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STUDIES IN NON-LINEAR MODELING-IV: FAR FIELD SCATTERING BY SIMPLE SHAPES AT LOW FREQUENCIES

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I
INTRODUCTION

This is the final report on Contract AF-19(604)-8030 which was concerned with the theoretical investigation of electromagnetic scattering problems. The objectives of the contract were two-fold. One goal was to obtain the solution for scattering by a body formed by the union or intersection of shapes for each of which solutions were known. This solution was to avoid the problem of inverting infinite matrices.

The second goal was the extension of the method of non-linear modeling to a particular and practical problem, namely, modeling the solution for scattering from a prolate spheroid by that for scattering from a sphere.

This work is an outgrowth of previous investigations carried out by the Radiation Laboratory supported largely by the Air Force Cambridge Research Laboratories under Contract AF-19(602)-4993.

The effort on the problem of scattering by complicated shapes is part of a long range program, the ultimate goal of which is the theoretical determination of the electromagnetic (and acoustic) scattering properties of an arbitrarily shaped target to any desired degree of accuracy. Though this goal may never be fully attained, considerable activity, both in the Radiation Laboratory and in the field in general, has produced a variety of approximate techniques as well as exact solutions, thus considerably enlarging the class of shapes whose scattering properties can be said to be known. Under the contract of which this is the final report, one concern has been refining techniques applicable for low frequencies (wavelength large with respect to scatterer) to extend their range of validity. This effort has met with considerable success.

Darling (1960) developed a static approximation which employs probablistic techniques for solving Laplace's equation with Dirichlet boundary conditions on a surface which is the intersection of two (or more) surfaces for which both the Dirichlet and Neumann solutions are known. Under the present contract these results have been extended to the scalar wave equation. In addition the Neumann boundary condition is treated. These results together with extensions of the low frequency expansions developed by Senior (1961) form the subject of an invited paper at the URSI Symposium on Electromagnetic Theory and Antennas, June 1962, and will appear in the proceedings of that meeting. In addition the work on Rayleigh scattering from complex targets described by Siegel (1958,1959) has been extended to include higher order terms.

These low frequency approximations can be used to penetrate the resonance region, which for the present purpose might be defined as that middle range of frequency where both low and high frequency approximations are no longer valid. This region of theoretical inadequacy is narrowed by extending the range of validity of available approximations. In some cases, the resonance region can be eliminated in the sense that the ranges of validity of the low and high frequency techniques adjoin or overlap.

The results of these investigations are reported in the literature; Siegel (1962), and Senior et.al.(1962), and it would appear redundant to further dwell on them here.

Non-linear modeling as conceived by Ritt (1956) and developed by Belyea et al (1959), (1960), and Chen et al (1961) can be characterized as an attempt to transform by analytic methods a physical problem for which neither experimental nor direct theoretical techniques is feasible into another problem more amenable to either experiment or theory, or both. Under the present contract, two facets of the problem have received attention. One effort consists of a systematic investigation of the consequences of a basic assumption of non-linear modeling, i.e. there exists a functional relation (modeling function) of the form $f_1=f_1(f_2)$ when f_1 and f_2 are solutions of the same partial differential equation for different parameter values. That is, if D_k is the differential operator (in n dimensions) containing a parameter k, and

$$D_{k_i} \circ f_i (x_1, x_2, \ldots x_n) = 0 \qquad i = 1 \text{ or } 2 ,$$

does the assumption that f_1 be a function of f_2 constitute a sensible restriction and if so what can be said about the functional form? These questions have been thoroughly investigated for particular elliptic,parabolic and hyperbolic operators. The results are highly dependent on the choice of operator. In the case of the hyperbolic operator chosen, the time dependent wave equation, the assumption essentially did not constitute a restriction. Whereas, in the case of the Helmholtz equation (the elliptic operator chosen), not only is the restriction severe but explicit power series representations of the modeling functions can be found. This work is the subject of a separate technical report, Ruehr (1962).

In view of the considerable progress in these general theoretical treatments, it was hoped that the techniques of non-linear modeling had developed sufficiently so as to be applicable to the solution of specific practical problems. To this end considerable effort was devoted to the modeling of the field scattered by a prolate spheroid with that scattered by a sphere. Although the prolate spheroid is a shape for which the theoretical treatment is known, numerical calculations are quite difficult and accurate experiments are quite expensive. The hope was to transform the problem to one involving a spherical scatterer which is much more amenable to theoretical treatment.

Sections II, III and IV present the results of three different methods of attack. In Section II a precise formulation of the problem is given. With the help of a representation of spheroidal functions in terms of spherical functions presented in Appendix A, the problem is solved but unfortunately in an inappropriate form.

In Section III, it is shown that non-linear modeling can be achieved after relaxing one of the requirements, and the modeling function is explicitly derived. By making a different assumption about the functional relationship between the sphere and spheroid fields in Section IV, a different modeling function is derived which is in fact linear. Since the coefficients in any expansion in spherical harmonics of the far zone term of any field are precisely the coefficients in a multipole expansion of that field, a result derived explicitly in Appendix B, this linear modeling function is shown to be an extrapolation from that case when two terms of a multipole expansion suffice to describe the spheroid field.

A numerical example is presented in Section V which compares the spheroid field obtained from each of the two modeling functions with the exact result. The comparison does not provide a criterion for preferring either modeling function and suggests that their range of applicability is severely limited. Thus the promise of the more general results, another example of which (non-linear modeling of quantum mechanical systems, written by J.E.Belyea) appears in Appendix C, has not yet been completely fulfilled in the sphere-spheroid modeling problem.

II
A BOUNDARY VALUE PROBLEM RELATING THE SPHERE AND THE PROLATE SPHEROID

In most previous work on non-linear modeling, problems were chosen to illustrate the technique rather than out of inherent interest. Here we attempt to go to the next stage of useful rather than illustrative application.

The particular problem we shall consider is that of finding the field scattered by a rigid (Neumann boundary condition) prolate spheroid when a plane wave is incident along the axis of symmetry. In an attempt to avoid the use of series of products of prolate-spheroidal wave functions, we shall try to solve an equivalent problem exterior to a sphere.

