\underline{\varepsilon}_\lambda(\underline{k}) \cdot \underline{v})^2 k^2}{(\omega_{\lambda k} - \underline{v} \cdot \underline{k})^2 - (\frac{\hbar k^2}{2m})^2} \right] . \qquad \text{(III-9)}$$

A quantum correction to the classical result is in evidence in the integrand. If the particle distribution function is isotropic in velocity space, the frequency $\omega_{\lambda k}$ is independent of which polarization is chosen if linear polarization is used. This is the same result obtained by Bohm and Pines[(12)]. In the zero temperature limit, equation (III-9) reduces to the Langmuir dispersion relation

$$\omega^2 = c^2 k^2 + \omega_p^2 .$$

Dropping the quantum corrections and assuming a symmetric velocity distribution, equation (III-9) can be written, by straightforward manipulation, in the form

$$\omega^2 = c^2 k^2 + \omega \omega_p^2 \; P \int d^3 v \; \frac{\bar{f}(v)}{(\omega - \underline{k} \cdot \underline{v})} \quad . \qquad \text{(III-10)}$$

This transverse dispersion relation for the fully ionized gas was obtained by Bernstein[15] by solving the linearized set of equations consisting of Vlassov's equation and Maxwell's equations. Our dispersion relation (with no external fields) is obtained from his equation (32) when his Laplace variable s is set equal to $- i\omega$.

C. Fully Ionized Gas in A Constant, Uniform External Magnetic Field

When an external magnetic field H is present, the medium becomes anisotropic. In general, this will introduce coupling between the transverse and longitudinal modes. Radiation then propagates in a direction different from the normal to the wave front. The concept of the photon does not enter naturally (or, perhaps, even legitimately) in this context.

Nonetheless, under certain circumstances the different modes decouple and $\epsilon^T = \mu_\lambda^2$, where ϵ^T is the transverse dielectric constant for photons of polarization λ. One such case is propagation parallel to the magnetic field. For convenience the magnetic field is chosen along the z axis. For particle eigenstates we choose the magnetic states of Johnson and Lippmann[28]. When circular polarization is employed, the waves propagate independently and we obtain [3]

$$\omega_{\pm}^2 = c^2k^2 + \omega_p^2 \; P\int dv_z \bar{f}(v_z) \frac{\omega_{\pm}\left(1 - \frac{v_z k_z}{\omega_{\pm}}\right)}{(\omega_{\pm} \pm \omega_c)\left(1 - \frac{v_z k_z}{\omega_{\pm} \pm \omega_c}\right)} \tag{III-11}$$

where $\omega_c = \frac{eH}{mc}$. The positive sign is associated with a wave whose electric vector rotates in the same direction as the electrons gyrate. This is called[13] the "extraordinary" wave. The negative sign refers to the "ordinary" wave, for which the electric vector rotates in the opposite sense.

In the zero temperature limit equation (III-11) reduces to

$$\omega_{\pm}^2 = c^2k^2 + \frac{\omega_p^2}{1 \pm \frac{\omega_c}{\omega_{\pm}}} \quad . \tag{III-12}$$

The apparent difference between this result and that given by Spitzer[13] is accounted for by our associating a sign with the charge e, while Spitzer uses $|e| = -e$ for electrons.

A first order temperature correction can be obtained from equation (III-11) by expansion of $\left(1 - \frac{v_z k_z}{\omega_{\pm} \pm \omega_c}\right)^{-1}$, assuming we are not near a resonance. We find

$$\mu^2 = \frac{1 - \frac{\omega_p^2}{\omega_{\pm}(\omega_{\pm} \pm \omega_c)}}{1 - \frac{\lambda_D^2}{c^2} \frac{\omega_p^4}{(\omega_c \pm \omega_{\pm})^2}} \tag{III-13}$$

3. Appendix B

where $\lambda_D = \left(\frac{k_o T}{4\pi n e^2}\right)^{1/2}$ is the Debye shielding distance. This is the same result as obtained by Pradhan(14) for the ordinary wave. He does not give a result for the extraordinary wave.

The inadequacies of the present method for anisotropic media have been mentioned. Nevertheless, it is of some interest to consider propagation of radiation at some angle θ with respect to the magnetic field and compare the results achieved through our perturbation procedure with those obtained by the more conventional approach(15). For simplicity we consider propagation perpendicular to the magnetic field along the y axis in a left-handed coordinate system. The polarization vectors are chosen as $\underline{\mathcal{E}}_z$ and $\underline{\mathcal{E}}_x$. For polarization parallel to the magnetic field, there is no longitudinal-transverse coupling and we find[4]

$$\omega_z^2 = \omega_{\parallel}^2 = c^2k^2 - \omega_p^2 \sum_{n=-\infty}^{\infty} \frac{\omega_{\parallel}}{n\omega_c - \omega_{\parallel}} \Upsilon_n \tag{III-14}$$

where

$$\Upsilon_n = \int_0^{\infty} dv_{\perp}\ \bar{f}(v_{\perp}) \left[J_n \left(\frac{v_{\perp}}{c} \frac{ck}{\omega_c}\right)\right]^2 . \tag{III-15}$$

For polarization perpendicular to the magnetic field, we obtain

$$\omega_x^2 = \omega_{\perp}^2 = c^2k^2 - \omega_p^2 \left(\frac{m}{k_o T}\right) \sum_{n=-\infty}^{\infty} \frac{\omega_{\perp}}{n\omega_c - \omega_{\perp}} \mathrm{X}_n \tag{III-16}$$

where

k_o = Boltzmann's constant

---

4. Appendix C

$$\mathfrak{X}_n = \int dv_\perp \, f(v_\perp) v_\perp^2 \left[J'_n \left(\frac{v_\perp}{c} \frac{ck}{\omega_c} \right) \right]^2 \qquad \text{(III-17)}$$

For comparison, consider the solution of the linearized set of equations containing the Vlassov equation and Maxwell's equation[15]. Bernstein does not exhibit the transverse dispersion relations in a form which is easily compared with our results. However, by performing the ϕ and v_z integrations in his equation (21), we can, from his equations (21), (23), and (27) obtain these dispersion relations. We find first that equation (III-14), which represents a wave whose polarization vector is parallel to the magnetic field, exhibits no coupling and is in agreement with Bernstein without further approximation; and second that, because of longitudinal-transverse coupling, equation (III-16) is in agreement only for $ck >> \omega_c$ and $ck >> \omega_p$. At these frequencies the coupling is negligible.

D. The Neutral, Single Species Gas

As a final application of our method of determining dispersion relations, we consider the neutral gas of one species. The particle eigenstates are the eigenfunctions of $H_{mol}^{(\alpha)}$ where $H_{mol}^{(\alpha)}$ is the unperturbed Hamiltonian for the αth molecule, and where we have written $H^M = \sum_\alpha H_{mol}^{(\alpha)}$.

For the dipole approximation we find

$$c^2 k^2 = \omega_{\lambda k}^2 \left[1 - \omega_p^2 \sum_{KK_1} \bar{f}(\underline{K}_1) \frac{b_{KK_1}}{\omega_{\lambda k}^2 - \omega_{KK_1}^2} \right] \qquad \text{(III-18)}$$

where b_{KK_1}, the oscillator strength, is given by

$$b_{KK_1} = \frac{2m}{\hbar} \omega_{KK_1} \left| x_{KK_1} \right|^2 \tag{III-19}$$

with

$$x_{KK_1} = (\underline{\varepsilon}_\lambda(\underline{k}) \cdot \underline{x})_{KK_1}$$

$$= \underline{\varepsilon}_\lambda(\underline{k}) \cdot \langle \underline{K} \left| \sum_i^n \underline{x}_i \right| \underline{K}_1 \rangle .$$

If $f(\underline{K}_1) = \delta_{K_1 0}$ (all molecules in the ground state), we obtain

$$\epsilon^T = \frac{c^2 k^2}{\omega^2} = 1 - \omega_p^2 \sum_K \frac{b_{K0}}{\omega_\lambda^2 - \omega_{K0}^2} \tag{III-20}$$

which is the Sellmeyer-Drude formula for the dielectric constant of a medium of undamped oscillators with frequency ω_{K0}. When equation (III-20) is written in the form

$$\mu_\lambda^2 - 1 = \frac{8\pi e^2 N}{V} \sum_K \frac{\omega_{K0} \left| (\underline{\varepsilon}_\lambda(\underline{k}) \cdot \underline{x})_{K0} \right|^2}{\hbar(\omega_{K0}^2 - \omega_\lambda^2)} \tag{III-21}$$

we have the Kramers-Heisenberg dispersion relation. With respect to this result our procedure is equivalent to the calculation of Mead[10].

IV. A MODIFIED DERIVATION OF THE PHOTON TRANSPORT EQUATION

In the derivation of the transport equation presented in Chapter II, a certain ambiguity arose in connection with specifying the coarse-grained transport term in an inhomogeneous medium, i.e. a medium in which the photon speed v^γ and the frequency ω are functions of position. In this chapter we will present an alternative derivation of the photon balance equation in an effort to shed more light on the question of photon transport.

A coarse-grained distribution function will be introduced at the outset, and a formal cell procedure developed[29]. This will be in contrast to Chapter II, where the decomposition of the Hamiltonian was carried out on a somewhat intuitive basis. Other aspects of this derivation will also be of interest. For instance, it will be shown that the condition $\frac{sv^\gamma}{L} << 1$, where s is the time employed in connection with coarse-graining and L^3 is the cell volume, is necessary to eliminate terms describing combinations of interaction and transport.

The statistical postulate for the present development is introduced by defining

$$F_\lambda(\underline{X}, \underline{k}, t) \equiv \frac{1}{L^3} \operatorname{Tr} D(t)\, \rho_\lambda(\underline{X}, \underline{k}) \tag{IV-1}$$

as the expected number of photons per cm^3 of momentum $\hbar \underline{k}$ in the cell of volume L^3 located at $\underline{X}$. The density matrix D is again defined as $D_{n\eta n'\eta'} = b^*_{n'\eta'} b_{n\eta}$ where b is an expansion coefficient of the expansion

$$\underline{\Psi} = \sum_{n\eta} b_{n\eta} \mid n\eta >$$

and $\{| n\eta > \}$ is a set of orthonormal base vectors presumed to be complete. The meaning of the labels n and η will be given later.

The singlet density cell operator is given by

$$\rho_\lambda(\underline{X}, \underline{k}) = \alpha^+_\lambda(\underline{X}, \underline{k})\,\alpha_\lambda(\underline{X}, \underline{k}) \qquad \text{(IV-2)}$$

where the cell creation and destruction operators obey the commutation relations

$$\left[\alpha_\lambda(\underline{X}, \underline{k}), \alpha^+_{\lambda'}(\underline{X}', \underline{k}')\right] = \delta_{\lambda,\lambda'}\,\delta_{\underline{X},\underline{X}'}\,\delta(\underline{k}-\underline{k}'). \qquad \text{(IV-3)}$$

The equation of motion for F_λ is

$$\dot{F}_\lambda = \frac{1}{V}\left(\frac{i}{\hbar}\right) \mathrm{Tr}\,\rho\left[D, H\right] = \left(\frac{1}{V}\right)\left(\frac{i}{\hbar}\right) \mathrm{Tr}\,D\left[H, \rho\right] \qquad \text{(IV-4)}$$

where the Hamiltonian is the same as equations (II-3) except that the second quantized formalism will be used for the particles[9].

In order to accomplish the averaging of $[H, \rho]$ we expand

$$\underline{A}(\underline{x}) = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \sum_{\underline{X}, \underline{k}, \lambda} \frac{e^{-i\underline{k}\cdot\underline{x}}}{\sqrt{\omega_\lambda(\underline{X}, \underline{k})}}\, E(\underline{X}, \underline{x})\,\underline{\xi}^+_\lambda(\underline{X}, \underline{k})$$

(IV-5a)

$$\underline{P}(\underline{x}) = \sqrt{\frac{\hbar}{8\pi c^2 L^3}} \sum_{\underline{X}, \underline{k}, \lambda} \sqrt{\omega_\lambda(\underline{X}, \underline{k})}\, e^{-i\underline{k}\cdot\underline{x}}\, E(\underline{X}, \underline{x})\,\underline{\xi}^-_\lambda(\underline{X}, \underline{k})$$

(IV-5b)

$$\psi_\sigma(\underline{x}) = \frac{1}{\sqrt{V}} \sum_{\underline{X}, \underline{K}} a_\sigma(\underline{X}, \underline{K})\, u_{\sigma\underline{K}}(\underline{x})\, E(\underline{X}, \underline{x}) \qquad \text{(IV-5c)}$$

where

$$\underline{\xi}^\pm_\lambda(\underline{X}, \underline{k}) = \alpha^+_\lambda(\underline{X}, \underline{k})\,\underline{\varepsilon}_\lambda(\underline{k}) \pm \alpha_\lambda(\underline{X}, -\underline{k})\,\underline{\varepsilon}_\lambda(-\underline{k});$$

$\omega_\lambda(\underline{\mathbf{X}}, k)$ is the oscillation frequency associated with the cell located at $\underline{\mathbf{X}}$; $\left\{ u_{\sigma \underline{K}}(\underline{x}) \right\}$ are a complete set of particle eigenstates that will be chosen later in accordance with the radiation mechanism under discussion; and $a_\sigma^+(\underline{\mathbf{X}}, \underline{K})$ and $a_\sigma(\underline{\mathbf{X}}, \underline{K})$ are creation and destruction operators for particles of type σ and momenta $\hbar \underline{K}$ in the cell located at $\underline{\mathbf{X}}$. They have commutation relations

$$\left[a_\sigma(\underline{\mathbf{X}}, \underline{K}), a_{\sigma'}^+(\underline{\mathbf{X}}', \underline{K}') \right]_{\pm} = \delta_{\sigma\sigma'} \, \delta_{\underline{\mathbf{X}}, \underline{\mathbf{X}}'} \, \delta(\underline{K} - \underline{K}') \qquad \text{(IV-6)}$$

where the plus and minus signs refer to fermions and bosons respectively. The step function $E(\underline{\mathbf{X}}, \underline{x})$ can be written

$$E(\underline{\mathbf{X}}, \underline{x}) = E(\mathbf{X}_1, x_1)\, E(\mathbf{X}_2, x_2)\, E(\mathbf{X}_3, x_3) \qquad \text{(IV-7)}$$

where

$$\begin{aligned} E(\mathbf{X}_j, x_j) &= 1 \quad \text{for } \mathbf{X}_j - \frac{L}{2} < x_j < \mathbf{X}_j + \frac{L}{2} \\ &= \frac{1}{2} \quad \text{for } x_j = \begin{cases} \mathbf{X}_j - \frac{L}{2} \\ \mathbf{X}_j + \frac{L}{2} \end{cases} \\ &= 0 \quad \text{otherwise.} \end{aligned} \qquad \text{(IV-8)}$$

Two properties of E which are of special interest are

1) $E(\mathbf{X}_j, x_j)\, E(\mathbf{X}_j', x_j) = E(\mathbf{X}_j, x_j)\, \delta_{\mathbf{X}_j, \mathbf{X}_j'}$ (IV-9a)

2) $\dfrac{d\,E(\mathbf{X}_j, x_j)}{dx_j} = \delta(x_j - \mathbf{X}_j + \frac{L}{2}) - \delta(x_j - \mathbf{X}_j - \frac{L}{2}).$ (IV-9b)

The cell procedure, as introduced here, has some unconventional aspects. It is seen, for instance, that

$$\underline{\nabla} \cdot \underline{A} = \sqrt{\frac{2\pi \hbar c^2}{L^3}} \sum_{\underline{X}, \underline{k}\lambda} \frac{e^{-i\underline{k} \cdot \underline{x}}}{\sqrt{\omega_\lambda(\underline{X}, k)}} \underline{\xi}^{+}_{\lambda}(\underline{X}, \underline{k}) \cdot \underline{\nabla} E(\underline{X}, \underline{x}),$$

which vanishes within the cell, but not on the boundary. In a similar manner, the $(\underline{\nabla} \times \underline{A})^2$ term in the Hamiltonian will result in the appearance of new terms which we will call H^T. It must be kept in mind that the cell procedure was introduced to perform the operation $\mathrm{Tr}\, D\left[H, \rho_\lambda\right]$, so that the effect of these additional terms should be judged with respect to the equation for $\dot{F}_\lambda$ rather than on some intermediate stage where it may appear unconventional. In fact, the terms in H^T will lead to photon transport, as well as other meaningful effects which will be discussed later.

After a considerable amount of straightforward calculation we can write the Hamiltonian as

$$H = H^o + H^{(1)} + H^{(2)} + H^{\gamma T} + H^{PT} + H^{IT} + V^{cc} \qquad \text{(IV-10a)}$$

where

$$H^o = T^{\gamma o} + H^M$$

$$H^M = T^P + H^{Pe} + V^c \qquad \text{(IV-10c)}$$

$$H^{(1)} = H^{P\gamma 1} + H^{P\gamma e} \qquad \text{(IV-10d)}$$

$$H^{(2)} = (H_o^{P\gamma 2} + T^{\gamma 1}) + (H_1^{P\gamma 2} + T^{\gamma 2}) + H_2^{P\gamma 2} \qquad \text{(IV-10e)}$$

$$H^{\gamma T} = H_1^{\gamma T} + H_2^{\gamma T} . \qquad \text{(IV-10f)}$$

The terms H^o, H^M and $H^I = H^{(1)} + H^{(2)}$ are the same as those discussed in Chapter II. They refer to photon and particle energies and photon-particle interactions within cells, and can be written

$$H^o + H^M + H^I = \sum_{\underline{X}} \left[H^o(\underline{X}) + H^M(\underline{X}) + H^I(\underline{X}) \right] .$$

The terms $H^{\gamma T}$ have already been mentioned. The term H^{PT} is associated with particle transport and H^{IT} will describe an interaction in which a particle moves from one cell to another during an emission or absorption of a photon. Lastly, the term V^{cc} is the Coulomb interaction for particles in different cells. All of the terms are shown in Appendix D.

The Schroedinger equation can now be written

$$i\hbar \frac{\partial \Psi}{\partial t} = (H^o + H^I + H^{\gamma T} + H^{PT} + H^{IT} + V^{cc}) \quad . \qquad \text{(IV-11)}$$

The set of eigenfunctions $\left\{ |n\eta> \right\}$ mentioned earlier are chosen such that

$$H^o \, | \, n\eta> = \sum_{\underline{X}} \left(E_n(\underline{X}) + \mathcal{E}_\eta(\underline{X}) \right) | \, n\eta> .$$

The eigenlabels $n\eta$ specify the number of particles and photons of all possible momenta (and of different kinds of particles and different photon polarization) within every cell; i. e.

$$|n\eta> = |n_{\underline{X}_1\sigma_1 K_1} \ldots n_{\underline{X}_1\sigma_j K_j}\ldots ; n_{\underline{X}_2\sigma_1 K_1}\ldots n_{\underline{X}_2\sigma_j K_j}\ldots ;$$

$$\ldots ; \eta_{\underline{X}_1\lambda_1 k_1} \ldots \eta_{\underline{X}_1\lambda_j k_j}\ldots ; \eta_{\underline{X}_2\lambda_1 k_1}\ldots ; \ldots > .$$

Expanding $\Psi = \sum_{n\eta} b_{n\eta}(t) \, | \, n\eta >$, we observe that the diagonal elements of the density matrix $D_{n\eta n\eta} = | \, b_{n\eta} |^2$ give the probability of finding the entire system in the state $| n\eta >$.

