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STUDY OF PROBLEM AREAS IN OPTICAL COMMUNICATIONS

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ABSTRACT

Progress in the study of absorption by molecular bands and attenuation due to fog is briefly reported. Attention is directed to another factor in infrared propagation and detection; the emission of 'sky noise' by the upper atmosphere, which is as yet incompletely understood.

The detection of a pulse or **arbitrary** group of samples of an optical wave form by means of a photon counter is analyzed by the conventional procedure of statistical decision theory. The relation between the likelihood ratio and a non-linear correlation function is derived. The parameters which determine the likelihood ratio and the error probabilities are discussed.

The laser amplifier is investigated theoretically. The relations between input and output wave forms are derived. The fluctuations of the output are determined in terms of the fluctuations of the input and those added in the amplifier by spontaneous emission.

I INTRODUCTION

An outline of the problem areas subject to theoretical investigation under this contract and the significant ranges of the various parameters involved in each case, as well as a tentative strategy selecting particular problems for detailed study was presented in the Introduction to the First Interim Report (Barasch, et al 1964).

Some results of this work were given in detail in that report as well as in the subsequent Second and Third Interim Reports (Hok et al, 1964, 1965). In one problem area, designated as the third problem area of the contract, and concerned with highly selective, preferably tunable filters for use with photon counting receivers, the work was terminated with the Second Interim Report. The conclusion reached was that such filters are theoretically feasible, but the available materials and technological state of the art do not justify, at present, attempts to develop such filters with losses, dimensions, operating voltages, etc., reasonable for use in a satellite communication system.

In the first problem area, which concerns propagation, scattering and absorption of visible and infrared radiation between deep space and points on the earth, the theory of such attenuation or extinction by various constituents, permanent as well as temporary, has been reviewed and various empirical as well as computed data obtained from the literature tabulated in the Interim Reports. This work is continued in the present report. Some attention has recently been directed to another phenomenon of interest in this problem area referred to as 'sky noise', a random emission of infrared radiation of uncertain origin, which has been found a limiting factor in certain infrared astronomical observations. It extends into the wavelength range specified for this project and will require some future attention.

The second problem area is concerned with the operation of an optical communication channel under conditions of very low signal level at the receiver. In the previous reports we discussed the quantum-mechanical extensions of statistical communication theory, both in general and for specific applications within the scope of this project. The detection of consecutive samples of a modulated wave form may be considered a string of binary digits, which at low power level will contain a large percentage of errors. The transmission of information without errors then becomes a problem of choosing an appropriate code. The present report discusses a preliminary approach to the coding problem: the reduction of the error percentage by the simultaneous detection of a multi-sample pulse or code group by means of a photon counter. Conventional statistical decision theory is used. The discussion of the remaining coding problem is postponed to a future report.

As an alternative to the simple photon-counter receiver we have, in previous reports, suggested a receiver using a laser amplifier as its first component and discussed some of the properties of such an amplifier. In the present report the fundamental theory of a laser amplifier is derived from quantum mechanics, giving the basic relationships between gain and noise output, on the one hand, and the operating parameters of the laser on the other. The effectiveness of the amplifier as a filter against a broad spectrum of background noise radiation will be considered in a future report.

II. FIRST PROBLEM AREA: OPTICAL COMMUNICATION UNDER ADVERSE METEOROLOGICAL CONDITIONS.

2.1 Introduction

The influence of the atmosphere on electromagnetic radiation propagating through it is of interest not only in the area of space communication but also in astronomy because of the limitations imposed on observations of radiation from astronomical objects. In evaluating these limitations and in the choice of sites for optical, infrared, and radio observatories, the astronomer faces some of the same problems that are under investigation under this project. A recent memorandum (Augason and Spinrad, 1965) which is of considerable interest, reported the results of a survey undertaken for the Astronomy Subcommittee for NASA. One empirical fact emphasized in this survey is the 'sky noise' observed in the infrared windows between the wavelengths of 1μ and 13μ. The origin of this noise radiation appears to be uncertain at the present time. We have not so far devoted any attention to to this sky noise, and we are not ready to do so in the present report. Observations indicate, however, that it constitutes an important limiting factor for communication in the infrared range; for this reason we hope to investigate and discuss this subject in a future report.