Specifically, let S be a prolate spheroid of semi-major axis a, lying in the polar axis of a spherical coordinate system (ρ, θ, ϕ) (Figure 2.1). Let $\sum$ be

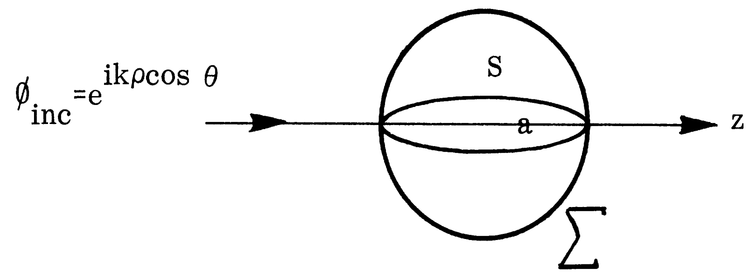

FIGURE 2.1

the sphere $\rho = a$, and let the plane wave $e^{ik\rho\cos\theta}$ be incident on S. Let $\phi(\rho, \theta)$ be the solution of the following boundary value problem:

$$\nabla^2\phi + k^2\phi = 0 , \qquad \text{exterior to S,} \tag{2-1}$$

$$\frac{\partial\phi}{\partial n} = 0 , \qquad \text{on S,} \tag{2-2}$$

$$\rho\left(\frac{\partial\phi_s}{\partial\rho} - ik\,\phi_s\right) \xrightarrow[\rho\to\infty]{} 0, \quad \text{uniformly in } \theta, \tag{2-3}$$

where $\phi_s = \phi - e^{ik\rho\cos\theta}$. It is then required to find a function $h(\theta)$ such that if ϕ satisfies (2-1), (2-2), and (2-3) above, then

$$\frac{\partial\phi}{\partial\rho} + h\,\phi = 0, \quad \text{on} \sum . \tag{2-4}$$

It was hoped that $h(\theta)$ could be determined from (2-1) - (2-4) above <u>without</u> explicit knowledge of ϕ. That such an $h(\theta)$ exists and is unique follows from the fact that (2-1) - (2-3) define $\phi(\rho,\theta)$ uniquely everywhere exterior to S. So that one has only to compute the ratio $-\left(\frac{1}{\phi}\frac{\partial\phi}{\partial\rho}\right)_{\rho=a}$, the result being the desired $h(\theta)$ as can be seen from (2-4).

By defining a function $\psi(\rho,\theta)$ to be

$$\psi(\rho,\theta) = -\ln\phi(\rho,\theta) \quad , \tag{2-5}$$

then one has

$$\psi_\rho(a,\theta) = -\frac{\phi_\rho(a,\theta)}{\phi(a,\theta)} = h(\theta) \quad . \tag{2-6}$$

Thus if we can formulate and solve the boundary problem for $\psi(\rho,\theta)$, then (2-6) gives the desired $h(\theta)$. From (2-5), we see that $\phi = e^{-\psi}$ and then (2-1) implies that ψ be a solution of

$$\nabla^2\psi - (\nabla\psi)^2 = k^2 \quad , \tag{2-7}$$

where $(\nabla\psi)^2$ means $\nabla\psi\cdot\nabla\psi$. The boundary conditions on ψ follow from those on ϕ, and we have

$$\frac{\partial\psi}{\partial n} = 0, \text{ on S} \tag{2-8}$$

and

$$\psi \xrightarrow[\rho\to\infty]{} -ik\rho\cos\theta, \text{ uniformly in } \theta \quad . \tag{2-9}$$

Unfortunately the only method that was found to solve this transformed problem depended on the use of spheroidal functions and represented no improvement over directly computing $h(\theta)$ from (2-6) as a ratio of two infinite series. This expression may be of interest in itself since, with the help of the expansion derived in Appendix A, it is given in terms of spherical rather than spheroidal variables.

Hence we include it here.

In terms of the spheroidal coordinates

$$z = c\,\xi\,\eta\ ,$$
$$\sqrt{x^2+y^2} = c\sqrt{(\xi^2-1)(1-\eta^2)},$$

where $\qquad 2c$ = interfocal distance ,

the solution of the hard spheroid problem is, e.g. Flammer (1957), where the notation is that of Morse and Feshbach (1953),

$$\phi\left[\xi,\eta\right] = e^{ikc\xi\eta} - 2\sum_{n=o}^{\infty} \frac{i^n\, je'_{on}(\xi_o)}{\Lambda_{on}\, he'_{on}(\xi_o)}\, S_{on}(\eta) he_{on}(\xi)\ . \qquad (2\text{-}10)$$

Now, as derived in Appendix A,

$$S_{on}(\eta) he_{on}(\xi) = {\sum_{m=o,1}^{\infty}}' i^{m-n}\, d_m(o,n) P_m(\cos\theta) h_m(k\rho)\ , \qquad (2\text{-}11)$$

where the summation includes only odd values of m for n odd and even values of m for n even. Recognizing that

$$e^{ikc\xi\eta} = e^{ik\rho\cos\theta}$$

and substituting (2-11) in (2-10) we obtain

$$\phi(\rho,\theta) = e^{ik\rho\cos\theta} - 2\sum_{n=o}^{\infty} \frac{i^n je'_{on}(\xi_o)}{\Lambda_{on} he'_{on}(\xi_o)} {\sum_{m=o,1}^{\infty}}' i^{m-n} d_m(o,n) P_m(\cos\theta) h_m(k\rho)$$

$$= \sum_{m=o}^{\infty} \left[(2m+1) i^m j_m(k\rho) - 2 i^m h_m(k\rho) {\sum_{n=o,1}^{\infty}}' \frac{d_m(o,n)\, je'_{on}(\xi_o)}{\Lambda_{on}\, he'_{on}(\xi_o)} \right] P_m(\cos\theta).$$

Hence, if one computes $\partial\phi/\partial\rho$ and forms the ratio $(-\phi_\rho/\phi)_{\rho=a}$, there results

$$h(\theta)=-k\frac{\sum_{m=o}^{\infty}\left[(2m+1)i^m j_m'(ka)-2i^m h_m'(ka)\sum_{n=o,1}^{\infty}{}' \frac{d_m(o,n)je_{on}'(\xi_o)}{\Lambda_{on}he_{on}'(\xi_o)}\right]P_m(\cos\theta)}{\sum_{m=o}^{\infty}\left[(2m+1)i^m j_m(ka)-2i^m h_m(ka)\sum_{n=o,1}^{\infty}{}' \frac{d_m(o,n)je_{on}'(\xi_o)}{\Lambda_{on}he_{on}'(\xi_o)}\right]P_m(\cos\theta)} . \qquad (2\text{-}12)$$

The fact that we were unable to find $h(\theta)$ without making use of the spheroid solution suggests that the problem as formulated in this section represents too severe a test of modeling techniques in their present state of development. By modifying our demands, we are somewhat more successful in finding modeling functions relating to the sphere and the spheroid. These efforts are discussed in the following sections.

III

NON-LINEAR MODELING OF SCALAR SCATTERING BY A PROLATE SPHEROID

In this section we make use of the known far field behavior of any scattered field to explicitly derive a modeling function relating the sphere and spheroid fields.

Let us denote by ϕ_s the field scattered by a prolate spheroid when a plane wave is incident upon it along the axis of symmetry. Then the Helmholtz equation, radiation condition and boundary condition on the spheroid serve to uniquely determine ϕ_s. Similarly let us denote by ψ_s the field scattered by a sphere where we again defer specifying the boundary condition.

Assuming that the wavelength of the incident field is the same in both cases, then, exterior to both the sphere and the spheroid, both ϕ_s and ψ_s satisfy the same differential equation

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k^2\right\}u=0 \tag{3-1}$$

where there is no angular variation in planes perpendicular to the axis of symmetry.