It will be convenient for further calculations to transform to an interaction representation. Let

$$\Psi = U \Phi \tag{IV-12}$$

where $U = e^{-i(H^o + H^{\gamma T})t/\hbar}$ satisfies the equation

$$i\hbar \frac{\partial U}{\partial t} = (H^o + H^{\gamma T}) U(t) . \tag{IV-13}$$

Because U is unitary, we can write

$$F_\lambda = \frac{1}{V} \mathrm{Tr}\, \bar{\rho}\, \bar{D}, \tag{IV-14}$$

where

$$\bar{D} = U^+ DU \text{ and } \bar{\rho} = U^+ \rho U . \tag{IV-15}$$

Then

$$\dot{F}_\lambda = \frac{1}{V} \mathrm{Tr} \left[\dot{\bar{\rho}}_\lambda \bar{D} + \bar{\rho}_\lambda \dot{\bar{D}} \right]. \tag{IV-16}$$

It follows from equation (IV-13) that

$$\dot{\bar{\rho}} = (\frac{i}{\hbar}) \left[\overline{H^o + H^{\gamma T}} , \bar{\rho}_\lambda (\mathbf{X}, \underline{k}) \right]. \tag{IV-17}$$

The term $\mathrm{Tr}\, \dot{\bar{\rho}}\, \bar{D}$ leads to photon transport and will be designated $\dot{F}_\lambda^T$. From equation (IV-17), and noting that

$$\left[H^o, \rho_\lambda (\mathbf{X}, \underline{k}) \right] = 0,$$

we find

$$\dot{F}_\lambda^T = \frac{1}{V}\left(\frac{i}{\hbar}\right) \mathrm{Tr}\, D \left[H^{\gamma T}, \rho_\lambda(\underline{\mathbf{X}}, \underline{k}) \right]$$

(IV-18)

$$= \left(\frac{i}{\hbar}\right) \frac{1}{V} \sum_{\substack{n\eta \\ n'\eta'}} D_{n\eta n'\eta'} \left[H^{\gamma T}, \rho_\lambda(\underline{\mathbf{X}}, \underline{k}) \right]_{n'\eta' n\eta} .$$

By straightforward calculation, we find

$$\left[H_1^T, \rho_\lambda(\underline{\mathbf{X}}, \underline{k}) \right] \simeq i\hbar c^2 k_i \frac{\partial}{\partial \mathbf{X}_i} \frac{F_\lambda(\underline{\mathbf{X}}, \underline{k}, t)}{\omega_\lambda(\underline{\mathbf{X}}, k)}$$

$$+ i\hbar \frac{c^2}{2} \left[\sum_{\lambda' k'}{}' \frac{(k_1 + k_1') \underline{\mathcal{E}}_{\lambda}(\underline{k}') \cdot \underline{\mathcal{E}}_\lambda(\underline{k})}{\sqrt{\omega_\lambda(\underline{\mathbf{X}}, k)}} \delta(k_2 - k_2') \delta(k_3 - k_3') \right.$$

$$(x) \left\{ \alpha_\lambda(\underline{\mathbf{X}}, \underline{k}) \frac{\partial}{\partial \mathbf{X}_i} \frac{\alpha^+_{\lambda'}(\underline{\mathbf{X}}, \underline{k}')}{\omega_{\lambda'}(\underline{\mathbf{X}}, k')} \right.$$

$$\left. + \alpha^+_\lambda(\underline{\mathbf{X}}, \underline{k}) \frac{\partial}{\partial \mathbf{X}_i} \frac{\alpha_{\lambda'}(\underline{\mathbf{X}}, \underline{k}')}{\sqrt{\omega_{\lambda'}(\underline{\mathbf{X}}, k')}} \right\}$$

$$\left. + \text{cyclic permutation} \right] ,$$

(IV-19)

where the operation $\frac{\partial}{\partial \mathbf{X}_i}$ operating on a function h is defined by

$$\frac{\partial h(\underline{\mathbf{X}})}{\partial \mathbf{X}} \equiv \frac{h(\mathbf{X}_1 + L, \mathbf{X}_2 \mathbf{X}_3) - h(\underline{\mathbf{X}})}{L} .$$

(IV-20)

Equation (IV-19) is not exact. Letting $h(\mathbf{X}_j)$ be each of the above functions which follow a differentiation, we have approximated

$$\frac{\partial h(\mathbf{X}_j - L)}{\partial \mathbf{X}_j} \simeq \frac{\partial h(\mathbf{X}_j)}{\partial \mathbf{X}_j} - L \frac{\partial^2 h(\mathbf{X}_j)}{\partial \mathbf{X}_j^2} + \ldots$$

by $\frac{\partial h(\mathbf{X}_j)}{\partial \mathbf{X}_j}$, neglecting all higher derivatives. This implies, for instance, that $\frac{F}{\omega}$ must be slowly varying, and partially specifies the upper bound on the cell size.

Assuming the off-diagonal elements of the density matrix in equation (IV-18) give negligible contribution, we find that only H_1^T is of importance, and then only the first term. Thus

$$\dot{F}_\lambda^T \simeq \underline{\Omega} \cdot \underline{\nabla} v^\gamma(\underline{\mathbf{X}}, k)\, F_\lambda(\underline{\mathbf{X}}, \underline{k}, t) \qquad \text{(IV-21)}$$

where

$$v^\gamma = \frac{c^2 k}{\omega_\lambda(\underline{\mathbf{X}}, k)} = c\,\mu_\lambda(\underline{\mathbf{X}}, k)\,.$$

Equation (IV-21) indicates that v^γ should be operated on by the divergence. Note that for the nondispersive medium we obtain the expected result $v^\gamma = c$.

The photon speed obtained in Chapter II was given by $v^\gamma \equiv v_g = \frac{\partial\omega}{\partial k}$ whereas in equation (IV-21) we find $v^\gamma = c\mu_\lambda(\underline{\mathbf{X}}, k)$. In order to determine the difference in these results, let us consider two isotropic dispersive media, the fully ionized gas and the neutral, one species gas.

The dispersion relation for the fully ionized gas was given by equation (III-10) as

$$\omega^2 = c^2k^2 - \omega\omega_p^2 \int d^3 v \, \frac{f(v)}{(\omega - \underline{k} \cdot \underline{v})} \qquad \text{(III-10)}$$

We find, with a few manipulations, that

$$v_g = \frac{\partial \omega}{\partial k} = c\mu \left[\frac{1 + \frac{\omega_p^2}{2c}\, \mu\, P \int d^3 v\, f(v)\, \frac{\hat{\underline{k}} \cdot \underline{v}}{(\omega - \underline{k}\cdot\underline{v})^2}}{1 + \frac{\omega_p^2}{2c}\, \frac{1}{\mu}\, P \int d\; v\, f(v)\, \frac{\hat{\underline{k}} \cdot \underline{v}}{(\omega - \underline{k}\cdot\underline{v})^2}} \right] \qquad \text{(IV-22)}$$

where $\hat{\underline{k}}$ is the unit vector in the direction k. It is seen, then, that if the index of refraction μ is not appreciably different from one, $(\omega >> \omega_p)$, then $v_g \simeq v^\gamma = c\mu$. In the zero temperature limit, $v_g = c\mu$ is exact. As we will see in Chapter VII, when $\mu \to 0$ the absorption coefficient gets very large and the present analysis becomes of questionable validity, so that the frequency region for which μ does not differ greatly from one is the region where our development is expected to have greatest validity.

For the monomolecular gas, we found in Chapter III that

$$\mu_\lambda^2 - 1 = \frac{8\pi N e^2}{V} \sum_K \frac{\omega_{K0} \left| (\underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{x})_{K0} \right|^2}{\hbar(\omega_{K0}^2 - \omega_\lambda^2)} . \qquad \text{(III-21)}$$

If $\omega_\lambda >> \omega_{K0}$, this becomes

$$\frac{c^2 k^2}{\omega_\lambda^2} \simeq 1 - \frac{8\pi N e^2}{\omega_\lambda^2 V} \sum_K \frac{\omega_{K0}}{\hbar} \left| (\underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{x})_{K0} \right|^2$$

so that, multiplying by ω_λ^2, we obtain

$$\frac{\partial \omega_\lambda}{\partial k} = v_g = c\mu_\lambda .$$

The frequencies for which $\omega_\lambda >> \omega_{K0}$ are above the resonance region.

For $\omega_\lambda << \omega_{K0}$ we are below the resonance region where, from equation (III-21), it is found that $v_g \neq v^\gamma = c\mu$. However, we have no assurance from either derivation of the transport term that it is valid to pass through the resonance region where, because of strong absorption, the transport equation is not expected to be appropriate.

We see then that in the frequency region where the present analysis is expected to be valid, the photon speed given in Chapter II and the one obtained in this chapter are approximately the same and it is not obvious, from the point of view of measurement, which is more appropriate.

The second term in equation (IV-16) accounts for interactions, and we will write

$$\dot{F}^{I}_{\lambda} = \frac{1}{V} \operatorname{Tr} \bar{\rho}_{\lambda} \dot{\bar{D}} . \qquad \text{(IV-23)}$$

The time coarse-graining will be employed in a manner similar to that already used in Chapter II. Thus

$$\dot{\bar{D}} \simeq \frac{\bar{D}(t+s) - \bar{D}(t)}{s} , \qquad \text{(IV-24)}$$

We will again require that $s\omega >> 1$ and $s/\tau << 1$, where ω is the frequency of the photon which was emitted, absorbed or scattered, and τ is the lifetime of a state.

Equation (IV-23) now becomes

$$\dot{F}^{I}_{\lambda} \simeq \frac{1}{Vs} \operatorname{Tr} \bar{\rho} \left[\bar{D}(t+s) - \bar{D}(t) \right]$$

$$= \frac{1}{Vs} \left[\operatorname{Tr} \bar{\rho}_{\lambda} \bar{D}(t+s) - \operatorname{Tr} \rho_{\lambda} D(t) \right] . \qquad \text{(IV-25)}$$

Recall that

$$\bar{D}(t+s) = U^{+}(t+s)\, D(t+s)\, U(t+s)$$

so that

$$\mathrm{Tr}\,\bar{\rho}_\lambda\;\bar{D}(t+s) = \mathrm{Tr}\,U(t+s)\,U^{+}(t)\,\rho_\lambda\,U(t)\,U^{+}(t+s)\,D(t+s)$$

$$= \mathrm{Tr}\,U(s)\,\rho_\lambda\,U^{+}(s)\,D(t+s)\,. \tag{IV-26}$$

But

$$D(t+s) = \bar{U}(s)\,D(t)\,\bar{U}^{+}(s) \tag{IV-27}$$

where

$$\bar{U}(s) = e^{-iHs/\hbar}\,, \tag{IV-28}$$

Utilizing the Feynmann calculus[30] , we write

$$\bar{U}(s) = U(s)\,(I + Q^{I}) \tag{IV-29}$$

where

$$Q^{I} = \sum_{j=1}^{\infty} \left(-\frac{i}{\hbar}\right)^{j} \int_{0}^{s} ds_1 \ldots \int_{s_{j=0}}^{s_{j-1}} ds_j\; \overline{(H^{I}+H^{IT})}_1 \ldots \overline{(H^{I}+H^{IT})}_j \tag{IV-30}$$

and

$$\overline{(H^{I}+H^{IT})}_1 = e^{i(H^{o}+H_1^{T})s_1/\hbar}\,(H^{I}+H^{IT})\,e^{-i(H^{o}+H_1^{T})s_1/\hbar}\,. \tag{IV-31}$$

The terms H^{PT} and V^{cc} have been dropped from the discussion because they do not account for any photon processes. With the aid of equation (IV-29), equation (IV-26) becomes

$$\mathrm{Tr}\,\bar{\rho}_\lambda\;\bar{D}(t+s) = \mathrm{Tr}\,\rho_\lambda\,(I+Q^{I})\,D(I+Q^{I^{+}}) \tag{IV-32}$$

so that

$$\mathrm{Tr}\left[\bar{\rho}\ \bar{D}(t+s) - \rho\ D(t)\right]$$

$$= \mathrm{Tr}\left[Q^I D + D\, Q^{I^+} + Q^I D\, Q^{I^+}\right]$$

$$= \sum_{n\eta} \ \underset{\lambda \underline{k}}{\overset{\eta}{\mathbf{\Sigma}}} \ \sum_{n'\eta'} \left[D_{n'\eta' n'\eta'} \left| Q^I_{n\eta n'\eta'} \right|^2 \right.$$

$$\left. - D_{n\eta n\eta} \left| Q^I_{n'\eta' n\eta} \right|^2 \right] + \left\{ \text{terms proportional to the off-diagonal elements of the density matrix.} \right\} \quad \text{(IV-33)}$$

These latter terms will be assumed to give negligible contribution.

We turn now to a calculation of $Q^I_{n\eta n'\eta'}$. There are two contributions to this matrix element, one from H^I and the other from H^{IT}. In order to indicate the relative importance of these terms, we calculate H^I and H^{IT} using plane wave states for the particles and find,

$$\frac{H^{IT}}{H^I} \sim \frac{1}{KL} \ .$$

This quantity is much smaller than one for all cell sizes of interest. Thus, we may expect that H^{IT} will give a negligible contribution and it will be dropped henceforth.

The interaction processes of interest are contained in $Q^{I(1)}$, where

$$Q^{I(1)}_{n\eta n'\eta'} = \left(-\frac{i}{\hbar}\right) \int_0^s ds_1 \left(e^{i(H^o + H_1^T) s_1/\hbar}\ H^I\ e^{-i(H^o + H_1^T) s_1/\hbar} \right)_{n\eta n'\eta'} \ . \quad \text{(IV-34)}$$

Writing

$$e^{-i(H^o + H_1^T) s_1/\hbar} = e^{-iH^o s_1/\hbar} \left(I + Q^T(s_1)\right) \quad \text{(IV-35)}$$

with

$$Q^T(s_1) = \sum_{j=1}^{\infty} (-\frac{i}{\hbar})^j \int_0^{s_1} \cdots \int_{s'_j=0}^{s'_{j-1}} ds'_1 \cdots ds'_j \left(e^{iH^0 s'_1/\hbar} H_1^T e^{-iH^0 s'_1/\hbar}\right)$$

$$(x) \ldots (x) \left(e^{iH^0 s'_j/\hbar} H_1^T \; e^{-iH^0 s'_j/\hbar}\right) \qquad \text{(IV-36)}$$

the expression (IV-34) can be written

$$Q^{I(1)}_{n\eta n'\eta'} = (-\frac{2i}{\hbar}) H^I_{n\eta n'\eta'} \left(e^{i\frac{\omega_{n\eta n'\eta'}}{2} s}\right) \frac{\sin \frac{\omega_{n\eta n'\eta'} s}{2}}{\omega_{n\eta n'\eta'}} + J_{n\eta n'\eta'} \qquad \text{(IV-37)}$$

where

$$J_{n\eta n'\eta'} = (-\frac{i}{\hbar}) \int_0^s ds_1 \left[Q^{T^+}(s_1)\, U^{0^+} H^I U^0 + U^{0+} H^I U^0 Q^T (s_1)\right.$$

$$\left. + Q^{T^+}(s_1)\, U^{0^+} H^I\, U^0\, Q^T(s_1)_{\,n\eta n'\eta'}\right] . \qquad \text{(IV-38)}$$

The first term is the usual interaction term for photons within a cell. The remaining terms describe processes in which both transport and interactions take place, e. g. creation in the cell $\mathbf{X}$ - L and transport to the cell $\mathbf{X}$. Let us examine the first of these "correction" terms. To lowest order

$$J^{(1)}_{n\eta n'\eta'} = (-\frac{i}{\hbar}) \sum_{n''\eta'} \int_0^s ds_1 Q^{T(1)^+}_{n\eta n''\eta'} (s_1)\, H^T_{1\, n''\eta' n'\eta'}\, e^{i\omega_{n''\eta' n'\eta'} s_1} \qquad \text{(IV-39)}$$

where

$$Q^{T(1)^+}_{n\eta n''\eta'} (s_1) = Q^{T(1)^*}_{n''\eta' n\eta} (s_1) = (-\frac{i}{\hbar}) H^T_{1n''\eta' n\eta} \int_0^{s_1} ds_2\, e^{i(\omega_{n\eta} - \omega_{n''\eta'}) s_2} .$$

The exponent of the exponential contains the factor $s(\omega_{n\eta}-\omega_{n''\eta'})$, where $\omega_{n\eta}$ is the frequency of the photon in the cell $\underline{X}$ and $\omega_{n''\eta'}$ is the frequency of this photon in an adjacent cell. This change of frequency reflects the condition that in an inhomogeneous medium, a photon of a given k will have a different energy in each cell. Further discussion of this point will be given in Chapter VI. If we restrict our systems of interest to be those for which

$$\frac{L}{\omega}\,\frac{\partial\omega}{\partial \underline{X}} << 1$$

we may expect that $\omega_{n\eta} \simeq \omega_{n''\eta'}$ and, consequently

$$s(\omega_{n\eta} - \omega_{n''\eta'}) << 1 .$$

Using this condition we obtain

$$Q^{T(1)^*}_{n''\eta' n\eta}(s_1) \cong \left(\frac{-i s_1}{\hbar}\right) H^T_{1n''\eta' n\eta} . \qquad \text{(IV-40)}$$

The magnitude of this term (as seen from Appendix D) is indicated by the quantity $\frac{s_1 c^2 k}{\omega L} = \frac{s_1 v^\gamma}{L}$ where v^γ is the photon speed. If now we require the cell to be of such size that $\frac{s_1 v^\gamma}{L} << 1$ then Q^T will be very small and the terms J can be neglected. The implication of this condition is that s should be small compared to the time for a photon to cross a cell. This restriction was also used by Ono[(29)] in a similar context.

Note that $\frac{s v^\gamma}{L} = \frac{s\omega}{kL}\mu^2$, so that for μ not appreciably different from one, $s\omega << kL$. But $s\omega >> 1$, so that $\frac{1}{kL} << 1$. This implies that the wavelength of the photon must be much smaller than the cell size.

From equation (IV-25), (IV-33) and (IV-37), we obtain, finally

$$\dot{F}^{I}_{\lambda}(\underline{\mathbf{X}}, \underline{k}, t) = \sum_{n\eta} \eta_{\underline{\mathbf{X}}\lambda\underline{k}} \sum_{n'\eta'} \left[D_{n'\eta' n'\eta'} \frac{W^{(1)}_{n\eta n'\eta'}}{V} - D_{n\eta n\eta} \frac{W^{(1)}_{n'\eta' n\eta}}{V} \right], \tag{IV-41}$$

where

$$W^{(1)}_{n\eta n'\eta'} = \lim_{s\to\infty} \frac{4}{\hbar^2} \left| H^{I}_{n\eta n'\eta'} \right|^2 \frac{\sin^2 \frac{\omega_{n\eta} - \omega_{n'\eta'}}{2} s}{s(\omega_{n\eta} - \omega_{n'\eta'})^2}$$

$$= \frac{2\pi}{\hbar^2} \left| H^{I}_{n\eta n'\eta'} \right|^2 \delta(\omega_{n\eta} - \omega_{n'\eta'}) . \tag{IV-42}$$

The quantity $\dot{F}^{I}_{\lambda}$ indicates the rate of change of the number of photons in the cell $\mathbf{X}$ by interactions. It would appear from equation (IV-33), however, that the sum over $n'\eta'$ implies we are summing over the interactions in all cells. Except for the cell $\mathbf{X}$, the sum over $n\eta$ can be taken to the right of $\eta_{\underline{\mathbf{X}}\lambda\underline{k}}$ and all of the terms will cancel. Thus the states $|n\eta\rangle$ and $|n'\eta'\rangle$ can be considered the same except for photon and particle numbers in the cell $\mathbf{X}$.

Lastly, it should be noted that the procedure used here does not provide us with a condition for determining $\omega_{\lambda}(\underline{\mathbf{X}}, k)$. This was not because of any fundamental difficulty in developing such a procedure, but rather because of expediency. The present procedure took us quickly to a discussion of Q^{T} and of the relative importance of the interaction terms. The alternative procedure, in which a level shift analysis is introduced, is quite lengthy for considering the additional terms arising from a combination of interactions and transport. It leads to the same final result, however, and the condition for determining $\omega_\lambda(\underline{X}, k)$ is the same one discussed in Chapter II.

The final reduction to a photon balance equation is the same as we have already shown in Chapter II, and will not be repeated here. Our final result is the same as equation (II-47).

V. THERMODYNAMICS

The deduction of equilibrium properties for our system is conveniently achieved by first obtaining an H-theorem to describe the irreversible nature of the approach to equilibrium. The development of such a theorem for the system of particles and photons is not possible from equation (II-47) since this equation does not describe the rate of change of particles. An additional equation would be required. This procedure has been employed by Osborn (19) to obtain an H-theorem for the complete system.