We now make the assumption that after specifying boundary conditions on the sphere and spheroid (not necessarily the same condition on both) which uniquely determine the fields, there exists a functional relation between the two scattered fields, i.e.

$$\phi_s = \phi_s(\psi_s) \quad . \tag{3-2}$$

Denoting $\partial\phi_s/\partial\psi_s$ by ϕ'_s it is clear that

$$\frac{\partial\phi_s}{\partial r} = \phi'_s\frac{\partial\psi_s}{\partial r} \qquad \text{and} \qquad \frac{\partial\phi_s}{\partial\theta} = \phi'_s\frac{\partial\psi_s}{\partial\theta} \, . \tag{3-3}$$

Making use of (3-3) in the differential equation (3-1) satisfied by ϕ_s we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\phi'_s\frac{\partial\psi_s}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\phi'_s\frac{\partial\psi_s}{\partial\theta}\right)+k^2\phi_s = 0 \quad . \tag{3-4}$$

Upon further differentiation and rearrangement of terms, (3-4) becomes

$$\phi''_s\left[\left(\frac{\partial\psi_s}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\psi_s}{\partial\theta}\right)^2\right]+\phi'_s\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi_s}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_s}{\partial\theta}\right)\right]+k^2\phi_s = 0 \, . \tag{3-5}$$

However, since ψ_s is also a solution of (3-1) this becomes

$$\phi_s'' \left[\left(\frac{\partial \psi_s}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \psi_s}{\partial \theta}\right)^2\right] -k^2 \psi_s \phi_s' + k^2 \phi_s = 0 \quad . \tag{3-6}$$

Finding the coefficient of ϕ_s'' as a function of ψ_s is the point of departure for the exact analysis of Ruehr (1962). However, since the behavior of ψ_s for large values of r is known, we can approximate this coefficient as follows:

$$\psi_s \sim \frac{e^{ikr}}{r} f(\theta)^* \tag{3-7}$$

Hence to this order in $\frac{1}{r}$,

$$\frac{\partial \psi_s}{\partial r} = ik \psi_s \tag{3-8}$$

and

$$\frac{1}{r} \frac{\partial \psi_s}{\partial \theta} = 0 ,$$

thus

$$\left(\frac{\partial \psi_s}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \psi_s}{\partial \theta}\right)^2 \sim -k^2 \psi_s^2 \tag{3-9}$$

and (3-6) becomes

$$\psi_s^2 \phi_s'' + \psi_s \phi_s' - \phi_s = 0 \quad . \tag{3-10}$$

The general solution of (3-10) is written immediately as

$$\phi_s = A \psi_s + \frac{B}{\psi_s} \tag{3-11}$$

---

*The time dependence $e^{-i\omega t}$ is suppressed.

where A and B are constants to be determined. Unless B = 0, this is indeed a non-linear modeling function relating the spheroid field ϕ_s and the sphere field ψ_s.

The question remains, exactly which sphere and spheroid fields are related, or to put it more explicitly, what boundary conditions on the sphere and spheroid validate the assumption that the scattered fields are functionally related? Certainly it is valid when we take the example of Section II, i.e. apply a Neumann condition on the spheroid and the appropriate impedance type condition on the sphere so that the fields are identical. Then, of course, the modeling function (3-11) reduces to its simplest form since A = 1 and B = 0, bearing in mind the results of the previous section.

Now we wish to determine whether there are cases other than this one for which the modeling function (3-11) is applicable. Although a simple argument makes this appear unlikely the question remains open. This follows by substituting (3-7) and a corresponding far field expression for ϕ_s,

$$\phi_s = \frac{e^{ikr}}{r} \; g(\theta), \tag{3-12}$$

in (3-11) obtaining

$$g(\theta) = A\; f(\theta) + B\; r^2 \; \frac{e^{-2ikr}}{f(\theta)}, \tag{3-13}$$

From (3-13) it is clear that B must be zero or a function of r, hence not constant. It would be erroneous however, to conclude that B must be zero and hence that (3-11) only leads to the case mentioned above. By making the far field approximation we have essentially said that

$$\frac{e^{ikr}}{r^n} = 0, \quad n > 1 \tag{3-14}$$

thus if

$$B = B_1 \; \frac{e^{2ikr}}{r^2} \; ,$$

we not only satisfy the requirement that B be zero, to the order in 1/r with which we are concerned, but also obtain the non-linear modeling function

$$g(\theta)=A\ f(\theta)+\frac{B_1}{f(\theta)} \tag{3-15}$$

relating the far field pattern factors. From (3-15) it is clear that by measuring or computing two pairs of points in the polar diagrams of the sphere and spheroid far fields, the constants A and B can be determined and thus the entire spheroid polar diagram can be produced from the corresponding sphere data.

Just when (3-15) is valid is of course a crucial question. In Section V a calculation is carried out where Dirichlet boundary conditions are assumed on both sphere and spheroid and the spheroid pattern factor predicted by (3-15) for a particular wavelength, sphere, and spheroid is compared with the exact result. The agreement is quite good but this might be attributed to a fortuitous choice of parameters rather than the efficacy of (3-15). Since the lengthy computations involving spheroidal functions are precisely what we are striving to avoid, we still have the problem of determining under which, if any, boundary conditions other than those of the previous section the modeling function (3-15) provides significant results.

IV

AN ALTERNATE MODELING PROCEDURE

The importance of knowing the far field behavior of scattered fields is clearly illustrated in Appendix B. There we find that a knowledge of the far field essentially determines the field everywhere. In the previous section we used the known functional form of the far field to derive a modeling function relating the sphere and spheroid fields. In this section we again make use of this knowledge to derive a modeling function. This time we do not assume the functional relation between fields scattered by sphere and spheroid until after deleting the factor e^{ikr}/r. What results then is an assumption of functional dependence between the pattern factors. A modeling function relating pattern factors is derived which in fact is linear.

Again denoting the sphere and spheroid fields by ψ_s and ϕ_s where both satisfy equation (3-1), we can write these fields as convergent series expansions (see Sommerfeld (1949)) in inverse powers of r as follows

$$\psi_s = \frac{e^{ikr}}{r} \sum_{n=o}^{\infty} \frac{f_n(\theta)}{r^n} \quad , \quad \phi_s = \frac{e^{ikr}}{r} \sum_{n=o}^{\infty} \frac{g_n(\theta)}{r^n} \tag{4-1}$$

where $f_o(\theta)$ and $g_o(\theta)$ are the pattern factors of Section III appearing here with subscripts. For convenience we denote the sums as follows

$$\sum_{n=o}^{\infty} \frac{f_n(\theta)}{r^n} = \psi \quad , \qquad \sum_{n=o}^{\infty} \frac{g_n(\theta)}{r^n} = \phi \tag{4-2}$$

and derive the differential equation satisfied by ψ and ϕ .