Our deduction will follow the argument presented in reference (9) which is based upon the use of the Pauli equation,

$$\dot{D}_{n\eta n\eta} = \sum_{m\alpha} W_{n\eta m\alpha} \left[D_{m\alpha m\alpha} - D_{n\eta n\eta} \right] . \qquad \text{(II-37)}$$

Defining

$$H = \sum_{n\eta} D_{n\eta n\eta} \ln D_{n\eta n\eta} \qquad \text{(V-1)}$$

it is easily shown by use of equation (II-37) that

$$\frac{dH}{dt} \leq 0 , \qquad \text{(V-2)}$$

the equality holding when

$$D_{n\eta n\eta} = D_{m\alpha m\alpha} .$$

Since $D_{n\eta n\eta}(t) = P(n\eta, t)$ is the probability of finding the system in the state $|n\eta>$ at time t, the H-theorem suggests the identity

$$S = -k_0 H$$

where k_0 is the Boltzmann constant and where S, the entropy, obeys

$$\frac{dS}{dt} \geq 0$$

with S a maximum for the equilibrium state.

By standard thermodynamic argument (44), it is readily established that

$$F_\lambda(\underline{k}) = \sum_\eta \eta_{\lambda k} P(\eta) = \frac{1}{e^{\beta\hbar\omega_\lambda} - 1} \qquad \text{(V-3a)}$$

$$f(\underline{K}) = \sum_n n_K P(n) = \frac{1}{Be^{\beta E_K} \pm 1} \qquad \text{(V-3b)}$$

where $\beta = 1/k_0T$, and where the $(\pm)$ in equation (V-3b) indicates fermions and bosons, respectively.

The equilibrium radiation law is obtained by converting to the continuum in photon momentum space, i.e.

$$\sum_{\underline{k}\epsilon d^3k} F_\lambda(\underline{k}) = F_\lambda^0\, d\,k\, d\,\Omega(k) = \frac{F_\lambda \rho_k}{V}\, dk d\Omega$$

where

$$\rho_k = \frac{Vk^2}{(2\pi)^3}$$

is the density of states. Defining the energy density $U_\lambda(\underline{k})$ by $U_\lambda(\underline{k}) \equiv \hbar\omega_\lambda F_\lambda^0$ and transforming to ν_λ space ($\nu_\lambda = \frac{\omega_\lambda}{2\pi}$), we find, after integrating over

$\underline{\Omega}$, that

$$U_\lambda(\nu_\lambda)\,d\nu_\lambda = \frac{4\pi h}{c^3}\,\frac{\partial\nu_o}{\partial\nu_\lambda}\,\frac{\nu_\lambda\,\nu_o^2}{(e^{h\nu_\lambda/kT}-1)}\,d\nu_\lambda \qquad \text{(V-4)}$$

where $\nu_o = \frac{ck}{2\pi}$. This is the same result obtained by Landau and Lifshitz (16) for black body radiation in a transparent medium. Letting $\nu_\lambda = \nu_o$ and summing over polarization, we obtain the usual black body radiation spectrum

$$U(\nu_o) = \frac{8\pi h}{c^3}\,\frac{\nu_o^3}{e^{h\nu_o/kT}-1} \,. \qquad \text{(V-5)}$$

In radiation transfer problems, the quantity $I_\lambda = v_g\, U(\nu_\lambda)$ is of more direct interest. It is seen from equation (V-4) that

$$I_\lambda\, d\nu_\lambda = \frac{4\pi h\nu_\lambda^3\,\mu_\lambda^2}{(e^{h\nu_\lambda/kT}-1)\,c^2}\,d\nu_\lambda \qquad \text{(V-6)}$$

where $\mu_\lambda = \frac{ck}{\omega_\lambda}$ is the index of refraction. Note that I_λ/μ_λ^2 is independent of the medium.

VI. THE RADIATIVE TRANSFER EQUATION

A. Reduction of the Photon Balance Equation

The photon balance equation (II-47) describes the rate of change of the expected number of photons in the cell located at the position $\mathbf{X}$, with wave number k in dk, going into direction $\underline{\Omega}$ in $d\Omega$. As pointed out in Chapter I, the quantity usually studied in astrophysics is the radiation intensity $I_\lambda(\underline{x}, \omega, \underline{\Omega}, t)$. In this chapter, equation (II-47) will be reduced to an equation for the intensity I_λ. The resultant equation will differ in appearance from the phenomenological equation (I-3). In order to compare them, we will calculate the radiation from a plane slab of plasma and compare the results.

It will be assumed that the number of occupied particle states in a given energy range is small compared to the number of states available, so the degenerate particle systems are excluded and Boltzmann statistics apply. Recalling equations (II-46), (II-47) and (II-48), we define

$$\epsilon_\beta^\lambda \equiv \sum_{KK_1} V^2 T_{\beta K}^{K_1} (\lambda \underline{k}) f(\underline{K}) \tag{VI-1a}$$

$$\gamma_\beta^\lambda \equiv \sum_{KK_1} V^2 T_{\beta K}^{K_1} (\lambda \underline{k}) f(\underline{K}_1) \tag{VI-1b}$$

$$\alpha_\beta^\lambda \equiv \sum_{KK_1} V^2 T_{\beta K}^{K_1} (\lambda \underline{k}) \left[f(\underline{K}_1) - f(\underline{K})\right] \tag{VI-1c}$$

$$s_i^\lambda \equiv \sum_{\lambda'} \int_{\underline{K}, \underline{K}_1, \underline{k}'} d^3K \, d^3K_1 dk' \, d\Omega' \, V^3 \, S_{K_1, \lambda' k'}^{K, \lambda k} \, F_{\lambda'}^{o}(\underline{k}') f^{o}(\underline{K}_1) \tag{VI-1d}$$

$$s_o^{\lambda} \equiv \sum_{\lambda'} \int_{\underline{K}, \underline{K}_1, \underline{k}'} d^3K\, d^3K_1\, dk'\, d\Omega'\, V^3\, S_{K_1, \lambda' k'}^{K, \lambda k} \left[F_{\lambda'}^{o}(\underline{k}') + \frac{\rho_{k'}}{V} \right] f^o(\underline{K}) \qquad \text{(VI-1e)}$$

where $F_{\lambda}^{o}(\underline{k})$, ρ_k and $f^o(\underline{K})$ are defined by

$$\sum_{\underline{k} \epsilon d^3 k} F_{\lambda}(\underline{k}) = F_{\lambda}^{o}\, dk\, d\Omega(k) \qquad \text{(VI-2a)}$$

$$\sum_{\underline{k} \epsilon\, d^3 k} (1) = \rho_k dk\, d\Omega(k) = \frac{V\, k^2 dk\, d\Omega(k)}{(2\pi)^3} \qquad \text{(VI-2b)}$$

$$\sum_{\underline{K} \epsilon\, d^3 K} f(\underline{K}) = f^o(\underline{K})\, d^3K. \qquad \text{(VI-2c)}$$

Then equation (II-47), the photon balance relation, can be written

$$\dot{F}_{\lambda}^{o}(\underline{k}) + \underline{\Omega} \cdot \underline{\nabla}\, v^{\gamma} F_{\lambda}^{o}(\underline{k})$$

$$= \sum_{\beta} \left[\frac{\epsilon_{\beta}^{\lambda} \rho_k}{V} - \alpha_{\beta}^{\lambda}(\underline{k})\, F_{\lambda}^{o}(\underline{k}) \right] \qquad \text{(VI-3)}$$

$$+ \frac{s_i \rho_k}{V} - (s_o - s_i)\, F_{\lambda}^{o}(\underline{k}).$$

For nonrelativistic plasma systems, the scattering rate is usually small in comparison with at least one of the absorption and emission processes. Even with free electron densities of the order 10^{18} electrons per cm, the mean free path is of order 10^6 cm. For large stellar systems, scattering can still become important and a considerable body of literature exists treating this case[2, 3].

For laboratory-size systems the effect of scattering is expected to be negligible, and for the remainder of our discussion, the scattering process will not be given further consideration.

The radiation transport equation is usually exhibited as a function of frequency rather than k. If the medium is spatially homogeneous with respect to particle distributions, ω, v^{γ} and μ_{λ} are independent of position. When the medium is inhomogeneous, however, ω, for a given k, will be different at different places in the medium. It was seen in Chapter IV that k rather than ω, was the natural variable for our description, for the assumption was made that only terms in transport for which k remains unchanged as we move across the boundary are important. Modifications to this are possible in principle, but rather impracticable. It was also pointed out in Chapter IV, in connection with terms for which we have photon creation in cell X-L and transport to X (or the inverse) that ω, for a given k, could change only by a small amount from one cell to the next.

The physical origin of the change of ω from one cell to the next was given in Chapter II where the method for choosing ω was developed. The radiation present induces collective particle behavior which can be associated with the polarization of the medium. The "dressed" photons being considered have been defined so that the collective effects are included in the photons. In the inhomogeneous medium, the "dressing" is changing from cell to cell.

Consider the low temperature plasma, for instance where, to a good approximation, the dispersion relations for transverse oscillations is given by

$$\omega^2(\underline{x}) = c^2k^2 + \omega_p^2(\underline{x}).$$

Photons which in free space have energy $\omega = ck$ are now modified as above and ω varies slowly in space as a result of the space dependence of ω_p. Thus, although we have total energy conservation for the system, the energy of the photons changes from cell to cell.

Consider the transformation from the variables $\underline{x}$, k, $\underline{\Omega}$, t to $\underline{x}'$, ω, $\underline{\Omega}'$, t' where

$$\underline{x}' = \underline{x}$$

$$\omega = \omega(\underline{x}\,,\ k)$$

$$\underline{\Omega}' = \underline{\Omega}$$

$$t' = t.$$

Thus

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j'} + \left(\frac{\partial \omega}{\partial x_j}\right) \frac{\partial}{\partial \omega}$$

where $(\partial\omega/\partial x_j)$ is independent of $\underline{x}'$. Defining

$$\phi_\lambda(\underline{x}',\ \omega,\ \underline{\Omega}',\ t')\ d^3x'\ d\omega\ d\Omega' = F_\lambda(\underline{x},\ k,\ \underline{\Omega}, t)\ d^3x\ dk\ d\Omega \qquad \text{(VI-4)}$$

we obtain

$$\frac{\partial \phi_\lambda}{\partial t'} + \underline{\Omega}' \cdot \underline{\nabla}'\ v^\gamma(\underline{x}',\ \omega)\ \phi_\lambda + \underline{\Omega}' \cdot (\underline{\nabla}\ \omega)\ \frac{\partial}{\partial \omega}\ v^\gamma \phi_\lambda$$

$$= \sum_\beta \left[\frac{\epsilon_\beta^\lambda(\underline{x}', \omega)\mu_\lambda^2(\underline{x}', \omega)\ \omega_\lambda^2}{c^2\ (2\pi)^3} \left(\frac{\partial k}{\partial \omega}\right) - \alpha_\beta^\lambda(\underline{x}', \omega)\phi_\lambda \right]. \qquad \text{(VI-5)}$$

The intensity of radiation is given by

$$I_\lambda(\underline{x}, \omega, \underline{\Omega}, t) \equiv \hbar \omega_\lambda v^\gamma \phi_\lambda(\underline{x}, \omega, \underline{\Omega}, t) \qquad \text{(VI-6)}$$

so that (dropping the prime)

$$\frac{1}{v^\gamma} \frac{\partial I_\lambda}{\partial t} + \underline{\Omega} \cdot \underline{\nabla} I_\lambda + \omega \underline{\Omega} \cdot \underline{\gamma} \frac{\partial}{\partial \omega}\left(\frac{I_\lambda}{\omega}\right)$$

$$= \sum_\beta \left[j_\beta^\lambda (\underline{x}, \omega, \underline{\Omega}) - \bar{\alpha}_\beta^\lambda (\underline{x}, \omega, \underline{\Omega}) I_\lambda \right] \qquad \text{(VI-7)}$$

where

$$\underline{\gamma} = (\underline{\nabla}\ \omega)$$

$$j_\beta^\lambda (\underline{x}, \omega, \underline{\Omega}) \equiv \frac{\hbar \epsilon_\beta^\lambda \mu_\lambda^2 \omega_\lambda^3}{c^2 (2\pi)^3} \left(\frac{\partial k}{\partial \omega}\right) \qquad \text{(VI-8a)}$$

$$\bar{\alpha}_\beta^\lambda (\underline{x}, \omega, \underline{\Omega}) \equiv \frac{\alpha_\beta^\lambda}{v^\gamma}\ , \qquad \text{(VI-8b)}$$

For the homogeneous, stationary state system, $\gamma_j = 0$ and

$$\underline{\Omega} \cdot \underline{\nabla} I_\lambda = \sum_\beta \left[j_\beta^\lambda - \bar{\alpha}_\beta I_\lambda \right] \qquad \text{(VI-9)}$$

which is of the same form as equation (I-3) when the index of refraction μ is not a function of position. In thermodynamic equilibrium, with a Maxwellian particle distribution, we have, after integrating over $\underline{\Omega}$,

$$j_\beta^\lambda / \bar{\alpha}_\beta^\lambda = \mu_\lambda^2 B_\lambda(\omega) \qquad \text{(VI-10a)}$$

where

$$B_\lambda(\omega)\, d\omega_\lambda = \frac{d\omega_\lambda}{e^{\hbar \omega_\lambda / k_o T} - 1} \frac{\hbar \omega_\lambda^3}{4\pi^3 c^2}\ . \qquad \text{(VI-10b)}$$

The quantity $\mu_\lambda^2 B_\lambda$ is the same equilibrium distribution obtained in Chapter V from a somewhat different approach.

B. Radiation From a Plane Slab of Plasma

For the stationary state, equation (VI-7) can be written

$$\frac{dI_\lambda}{ds} + \omega_\lambda \left(\frac{d\omega_\lambda}{ds}\right) \frac{\partial}{\partial \omega_\lambda}\left(\frac{I_\lambda}{\omega_\lambda}\right) = \sum_\beta \left[j_\beta^\lambda(\underline{x}, \omega, \underline{\Omega}) - \alpha_\beta^\lambda(\underline{x}, \omega, \underline{\Omega}) I_\lambda \right] \tag{VI-11}$$

where, employing the summation convention,

$$\frac{d}{ds} = \frac{dx_j}{ds} \frac{\partial}{\partial x_j} = \Omega_j \frac{\partial}{\partial x_j} \quad .$$

The comparison of equation (VI-11) with the phenomenological equation

$$\frac{dI}{ds} - \frac{2I}{\mu} \frac{d\mu}{ds} = j - \alpha I \tag{I-3}$$

is not easily accomplished without a knowledge of the intensity.

Rather than attempting to compare these equations as they stand, we will calculate the radiation from a plane slab of plasma and compare the results. Instead of equation (VI-11), however, we will solve equation (VI-3) (without scattering) and then transform to frequency space after obtaining the solution.

In order that this be done unambiguously, we will subdivide the slab into two regions A and B. (See Figure 2a). In region A, of thickness L_1, the absorption and emission coefficients are assumed constant. In region B, of thickness L_2, the density of the particles decreases monatonically from the density in region A to zero. Thus, at the point P, where the radiation emerges going in direction $\underline{\Omega}$ in $d\Omega$, we are in free space. In the end we will take the limit $L_2 \longrightarrow 0$.

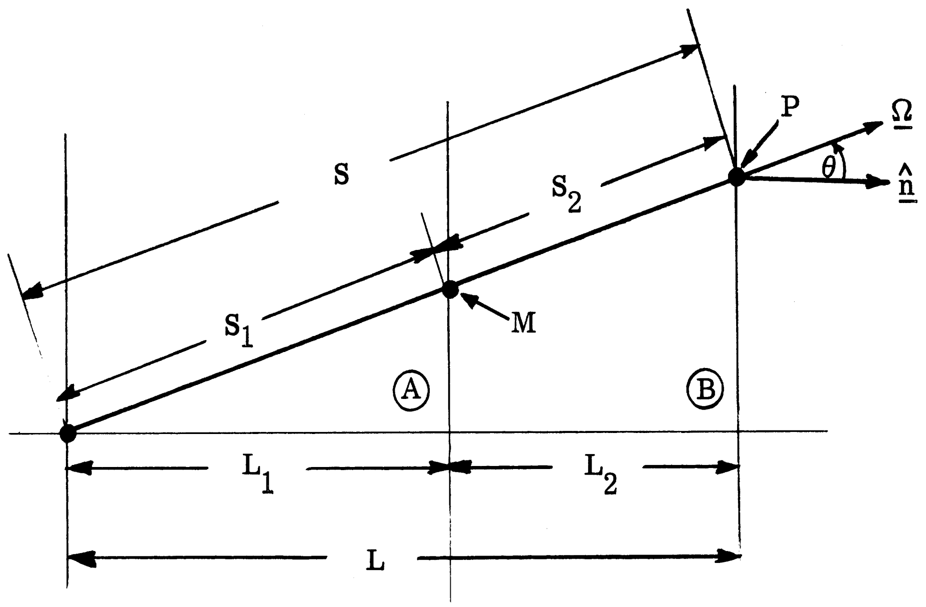

Figure 2a. Plane Two-Region Plasma Slab

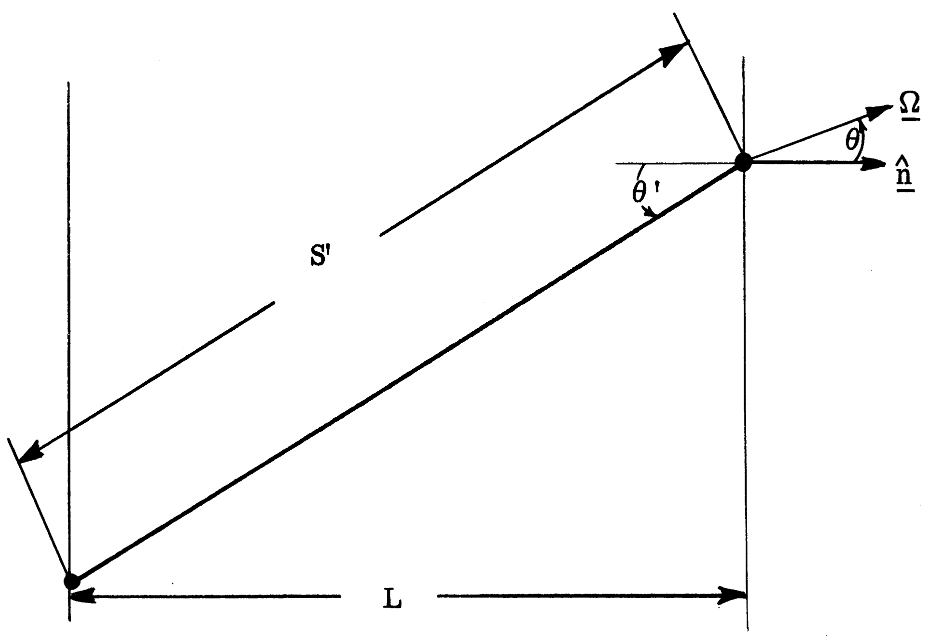

Figure 2b. Plane One-Region Plasma Slab

In the plasma with no external fields, the frequency and the emission and absorption coefficients are independent of the choice of polarization (for linear polarization). Assuming random polarization of the photons, we take $F^o_\lambda = \frac{1}{2} F^o$ and obtain, for the stationary state,

$$\frac{d}{ds} v^\gamma F^o = \frac{2\rho_k \epsilon}{V} - \bar{\alpha} v^\gamma F^o \qquad \text{(VI-12)}$$

where we have dropped the label λ from the emission and absorption coefficients. We have also assumed that bremsstrahlung was the only radiation mechanism. Calculation of these coefficients for bremsstrahlung will be given in Chapter VII, but will not be necessary for our purposes here.

The path of integration for the solution of equation (VI-12) is shown in Figure 2a. The length S is related to L by

$$S = L/\cos\theta$$

where $\cos\theta = \hat{\underline{n}} \cdot \underline{\Omega}$ and $\hat{\underline{n}}$ is the outward unit normal to the slab surface at P. The boundary condition $F^o(0, k, \underline{\Omega}) = 0$ seems appropriate for the present problem.

Assuming a uniform temperature, the solution of equation (VI-12) can now be written as

$$c\,F^o(P, k, \theta) = \int_0^P ds\, \frac{2\epsilon\,\rho_k}{V}\, e^{-\int_s^P ds' \bar{\alpha}(s')}$$

$$= \frac{2\epsilon_a \rho_k}{V\bar{\alpha}_a}\left[1 - e^{-\bar{\alpha}_a S_1}\right] e^{-\int_M^P ds\, \bar{\alpha}_b(s)}$$

$$+ \frac{2\rho_k}{V}\int_M^P ds\, \epsilon_b(s)\, e^{-\int_s^P ds' \bar{\alpha}_b(s')}, \qquad \text{(VI-13)}$$

To perform the remaining integrations, we assume the space dependence

$$\bar{\alpha}_b(s) = \frac{\bar{\alpha}_a(s)}{S_2} (P - s)$$

$$\epsilon_b(s) = \frac{\epsilon_a}{S_2} (P - s).$$

The result is given as

$$c\, F^o(P, k, \theta) = \frac{2\, \rho_k \epsilon_a}{\bar{\alpha}_a V} \left[1 - e^{-\bar{\alpha}_a S_1 - \frac{1}{2} \bar{\alpha}_a S_2} \right]. \qquad \text{(VI-14)}$$

Finally, taking $S_2 \longrightarrow 0$ and $S_1 \longrightarrow S$ we obtain

$$F^o(P, k, \theta)\, dk = \frac{k^2\, dk\, \epsilon_a}{4\, \pi^3\, \bar{\alpha}_a\, c} \left[1 - e^{-\bar{\alpha}_a S} \right]. \qquad \text{(VI-15)}$$

At the point P, we still use free space conditions, so we can write

$$F^o(P, k, \theta)\, dk = \phi\, (P, \omega, \theta)\, d\omega = d\omega\, \frac{\omega^2}{4\, \pi^3\, c^3} \left(\frac{\epsilon_a}{\bar{\alpha}_a} \right) \left[1 - e^{-\bar{\alpha}_a S} \right].$$

Assuming kinetic equilibrium for the particles, it can be shown that

$$\frac{\epsilon_a}{\bar{\alpha}_a} = \frac{v_a^{\gamma}}{e^{\hbar \omega / k_o T} - 1}$$

so that

$$I(P, \omega, \theta) = c\, \hbar\, \omega\, \phi(P, \omega, \theta) = B(\omega) \left(\frac{v_a^{\gamma}}{c} \right) \left[1 - e^{-\bar{\alpha}_a S} \right]. \qquad \text{(VI-16)}$$

For $v_a^{\gamma} = c\mu$ we have

$$I(P, \omega, \theta) = \mu\, B(\omega) \left[1 - e^{-\bar{\alpha}_a S} \right]. \qquad \text{(VI-17)}$$

Assuming that the dispersion relation for the plasma is given with sufficient accuracy by

$$\omega^2 = c^2 k^2 + \omega_p^2$$

it is seen that

$$\mu = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad .$$

Thus, as we approach the plasma frequency (where, in fact, $\bar{\alpha}_a$ gets large) the radiation appearing outside the plasma is strongly reduced and approaches zero.

Before giving further discussion to our results, let us consider the solution of equation (I-3) for the same problem. This time, however, it will not be necessary to consider two regions. (See Figure 2b). The boundary condition at P is[(2)]

$$I(P^+, \omega, \theta) = \frac{I(P^-, \omega, \theta')}{\mu_a^2}$$

where the plus and minus signs indicate that we are approaching the boundary from the right (free space) and left (plasma) respectively. The angles θ and θ' are related by Snell's law

$$\mu_a \sin \theta' = \sin \theta.$$

Note that the path length $S' = L/\cos \theta'$ is different from S. The boundary condition at the other boundary will be taken as $I(0, \omega, \theta') = 0$.

For the same conditions discussed for the previous problem, it is a straightforward calculation to show that

$$I(P^+, \omega, \theta) = B(\omega) \left[1 - e^{-\alpha S'}\right]. \qquad \text{(VI-18)}$$

Equations (VI-17) and (VI-18) are quite different and would predict entirely different results as we approached the plasma frequency. However, as pointed out by Bekefi and Brown[17], equation (VI-18) must be modified to account for internal reflection at the boundaries. When this is done equation (VI-18) becomes, to a good approximation,

$$I(P^{+}, \omega, \theta) = (1 - \Gamma)\, B(\omega) \left[1 - e^{-\alpha S'} \right] \qquad \text{(VI-19)}$$

where Γ is the reflection coefficient. As a reasonable low order approximation, we take[31]

$$\Gamma = \left| \frac{\mu - 1}{\mu + 1} \right|^{2} .$$

With this expression for Γ, equations (VI-19) and (VI-17) are in qualitative agreement, although they can differ by as much as a factor of four. Nonetheless, both of them give reasonably good agreement with the experimental results of Bekefi and Brown[17], and over the range where our equation would be expected to be valid, they are essentially in agreement. The experimental results do not suggest a clear preference for either equation (VI-17) or (VI-19).

As a consequence of the analysis presented here, we feel that the validity of equation (I-3) is still open to question. The limitations of the phenomenological equation are not obvious and we have not been able to obtain this equation from the postulate-deduction method described in Chapters II and IV. On the other hand, we have obtained an equation by this method which is to represent the transport of photons in a dispersive medium. Many approximations and assumptions have been invoked, but we have attempted at each stage of the development, to point these out and to suggest the limitations within which we expect this equation to be valid. The experimental work carried out to this time does not demonstrate which result, the one obtained from the phenomenological equation or the one we have developed, is better.

VII. EMISSION AND ABSORPTION

A. General Considerations

In this chapter we will discuss the modifications to the emission and absorption coefficients resulting from collective particle behavior. Recall that, for the βth type of process

$$j_\beta^\lambda (\omega, \underline{\Omega}) = \frac{\hbar \epsilon_\beta^\lambda \mu_\lambda^2 \omega_\lambda^3}{c^2 v^\gamma (2\pi)^3} \qquad \text{(VI-8a)}$$

$$\bar{\alpha}_\beta^\lambda (\omega, \underline{\Omega}) = \frac{1}{v^\gamma} \sum_{KK_1} V^2 T_{\beta K}^{K_1} (\lambda \underline{k}) \left[f(\underline{K}_1) - f(\underline{K}) \right] \qquad \text{(VI-8b)}$$

$$\epsilon_\beta^\lambda = \sum_{KK_1} V^2 T_{\beta K}^{K_1} (\lambda \underline{k})\, f(\underline{K}) \qquad \text{(VI-1a)}$$

$$V^2 T_{\beta K}^{K_1} (\lambda \underline{k}) = \frac{4\pi^2 c^2}{\hbar} \left(\frac{e}{mc}\right)^2 \frac{1}{\omega_{\lambda k}} \delta(\omega_{K_1 k} - \omega_K)$$

$$(x) \left| \langle \underline{K}_1 | \, e^{-i\underline{k} \cdot \underline{x}} \, \underline{\mathcal{E}}_\lambda (\underline{k}) \cdot \underline{\Pi} \, | \underline{K} \rangle \right|^2 . \qquad \text{(II-46)}$$

Aside from changes which may appear in ϵ_β^λ as a result of the medium behavior, it is seen that j_β^λ differs from the "free space" result by a factor $\frac{\mu_\lambda^2 c}{v^\gamma}$, which for $v^\gamma = c\mu_\lambda$, gives

$$j_\beta^\lambda = \mu_\lambda \left(\frac{\hbar\, \epsilon_\beta^\lambda\, \omega_\lambda^3}{c^3\, (2\pi)^3} \right), \qquad \text{(VII-1)}$$

In the fully ionized gas with no external fields, the index of refraction μ is always less than one. Consequently, as we approach the plasma frequency, where $\mu \to 0$, we may expect the emission inside the body to be strongly reduced. Furthermore, since $v^\gamma \to 0$ as $\omega \to \omega_p$ and $\bar{\alpha}_\beta^\lambda$ is inversely proportional to v^γ, we anticipate that the absorption coefficient would get very large. However, this analysis is not appropriate to investigate these coefficients in the vicinity of the plasma frequency.

B. Bremsstrahlung

The content of these remarks can be illustrated by consideration of the bremsstrahlung process. Taking positive energy Coulomb states for the particles and going to the continuum for the particle momenta distribution, it can be shown[5] that

$$\epsilon_\beta^\lambda = \frac{8\pi c^4 r_o^2 e^2 Z^2 n_I}{\omega_\lambda^3} \int d^3 K\, f\, (\underline{K})$$

$$(x) \int d^3 K_1\, \delta\, (E_{K_1 k} - E_K)\, \frac{\left| \underline{\mathcal{E}}_\lambda (\underline{k}) \cdot \underline{K}_1 - \underline{\mathcal{E}}_\lambda (\underline{k}) \cdot \underline{K} \right|^2}{\left| \underline{K}_1 - \underline{K} \right|^4} \qquad \text{(VII-2)}$$

where $r_o = \frac{e^2}{mc^2}$ and n_I is the number density of ions. Averaging over polarization and integrating over E_{K_1}, we find

---

5. Appendix E

$$\bar{\epsilon}_B = \frac{1}{2} \sum_\lambda \epsilon_B^\lambda = 2\sqrt{2}\,\pi\, n_I r_o^2 Z^2 \frac{\alpha c^5 m^{1/2}}{\omega^3}$$

$$(x) \int d E\, d\Omega\, d\Omega_1 f_\sigma (E) \sqrt{E - \hbar\omega}\;\; \xi\, (E, \theta, \phi, \theta_1, \phi_1) \qquad \text{(VII-3)}$$

where

$$\xi\, (E, \theta, \phi, \theta_1, \phi_1)$$

$$= \frac{E \sin^2 \theta + (E - \hbar\omega) \sin^2 \theta_1 - 2\sqrt{E\,(E - \hbar\omega)} \sin\theta \sin\theta_1 \cos(\phi - \phi_1)}{\left[E + (E - \hbar\omega) - 2\,\mu_o \sqrt{E\,(E - \hbar\omega)}\right]^2}$$

$$\mu_o = \cos\theta \cos\theta_1 \sin\theta \sin\theta_1 \cos(\phi - \phi_1); \quad \alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137} .$$

Define the quantity $\sigma_B\, d\omega\, d\Omega_1$ as the cross section for emission of a photon of frequency ω in $d\omega$ in which an electron with initial energy E and direction $\underline{\Omega}$ has a final state in which the electron is going into $\underline{\Omega}_1$ in $d\Omega_1$. The source term

$$\bar{j}_B\, d\omega \equiv \frac{1}{2} \sum_\lambda j_B^\lambda\, d\omega = \bar{\epsilon}_B \frac{2\,\omega^2 \mu^2}{(2\pi)^3 v^\gamma c^2} \hbar\,\omega\, d\omega \qquad \text{(VII-4a)}$$

is related to the cross section by

$$\bar{j}_B\, d\omega = d\omega\, n_e\, n_I\, \hbar\,\omega \int_{E,\underline{\Omega}} v\, \bar{f}_e\, d E\, d\Omega \int_{\underline{\Omega}_1} \sigma_B\, d\Omega_1 \qquad \text{(VII-4b)}$$

where n_e is the number density of electrons and $\bar{f}_e$ is the electron tribution normalized to unity. From equations (VII-3), (VII-4a) and (VII-4b) find that

$$\sigma_B \, d\,\Omega_1 \, d\,\omega = \frac{d\omega d\Omega_1}{2\pi^2} \; \frac{r_o^2 \, Z^2}{137} \; \frac{m\,c^3\,\mu^2}{v^\gamma \, \omega} \; \xi \; \sqrt{\frac{E - \hbar\omega}{E}} \; . \qquad \text{(VII-5)}$$

This differs from the corresponding result obtained in reference (9) by the factor $\frac{\mu^2 c}{v^\gamma}$ which is just the factor suggested by equation (VI-8a) and discussed earlier. Note that as a function of frequency, $\bar{\epsilon}_B$ is not altered by collective particle behavior of the medium.

C. Cyclotron Radiation

It was pointed out in Chapter III that a plasma in a constant, uniform magnetic field is highly anisotropic and that our procedure is not sufficiently general to treat this case properly. Nonetheless it is both qualitatively and quantitatively interesting to consider this system. For instance, we will observe the possible appearance of Cerenkov radiation. Also, we will indicate how the results of this analysis differ from those presented previously (9).

For magnetic transitions, $V^2 \, T^{K_1}_{mK} (\lambda \, \underline{k})$ can be written

$$V^2 \, T^{K_1}_{mK} \; (\lambda \underline{k}) = \frac{4\pi^2 c^2}{\hbar} \; \left(\frac{e}{mc}\right)^2 \; \frac{1}{\omega_{\lambda k}} \quad \delta\,(\omega_{K_1 k} - \omega_K)$$

$$(x) \;\; I \, (j' \ell' \, K'_z \, | \, j \, \ell \, K_z) \qquad \text{(VII-6)}$$

where

$$I \, (j' \ell' \, K'_z \, | \, j \, \ell K_z) = \langle \, j' \, \ell' \, K'_z \, | \, e^{-i\underline{k} \cdot \underline{x}} \; \mathcal{E}_\lambda \; (\underline{k}) \cdot \underline{\Pi} \, | \, j \, \ell K_z \rangle$$

and the eigenfunctions $|j \ell K_z>$ are given in Appendix B. Transitions for which $j' \neq j$ are associated with the emission and absorption of cyclotron radiation. Relevent matrix elements for the transitions are given in Appendix C. From equations (C-7), (C-8), (VI-8b) and (VII-6) we obtain

$$\bar{\alpha}_c^{\phi}(\omega, \underline{\Omega}) \simeq \frac{\omega_p^2 \pi}{v^{\gamma}} \sum_n \frac{m}{k_o T} \int d^3 v \, \bar{f}(v) \, v_{\perp}^2$$

$$(x) \left[J_n'(v_{\perp} \gamma_{\phi} \sin \theta)\right]^2 \delta\left[\omega_{\phi} - n\omega_c - \omega_{\phi}\left(\frac{\mu_{\phi} v_z \cos\theta}{c} - \frac{\mu_{\phi} \hbar \omega_{\phi} \cos^2\theta}{2mc^2}\right)\right] \qquad \text{(VII-7a)}$$

$$\bar{\alpha}_c^{\theta}(\omega, \underline{\Omega}) \simeq \frac{\omega_p^2 \pi}{v^{\gamma}} \sum_n \left(\frac{m}{k_o T}\right) \frac{(n\omega_c)^2 c^2}{\mu_\theta^2 \omega_\theta^2}$$

$$(x) \int d^3 v \, \bar{f}(v) \left[\cot\theta - \frac{v_z}{c}\left(\frac{\mu_\theta \omega_\theta}{n\omega_c}\right) \sin\theta\right]^2 \left[J_n(v_{\perp} \gamma_\theta \sin\theta)\right]^2$$

$$(x) \ \delta\left[\omega_\theta - n\omega_c - \omega_\theta\left(\frac{\mu_\theta v_z \cos\theta}{c} - \frac{\mu_\theta^2 \hbar \omega_\theta \cos^2\theta}{2mc^2}\right)\right] \qquad \text{(VII-7b)}$$

where $\gamma_\lambda = \dfrac{\mu_\lambda \omega_\lambda}{c\,\omega_c}$ and where the initial and final particle distributions have been taken as Maxwellian. The θ and ϕ polarizations are associated with the polarization vectors which have been chosen as the spherical base vectors in the polar and azimuthal directions.

In the limit $v^{\gamma} \rightarrow c$ and $\mu_{\lambda} \rightarrow 1$ we find the results given in reference (9). The absorption coefficient for this case has received a great deal of attention (32, 33, 34) in the past few years because of the importance of cyclotron radiation as an energy loss mechanism for laboratory thermonuclear systems. Numerical calculations of the absorption coefficients (34) have indicated that only $\bar{\alpha}_c^{\phi}$ is of importance for systems where cyclotron radiation is important and then only for propagation nearly perpendicular to the magnetic field. For $\theta = \frac{\pi}{2}$,

$$\bar{\alpha}_c^{\phi}(\omega, \underline{\Omega}) = \left(\frac{c}{v^{\gamma}}\right) \frac{\omega_p^2 \pi}{c} \sum_n \left(\frac{m}{kT}\right) \int d^3 v \bar{f}(v)\, v_{\perp}^2$$

$$(x)\; J_n'^2 \left(v_{\perp} \frac{\mu_{\phi}\omega_{\phi}}{c\,\omega_c}\right) \delta\,(\omega_{\phi} - n\,\omega_c)\ . \qquad \text{(VII-8)}$$

It is seen from this equation that the deviation from $\mu = 1$ is reflected in the factor $\frac{c}{v^{\gamma}}$ and in the argument of the Bessel function.

D. Cerenkov Radiation

Cerenkov radiation results from the matrix element

$$< j\ell K_z' \,|\, e^{-i\underline{k}\cdot\underline{x}}\; \mathcal{E}_{\lambda}(\underline{k})\cdot\underline{\Pi} \,|\, j\, 0\, K_z > .$$

This element was not considered in the calculations of Osborn and Klevans (9) because when $\mu = 1$, it is not possible to simultaneously conserve energy and momentum and the term must vanish. For a plasma in a magnetic field, however, it is possible for $\mu > 1$ and this implies the possibility of simultaneously conserving energy and momentum. Thus, the radiation associated with this matrix element should now be included in the discussion.

Since our analysis can at best give qualitative results we will consider the simple case of a single particle moving along the magnetic field. Then, taking k propagating in the y - z plane (using a left handed coordinate system), we need only calculate $I(0\,\ell\,K'_z \mid 0\,0\,K_z)$. The index ℓ refers to the position of the orbit.[6] Because the photon is emitted at some angle θ with respect to the z axis, it is necessary for momentum conservation that the particle make some movement in the x - y plane. Such motion is accomplished by a transition to a different ℓ state.

We then find

$$I(0\,\ell\,K'_z \mid 0\,0\,K_z) = \delta(K'_z + k_z - K_z)\,\frac{(-i)^{\ell}}{\sqrt{\ell!}}\left(\frac{\alpha^2}{2}\right)^{\ell/2} e^{-\frac{\alpha^2}{2}}\left(\mathcal{E}^3_\lambda \hbar K_z\right)^2 \qquad \text{(VII-9)}$$

where $\alpha = k\left(\frac{\hbar c}{eH}\right)^{1/2} \sin\theta$. Substituting equation (VII-9)into equation (II-46), we obtain

$$V^2\, T^{K_1}_{ce\,K}(\lambda\,\underline{k}) = \frac{4\pi^2 c^2}{\hbar}\left(\frac{e}{mc}\right)^2 \frac{1}{\ell!}\left(\frac{\alpha^2}{2}\right)^{\ell} e^{-\alpha^2}$$

$$(x)\ \frac{1}{\omega_\lambda}\left(\mathcal{E}^3_\lambda \hbar K_z\right)^2 \delta(\omega_{K_1 k} - \omega_K)\,\delta(K'_z + k_z - K_z). \qquad \text{(VII-10)}$$

Combining energy and momentum conservation, we find

6 The position coordinates of the center of the orbit do not commute and cannot be simultaneously specified (28). The center of the orbit is quantized with eigenvalues $\frac{\hbar c}{eH}(2\ell + 1)$.

$$\omega_{K_1 k} - \omega_K = \left(\frac{\mu_\lambda v_z \cos\theta}{c} - \frac{\hbar \omega_\lambda}{2mc^2} - 1 \right) \omega_\lambda$$

$$\simeq \left(\frac{\mu_\lambda v_z \cos\theta}{c} - 1 \right) \omega_\lambda = 0 , \qquad \text{(VII-11)}$$

where θ is the angle between $\underline{k}$ and the z axis. Since $\omega \neq 0$, equation (VII-11) is simply the Cerenkov condition

$$\frac{c}{\mu_\lambda v_z} = \cos\theta . \qquad \text{(VII-12)}$$

From equations (VI-1a), (VII-10) and (VII-11), we obtain, after carrying out the sum over ℓ,

$$\epsilon_{ce}^{\lambda} = \left(\frac{4\pi^2 c^2}{\hbar} \right) \left(\frac{e}{mc} \right)^2 \frac{\left(\mathcal{E}_\lambda^3 p_z \right)^2}{\omega_\lambda^2} e^{-\frac{\alpha^2}{2}} .$$

$$(x)\ \delta \left(\frac{\mu_\lambda v_z \cos\theta}{c} - 1 \right) . \qquad \text{(VII-13)}$$

Noting that $e^{-\frac{\alpha^2}{2}} \simeq 1$, substitution of equation (VII-13) into (VI-8a) gives

$$j_{ce}^{\lambda} (\omega, \underline{\Omega}) = \frac{\mu_\lambda^2 \omega_\lambda e^2}{c^2 \, 2\pi \, v^\gamma} \left(\mathcal{E}_\lambda^3 v_z \right)^2 \delta \left(\frac{\mu_\lambda v_z \cos\theta}{c} - 1 \right) . \qquad \text{(VII-14)}$$

Using the spherical base vectors $\hat{\theta}$ and $\hat{\phi}$ as polarization vectors, we have $\mathcal{E}_\theta^3 = \sin\theta$ and $\mathcal{E}_\phi^3 = 0$. Thus

$$j_\theta \, d\omega = d\omega \int d\Omega \, j_\theta (\omega, \underline{\Omega}) = \frac{\mu_\theta \omega e^2 v_z}{v^\gamma c} \left(1 - \frac{c^2}{\mu_\theta^2 v_z^2}\right) d\omega . \qquad \text{(VII-15)}$$

With $v^\gamma = c\mu_\theta$, this gives the classical result for Cerenkov radiation (18).