Substituting

$$\phi_s = \frac{e^{ikr}}{r} \phi$$

into the scalar Helmholtz equation (3-1), we find, since

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{e^{ikr}}{r} \phi\right) = \frac{e^{ikr}}{r} \left(\frac{\partial^2 \phi}{\partial r} + 2ik \frac{\partial \phi}{\partial r} - k^2 \phi \right),$$

that

$$\frac{\partial^2 \phi}{\partial r^2} + 2ik \frac{\partial \phi}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0 \ . \tag{4-3}$$

Whereas in the previous section we assumed a functional relation between ψ_s and ϕ_s, now we assume that ψ and ϕ are related. That is, assume

$$\phi = \phi(\psi), \text{ and denote } \frac{\partial\phi}{\partial\psi} \text{ by } \phi'$$

hence

$$\frac{\partial\phi}{\partial r} = \phi' \frac{\partial\psi}{\partial r}, \quad \frac{\partial\phi}{\partial\theta} = \phi' \frac{\partial\psi}{\partial\theta}, \quad \text{and} \quad \frac{\partial^2\phi}{\partial r^2} = \phi''\left(\frac{\partial\psi}{\partial r}\right)^2 + \phi' \frac{\partial^2\psi}{\partial r^2} \ . \tag{4-4}$$

Substituting (4-4) in (4-3) we obtain

$$\phi''\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)\right]^2 + \phi'\left[\frac{\partial^2\psi}{\partial r^2} + 2ik\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\right] = 0\,. \tag{4-5}$$

However, since ψ satisfies the same differential equation as does ϕ, namely (4-3), the coefficient of ϕ' vanishes and we have simply

$$\phi''\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2\right] = 0 \ , \tag{4-6}$$

and ϕ'' or its coefficient must be zero. From (4-2), however, we find that

$$\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 = \frac{1}{r^2}\sum_{n=o}^{\infty}\frac{1}{r^n}\sum_{m=o}^{\infty} m(n-m)f_m(\theta)f_{n-m}(\theta) + f'_m(\theta)\,f'_{n-m}(\theta) \tag{4-7}$$

and if this factor is to vanish the coefficient of $1/r^n$ must vanish for all n. In particular, for n=o , we see that

$$f_o'(\theta) = 0 \qquad \text{or} \qquad f_o(\theta) \quad \text{is constant.}$$

But, as shown in Appendix B (equation (B-4) with no ϕ-dependence) ,

$$f_{n+1}(\theta) = \frac{1}{2ik(n+1)}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,\frac{\partial f_n}{\partial\theta}(\theta)\right] + n(n+1)f_n(\theta)\right\} \tag{4-8}$$

hence if $f_o(\theta)$ is constant, $f_n(\theta) = 0$ for $n > 0$.

We have thus established that if

$$\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 = 0, \text{ then } \psi \text{ is constant.}$$

On the other hand (4-6) can be satisfied if $\phi'' = 0$ which implies the linear relation

$$\phi = A\,\psi + B \quad , \tag{4-9}$$

where A and B are constants. In the far zone this clearly becomes (compare equation (3-15)),

$$g_o(\theta) = A\, f_o(\theta) + B \quad . \tag{4-10}$$

Thus we have shown that the assumed functional relation between sphere and spheroid pattern factors is a severe restriction and can be met by requiring either that the sphere field be constant or that (4-10) be valid. Both choices can be shown to be very unlikely (not to say impossible) unless the scatterers are very small by examining expansions of the far fields in series of Legendre polynomials. Thus by writing

$$f_o(\theta) = \sum_{n=o}^{\infty} a_n P_n(\cos\theta) \quad \text{and} \quad g_o(\theta) = \sum_{n=o}^{\infty} b_n P_n(\cos\theta) \tag{4-11}$$

we see that if $f_o(\theta)$ is constant then $a_n = 0$, $n > 0$. Furthermore, if f_o and g_o can be approximated by the first two terms (rather than just one) of the series we have, since $P_o = 1$ and $P_1(x) = x$,

$$\begin{aligned} f_o &= a_o\, P_o(\cos\theta) + a_1\, P_1(\cos\theta) \\ &= a_o + a_1 \cos\theta \end{aligned} \tag{4-12}$$

and

$$g_o = b_o P_o(\cos\theta) + b_1 P_1(\cos\theta)$$
$$= b_o + b_1 \cos\theta \quad . \qquad (4\text{-}13)$$

We can immediately derive a linear relation between f_o and g_o by eliminating $\cos\theta$ in (4-12) and (4-13), obtaining

$$g_o = \frac{b_1}{a_1} f_o + \frac{b_o a_1 - b_1 a_o}{a_1} \quad . \qquad (4\text{-}14)$$

Except for a different notation of the constants, this is precisely the same as (4-10). These results strongly suggest that the assumed functional relation between sphere and spheroid fields is valid only when the pattern factors can be approximated by the first (one or two) terms of a Legendre function expansion. This effectively limits the discussion to very low frequencies, the Rayleigh region, where other techniques are available (Siegel, 1958, 1959). Although it may be possible that for some particular boundary conditions on sphere and spheroid, the linear relation (4-10) can be used outside the very long wavelength region, no method for determining these conditions has been found. In the next section a calculation is carried out for a particular case using (4-10) and the results are compared with the exact pattern as well as that predicted by the non-linear modeling function (3-15) .

V

A NUMERICAL EXAMPLE

In this section we carry out an explicit calculation of the pattern factor of a prolate spheroid with length-to-width ratio $\frac{a}{b} = 10$ for ka = 1 where a is the semi-major axis. For this example, Dirichlet boundary conditions are assumed on both sphere and spheroid and the primary excitation is taken to be a plane wave incident along the axis of symmetry. Both real and imaginary parts of the far field are calculated from the exact series as well as the modeling formulas developed in Sections III and IV.

The constants in equations (3-15) and (4-10) are evaluated by using the exact results for forward and backscattering. Thus the pattern factor $g(\theta)$ for the spheroid is calculated from the formulae:

$$g(\theta) = A\, f(\theta) + \frac{B}{f(\theta)} \tag{5-1}$$

where

$$A = \frac{g(0)f(0)-g(\pi)f(\pi)}{f^2(0) - f^2(\pi)} \quad \text{and} \quad B = \frac{f(0)f(\pi)\left[f(0)g(\pi)-f(\pi)g(0)\right]}{f^2(0) - f^2(\pi)} ,$$

$$g(\theta) = a\, f(\theta) + b \tag{5-2}$$

where

$$a = \frac{g(0)-g(\pi)}{f(0)-f(\pi)} , \quad b = \frac{f(0)g(\pi)-f(\pi)g(0)}{f(0)-f(\pi)} ,$$

and

$$g(\theta) = 2i \sum_{n=o}^{\infty} \frac{S_{on}(c,-1)}{N_{on}(c)} \frac{R^{(1)}_{on}(c,\xi_o)}{R^{(3)}_{on}(c,\xi_o)} S_{on}(c, \cos\theta) \tag{5-3}$$

where the notation and tables of Flammer (1957) are employed. Equation (5-3) was also used to calculate $g(0)$ and $g(\pi)$ of (5-1) and (5-2). The sphere pattern $f(\theta)$ was calculated from the formula

$$f(\theta) = i \sum_{n=o}^{\infty} (-1)^n (2n+1) \frac{j_n(ka)}{h^{(1)}_n(ka)} P_n(\cos\theta) , \tag{5-4}$$

using the tables of Gumprecht and Sliepcevich (1951). The results of these calculations are shown in Figures 5.1 and 5.2. It is seen that the linear relation (5-2) yields better results for the imaginary part while the non-linear relation (5-1) is