It must again be emphasized, however, that the applicability of this result is limited since we cannot obtain μ_θ within the context of this analysis, except for frequencies sufficiently high that the index of refraction is nearly equal to one.

E. De-excitation Radiation

For de-excitation radiation from an impurity in a highly ionized gas, frequencies are too high and densities too low for the present discussion to be of much interest. The results presented by Osborn and Klevans (9) are sufficiently accurate, and for completeness will be included here.

For a dipole approximation

$$V^2 \, T_{dK}^{K_1} (\lambda \, \underline{k}) = \frac{4\pi^2 c^2}{\hbar} \left(\frac{e}{mc}\right)^2 \frac{1}{\omega_\lambda} \delta(\omega_{K_1 k} - \omega_K)$$

$$(\times) \; \left| \langle \underline{K}_1 \right| \underline{\varepsilon}_\lambda (\underline{k}) \cdot \underline{p} \left| \underline{K} \rangle \right|^2 . \qquad \text{(VII-16)}$$

From the commutation relations

$$\left[x_j, p_k^2 \right] = i\hbar \, \delta_{jk} \, p_k \quad \text{and} \quad \left[x_j, V(x) \right] = 0$$

we obtain

$$\left| \underline{\mathcal{E}}_{\lambda}(\underline{k}) \cdot < \underline{K}_1 \middle| \underline{p} \middle| \underline{K} > \right|^2 = \left| \underline{\mathcal{E}}_{\lambda}(\underline{k}) \cdot \left(\frac{m}{i\hbar}\right) < \underline{K}_1 \middle| \left[\underline{x}, H_M\right] \middle| \underline{K} > \right|^2$$

$$= \frac{m^2}{\hbar^2} \left| (E_K - E_{K_1}) \right|^2 \left| < \underline{K}_1 \middle| \underline{\mathcal{E}}_{\lambda}(\underline{k}) \cdot \underline{x} \middle| \underline{K} > \right|^2$$

so that

$$V^2 T_{dK}^{K_1}(\lambda \underline{k}) = \frac{4\pi^2 c^2}{\hbar} \left(\frac{e^2}{c^2}\right) \omega_{\lambda} \left| < \underline{K}_1 \middle| \underline{x} \middle| \underline{K} > \right|^2 \cos^2\theta \, \delta(\omega_{K_1 k} - \omega_K)$$

(VII-17)

where θ is the angle between the direction of polarization and the vector $\underline{x}$. From equation (VII-17) we can obtain the source term j^{λ} or the absorption coefficient $\bar{\alpha}^{\lambda}$. Or we can relate it to the rate of spontaneous de-excitation $\gamma d\Omega$ presented by Heitler (35) by

$$\gamma d\Omega = \frac{1}{V} \sum_{\underline{K}_1} V^2 T_{dk}^{K_1}(\lambda \underline{k}) = \frac{e^2 \omega_{\lambda}^3}{2\pi\hbar c^3} \, d\Omega \left| < \underline{K}_1 \middle| \underline{x} \middle| \underline{K} > \right|^2 \cos^2\theta \, .$$

(VII-18)

Further calculations with this quantity are presented by Berman (36).

F. Recombination Radiation

Lastly, we consider recombination radiation, which is important for hydrogen plasma systems with kinetic temperature from 3 ev. to 200 ev. We choose the electron and atomic wave functions as

$$\left| \underline{K} > = \frac{1}{\sqrt{V}} e^{i\underline{K}\cdot\underline{x}} \; ; \; \right| \underline{K}_1 > = \psi_n \qquad \text{(VII-19)}$$

where n represents a sufficient set of labels to completely specify an atomic state.

After converting to the continuum for the electron's initial momenta, the emission coefficient for recombination radiation can be written

$$\epsilon_r^\lambda = \frac{\mu_\lambda^2}{v^\gamma} \pi^2 n_I \left(\frac{e^2}{mc^2}\right) \frac{\hbar c^3}{m} \frac{1}{\omega_\lambda} \sum_n \int d^3 K f(K)$$

$$(x) \quad (\underline{\mathcal{E}}_\lambda \cdot \underline{K})^2 \quad \delta(\omega_{K_1 k} - \omega_K) \left| \psi_n(\underline{\kappa}) \right|^2 \qquad \text{(VII-20)}$$

where

$$\psi_n(\underline{\kappa}) = \int \psi_n^*(\underline{x})\, e^{-i\underline{\kappa}\cdot\underline{x}}\, d^3x \,,$$

and $\underline{\kappa} = \underline{k} - \underline{K}$. As a result of the electron normalization, the ion density is given by $n_I = \frac{1}{V}$. The frequency of such radiation is very high, and $\omega^2 \gg \omega_p^2$. Dispersive effects would be expected to be negligible and deviation from $\frac{c\mu_\lambda^2}{v^\gamma} = 1$ should be small. Additional calculations concerning recombination radiation are given in reference (36).

VIII. TRANSPORT IN CRYSTALS

The last subject to be discussed is radiation transport through crystals. It is of interest, not only because we obtain the anticipated x-ray scattering cross sections as a natural consequence of the analysis, but also because of the analogy that can be made with the problem of low energy neutron transport through moderators or filters. Since a quantum mechanical transport equation for cold neutrons has yet to be given, such an analogy is perhaps worth pursuing.

As mentioned in Chapter II, we will employ the relatively simple Einstein model of the crystal. In this model, the atoms (nuclei surrounded by electrons) execute independent isotropic oscillations about fixed equilibrium positions. The force constant of the Hooke's Law restoring force is identical for all nuclei.

In accordance with these remarks, and with the appropriate quantities shown on Figure 3, the Hamiltonian for the systemcan be written

$$H^M = \sum_{\sigma=1}^{N} \left[H_o^\sigma + H_e^\sigma\right] = \sum_{\sigma=1}^{N} \left[\frac{P_\sigma^2}{2M_\sigma} + \frac{1}{2} M_\sigma \omega_o^2 u_\sigma^2\right]$$

$$+ \sum_{\sigma=1}^{N} \sum_{j}^{n} \left[\frac{p_j^{\sigma\,2}}{2m_j} + V(\xi_1, \xi_2, \ldots \xi_n)\right] \qquad \text{(VIII-1)}$$

where H_o^σ is the Hamiltonian for the σth nucleus, and H_e^σ is the Hamiltonian

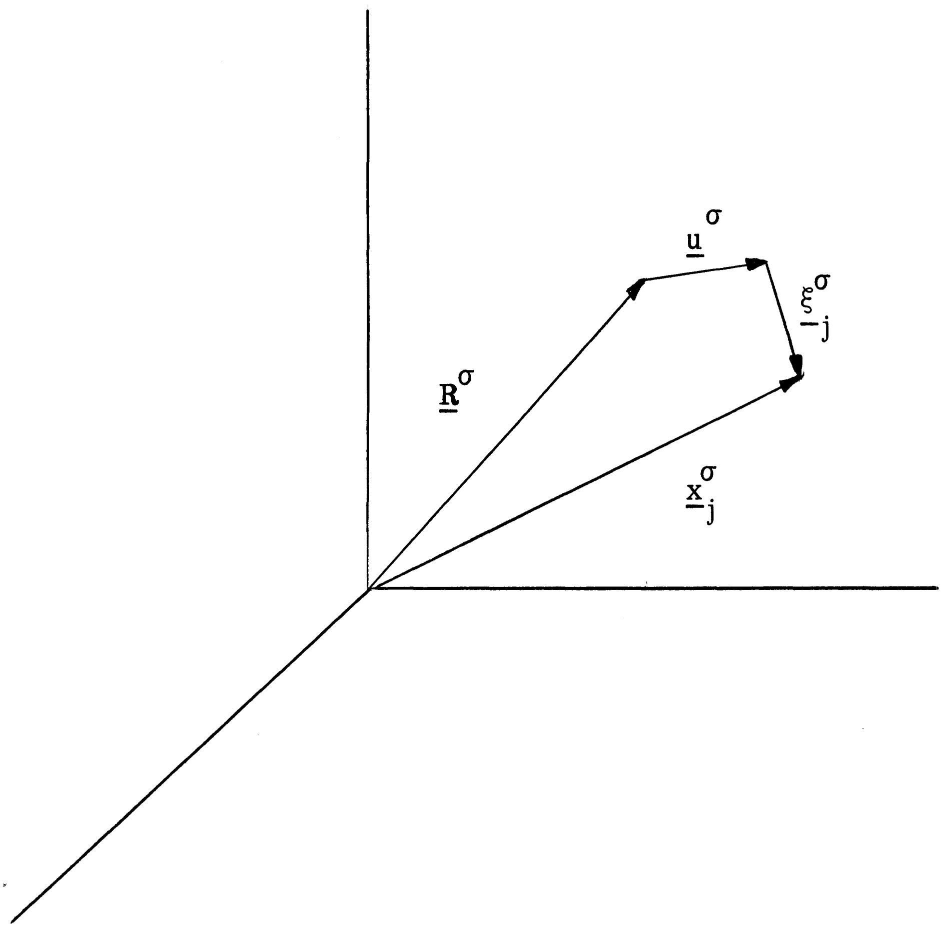

Figure 3. Location of the jth Particle of the σ^{th} Atom

for the electrons surrounding this nucleus. The natural oscillation frequency is given by ω_o. The nucleus has an equilibrium position $\underline{R}^\sigma$, and an instantaneous displacement $\underline{u}_\sigma$. The instantaneous position of the jth electron of the σth nucleus is $\underline{x}_j^\sigma$, displaced by $\underline{\xi}_j^\sigma$ from the position of the nucleus.

We have assumed a complete, orthonormal set of medium eigenstates $\{|m>\}$ such that

$$|m> = \prod_\sigma \psi_{m_\sigma}(\underline{u}_\sigma)\,\phi_{n_\sigma}(\underline{\xi}^\sigma) \qquad \text{(VIII-2)}$$

where

$$H^M |m> = E_m |m> \qquad \text{(VIII-3a)}$$

$$H_o^\sigma \psi_{m_\sigma}(\underline{u}_\sigma) = E_{m_\sigma}^\sigma \psi_{m_\sigma}(\underline{u}_\sigma) \qquad \text{(VIII-3b)}$$

$$H_e^\sigma \phi_{n_\sigma}(\underline{\xi}^\sigma) = \epsilon_{n_\sigma} \phi_{n_\sigma}(\underline{\xi}^\sigma) \qquad \text{(VIII-3c)}$$

and $\underline{\xi}^\sigma = \{\underline{\xi}_1^\sigma, \underline{\xi}_2^\sigma, \ldots, \underline{\xi}_n^\sigma\}$. From equations (VIII-1) and (VIII-2), it is seen that $\psi_{m_\sigma}(\underline{u}_\sigma)$ is a harmonic oscillator wave function. For notational convenience, we denote

$$\psi_m(\underline{u}_\sigma) \equiv |\bar{m}_\sigma> \qquad \text{(VIII-4a)}$$

$$\phi_{n_\sigma}(\underline{\xi}_\sigma) \equiv |\bar{n}_\sigma>. \qquad \text{(VIII-4b)}$$

We will only be concerned here with scattering of radiation from electrons, and absorption will be neglected. For the high frequencies under consideration, it is also anticipated that we may assume an index of refraction of unity with little error. In accordance with these remarks we take

$$H = H^M + H_2^{P\gamma 2}$$

where

$$H_2^{P\gamma 2} \simeq \sum_{\substack{\lambda \underline{k} \\ \lambda' \underline{k}'}}{}' \frac{\pi \hbar c}{V}\left(\frac{e^2}{mc^2}\right)\frac{1}{[k\ k']^{1/2}} \underline{\xi}^{+}_{\lambda}(\underline{k})\cdot\underline{\xi}^{+}_{\lambda'}(-\underline{k}') \sum_{\sigma j} e^{-i\underline{K}\cdot\underline{x}^\sigma_j}; \qquad \text{(VIII-5)}$$

with $\underline{K} = \underline{k} - \underline{k}'$. Charge and mass are not subscripted with the index σ since only photon-electron interactions are considered. The derivation of the transport equation will be assumed valid. For some scattering problems, the spatial coarse-graining might become controversial, but we will not concern ourselves with this possibility.

Writing

$$\delta(\omega_{n\eta} - \omega_{m\alpha}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{it(\epsilon_\eta - \epsilon_\alpha + E_n - E_m)/\hbar}$$

and defining

$$A_{\lambda k}^{\lambda' k'} = \frac{8\pi^3 c}{V^3 kk'} \left(\frac{e^2}{mc^2} \right)^2 \left| \underline{\xi}_{\lambda}(\underline{k}) \cdot \underline{\xi}_{\lambda'}(\underline{k}') \right|^2$$

$$\Omega^{\sigma} = e^{-i\underline{\kappa} \cdot (\underline{R}^{\sigma} + \underline{u}^{\sigma})} \sum_j e^{-i\underline{\kappa} \cdot \underline{\xi}_j^{\sigma}}$$

$$\Omega^{\sigma}(t) = e^{i(H_o^{\sigma} + H_e^{\sigma}) t/\hbar} \, \Omega^{\sigma} \, e^{-i(H_o^{\sigma} + H_e^{\sigma}) t/\hbar}$$

$$\bar{\omega} = \frac{\epsilon_{\eta} - \epsilon_{\alpha}}{\hbar}$$

it can be shown from equations (II-24), (II-37), and (VIII-5) (after replacing averages of products of photon occupation numbers and particle matrix elements by products of averages) that

$$\frac{1}{c} \frac{\partial F_{\lambda \underline{k}}}{\partial t} + \underline{\Omega} \cdot \underline{\nabla} \, F_{\lambda \underline{k}} = \sum_{\lambda' \underline{k}'} \Big[I_{\lambda k}^{\lambda' k'} \, V F_{\lambda' \underline{k}'} \, (V F_{\lambda \underline{k}} + 1)$$

$$- \bar{I}_{\lambda k}^{\lambda' k'} \, V F_{\lambda \underline{k}} (V F_{\lambda' \underline{k}'} + 1) \Big] \qquad \text{(VIII-6)}$$

where

$$I_{\lambda k}^{\lambda' k'} \equiv A_{\lambda k}^{\lambda' k'} \frac{1}{2\pi} \int dt \; e^{i\bar{\omega} t} \sum_{\sigma \sigma'} \sum_{m} P(m) < m \,|\, \Omega^{\sigma +} \, \Omega^{\sigma'}(t) \,|\, m > \qquad \text{(VIII-7a)}$$

$$\bar{I}_{\lambda k}^{\lambda' k'} \equiv A_{\lambda k}^{\lambda' k'} \frac{1}{2\pi} \int dt \; e^{i\bar{\omega} t} \sum_{\sigma\sigma'} \sum_{m} P(m) < m \mid \Omega^{\sigma'}(t)\, \Omega^{\sigma^+} \mid m > .$$

(VIII-7b)

The quantity $P(m) = D_{mm}$ is the probability of finding the particle system in the state $| m >$. Note that $I_{\lambda k}^{\lambda' k'} \neq \bar{I}_{\lambda k}^{\lambda' k'}$ since $\Omega^{\sigma'}(t)$ and Ω^{σ^+} do not commute.

Assuming random polarization for the radiation, we take $F_\lambda = \frac{1}{2} F$. Then, summing equation (VIII-6) over polarization and converting to the continuum in photon momentum space, we find

$$\frac{1}{c} \frac{\partial F^{o}(\underline{k})}{\partial T} + \underline{\Omega} \cdot \underline{\nabla} \; F^{o}(\underline{k})$$

$$= N\left(1 + \frac{V F^{o}(k)}{2 \rho_k}\right) \int dk' \; d\Omega' \; F^{o}(\underline{k}') \, \sigma\,(k', \underline{\Omega}'; \, k, \underline{\Omega})$$

$$- N F^{o}(\underline{k}) \int dk' \, d\Omega' \left(1 + \frac{V F^{o}(\underline{k}')}{2 \rho_{k'}}\right) \sigma\,(k, \underline{\Omega}; \, k', \underline{\Omega}') \qquad \text{(VIII-8)}$$

where

$$\sigma(k', \underline{\Omega}'; k, \underline{\Omega}) = \sigma_T \frac{ck}{k'} \left(\frac{1}{2\pi}\right) \int dt \; e^{i\bar{\omega} t} \frac{1}{N} \sum_{\sigma\sigma'} \sum_{m} P(m) < m \mid \Omega^{\sigma^+} \; \Omega^{\sigma'}(t) \mid m >$$

(VIII-9a)

$$\sigma(k, \underline{\Omega}; k' \underline{\Omega}') = \sigma_T \frac{ck'}{k} \left(\frac{1}{2\pi}\right) \int dt \; e^{i\bar{\omega} t} \frac{1}{N} \sum_{\sigma\sigma'} \sum_{m} P(m) < m \mid \Omega^{\sigma'}(t)\, \Omega^{\sigma^+} \mid m >$$

(VIII-9b)

$$\sigma_T = \frac{1}{2} r_o^2 \left[1 + (\underline{\hat{k}} \cdot \underline{\hat{k}}')^2\right]$$

$$r_o = \frac{e^2}{mc^2}$$

N = number of nuclei .

The quantity $\sigma(k', \underline{\Omega}'; k\underline{\Omega})\, dk'\, d\Omega'$ is a cross section.

For neutron transport, even near thermal equilibrium, the number of occupied states is small compared to the number available for occupation. Thus, scattering terms dependent upon the final state particle density could be dropped. For a crystal at room temperature, this approximation is also valid for x-ray scattering, since we are on the high frequency tail of the Planck distribution where the above remarks on occupation of available states again applies.

We can obtain, by a rather tedious calculation, the result[7]

$$\sigma(k', \underline{\Omega}'; k, \underline{\Omega}) = \sigma_T\, c\, \delta(\bar{\omega})\, |\bar{f}|^2\, e^{-DK^2}\, \frac{1}{N} \left| \sum_{\sigma} e^{-i\underline{K}\cdot\underline{R}\sigma} \right|^2$$

$$+ \sigma_T\, c\, \delta(\bar{\omega})\, e^{-DK^2} \left[\overline{|f|^2}\, I_0(BK^2) - |\bar{f}|^2 \right]$$

$$+ \sigma_T \frac{k'}{k}\, c\, e^{-DK^2}\, \overline{|f|^2} \left\{ \sum_{n=1}^{\infty} I_n(BK^2) \left[e^{\frac{M\hbar\omega_0}{2k_0T}}\, \delta(\bar{\omega} + n\omega_0) \right.\right.$$

$$\left.\left. + e^{\frac{-M\hbar\omega_0}{2k_0T}}\, \delta(\bar{\omega} - n\omega_0) \right] \right\} \qquad \text{(VIII-10)}$$

where

$$D = \frac{\hbar}{2M\omega_0} \coth \frac{\hbar\omega_0}{2\,k_0T} \qquad \text{(VIII-11a)}$$

$$B = \frac{\hbar}{2M\omega_0} \operatorname{csch} \frac{\hbar\omega_0}{2\,k_0T} \qquad \text{(VIII-11b)}$$

---

7. Appendix F

$$\bar{f} = \sum_{\bar{n}} P(\bar{n})\, f_{\bar{n}\bar{n}} \qquad \text{(VIII-11c)}$$

$$f_{\bar{n}\bar{n}} = \langle \bar{n} \,|\sum_{j} e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_j}\,|\, \bar{n} \rangle \qquad \text{(VIII-11d)}$$

and where I_n is the modified Bessel function of the first kind of order n. We have made the approximation that there are no transitions between electronic states. The quantity $f_{\bar{n}\bar{n}}$ is called the atomic scattering factor[37]. It has the meaning that if the electronic system is in the state $|\bar{n}\rangle$, then $|f_{\bar{n}\bar{n}}|^2 \sigma_T$ gives the scattering from the electrons surrounding the nuclei. The cross section σ_T is the Thomson cross section for scattering from a free electron.

The first term of equation (VIII-10) represents interference scattering. (Note that such scattering is elastic.) The remainder of the terms comprise what is called direct scattering. In the second term no oscillator states are changed during the scattering, while in the last term the oscillator state is changed. The last term accounts for inelastic scattering.

It should be noted also that $e^{-D\kappa^2}$ is the Debye-Waller factor and $\frac{1}{N}|\sum_{\sigma} e^{-i\boldsymbol{\kappa}\cdot\mathbf{R}^{\sigma}}|^2$ will give rise to the Bragg scattering condition[37] in an infinite homogeneous crystal.

Equation (VIII-10) reduces to the cross section for neutron scattering from a monoisotropic crystal of spinless nuclei[38] if $f \rightarrow 1$ and $\sigma_T \rightarrow a^2$, the nuclear scattering length. A thorough discussion of this formula is given in reference[38].

It is perhaps worth a final observation that when inelastic scattering can be neglected, we find that

$$\frac{1}{c}\frac{\partial F^{Q}(\underline{k})}{\partial t} + \underline{\Omega}\cdot\underline{\nabla}\, F^{Q}(\underline{k})$$

$$= N\int d\Omega'\, F^{Q}(k, \underline{\Omega}')\, \bar{\sigma}(k, \underline{\Omega}, \underline{\Omega}')$$

$$- N\, F^{Q}(k, \underline{\Omega})\int d\Omega'\, \bar{\sigma}(k, \underline{\Omega}, \underline{\Omega}') \qquad \text{(VIII-12)}$$

where

$$\bar{\sigma}(k, \underline{\Omega}, \underline{\Omega}') = \sigma_T\, e^{-DK^2}\Big\{|\bar{f}|^2$$

$$(x)\left[\frac{1}{N}\Big|\sum_{\sigma} e^{-i\underline{K}\cdot\underline{R}^{\sigma}}\Big|^2 - 1\right] + \overline{|f|^2}\, I_0(BK^2)\Big\}. \qquad \text{(VIII-13)}$$

Thus, within the context of the Einstein model (and the additional approximations we have made), elastic scattering terms which are dependent upon the final state photon density cancel.

IX. CONCLUSIONS

In the foregoing, a transport description of electromagnetic phenomena in terms of creation, destruction, scattering and flow of photons has been developed. The usefulness of characterizing electromagnetic phenomena in such terms is limited by several factors. First, it is limited in a practical sense to physical problems for which one-photon processes give the only significant contribution to the rate of change of the photon density. It is also restricted to those systems for which spatial variation of particle and photon distributions is slow over regions characterized by a length many times larger than the longest photon wavelength under consideration. This geometric optics condition suggests that our description is applicable for investigating the energy balance of short wavelength radiation, whereas the full set of Maxwell's equations is needed to describe electromagnetic phenomena for which longer wavelengths are of interest.

There are additional conditions which tend to restrict the expected range of validity of our transport equation. These conditions are intended only as sufficiency conditions, however, and it is not claimed that they are necessary. First, the medium should be isotropic. In the anisotropic medium, there is in general a coupling between the various transverse and longitudinal modes and our procedure is not appropriate to treat these systems. In a few special cases, e.g. photon propagation in a plasma, parallel to a uniform external magnetic field, the different modes decouple and our procedure can still be applied.

A second restriction is that the photon frequencies should be sufficiently far above the resonance regions that level broadening effects can be neglected.

It may also be possible to apply the equation between resonance regions or below them, but it is not obvious from the derivation that it is legitimate to pass through the resonance region, where the equation is not expected to be valid.

The radiation transport equation which we have obtained is not in agreement with the equation obtained phenomenologically (equation (I-3)). However, when the solution of equation (I-3) is modified to account for internal reflection, both equations yielded solutions which were in reasonably good agreement with the experimental results of Bekefi and Brown (17). Thus it does not yet appear possible to reject either of the equations.

Because there have been many approximations employed without investigation, and because there is an unresolved descrepancy for the inhomogeneous medium between our transport equation and the more conventional phenomenological equation, we suggest that further study in the development of a radiation transport equation for the dispersive medium is in order. This thesis offers a beginning, and it is hoped that further work can shed light on some of the difficulties encountered in our derivation.

APPENDIX A

CALCULATION OF THE TRANSITION MATRIX ELEMENT $U_{n'\eta' n\eta}$

We wish to apply radiation damping theory to the study of

$$U(s) = \exp\left[-iH\ s/\hbar\right].$$

Define

$$K(z) \equiv \frac{1}{z + i(H^{o} + H^{I})} . \tag{A-1}$$

Then we can write

$$U_{n'\eta' n\eta}(s) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dz\ e^{zs/\hbar}\ K_{n'\eta' n\eta}(z) . \tag{A-2}$$

From equation (A-1) we obtain (suppressing the index η for simplicity)

$$(z + i\epsilon_n) K_{nn} + i H^I_{nn} K_{nn} + i \sum_{n' \neq n} H^I_{nn'} K_{n'n} = 1 \tag{A-3}$$

$$(z + i\epsilon_{n'}) K_{n'n} + i H^I_{n'n} K_{nn} + i H^I_{n'n'} K_{n'n} + i \sum_{n'' \neq n, n'} H^I_{n'n''} K_{n''n'} = 0 . \tag{A-4}$$

Define $Q\ (z)$ by

$$K_{n'n} = K_{n'n'} Q_{n'n} K_{nn} \qquad n' \neq n \tag{A-5a}$$

$$Q_{n'n'} \equiv 0 \qquad \text{all } n' . \tag{A-5b}$$

Define $\Gamma(z)$ by

$$\frac{1}{2} i\hbar \Gamma_{nn}(z) = i H^I_{nn} + i \sum_{n' \neq n} H^I_{nn'} K_{n'n'} Q_{n'n} \qquad \text{(A-6a)}$$

$$\frac{1}{2} i\hbar \Gamma_{nn'}(z) \equiv 0 \; . \qquad \text{(A-6b)}$$

From equations (A-3) and (A-5) we see that

$$\frac{1}{2} i\hbar \Gamma_{nn}(z) K_{nn} = 1 - (z + i\epsilon_n) K_{nn} \; .$$

Thus

$$K_{nn} = \frac{1}{z + i\epsilon_n + \frac{1}{2} i\hbar \Gamma_{nn}} \; . \qquad \text{(A-7)}$$

From equations (A-4), (A-5), and (A-7) we obtain

$$Q_{n'n} = -i H^I_{n'n} + \left(\frac{1}{2} i\hbar \Gamma_{n'n'} - i H^I_{n'n'}\right) K_{n'n'} Q_{n'n}$$

$$- i \sum_{n'' \neq n', n} H^I_{n'n''} K_{n''n''} Q_{n''n} \; , \qquad \text{(A-8)}$$

From equation (A-6) we see that equation (A-8) can be written

$$Q_{n'n} = -i H^I_{n'n} + i \sum_{n'' \neq n'} H^I_{n'n''} K_{n''n''} Q_{n''n'} K_{n'n'} Q_{n'n}$$

$$- i \sum_{n'' \neq n, n'} H^I_{n'n''} K_{n''n''} Q_{n''n} \; , \qquad \text{(A-9)}$$

We note that Q first appears to first order in H^I, which is equivalent to first order in the interaction strength. We wish to obtain $Q_{n'n}$ in an expansion in powers of e in terms of H^I and K. To first order, we would have

$$Q_{n'n} = -i H^I_{n'n} + R$$

where R is the remainder and is assumed small compared with the first term. R is still a function of Q. Substitution for $Q_{n''n}$ into equation (A-9) would give

$$Q_{n'n} = -i H^I_{n'n} + (-i)^2 \sum_{n'' \neq n, n'} H^I_{n'n''} K^I_{n''n'} + R . \qquad \text{(A-10)}$$

Thus, we have exhibited $Q_{n'n}$ to second order in powers of H^I, plus a remainder. By similar substitution into equation (A-9) whenever a Q appears on the right hand side, we can express $Q_{n'n}$ in terms of H^I and K up to any desired order, plus a remainder.

By a similar procedure, we find, from equations (A-6a) and (A-9),

$$\frac{1}{2} i\hbar \Gamma_{nn}(z) = i H^I_{nn} + i(-i) \sum_{n' \neq n} H^I_{nn'} K_{n'n'} H^I_{n'n}$$

$$+ (i)(-i)^2 \sum_{\substack{n' \neq n \\ n'' \neq n, n'}} H^I_{nn'} K_{n'n'} H^I_{n'n''} K_{n''n''} H^I_{n''n}$$

$$+ R'' . \qquad \text{(A-11)}$$

Except for the remainders R' and R'', equations (A-10) and (A-11) are the same as the first few terms of equations (16) and (15), respectively, of reference (24), except for differences in notation.

For the purposes of this paper we will need only the first term of equation (A-10) and the first two terms of equation (A-11).

As a consequence of the above, equation (A-2) now becomes

$$U_{n'\eta'n\eta}(s) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dz \; e^{z s/\hbar} \; K_{n'\eta'n'\eta'} \; Q_{n'\eta'n\eta} \; K_{n\eta n\eta} \, . \tag{A-12}$$

Writing the first term in the expansion of $U_{n'\eta'n\eta}(s)$ as $U^{(1)}_{n'\eta'n\eta}(s)$ and noting that all first order processes and scattering are contained in this term, we have

$$U^{(1)}_{n'\eta'n\eta}(s) = (-i)\, H^{I}_{n'\eta'n\eta}$$

$$(x)\ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dz \, \frac{e^{z s/\hbar}}{\left(z + i\epsilon_{n\eta} + \frac{i\hbar}{2}\Gamma_{n\eta n\eta}(z)\right)\left(z + i\epsilon_{n'\eta'} + \frac{i\hbar}{2}\Gamma_{n'\eta'n'\eta'}(z)\right)} \, .$$

(A-13)

APPENDIX B

CALCULATION OF THE PLASMA DISPERSION RELATION FOR PROPAGATION ALONG A CONSTANT EXTERNAL MAGNETIC FIELD

From Johnson and Lippmann[28], we take the result

$$|\underline{K}\rangle = |j\ell K_z\rangle = u_{j\ell}(\rho,\phi)\,\frac{e^{iK_z z}}{\sqrt{L}} \tag{B-1}$$

where

$$u_{j\ell} = \left(\frac{j!}{2\pi\ell!}\right)^{1/2} (-i)^j\, b^{-1}\, e^{-1/2\,\xi + i\phi(j-\ell)}$$

$$(x)\,\xi^{-\frac{(j-\ell)}{2}}\,\mathcal{L}_j^{\ell-j}(\xi) \tag{B-2}$$

and where $b^2 = \dfrac{\hbar c}{eH}$; $\xi = \dfrac{\rho^2}{2b^2}$; $L^{-1/2}$ is the box normalization; ϕ is the azimuthal angle with respect to the y axis in a left-handed coordinate system; and

$$\mathcal{L}_j^{\nu}(\xi) = \frac{\xi^{-\nu} e^{\xi}}{j!}\left(\frac{\partial}{\partial\xi}\right)^j e^{-\xi}\,\xi^{j+\nu} \tag{B-3}$$

is the associated Laguerre polynomial as defined in Magnus and Oberhettinger[39].

It will be useful to employ the operators

$$\Pi^{\pm} = \frac{1}{\sqrt{2}}\left[\Pi^y \pm i\,\Pi^x\right]$$

which have the properties:

$$\Pi^{+}\,|j\ell K_z\rangle = m\omega_c b\sqrt{j+1}\;|j+1\,\ell\,K_z\rangle \tag{B-4}$$

$$\Pi^{-}\,|j\ell K_z\rangle = m\omega_c b\sqrt{j}\;|j-1\,\ell\,K_z\rangle\,. \tag{B-5}$$

We can now write, for $k_{||}$ H, i.e. $\underline{k} = k\hat{\underline{z}}$,

$$I_\lambda(j'\ell' K_z' | j\ell K_z) = < j'\ell' \; K_z' \,|\, e^{-ik_z z} \, \underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{\Pi} \,|\, j\ell K_z >$$

$$= \delta(K_z' + k_z - K_z) < j'\ell' \,|\, \underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{\Pi} \,|\, j\ell > . \qquad \text{(B-6)}$$

Taking

$$\underline{\mathcal{E}}_1 = \frac{1}{\sqrt{2}} (\underline{e}_2 + i\,\underline{e}_1)$$

$$\underline{\mathcal{E}}_2 = \frac{1}{\sqrt{2}} (\underline{e}_2 - i\,\underline{e}_1)$$

where $\underline{e}_1$ and $\underline{e}_2$ are the unit vectors for the x and y axes, respectively, we have

$$\underline{\mathcal{E}}_1 \cdot \underline{\Pi} = \Pi^+$$

$$\underline{\mathcal{E}}_2 \cdot \underline{\Pi} = \Pi^- .$$

Thus

$$I_1(j'\ell' K_z' | j\ell K_z) = m\omega_c b \sqrt{j+1} \, \delta_{j', j+1} \, \delta_{\ell', \ell} \, \delta(K_z' + k_z - K_z) \qquad \text{(B-7a)}$$

$$I_2(j'\ell' K_z' \,|\, j\ell K_z) = m\omega_c b \sqrt{j} \, \delta_{j', j-1} \delta_{\ell', \ell} \, \delta(K_z' + k_z - K_z) . \qquad \text{(B-7b)}$$

We will complete the calculation of equation (III-11) by use of equation (III-7). In a spatially uniform, nondegenerate system, we find

$$\rho_\lambda = \frac{1}{m} \sum_{jj'} \int dK_z' \int dK_z \bar{f}(K_z) \bar{f}(j) \left[\frac{|I_\lambda(j\,|\,j')|^2 \, \delta(K_z' + k_z - K_z)}{(j'-j)\hbar\omega_c - \hbar\omega_\lambda + \frac{\hbar^2 K_z^2}{2m} - \frac{\hbar^2 K_z^2}{2m}} \right.$$

$$\left. + \frac{|I_\lambda(j'\,|\,j)|^2 \, \delta(K_z' + k_z - K_z)}{(j'-j)\hbar\omega_c + \hbar\omega_\lambda + \frac{\hbar^2 K_z'^2}{2m} - \frac{\hbar^2 K_z^2}{2m}} \right] . \qquad \text{(B-8)}$$

From the symmetry properties

$$I_1(j' \,|\, j) = I_2(j \,|\, j')$$

$$I_2(j' \,|\, j) = I_1(j \,|\, j')$$

and the assumption $\frac{\hbar k_z}{\hbar K_z} << 1$, it is straightforward to obtain the result

$$\omega_{\pm}^2 = c^2 k^2 + \omega_p^2 \; P \int d v_z \, \bar{f}(v_z) \; \frac{\omega_{\pm} \left(1 - \frac{v_z k_z}{\omega_{\pm}}\right)}{(\omega_{\pm} \pm \omega_c) \left(1 - \frac{v_z k_z}{\omega_{\pm} \pm \omega_c}\right)} \qquad \text{(B-9)}$$

where $\omega_1 = \omega_+$ and $\omega_2 = \omega_-$.

APPENDIX C

CALCULATION OF THE PLASMA DISPERSION RELATION FOR PROPAGATION PERPENDICULAR TO A CONSTANT, EXTERNAL MAGNETIC FIELD

In order to obtain equation (III-14), we must calculate $I(j'\ell'K'_z\,|\,j\ell K_z)$. Take $\ell = 0$ (the well-centered-orbit approximation (40, 41)) and write

$$\underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{\Pi} = \mathcal{E}^+_\lambda \Pi^- + \mathcal{E}^-_\lambda \Pi^+ + \mathcal{E}^3_\lambda \Pi^z$$

where $\mathcal{E}^{\pm}_\lambda = \frac{1}{\sqrt{2}}\left[\mathcal{E}^y_\lambda \pm i\,\mathcal{E}^x_\lambda\right]$. Then[8]

$$I(j'\ell'K'_z|j0K_z) = \mathcal{E}^3_\lambda \hbar K_z\, \delta(K'_z + k_z - K_z)\; I(j'\ell'|j0)$$

$$+ \delta(K'_z + k_z - K_z)\, m\omega_c b\sqrt{j+1}\left[\mathcal{E}^-_\lambda I(j'\ell'|j+1,0)\right.$$

$$\left. + \sqrt{\frac{j}{j+1}}\;\mathcal{E}^+_\lambda\, I(j'\ell'|j-1,0)\right] \qquad \text{(C-1)}$$

where

$$I(j'\ell'|j0) = \langle j'\ell'|e^{-ik_\perp \rho\cos\phi}|j0\rangle .$$

It was shown by Judd, et. al. (42) that

$$I(j'\ell'|j0) = \left[\frac{1}{\ell'!}\right]^{1/2}\left(-\frac{\alpha^2}{2}\right)^{\ell'/2} I(j'0|j0)$$

where $\alpha = kb\sin\theta$, θ being the angle between the z axis and the propagation vector $\underline{k}$, the latter being restricted to the y-z plane. Since

8. For the magnetic state vector and for properties of $\Pi^{\pm}$, see Appendix B. We also use $\Pi^z|j\ell K_z\rangle = \hbar K_z|j\ell K_z\rangle$.

$$\alpha^2 \lesssim 10^{-12} \mu^2 H (j-j')^2 << 1 ,$$

we can take $\ell' = 0$.

Writing $\xi = \rho^2/2b^2$, it is a straightforward calculation to obtain

$$I(j' 0 | j\ 0) = \left[\frac{1}{j!\, j'!}\right]^{1/2} \int_0^\infty d\xi\, e^{-\xi}\, \xi^{j-\frac{n}{2}}\, J_n\left[\alpha(2\xi)^{1/2}\right] \qquad \text{(C-2)}$$

where $n = j-j'$.