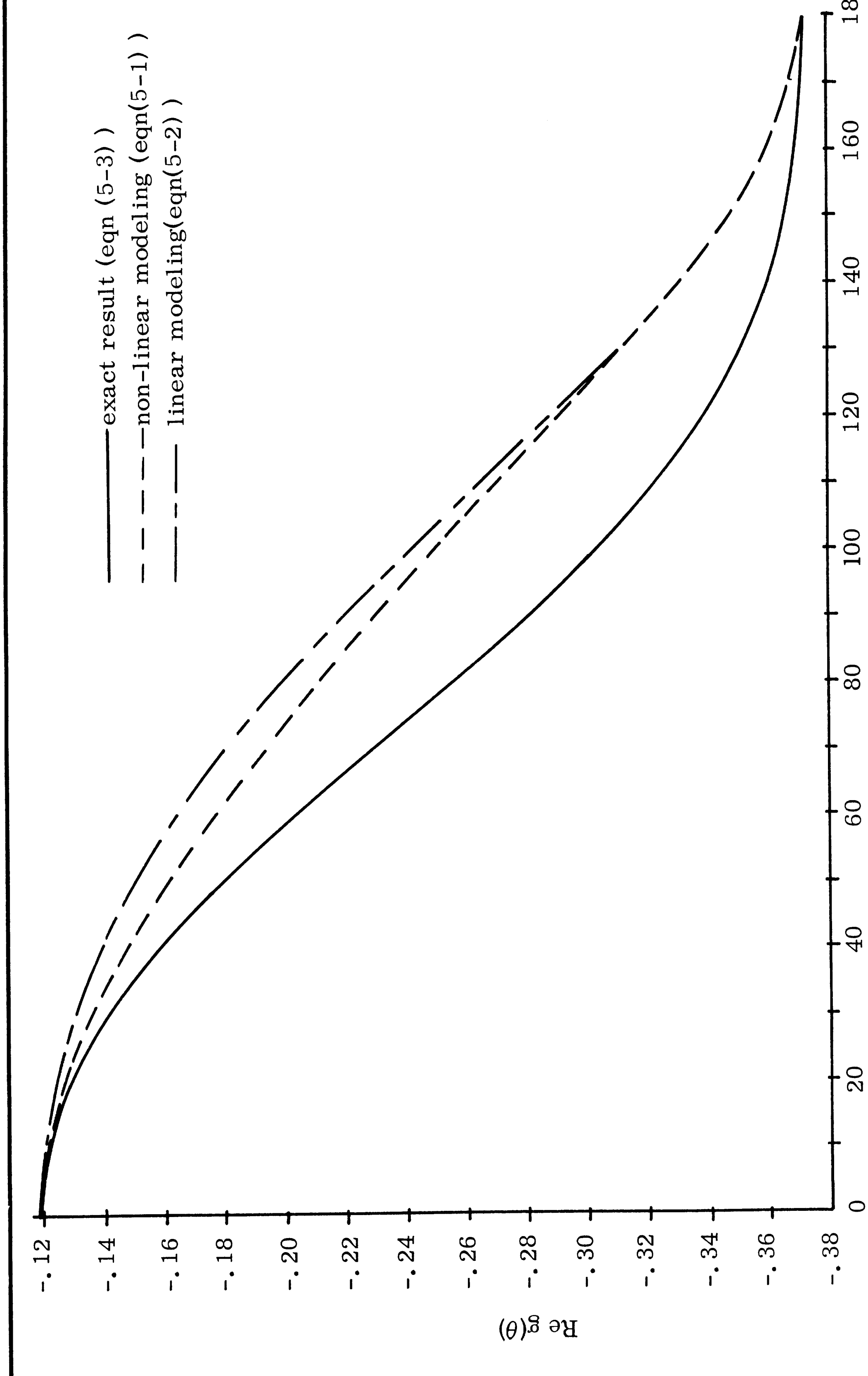

FIGURE 5.1: Real Part of the Pattern Factor for a Soft 10:1 Prolate Spheroid (ka=1)

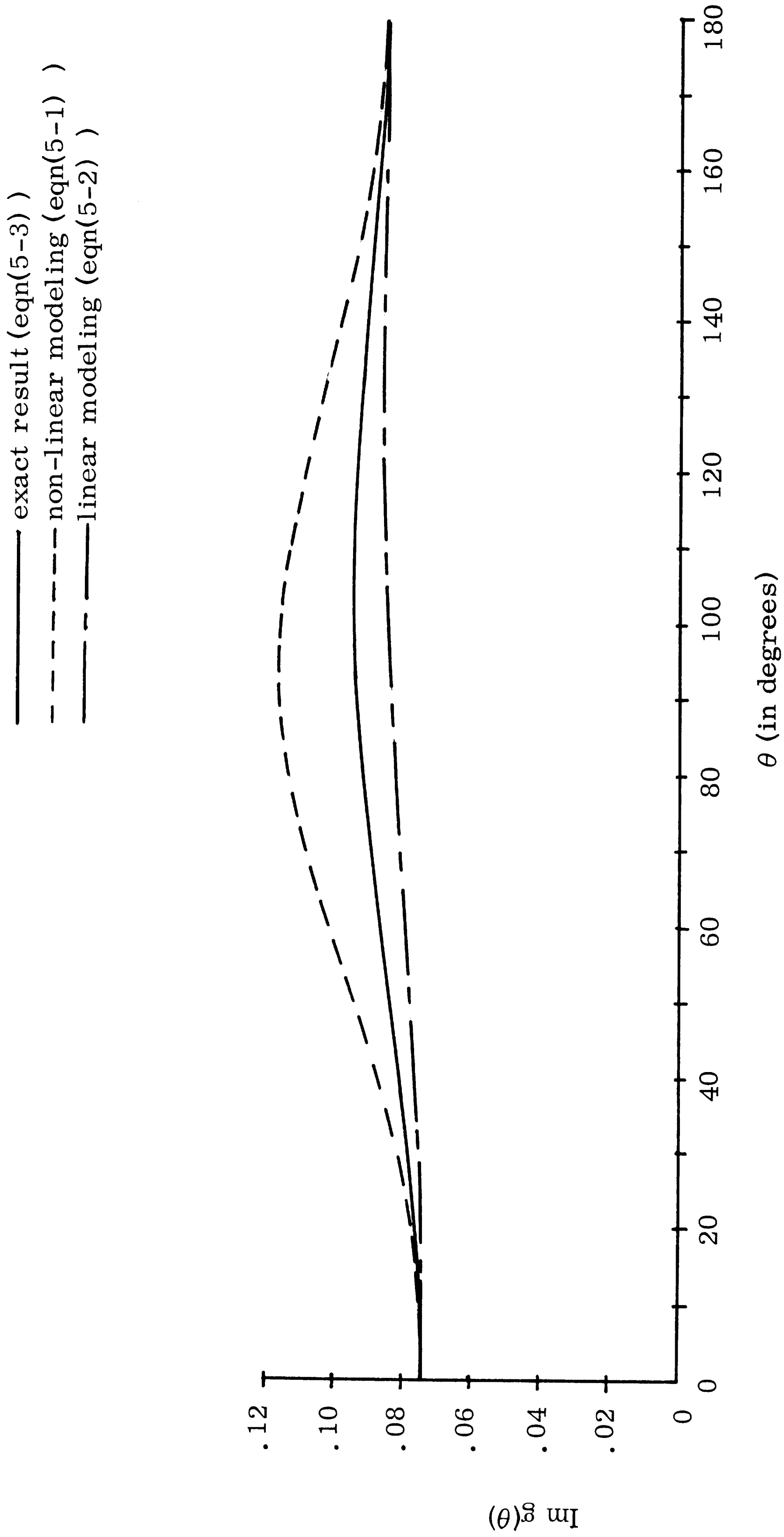

FIGURE 5.2: Imaginary Part of the Pattern Factor for a Soft 10:1 Prolate Spheroid (ka=1)