Substituting

$$J_n\left[\alpha(2\xi)^{1/2}\right] = \sum_{m=0}^{\infty} \frac{(-1)^m\, \alpha^{2m+n}\, \xi^{m+n/2}}{2^{m+n/2}\, m!\, (m+n)!}$$

into equation (C-2) and integrating, we find

$$I(j'\ 0 | j\ 0) = \left[\frac{1}{j!\ j'!}\right]^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m\, \alpha^{2m+n}\, (j+m)!}{2^{m+n/2}\, m!\, (m+n)!} . \qquad \text{(C-3)}$$

Using Stirling's approximation, and assuming $j >> m$, we have

$$(j + m)! \simeq j!\, j^m ,$$

so that

$$I(j'\ 0 | j\ 0) \simeq \left[\frac{(j-n)!}{j!}\right]^{-1/2} \frac{1}{j^{n/2}}\, J_n\left[\alpha(2j)^{1/2}\right] . \qquad \text{(C-4)}$$

Again applying Stirling's approximation, and assuming $j >> n$, equation (C-4) becomes

$$I(j'\ 0 | j\ 0) \simeq J_n\left[\alpha(2j)^{1/2}\right] . \qquad \text{(C-5)}$$

Similarly

$$I(j'\ 0 | j \pm 1,\ 0) \simeq J_{n \pm 1}\left[\alpha(2j)^{1/2}\right] . \qquad \text{(C-6)}$$

Specifying the polarization vectors as the spherical base vectors $\underline{\mathcal{E}}_1 = \underline{\mathcal{E}}_\theta$ and $\underline{\mathcal{E}}_2 = \underline{\mathcal{E}}_\phi$, we find

$$\mathcal{E}_\theta^{\pm} = \frac{1}{\sqrt{2}} \cos\theta ; \qquad \mathcal{E}_\theta^{3} = -\sin\theta$$

$$\mathcal{E}_\phi^{\pm} = \pm \frac{1}{\sqrt{2}} \qquad ; \qquad \mathcal{E}_\phi^{3} = 0 .$$

Then, noting that R_j, the radius of orbit of quantum number j, and v_j, the speed of an electron in that orbit are given by

$$R_j = (2j)^{1/2} \left(\frac{\hbar}{m\omega_c}\right)^{1/2} ; \quad v_j = (2j)^{1/2} \left(\frac{\hbar\omega_c}{m}\right)^{1/2}$$

we obtain, after using Bessel recursion formulas,

$$I_\phi(j'\,0\,K'_z \,|\, j\,0\,K_z) = m\,\omega_c\, b\sqrt{2j}\;\delta(K'_z + k_z - K_z)\, J'_n(v_j \gamma_\phi \sin\theta) \tag{C-7}$$

$$I_\theta(j'\,0\,K'_z \,|\, j\,0\,K_z) = -\hbar K_z \sin\theta\;\delta(K'_z + k_z - K_z)\, J_n(v_j \gamma_\theta \sin\theta)$$

$$+ m n \omega_c b\sqrt{2j}\,\cos\theta\;\delta(K'_z + k_z - K_z)\,\frac{J_n(v_j \gamma_\theta \sin\theta)}{v_j \gamma_\theta \sin\theta} \tag{C-8}$$

where $\gamma_\lambda = \dfrac{\mu_\lambda \omega_\lambda}{c\omega_c}$. Writing equation (III-7) in the form

$$\rho_{\lambda k} = \frac{1}{m} \sum_{jj'} \int d K_z \int dK'_z \left[\bar{f}(K'_z)\,\bar{f}(j') - \bar{f}(K_z)\,\bar{f}(j)\right] \tag{C-9}$$

$$(x)\; \frac{\left| I_{\lambda k}(j'\,K'_z \,|\, j\,K_z) \right|^2}{E_K - E_{K_1} - \hbar\,\omega_{\lambda k}} , \tag{C-9}$$

we find, by substitution of equation (C-7) into (C-9)

$$\rho_\theta = \sum_{j=0}^{\infty} \sum_{n=-\infty}^{j} \int d\,K_z\, \bar{f}(K_z)\, \bar{f}(j) \left[\frac{\bar{f}(K_z - k_z)}{\bar{f}(K_z)} \; e^{\hbar \omega_c n / k_o T} - 1 \right] \tag{C-10}$$

$$(\times) \left[\cot\theta - \frac{\hbar K_z}{mc} \left(\frac{ck}{n\omega_c} \right) \sin\theta \right]^2 \left[\frac{mc^2 \left(\frac{n\omega_c}{ck} \right)^2 \left[J_n(v_j \gamma_\theta \sin\theta) \right]^2}{n\hbar\omega_c - \hbar\omega_\theta - \frac{\hbar^2 k_z K_z}{m} + \frac{\hbar^2 k_z^2}{2m}} \right] .$$

To obtain this, we have written

$$\frac{\bar{f}(j')}{\bar{f}(j)} = e^{\hbar \omega_c n / k_o T}$$

a result which follows by taking an equilibrium distribution

$$\bar{f}(j) = (1 - e^{-\hbar\omega_c / k_o T})\, e^{-\hbar \omega_c j / k_o T} .$$

This distribution is obtained by conventional statistical mechanical methods assuming the energy levels are the magnetic state levels and that $\bar{f}(j, K_z) \simeq \bar{f}(j)\, \bar{f}(K_z)$. Observing that

$$\hbar \omega_c j = \frac{m}{2} v_j^2$$

it is seen that

$$\bar{f}(j)\, dj = f(j) \frac{m\, v_j}{\hbar\, \omega_c}\, d\, v_\perp$$

$$\simeq \frac{m\, v_\perp}{k_o\, T}\; e^{(m v_\perp^2/2)/k_o T}\; dv_\perp$$

$$= \bar{f}(v_\perp)\, dv_\perp .$$

The upper limit of the sum over n can be extended to infinity since the terms for j large give negligible contribution. Finally, for propagation along the y axis, it is readily established that

$$\rho_\theta \simeq \sum_n \frac{n\,\omega_c}{n\,\omega_c - \omega_\theta} \Upsilon_n \tag{C-11}$$

where

$$\Upsilon_n = \int_0^\infty d\,v_\perp\ \bar{f}(v_\perp)\ \left[J_n(v_\perp\,\gamma_\theta)\right]^2 \quad .$$

We can now write

$$\omega_\theta^2 = c^2k^2 - \omega_p^2 \sum_n \frac{\omega_\theta}{n\,\omega_c - \omega_\theta} \Upsilon_n + \omega_p^2 \left(1 - \sum_n \Upsilon_n\right) . \tag{C-12}$$

But

$$\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$$

so that

$$\sum_n \Upsilon_n = 1$$

and

$$\omega_\theta^2 = c^2k^2 - \omega_p^2 \sum_n \frac{\omega_\theta}{n\,\omega_c - \omega_\theta} \Upsilon_n \tag{C-13}$$

the result shown as equation (III-14).

Equa tion (III-16) can be obtained by similar considerations with the use of equation (C-7).

APPENDIX D

THE HAMILTONIAN AFTER DIVIDING THE SYSTEM INTO CELLS

We present here the various terms in the Hamiltonian. Recall equations (IV-10a) to (IV-10f).

$$H = H^{o} + H^{(1)} + H^{(2)} + H^{\gamma T} + H^{PT} + H^{IT} + V^{cc} \qquad \text{(IV-10a)}$$

$$H^{o} = T^{\gamma 0} + H^{M} \qquad \text{(IV-10b)}$$

$$H^{M} = T^{P} + H^{Pe} + V^{c} \qquad \text{(IV-10c)}$$

$$H^{(1)} = H^{P\gamma 1} + H^{P\gamma e} \qquad \text{(IV-10d)}$$

$$H^{(2)} = (H_{o}^{P\gamma 2} + T^{\gamma 1}) + (H_{1}^{P\gamma 2} + T^{\gamma 2}) + H_{2}^{P\gamma 2} \qquad \text{(IV-10e)}$$

$$H^{\gamma T} = H_{1}^{\gamma T} + H_{2}^{\gamma T} . \qquad \text{(IV-10f)}$$

Explicit calculation gives:

$$T^{\gamma 0} = \sum_{\underline{\mathbf{X}},\, \lambda \underline{k}} \hbar \omega_{\lambda} (\underline{\mathbf{X}} , k) \alpha_{\lambda}^{+} (\underline{\mathbf{X}} , \underline{k}) \alpha_{\lambda} (\underline{\mathbf{X}} , \underline{k}) \qquad \text{(D-1)}$$

$$T^{P} + H^{Pe} = \sum_{\sigma\, \underline{\mathbf{X}},\, \underline{K}\underline{K}'} \frac{1}{2m_{\sigma}} a_{\sigma}^{+} (\underline{K}) a_{\sigma} (\underline{K}') \frac{1}{V} \int d^{3}x\, \underline{\Pi}^{\sigma^{*}} u_{\sigma \underline{K}}^{*}(\underline{x}) \cdot$$

(D-2)

$$\cdot \ \underline{\Pi}^{\sigma} u_{\sigma \underline{K}'}(\underline{x})\, E(\underline{\mathbf{X}}, \underline{x})$$

$$V^c = \sum_{\substack{\sigma\sigma'\underline{X},\\ \underline{K},\underline{K}',\underline{K}'',\underline{K}'''}} a^+_\sigma(\underline{X},\underline{K})\, a^+_{\sigma'}(\underline{X},\underline{K}')\, a_\sigma(\underline{X},\underline{K}'')\, a_{\sigma'}(\underline{X},\underline{K}''')$$

(D-3)

$$(\times)\ \frac{1}{V^2}\int d^3x\, d^3x'\, E(\underline{X},\underline{x})E(\underline{X},\underline{x}')\ \frac{u^+_{\sigma\underline{K}}(\underline{x})\, u^+_{\sigma'\underline{K}'}(\underline{x}')\, u_{\sigma\underline{K}''}(\underline{x})\, u_{\sigma'\underline{K}'''}(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$H^{(1)} = -\sum_{\substack{\sigma\underline{X}\lambda\underline{k}\\ \underline{K},\underline{K}'}} \left(\frac{2\pi\hbar c^2}{V}\right)^{1/2} \left(\frac{e_\sigma}{m_\sigma c}\right) \frac{1}{\sqrt{\omega_\lambda(\underline{X},k)}}\ \xi^+_\lambda(\underline{X},\underline{k})$$

(D-4)

$$(\times)\ a^+_\sigma(\underline{X},\underline{K})a_\sigma(\underline{X},\underline{K}')\,\frac{1}{V}\int d^3x\, E(\underline{X},\underline{x})u^+_{\sigma\underline{K}}(\underline{x})\, e^{-i\underline{k}\cdot\underline{x}}\, \underline{\Pi}^\sigma u_{\sigma\underline{K}}(\underline{x})$$

$$H_0^{P\gamma2} + T^{\gamma1} = \frac{\hbar}{2}\sum_{\underline{X},\lambda\underline{k}}\left[\sum_\sigma \frac{4\pi e_\sigma^2}{m_\sigma\omega_\lambda(\underline{X},k)}\ a^+_\sigma(\underline{X},\underline{K})\, a_\sigma(\underline{X},\underline{K})\right.$$

(D-5)

$$\left. + \frac{c^2k^2-\omega^2_\lambda(\underline{X},k)}{\omega_\lambda(\underline{X},k)}\right]\alpha^+_\lambda(\underline{X},\underline{k})\,\alpha_\lambda(\underline{X},\underline{k})$$

$$H_1^{P\gamma2} + T^{\gamma2} = \frac{\hbar}{4}\sum_{\underline{X},\lambda\underline{k}}\left[\sum_\sigma \frac{4\pi e_\sigma^2}{m_\sigma\omega_\lambda(\underline{X},k)} + \frac{c^2k^2-\omega^2_\lambda(\underline{X},k)}{\omega_\lambda(\underline{X},k)}\right]$$

(D-6)

$$(\times)\ \left[\alpha^+_\lambda(\underline{X},\underline{k})\,\alpha^+_\lambda(\underline{X},-\underline{k}) + \alpha_\lambda(\underline{X},\underline{k})\,\alpha_\lambda(\underline{X},-\underline{k})\right]$$

$$H_2^{P\gamma 2} = \sum_{\substack{\underline{X}\sigma\lambda\underline{k}\lambda'\underline{k}' \\ \lambda'\underline{k}' \neq \lambda\underline{k} \\ KK', K \neq K'}} \frac{\pi\hbar e_\sigma^2}{m_\sigma} \frac{1}{\sqrt{\omega_\lambda(\underline{X}, k)\,\omega_{\lambda'}(\underline{X}, k')}} \xi_\lambda^+(\underline{X}, \underline{k}) \cdot \xi_{\lambda'}^+(\underline{X}, \underline{k}') \tag{D-7}$$

$$(x)\; a_\sigma^+(\underline{X}, \underline{K})\, a_\sigma(\underline{X}, \underline{K}') \frac{1}{V} \int d^3x\, E(\underline{X}, \underline{x})\, e^{-i(\underline{k}-\underline{k}')\cdot \underline{x}}\, u_{\sigma\underline{K}}^+(\underline{x})\, u_{\sigma\underline{K}'}(\underline{x})$$

$$H^{PT} = \sum_{\sigma \underline{K}\underline{K}'\underline{X}\underline{X}'} \frac{1}{2m_\sigma} a_\sigma^+(\underline{X}, \underline{K}) a_\sigma(\underline{X}, \underline{K}') \frac{1}{V} \int d^3x \Big[u_{\sigma\underline{K}}^+(\underline{x}) (\underline{\Pi}^{\sigma *} E(\underline{X}, \underline{x})) \cdot E(\underline{X}', \underline{x}) \underline{\Pi}^\sigma u_{\sigma\underline{K}'} \tag{D-8}$$

$$+ E(\underline{X}, \underline{x})(\underline{\Pi}^{\sigma *} u_{\sigma\underline{K}}^+(\underline{x}))\, u_{\sigma\underline{K}'} \underline{\Pi}^\sigma E(\underline{X}', \underline{x}) + u_{\sigma\underline{K}}^+ u_{\sigma\underline{K}'} (\underline{\Pi}^{\sigma *} E(\underline{X}, \underline{x})) \cdot (\underline{\Pi}^\sigma E(\underline{X}', \underline{x}))\Big]$$

$$H^{IT} = \sum_{\substack{\sigma\underline{X}\,\underline{X}'\lambda\underline{k} \\ \underline{K}\,\underline{K}'}} \left(\frac{2\pi\hbar c^2}{V}\right)^{1/2} \left(\frac{e_\sigma}{2m_\sigma c}\right) \frac{1}{\sqrt{\omega_\lambda(\underline{X}, k)}}\, \xi_\lambda^+(\underline{X}, \underline{k}) \cdot$$

$$(x) \left[a_\sigma^+(\underline{X}', \underline{K})\, a_\sigma(\underline{X}, \underline{K}') + a_\sigma^+(\underline{X}, \underline{K})\, a_\sigma(\underline{X}', \underline{K}') \right] \tag{D-9}$$

$$(x)\; \frac{1}{V} \int d^3x\, E(\underline{X}, \underline{x})\, u_{\sigma\underline{K}}^+(\underline{x})\, e^{-i\underline{k}\cdot\underline{x}}\, u_{\sigma\underline{K}'}(\underline{x}) (\underline{\Pi}^\sigma E(\underline{X}, \underline{x}))$$

$$V^{cc} = \sum_{\substack{\sigma\sigma' \underline{X}\underline{X}' \\ \underline{K}\underline{K}',\ \underline{K}''\underline{K}''' \\ \underline{X} \neq \underline{X}'}} a^{+}_{\sigma}(\underline{X},\underline{K})\, a^{+}_{\sigma'}(\underline{X}',\underline{K}')\, a_{\sigma}(\underline{X},\underline{K}'')\, a_{\sigma'}(\underline{X}',\underline{K}''') \tag{D-10}$$

$$(\times)\ \frac{1}{V^2}\int d^3x\, d^3x'\, E(\underline{X},\underline{x})E(\underline{X}',\underline{x}')\ \frac{u^{+}_{\sigma\underline{K}}(\underline{x})\, u^{+}_{\sigma'\underline{K}'}(\underline{x}')\, u_{\sigma\underline{K}''}(\underline{x})\, u_{\sigma'\underline{K}'''}(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$H_1^{\gamma T} = \frac{-i\hbar c^2}{4L}\Bigg[\sum_{\underline{X}\lambda\underline{k}\lambda'\underline{k}'} \frac{k_1 \underline{\xi}^{+}_{\lambda}(\underline{X},\underline{k})}{\sqrt{\omega_{\lambda}(\underline{X},k)}}\ \delta(k_2+k_2')\ \delta(k_3+k_3')\ ,$$

(D-11)

$$\cdot\left\{\frac{\underline{\xi}^{+}_{\lambda'}(X_1+L,X_2,X_3,-\underline{k}')}{\sqrt{\omega_{\lambda'}(X_1+L,X_2,X_3,k')}} - \frac{\underline{\xi}^{+}_{\lambda'}(X_1-L,X_2,X_3,-\underline{k}')}{\sqrt{\omega_{\lambda'}(X_1-L,X_2,X_3,k')}}\right\} + \text{cyclic permutation}\Bigg]$$

$$H_2^{\gamma T} = \frac{i\hbar c^2}{4L}\sum_{\underline{X}\lambda\underline{k}\lambda'\underline{k}'} \frac{\underline{\xi}^{+}_{\lambda}(\underline{X},\underline{k})}{\sqrt{\omega_{\lambda}(\underline{X},k)}}\ \delta(k_2+k_2')\,\delta(k_3+k_3')$$

(D-12)

$$(\times)\ \underline{k}\cdot\left\{\frac{\underline{\xi}^{+}_{\lambda'}(X_1+L,X_2,X_3,-\underline{k}')}{\sqrt{\omega_{\lambda'}(X_1+L,X_2,X_3,k')}} - \frac{\underline{\xi}^{+}_{\lambda'}(X_1-L,X_2,X_3,-\underline{k}')}{\sqrt{\omega_{\lambda'}(X_1-L,X_2,X_3,k')}}\right\} + \text{cyclic permutation}\Bigg]\ .$$

APPENDIX E

CALCULATION OF THE BREMSSTRAHLUNG EMISSION COEFFICIENT ϵ_B^λ

In order to calculate ϵ_B^λ, it is first necessary to consider

$$V^2 T_{BK}^{K_1} (\lambda \underline{k}) = \frac{4\pi^2 c^2}{\hbar} \left(\frac{e}{mc}\right)^2 \frac{1}{\omega_{\lambda k}} \delta(\omega_{K_1 k} - \omega_K)$$

$$(x) \quad |I(\underline{K}_1 | \underline{K})|^2 \tag{II-34}$$

where

$$I(\underline{K}_1 | \underline{K}) = \int d^3x \, u^*_{\underline{K}_1}(\underline{x}) \, e^{-i\underline{k}\cdot\underline{x}} (\underline{\mathcal{E}}_\lambda(\underline{k}) \cdot \underline{p}) u_{\underline{K}} \, . \tag{E-1}$$

Writing

$$u_{\underline{K}}(\underline{x}) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3K' \, e^{i\underline{K}'\cdot\underline{x}} \, \psi(\underline{K}') \tag{E-2}$$

equation (E-1) can be written

$$I(\underline{K}_1 | \underline{K}) = -i\hbar \int d^3K' \, \psi^*(\underline{K}') \, \psi(\underline{K}' + \underline{k}) \, (\underline{\mathcal{E}}_\lambda \cdot \underline{K}') \, . \tag{E-3}$$

The eigenstates $u_{\underline{K}}(\underline{x})$ are to satisfy the equation

$$(E - H_o - V) \, u = 0, \tag{E-4}$$

where $H_o = \dfrac{\hbar^2 K^2}{2m}$ and $V = -\dfrac{Ze^2}{r}$. (E-5)

Let u_o be a plane wave eigenstate of H_o with eigenvalue E. Then we can write[9]

$$u = u_o + \chi \; ; \qquad (E - H_o) \, u_o = 0 \tag{E-6}$$

---

9. The procedure used here for obtaining $u_{\underline{K}}(\underline{x})$ and $\psi(\underline{K})$ was originally developed by Lippmann and Schwinger(43). However, we will follow the presentation of this method of Bethe and Salpeter(26).

where χ satisfies the equation

$$(E - H_o - V)\ \chi - V u_o = 0 \ . \qquad \text{(E-7)}$$

It can be shown [26] that χ has a cut on the real axis and therefore represents two functions $\chi_\pm$, which satisfy the relation

$$\chi_\pm = \frac{1}{E - H_o \pm i\epsilon} V (u_o + \chi_\pm) \qquad \text{(E-8)}$$

where ϵ is an infinitesimally small quantity. (This integral equation representation of $\chi_\pm$ is not unique.) The eigenstates u can now be written

$$u_\pm = u_o + \chi_\pm \quad . \qquad \text{(E-9)}$$

It can be shown [26] that there is one unique state u_+ (and u_-) corresponding to each eigenstate of u_o of H_o. We will choose the eigenstates u_+ for the remainder of the discussion.

Define $\psi_+(\underline{K})$ as the Fourier transform of $u_+(\underline{x})$ i.e.

$$\psi_+(\underline{K}) = (\frac{1}{2\pi})^{3/2} \int d^3 x\, e^{-i\underline{K}\cdot \underline{x}} u_+ (\underline{x}) \ . \qquad \text{(E-10)}$$

From equation (E-9)

$$\psi_+(\underline{K}) = \delta(\underline{K} - \underline{K}') \frac{1}{\sqrt{V}} + \frac{1}{\frac{\hbar^2 K'^2}{2m} - \frac{\hbar^2 K^2}{2m} + i\epsilon}$$

$$(x) (\frac{1}{2\pi})^{3/2} \int d^3 x\, e^{-i\underline{K}\cdot\underline{x}} V(\underline{x}) u_+(\underline{x}) \qquad \text{(E-11)}$$

where u_o was chosen as

$$u_o = \frac{1}{\sqrt{V}} \left(\frac{1}{2\pi}\right)^{3/2} e^{-i\underline{K}'\cdot\underline{x}} \quad . \tag{E-12}$$

Substituting

$$V(\underline{x}) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3 K'' \; e^{i\underline{K}''\cdot\underline{x}} \; \bar{V}(\underline{K}'')$$

into equation (E-11), we obtain

$$\psi_+(\underline{K}) = \frac{1}{\sqrt{V}} \quad \delta(\underline{K} - \underline{K}') + \frac{1}{(K'+i\epsilon)^2 - K^2}$$

$$(x) \left(\frac{2m}{\hbar^2}\right) \int d^3 K'' \, \psi_+(\underline{K}'') \, \bar{V}(\underline{K} - \underline{K}'') \, . \tag{E-13}$$

In the first Born approximation we write

$$\psi_+(\underline{K}'') = \frac{1}{\sqrt{V}} \, \delta(\underline{K}'' - \underline{K}')$$

and equation (E-13) becomes

$$\psi_+(\underline{K}) = \frac{1}{\sqrt{V}} \left[\delta(\underline{K} - \underline{K}') + \left(\frac{2m}{\hbar^2}\right) \frac{\bar{V}(\underline{K} - \underline{K}')}{K'^2 - K^2} \right] \quad . \tag{E-14}$$

For the potential $V = -\dfrac{Z e^2}{r}$,

$$\bar{V}(|\underline{K}|) = - Z e^2 \, U(|\underline{K}|) \tag{E-15}$$

where

$$U(|\underline{K}|) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3 r \; \frac{e^{-i\underline{K}\cdot\underline{r}}}{r}$$

$$= \frac{1}{2\pi^2 \, K^2} \, (2\pi)^{3/2} \quad . \tag{E-16}$$

Substituting (E-14) and (E-15) into equation (E-3), we find

$$I(\underline{K}_1|\underline{K}) = \frac{-i\hbar}{V}\int d^3K' \, (\underline{\mathcal{E}}_\lambda \cdot \underline{K}') \left\{ \delta(\underline{K}'-\underline{K}_1)\;\delta(\underline{K}'+\underline{k}-\underline{K}) \right.$$

$$\left. - \left(\frac{2mZe^2}{\hbar^2}\right)\left[\delta(\underline{K}'-\underline{K}_1)\;\frac{U(|\underline{K}'+\underline{k}-\underline{K}|)}{K^2-|\underline{K}'+\underline{k}|^2} + \delta(\underline{K}'+\underline{k}-\underline{K})\;\frac{U(|\underline{K}'-\underline{K}_1|)}{K_1^2-K'^2}\right]\right\} . \tag{E-17}$$

The first term in brackets must vanish since $\underline{K}_1 + \underline{k} - \underline{K} = 0$ is not compatible with energy conservation. Thus

$$I(\underline{K}_1|\underline{K}) = \frac{2im\,Z\,e^2}{\hbar V}\;U(q)\left[\frac{(\underline{\mathcal{E}}_\lambda \cdot \underline{K}_1)}{K^2-|\underline{K}_1+\underline{k}|^2} + \frac{(\underline{\mathcal{E}}_\lambda \cdot \underline{K})}{K_1^2-|\underline{K}-\underline{k}|^2}\right] \tag{E-18}$$

where $\underline{q} = \underline{K}_1 + \underline{k} - \underline{K}$.

Assuming a nonrelativistic limit, we take $k << K_1 \simeq K$ and, substituting equation (E-18) into (II-46), obtain

$$V^2\,T^{K_1}_{BK}(\lambda\underline{k}) = \frac{8\pi n_I c^4 r_o^2 e^2 Z^2\;\delta(E_{K_1 k} - E_K)}{V\,\omega_\lambda^3}$$

$$(x)\;\frac{\left[\underline{\mathcal{E}}_\lambda \cdot \underline{K}_1 - \underline{\mathcal{E}}_\lambda \cdot \underline{K}\right]^2}{q^4} \tag{E-19}$$

where $r_o = \dfrac{e^2}{mc^2}$ and one factor of $\dfrac{1}{V}$ has been associated with the ion density n_I.

Converting to the continuum in particle momenta and observing that

$$\sum_{\underline{K}_1}(1) = \int d^3K_1\,\rho(\underline{K}) = V\int d^3K_1 \quad ,$$

ϵ^λ_B can now be written

$$\epsilon_B^\lambda = \frac{8\pi\, c^4 r_o^2\, e^2\, Z^2\, n_I}{\omega_\lambda^3} \int d^3\, K f(\underline{K}) \int d^3\, K_1 \;\; \delta(E_{K_1 k} - E_{\underline{K}})$$

$$(x) \quad \frac{|\underline{\mathcal{E}}_\lambda \cdot \underline{K}_1 - \underline{\mathcal{E}}_\lambda \cdot \underline{K}|^2}{|\underline{K}_1 - \underline{K}|^4} \quad . \qquad \text{(E-20)}$$

APPENDIX F

CALCULATION OF THE CROSS SECTION FOR PHOTON SCATTERING IN A CRYSTAL

We wish to reduce equation (VIII-9a)

$$\sigma(k',\underline{\Omega};k,\underline{\Omega}) = \sigma_T \frac{ck}{k'}\left(\frac{1}{2\pi}\right)\int dt\, e^{i\bar{\omega}t} \frac{1}{N}\sum_{\sigma\sigma'}\sum_{m} P(m)\left\langle m\left|\Omega^{\sigma^+}\Omega^{\sigma'}(t)\right|m\right\rangle \qquad \text{(F-1)}$$

to equation (VIII-10). Define

$$Y = Y^0 + Y^1 = \frac{1}{N}\sum_{\sigma\sigma'}\sum_{m} P(m)\left\langle m\left|\Omega^{\sigma^+}\Omega^{\sigma'}(t)\right|m\right\rangle$$

$$+\frac{1}{N}\sum_{\sigma}\sum_{m} P(m)\left\langle m\left|\Omega^{\sigma^+}\Omega^{\sigma}(t)\right|m\right\rangle \qquad \text{(F-2)}$$

and consider Y^0. Expanding $|m\rangle$ into the product wave function and utilizing orthogonality, we obtain

$$Y^0 = \sum_{\bar{m}_\sigma \bar{n}_\sigma} P(\bar{m}_\sigma)P(\bar{n}_\sigma)\left\langle \bar{m}_\sigma\left|e^{i\underline{\mathcal{K}}\cdot\underline{u}_\sigma}\right|\bar{m}_\sigma\right\rangle\left\langle \bar{n}_\sigma\left|\sum_j e^{i\underline{\mathcal{K}}\cdot\underline{\xi}_j}\right|\bar{n}_\sigma\right\rangle$$

$$(\times)\sum_{\bar{m}_{\sigma'},\bar{n}_{\sigma'}} P(\bar{m}_{\sigma'})P(\bar{n}_{\sigma'})\left\langle \bar{m}_{\sigma'}\left|e^{-i\underline{\mathcal{K}}\cdot\underline{u}_{\sigma'}}\right|\bar{m}_{\sigma'}\right\rangle\left\langle \bar{n}_{\sigma'}\left|\sum_j e^{-i\underline{\mathcal{K}}\cdot\underline{\xi}_j}\right|\bar{n}_{\sigma'}\right\rangle$$

$$(\times)\frac{1}{N}\sum_{\sigma\sigma'} e^{i\underline{\mathcal{K}}\cdot(\underline{R}^\sigma-\underline{R}^{\sigma'})} = \left|\sum_{\bar{m}_\sigma} P(\bar{m}_\sigma)\left\langle \bar{m}_\sigma\left|e^{-i\underline{\mathcal{K}}\cdot\underline{u}_\sigma}\right|\bar{m}_\sigma\right\rangle\right|^2 |\bar{f}|^2$$

$$(\times)\frac{1}{N}\sum_{\sigma\sigma'} e^{i\underline{\mathcal{K}}\cdot(\underline{R}^\sigma-\underline{R}^{\sigma'})} \qquad \text{(F-3)}$$

where

$$\bar{f} = \sum_{\bar{n}_\sigma} P(\bar{n}_\sigma) \left\langle \bar{n}_\sigma \left| \sum_j e^{-i\underline{\kappa}\cdot\underline{\xi}_j} \right| \bar{n}_\sigma \right\rangle . \tag{F-4}$$

By use of the Corallory to Bloch's theorem,

$$\sum_m P(m) \left\langle m \right| e^Q \left| m \right\rangle = e^{\frac{1}{2} \sum_m P(m) \langle m | Q^2 | m \rangle} \tag{F-5}$$

where Q is a harmonic oscillator variable. We take the thermal distribution for $P(\bar{m}_\sigma)$ so that

$$P(\bar{m}_\sigma) = 2 \sinh \frac{\hbar\omega_o}{2kT} \; e^{-\frac{\hbar\omega_o}{kT}(\bar{m} + \frac{1}{2})} . \tag{F-6}$$

From equations (F-3), (F-5) and (F-6) it is easily shown that

$$\left| \sum_{\bar{m}_\sigma} P(\bar{m}_\sigma) \left\langle \bar{m}_\sigma \right| e^{-i\underline{\kappa}\cdot\underline{u}_\sigma} \left| \bar{m}_\sigma \right\rangle \right|^2 = e^{-D\kappa^2} \tag{F-7}$$

where

$$D = \frac{\hbar}{2 M \omega_o} \coth \frac{\hbar\omega_o}{2kT} .$$

Finally, then, we have

$$Y^o = e^{-D\kappa^2} |\bar{f}|^2 \left\{ \frac{1}{N} \left| \sum_\sigma e^{i\underline{\kappa}\cdot\underline{R}^\sigma} \right|^2 - 1 \right\} . \tag{F-8}$$

Turning now to Y^1, we first rewrite it in the form

$$Y^1 = \frac{1}{N} \sum_\sigma \sum_{\bar{n}_\sigma \bar{n}'_\sigma} e^{\frac{i}{\hbar}(\epsilon_{\bar{n}'_\sigma} - \epsilon_{\bar{n}_\sigma})t} P(\bar{n}_\sigma) \left| \left\langle \bar{n}'_\sigma \left| \sum_j e^{-i\underline{\kappa}\cdot\underline{\xi}_j} \right| \bar{n}_\sigma \right\rangle \right|^2$$

$$(\times) \sum_{\bar{m}_\sigma} P(\bar{m}_\sigma) \left\langle \bar{m}_\sigma \right| \Lambda^{\sigma^+} \Lambda^\sigma(t) \left| \bar{m}_\sigma \right\rangle \tag{F-9}$$

where

$$\Lambda^{\sigma} \equiv e^{-i\underline{\kappa}\cdot\underline{u}^{\sigma}} .$$

After a somewhat tedious calculation, we find

$$\Lambda(t) = e^{-i\underline{\kappa}\cdot\left[\underline{u}_{\sigma}\cos\omega_{o}t + \underline{P}^{\sigma}\frac{\sin\omega_{o}t}{M\omega_{o}}\right]} .$$

Hence,

$$\langle \overline{m} | \Lambda^{+}\Lambda(t) | \overline{m} \rangle = \langle \overline{m} | e^{i\underline{\kappa}\cdot\underline{u}} e^{-i\underline{\kappa}\cdot\left[\underline{u}\cos\omega_{o}t + \underline{P}\frac{\sin\omega_{o}t}{M\omega_{o}}\right]} | \overline{m} \rangle ,$$

By use of the formula

$$e^{A} e^{B} = e^{A+B+\frac{1}{2}\left[A,B\right]}$$

which holds as long as A and B commute with $\left[A, B\right]$, we find

$$\sum_{\overline{m}} P(\overline{m}) \langle \overline{m} | \Lambda^{+} \Lambda(t) | \overline{m} \rangle = e^{i\hbar\kappa^{2}\sin\omega_{o}t/2M\omega_{o}}$$

$$(x) \quad P(\overline{m}) \langle \overline{m} | e^{i\underline{\kappa}\cdot\left[(1-\cos\omega_{o}t)\underline{u} - \frac{\sin\omega_{o}t}{M\omega_{o}}\underline{P}\right]} | \overline{m} \rangle . \qquad \text{(F-10)}$$

Again utilizing equations (F-5) and (F-6), we obtain for equation (F-10), after laborious computation,

$$e^{-D\kappa^{2}}\left[I_{o}(B\kappa^{2}) + \sum_{n=1}^{\infty} I_{n}(B\kappa^{2})\right.$$

$$(x) \left\{ e^{in(\omega_{o}t - \frac{i\hbar\omega_{o}}{2kT})} + e^{-in(\omega_{o}t - \frac{i\hbar\omega_{o}}{2kT})} \right\} , \qquad \text{(F-11)}$$

where

$$B = \frac{\hbar}{2M\omega_{o}} \operatorname{csch} \frac{\hbar\omega_{o}}{2kT} .$$

Writing

$$J \equiv \sum_{\bar{n},\bar{n}'} e^{\frac{i}{\hbar}(\epsilon_{\bar{n}'} - \epsilon_{\bar{n}})t} P(\bar{n}) \left| \left\langle \bar{n}' \right| \sum_{j} e^{-i\underline{\kappa}\cdot \underline{\xi}_j} \left| \bar{n} \right\rangle \right|^2$$

$$= \sum_{\bar{n},\bar{n}'}{}' e^{\frac{i}{\hbar}(\epsilon_{\bar{n}'} - \epsilon_{\bar{n}})t} P(\bar{n}) \left| f_{\bar{n}',\bar{n}} \right|^2 + \overline{|f|^2} \, . \qquad \text{(F-12)}$$

For scattering in which electronic states do not make any transitions, we obtain

$$J \simeq \overline{|f|^2} \, . \qquad \text{(F-13)}$$

Substituting equations (F-11) and (F-13) into (F-9), equation (VIII-10) follows directly from equations (F-1), (F-2), (F-8) and (F-9).

If we had also included electronic transitions, we would have approximated

$$\bar{\omega} + (\epsilon_{\bar{n}'} - \epsilon_{\bar{n}})/\hbar \simeq \bar{\omega}$$

and employed

$$\sum_{\bar{n}'} \left| f_{\bar{n}',\bar{n}} \right|^2 = 1$$

so that $J \simeq 1$. The modification of equation (VIII-10) is to replace $\overline{|f|^2}$ by unity.

BIBLIOGRAPHY

1. Goldstein, H. The Attenuation of Gamma Rays in Reactor Shields. Addison-Wesley, Reading, 1959.

2. Milne, E.A. "Thermodynamics of Stars." Handbuch der Astrophysik, 3, 158, Julius Springer, Berlin, 1930.

3. Chandrasekhar, S. Radiative Transfer. Dover, New York, 1960.

4. Woolley, R.v.d.R. "The Solar Corona." Australian J. Sci., Suppl. 10, No. 2., (1947).

5. Bogoliubov, N.N. Problems of Dynamic Theory in Statistical Physics. Federal Publishing House for Technical Theoretical Literature, Moscow Leningrad, 1946. (Translated by Lydia Venters, AEC-tr-3852) (19607)

6. Kirkwood, J.G. "The Statistical Mechanical Theory of Transport Processes I. General Theory." J. Chem. Phys., 14, (1946) 180.

7. Prigogine, I., and Balescu, R. "Irreversible Processes in Gases. III. Inhomogeneous Systems." Physica 26, (1960) 145.

8. Grad, H. "Principles of the Kinetic Theory of Gases" Handbuch der Physik, 12, 205, Springer-Verlag, Berlin, 1958.

9. Osborn, R.K., and Klevans, E.H. "Photon Transport Theory." Annals of Physics, 15, (1961) 105.

10. Mead, C.A. "Quantum Index of Refraction." Phys. Rev., 110, (1958) 359.

11. Tidman, D.A. "A Quantum Theory of Refractive Index, Cerenkov Radiation and the Energy Loss of a Fast Charged Particle." Nuc. Phys., 2, (1956-57) 289.

12. Bohm, D., and Pines, D. "A Collective Description of Electron Interactions. I. Magnetic Interactions." Phys. Rev., 82, (1951) 625.

13. Spitzer, L., Jr. Physics of Fully Ionized Gases. p. 53, Interscience Publishers, New York, 1956.

14. Pradhan, T. "Plasma Oscillations in a Steady Magnetic Field: Circularly Polarized Electromagnetic Modes." Phys. Rev., 107, (1957) 1222.

15. Bernstein, I.B. "Waves in a Plasma in a Magnetic Field." Phys. Rev., 109, (1958) 10.

16. Landau, L.D. and Lifshitz, E.M. Electrodynamics of Continuous Media, p. 368, Addison-Wesley, Reading, 1960.

17. Bekefi, G., and Brown, S.C. "Emission of Radio-Frequency Waves from Plasmas." Amer. J. Phys., 29, (1961) 404.

18. Schiff, L.I. Quantum Mechanics, Ch. 9, McGraw-Hill, New York, 1949.

19. Osborn, R.K. IRE Trans. PGAP AP-10, (1962) 8.

20. Van Hove, L. "Lectures on Statistical Mechanics of Non Equilibrium Phenomena." p. 149, Theory of Neutral and Ionized Gases. ed. Dewitt, C. and Detoeuf, J.F., Wiley, New York, 1960.

21. Mori, H. and Ross, J., "Transport Equation in Quantum Gases." Phys. Rev., 109, (1958) 1877.

22. O'Rourke, R.C. "Damping Theory." Naval Research Laboratory Report 5315, (1959).

23. Heitler, W. The Quantum Theory of Radiation, 3rd. ed., Oxford University Press, London and New York, 1954.

24. Arnous, E. and Heitler, W. "Theory of Line-breadth Phenomena." Proc. Royal Soc. London, Series A, 220, (1953) 290.

25. Van Hove, L. "Energy Corrections and Persistent Perturbation Effects In Continuous Spectra." Physica 22, (1956) 343.

26. Bethe, H. and Salpeter, E. Quantum Mechanics of One-and Two- Electron Atoms. p. 40, Academic Press, New York, 1954.

27. Landau, L.D. and Lifshitz, E.M. Quantum Mechanics, Non-Relativistic Theory, p. 215, Addison-Wesley, Reading, 1958.

28. Johnson, M.H. and Lippmann, B.A. "Motion in a Constant Magnetic Field." Phys. Rev., 76, (1949) 828.

29. Ono, S. "The Quantum-statistical Theory of Transport Phenomena, III." Prog. Theor. Phys. (Japan), 12, (1954) 113.

30. Fano, U. "Description of States in Quantum Mechanics by Density Matrix and Operator Techniques." Revs. of Mod. Phys., 29, (1957) 74.

31. Spitzer, L., Jr., op. cit., p. 49.

32. Trubnikov, B.A. and Bazhanova, A.E. "Magnetic Radiation from a Plasma Layer." Plasma Physics and the Problem of Controlled Thermonuclear Reactions. Vol. 3, 141, Pergamon Press, 1959.

33. Hirshfield, J. L., Baldwin, D. E. and Brown, S. C. "Cyclotron Radiation from a Hot Plasma." Phys. Fluids, 4, (1961) 198.

34. Drummond, W. E. and Rosenbluth, M. N. "Cyclotron Radiation from a Hot Plasma." Phys. Fluids, 3, (1960) 45.

35. Heitler, W., op. cit., p. 178, equation (10).

36. Berman, S. M. "Electromagnetic Radiation from an Ionized Hydrogen Plasma." Space Technology Laboratories, PRL 9-27 (1959).

37. James, R. W. The Optical Principles of the Diffraction of X-Rays. G. Bell and Sons Ltd., London, 1958.

38. Yip, S., Osborn, R. K. and Kikuchi, C. "Neutron Acoustodynamics." University of Michigan College of Engineering Industries Program Report IP-524.

39. Magnus, W. and Oberhettinger, F. Functions of Mathematical Physics. Chelsea, New York, 1954.

40. Parzen, G. "Radiation from an Electron Moving in a Uniform Magnetic Field." Phys. Rev., 84, (1951) 235.

41. Olsen, H. and Wergeland, H. "Radiation Loss of Electrons in the Syncrotron." Phys. Rev., 86, (1952) 123.

42. Judd, D. L., Lepore, J. V., Ruderman, M. and Wolff, P. "Radiation from an Electron in a Magnetic Field." Phys. Rev., 86, (1952) 123.

43. Lippmann, B. A. and Schwinger, J. "Variational Principles for Scattering Processes. I." Phys. Rev., 79, (1950) 469.

44. Tolman, R. C. The Principles of Statistical Mechanics. Oxford Press, London, (1938) 457.