preferable when calculating the real part. Roughly speaking both (5-1) and (5-2) produce values of the pattern factor which approximate the exact values equally well and this approximation is reasonably good. The maximum deviation for the entire θ-range of either real or imaginary parts is less than 25 percent of the exact values. It may be argued however, that the value of ka chosen (ka = 1) is too small for this agreement to be significant. Unfortunately a similar comparison for larger values of ka would involve a significant increase in computational difficulty. The fact that the non-linear modeling function does not appear significantly better than the linear one and that the linear modeling function can be reasonably limited to small ka values by the argument presented in Section IV make it appear doubtful that either modeling function has more than limited application.

APPENDIX A
AN IDENTITY INVOLVING SPHEROIDAL FUNCTIONS

A useful identity expressing products of spheroidal functions in terms of spherical functions of spherical arguments is derived here. Specifically, it is established that

$$S_{on}(\eta)he_{on}(\xi)=\sum_{m=o,1}^{\infty}{}' \, i^{m-n} d_m(o,n)h_m(k\rho)P_m(\cos\theta) \qquad \text{(A-1)}$$

where the spheroidal function notation of Morse and Feshbach (1953) is employed (S is the angular function, he the radial function, ξ and η are the radial and angular variables, respectively, $d_m(o,n)$ are expansion coefficients defined recursively in terms of the eigenvalues, h_m is a spherical Hankel function of the first kind, P_m a Legendre polynomial, and ρ, θ, ϕ are spherical coordinates). The relation (A-1) is given by Flammer (1953); however since his result apparently contains a misprint some details of the derivation are presented.

We establish (A-1) by first considering the intermediate result

$$\int_0^{\pi/2-i\infty} e^{ik\rho\cos t} P_n(\cos t)\sin t \, dt = i^n h_n(k\rho) \, , \qquad \text{(A-2)}$$

To prove (A-2) let $\chi = \cos t$ and then the left side, call it $F(\rho)$, becomes

$$F(\rho) = \int_{i\infty}^{1} e^{ik\rho\chi} P_n(\chi) \, d\chi \quad .$$

Now integrate by parts n times to obtain

$$F(\rho) = \frac{e^{ik\rho}}{ik\rho} \sum_{m=o}^{n} P_n^{(m)}(1) \left(\frac{i}{k\rho}\right)^m , \quad P_n^{(m)}(\chi) \equiv \frac{d^m}{d\chi^m} P_n(\chi) .$$

Now

$$P_n^{(m)}(1) = \frac{(n+m)!}{2^m \, m!(n-m)!} \quad , \text{ so one has}$$

$$F(\rho) = \frac{e^{ik\rho}}{ik\rho} \sum_{m=o}^{n} \frac{(n+m)!}{m(n-m)!} \left(\frac{i}{2k\rho}\right)^m$$

$$= i^n h_n(k\rho) .$$

Thus (A-2) is established.

We now show that if $\cos\gamma = \cos\theta\cos\theta_o + \sin\theta\sin\theta_o\cos\phi_o$, then

$$I(\rho, \theta) \equiv P_n(\cos\theta)h_n(k\rho) = \frac{1}{2\pi i^n} \int_0^{2\pi} d\phi_o \int_0^{\pi/2 - i\infty} e^{ik\rho\cos\gamma} P_n(\cos\theta_o)\sin\theta_o d\theta_o . \tag{A-3}$$

It can be shown that if $\gamma = \alpha + i\beta$ then the infinite integral in (A-3) exists if $\theta < \alpha < \pi - \theta$, and $\beta < 0$. We give a formal derivation of (A-3) by showing that both sides are solutions of $(\nabla^2 + k^2)I(\rho, \theta) = 0$, that both sides agree for $\theta = 0$, and that the normal derivatives (at $\theta=0$) of both sides are equal. That both sides agree for $\theta = 0$ is established by showing that

$$h_n(k\rho)=\frac{1}{2\pi i^n}\int_0^{2\pi} d\phi_o \int_0^{\pi/2-i\infty} e^{ik\rho\cos\theta_o} P_n(\cos\theta_o)\sin\theta_o d\theta_o ,$$

and that this is valid follows from (A-2) upon performing the integration with respect to ϕ_o. The normal derivative of the left side is

$$\frac{1}{\rho}\frac{\partial}{\partial\theta}\left[P_n(\cos\theta)h_n(k\rho)\right] ,$$

and this is clearly zero at $\theta = 0$. The corresponding derivative of the right side becomes

$$\frac{ik}{2\pi i^n}\int_0^{\pi/2-i\infty} e^{ik\rho\cos\theta_o} P_n(\cos\theta_o)\sin^2\theta_o d\theta_o \int_0^{2\pi} \cos\phi_o d\phi_o ,$$

which is obviously zero. Finally the operator $\nabla^2 + k^2$ annihilates the left side of (A-3) and when applied to the right side it gives

$$\frac{ik}{2\pi i^n}\int_0^{2\pi} d\phi_o \int_0^{\pi/2-i\infty}\left[(\nabla^2+k^2)e^{ik\rho\cos\gamma}\right] P_n(\cos\theta_o)\sin\theta_o d\theta_o .$$

Now it is easy to show that

$$(\nabla^2+k^2)e^{ik\rho\cos\gamma}=\left[k^2\sin^2\theta_o\sin^2\phi_o+\frac{ik}{\rho}\frac{\sin\theta_o\cos\phi_o}{\sin\theta}\right]e^{ik\rho\cos\gamma} ,$$

thus

$$\int_0^{2\pi}(\nabla^2+k^2)e^{ik\rho\cos\gamma}d\phi_o=k^2\sin^2\theta_o\int_0^{2\pi}\sin^2\phi_o e^{ik\rho\cos\gamma}d\phi_o+\frac{ik\sin\theta_o}{\rho\sin\theta}\int_0^{2\pi}\cos\phi_o e^{ik\rho\cos\gamma}d\phi_o . \tag{A-4}$$

From the known result

$$\int_0^{2\pi} e^{i\,t\cos\phi_o}\, d\phi_o = 2\pi\, J_o(t),$$

one finds, without difficulty, that

$$\int_0^{2\pi} \sin^2\phi\, e^{it\cos\phi_o}\, d\phi_o = 2\pi\left[J_o(t) + J_o''(t)\right]$$

and

$$\int_0^{2\pi} \cos\phi_o\, e^{it\cos\phi_o}\, d\phi_o = -2\pi i\, J_o'(t) \quad .$$

Thus, if one lets $t = k\rho \sin\theta \sin\theta_o$, it follows from (A-4) and the definition of $\cos\gamma$ that

$$\int_0^{2\pi} (\nabla^2 + k^2) e^{ik\rho\cos\gamma}\, d\phi_o = 2\pi k^2\, e^{ik\rho\cos\theta\cos\theta_o} \sin^2\theta_o \left[J_o''(t) + \frac{1}{t} J_o'(t) + J_o(t)\right] \equiv 0 \, .$$

Thus (A-3) is formally established.

Next, as stated by Flammer (1957)

$$S_{on}(\eta) he_{on}(\xi) = \frac{1}{2\pi i^n} \int_0^{2\pi} d\phi_o \int_0^{\pi/2 - i\infty} e^{ik\rho\cos\gamma}\, S_{on}(\cos\theta_o) \sin\theta_o\, d\theta_o \;, \qquad \text{(A-5)}$$

and if we substitute the known expansion

$$S_{on}(\cos\theta_o) = \sum_{m=o,1}^{\infty}{}' d_m(o,n) P_m(\cos\theta_o)$$

into (A-5) and utilize (A-3) there results

$$S_{on}(\eta) he_{on}(\xi) = \sum_{m=o,1}^{\infty}{}' \; i^{m-n} \, d_m(o,n) P_m(\cos\theta) h_m(k\rho) \; ,$$

which is the expansion (A-1). The misprint mentioned above is the omission of the factor i^{m-n} in Flammer's result.

APPENDIX B
SOME COMMENTS ON THE "FAR FIELD"

An item of much concern in three-dimensional scalar diffraction problems is the behavior of the scattered field in the far zone. This concept is more meaningful when the scatterer is finite in extent since then the scatterer can be considered as being concentrated at the center of a sphere for observation points exterior to the sphere provided its radius is sufficiently large. Much is known about the behavior of scattered fields in the far zone for a fairly general class of scatterers, at least for all finite convex bodies. For these scatterers in the far field, a diffracted field, ϕ_s, can be represented mathematically in spherical coordinates as

$$\phi_s = \frac{e^{ikr-i\omega t}}{r} \quad f(\theta, \phi) \quad . \tag{B-1}$$

Actually (B-1) results from neglecting all but the first term in an asymptotic expansion of ϕ_s. Nevertheless if $f(\theta, \phi)$, the pattern factor, is known in any particular case then the field ϕ_s, is determined everywhere, not only in the far zone, and an explicit representation can be given in terms of $f(\theta, \phi)$.

The proof of this statement is implicit in all standard treatments of scalar wave functions, e.g. Sommerfeld (1949), Stratton (1941), but a direct statement of the fact is curiously missing. At the risk of belabouring the obvious, we present here an outline of the proof and derive the representation alluded to.

What we shall do is essentially reverse the steps by which the far field is obtained from an expansion of any three-dimensional scattered field in spherical waves.

We know (Sommerfeld, 1949) that in any scalar scattering problem (Wilcox, 1956, also treats the vector case) the scattered field, ϕ_s, exterior to a sphere containing all scattering bodies (here assumed to be finite in extent) can be written in the absolutely, uniformly convergent series

$$\phi_s = \frac{e^{ikr}}{r} \sum_{n=o}^{\infty} \frac{f_n(\theta,\phi)}{r^n} \quad . \tag{B-2}$$

The time dependence $e^{-i\omega t}$ is assumed so that e^{ikr}/r represents an outgoing spherical wave. Furthermore, since

$$(\nabla^2+k^2)\phi_s = 0 \tag{B-3}$$

we have the following differential recursion relation between the f's:

$$f_n(\theta,\phi)= \frac{1}{2ikn}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\frac{\partial}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+n(n-1)\right\}f_{n-1}(\theta,\phi) \quad . \tag{B-4}$$

Using the abbreviation

$$\mathbf{L}_m = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\frac{\partial}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi}+m(m-1) \quad , \tag{B-5}$$

we can iterate (B-4) to obtain f_n in terms of f_o, as

$$f_n(\theta,\phi) = \frac{\prod_{m=1}^{n} L_m f_o(\theta,\phi)}{(2ik)^n\; n!} \quad . \tag{B-6}$$

With (B-6) and (B-2) we see that if we have an explicit analytic representation of the far zone field, $f_o(\theta,\phi)$, we can write down all the higher order terms in a convergent representation. Wilcox (1956) establishes the fact that (B-2) is absolutely and uniformly convergent.

A spherical wave expansion of the scattered field can be constructed from $f_o(\theta,\phi)$ as follows. We expand the pattern factor in spherical harmonics,

$$f_o(\theta,\phi) = \sum_{j=o}^{\infty} \sum_{s=-j}^{j} A_{js} P_j^s(\cos\theta)e^{is\phi} \qquad \text{(B-7)}$$

where

$$A_{js} = \frac{2j+1}{4\pi} \frac{(j-s)!}{(j+s)!} \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \, f_o(\theta,\phi) P_j^s(\cos\theta)e^{is\phi} \sin\theta \quad . \qquad \text{(B-8)}$$

Substituting (B-7) in (B-6) we find that

$$f_n(\theta,\phi) = \frac{\prod_{m=1}^{n} L_m}{(2ik)^n \, n!} \sum_{j=o}^{\infty} \sum_{s=-j}^{j} A_{js} P_j^s(\cos\theta)e^{is\phi} \quad . \qquad \text{(B-9)}$$

Since

$$L_m P_j^s(\cos\theta)e^{is\phi} = \left[m(m-1)-j(j+1)\right] P_j^s(\cos\theta)e^{is\phi} \qquad \text{(B-10)}$$

equation (B-9) can be rewritten as

$$f_n(\theta,\phi) = \frac{1}{(2ik)^n n!} \sum_{j=o}^{\infty} \sum_{s=-j}^{j} A_{js} \prod_{m=1}^{n} \left[m(m-1)-j(j+1)\right] P_j^s(\cos\theta)e^{is\phi} \quad .$$

(B-11)

However

$$m(m-1)-j(j+1)=(m+j)(m-j-1)$$

from which we find that

$$\prod_{m=1}^{n}\left[m(m-1)-j(j+1)\right] = \prod_{m=1}^{n} (m+j)(m-j-1)$$

$$= (-1)^n \frac{(j+n)!}{(j-n)!}$$

$$= (-j)_n (j+1)_n \quad , \qquad \text{(B-12)}$$

hence (B-11) becomes

$$f_n(\theta, \phi) = \frac{1}{(2ik)^n n!} \sum_{j=o}^{\infty} \sum_{s=-j}^{j} A_{js} (-j)_n (j+1)_n P_j^s(\cos\theta) e^{is\phi} \quad . \qquad \text{(B-13)}$$

Since $(-j)_n = 0$ if $j < n$ (B-13) can also be written

$$f_n(\theta, \phi) = \frac{1}{(2ik)^n n!} \sum_{j=n}^{\infty} \sum_{s=-j}^{j} A_{js} (-j)_n (j+1)_n P_j^s(\cos\theta) e^{is\phi} \quad . \qquad \text{(B-14)}$$

Substituting (B-14) in (B-2) we obtain

$$\phi_s = \frac{e^{ikr}}{r} \sum_{n=o}^{\infty} \sum_{j=n}^{\infty} \sum_{s=-j}^{j} A_{js} \frac{(-j)_n (j+1)_n P_j^s(\cos\theta)}{n!\ (2ikr)^n} e^{is\phi} \quad . \qquad \text{(B-15)}$$

Rearranging the terms, (B-15) can also be written

$$\phi_s = \frac{e^{ikr}}{r} \sum_{n=o}^{\infty} \sum_{j=o}^{n} \sum_{s=-n}^{n} A_{ns} \frac{(-n)_j (n+1)_j P_n^s(\cos\theta)}{j!\ (2ikr)^j} e^{is\phi} \quad . \qquad \text{(B-16)}$$

But

$$\frac{e^{ikr}}{r} \sum_{j=o}^{n} \frac{(-n)_j (n+1)_j}{j!\ (2ikr)^j} = k\, i^{n+1}\, h_n^{(1)}(kr) \quad , \qquad \text{(B-17)}$$

where $h_n^{(1)}$ is a spherical Hankel function, e.g. Magnus and Oberhettinger (1954). Making use of (B-17) in (B-16) we have

$$\phi_s = \sum_{n=o}^{\infty} \sum_{s=-n}^{n} k A_{ns} i^{n+1} h_n^{(1)}(kr) P_n^s(\cos\theta) e^{is\phi}, \tag{B-18}$$

which is the usual expansion in spherical wave functions convergent for $r > 0$. The coefficients A_{ns} are defined in terms of the far field in (B-8).

We have thus explicitly demonstrated that it is always possible to recover the near zone from the far zone. This inverted scattering problem is not without interest since in many practical situations, it is possible to measure the far zone field even though the precise nature of the scatterer is not known. Theoretically it would be possible to then compute the coefficients A_{ns} and even calculate level surfaces of $\phi_s + \phi_{incident}$, thus determining the configuration of the scatterer assuming Dirichlet boundary conditions. More complicated boundary conditions are not so simply handled. Even if it is known that the scatterer (or scatterers) could be characterized by Dirichlet conditions there is a question as to the practicality of this scheme since accurate measurements of the far field are often not available over the full range of θ and ϕ. Still this procedure could certainly be used as an approximation, by interpolating between measured values of the far field.

APPENDIX C
MODELING IN QUANTUM THEORY

Following the general treatment of Ritt (1956) we present another example of the existence of models of a given system which evolves in some variable t, according to

$$v(Q, t) = \exp A t \, v_0(Q) \quad , \qquad \text{(C-1)}$$

where A is an operator independent of t, $v_0(Q)$ is the given initial condition, and exp At is to be interpreted as

$$\exp At = \sum_{n=o}^{\infty} \frac{t^n A^n}{n!} \qquad \text{(C-2)}$$

Q symbolizes the remaining variables required to describe the system.

In the Heisenberg picture of time-dependent quantum theory, the wave function describing a quantum system evolves according to

$$\psi(\vec{r}, t) = \exp\left(\frac{-iHt}{\hbar} \right) \psi_0(\vec{r}) \quad . \qquad \text{(C-3)}$$

Here, H is the Hamiltonian operator of the system, and the exponential operator is to be interpreted as in (C-2). ψ_0 is the wave function describing the initial state of the system.

Thus for quantum systems, the operator A of (C-1) is $-iH/\hbar$. It is possible to say a great deal about this operator without specifying the system more completely than we have done already. Quantum systems are generally discussed (by physicists, at any rate) in the manifold of L_2^* composed of "finite" functions, i.e. functions non-zero only in some finite domain.+ In this manifold, H is always self-adjoint. As a result, the eigenvalues of H are all real, and its eigenvectors span

+This has the effect of limiting consideration to the point spectrum.

the manifold. Furthermore, the physical interpretation of the spectrum of H is as the set of allowed values of the energy of the system.

An essential point to be gleaned from the above is that the spectrum of A is purely imaginary. This introduces certain complications into the existence proof, as was noted by Ritt (1956). For, if we consider a model quantum system governed by a different Hamiltonian $\mathcal{H}$, then a necessary and sufficient condition for the existence of a modeling function relating it to the original (H) system is that

$$\exp\left(\frac{-iHt}{\hbar}\right)\psi_o(\vec{r}) = \psi_o(\vec{r}) \tag{C-4}$$

implies

$$\exp\left(\frac{-i\mathcal{H}t}{\hbar}\right)\phi_o(\vec{r}) = \phi_o(\vec{r}) \quad . \tag{C-5}$$

The fact that the spectrum of H (and $\mathcal{H}$) is real guarantees that (C-4) has a non-zero solution σ: the smallest number such that σE_n is an integer multiple of h for all n. Then a modeling function exists if $\sigma \mathcal{E}_n$ is also an integer multiple of h for all n and the same σ. Thus, whether or not a system may be used to model another depends on the allowed energy values of both. This seems very reasonable from a physical standpoint.

Actually constructing a modeling function, once we know it exists, is in general a complicated problem. However, when both model and prototype are in stationary states of the energy, it becomes quite simple. For, in that case,

$$\psi(\vec{r},t)=\exp\left(\frac{-iHt}{\hbar}\right)\psi_o(r) = \exp\left(\frac{-iE_rt}{\hbar}\right)\psi_o$$

while

$$\phi(\vec{r},t)=\exp\left(\frac{-i\mathcal{H}t}{\hbar}\right)\phi_o(r)=\exp\left(\frac{-i\mathcal{E}_st}{\hbar}\right)\phi_o .$$

The existence condition

$$\exp(-iE_r\sigma/\hbar) = \exp(-i\mathcal{E}_s\sigma/\hbar) = 1$$

is satisfied as long as $E_r/\mathcal{E}_s$ is the ratio of two integers. The modeling function relating the two systems is then clearly

$$\psi(\vec{r}, t) = \psi_o(\vec{r}) \left[\frac{\phi(\vec{r}, t)}{\phi_o(\vec{r})} \right]^{E_r/\mathcal{E}_s} . \tag{C-6}$$

Suppose that both model and prototype systems are in mixed states of energy. Let the representatives of ψ_o in the H-representation be $\{a_n\}$ and the representation of ϕ_o in the $\mathcal{H}$-representation be $\{b_n\}$. These are presumed to be given. Then

$$\psi(\vec{r}, t) = \sum_n a_n \exp\left(\frac{-iE_n t}{\hbar} \right) u_n(\vec{r})$$

while

$$\phi(\vec{r}, t) = \sum_n b_n \exp\left(\frac{-i\mathcal{E}_n t}{\hbar} \right) v_n(\vec{r}) .$$

Since $\{v_n\}$ is an orthonormal set,

$$t = + \frac{i\hbar}{\mathcal{E}_s} \ln \frac{1}{b_s} \int v_s^*(\vec{r})\, \phi(\vec{r}, t)\, \underline{dr} ,$$

for all s. By substitution, then

$$\psi(\vec{r}, t) = \sum_n a_n \left[\frac{1}{b_n} \int v_n^*(\vec{r})\, \phi(\vec{r}, t) d\underline{\vec{r}} \right]^{E_n/\mathcal{E}_n} u_n(\vec{r}) . \tag{C-7}$$

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