In the present report we continue the discussion of the calculation of absorption by molecular bands, particularly for more realistic conditions, where atmospheric composition,density, pressure and temperature vary widely along the path of propagation. Progress in the computation of attenuation by fogs will also be reported.

2.2 Absorption by Molecular Bands

The simple Lambert-Beer exponential law of attenuation is useful only for monochromatic radiation and homogeneous propagation paths (i.e. paths of constant atmospheric composition, pressure, density and temperature). For other situations it has been necessary, as many authors have realized recently, to develop more convenient formulations in order to predict attenuation. We shall discuss some of these approaches briefly here, and indicate the extent to which they promise to be applicable to situations of interest under this contract. Since the indicated computations are extremely lengthy, we have obtained no results as yet. After deciding what situations are of most importance, we hope to have predictions for the Final Report, or at least to quote results of other workers which may be pertinent.

It has proven fruitful to introduce the 'band model' and to deal with averaged absorptances over a spectral interval larger than the width of a molecular absorption band, thus eliminating the computation difficulties associated with the rapidly varying absorption coefficient derived from considering individual lines of the band. In conjunction with the band model approach, a method of introducing the variation of atmospheric properties over the propagation path is needed. Plass (1962) has derived an 'equivalent path' concept. Using this concept and some band parameters arising from the theory of molecular spectra he (Plass, 1963) has computed and tabulated a set of transmittances. Since no intermediate steps in the computations are reproduced or tabulated, these computations cannot easily be extended to situations other than those for which his results are tabulated. He deals with the effects of CO_2 bands in the wave number range 500 - 10,000 cm^{-1} and of water vapor bands (using both wet and dry stratospheric distributions) in the range 1000 - 10,000 cm^{-1}, tabulating transmittances averaged over 50 cm^{-1} regions for paths originating at altitudes of 15, 25, 30 and 50 km, traversing the entire atmospheric thickness at initial angles with the horizontal from 0^o to 90^o.

Because of the paucity of intermediate tabulations, the results of Plass (1963) cannot be applied to transmission over arbitrary slant paths wholly within the atmosphere. For ground-to-aerospace predictions, they require supplementation by a method which treats the segment of the path between ground level and the initial altitudes appearing in the tabulations.

Such a method has been constructed by Zachor (1961) and is detailed in that reference. He starts with band models which contain certain parameters. These are evaluated for the bands important at each wavelength region, by fitting laboratory data to the models. Zachor tabulates the appropriate model and parameters as a function of wavelength, essentially from 1.4μ to 10.8μ, for both CO_2 and H_2O. He invokes the Curtis-Godson approximation to for mulate his models in terms of mean concentrations and pressures over the propagation path, supplying instructions for obtaining these means associated with arbitrary slant paths (within some limits of location, length, etc). Summer and winter distributions of water vapor both enter the tabulations. Although much laborious computation would be involved, the material of Zachor (1961) is a satisfactory source for a method of predicting CO_2 and water vapor attenuation over paths wholly within the atmosphere. It may, of course, be used for the missing lower part to join with Plass' results in a prediction for ground-to-aerospace applications .

All of this material deals only with CO_2 and water vapor. It is suggested that transmitter sites be chosen to avoid sources of atmospheric pollution, since adequate predictions of attenuation due to them are not readily feasible. Alternatively, it will be advantageous to operate within spectral regions which, as far as the spectra of these contaminants are concerned, lie in 'windows'.

No quick or reliable estimate of attenuation due to minor but permanent consituents of the atmosphere is possible. This question requires more thought.

2.3 Attenuation by Fogs

We are in the process of deriving a new formulation, which should permit us to predict this attenuation while avoiding the well-known computational difficulties associated with Riccati-Bessel functions of large complex agruments. These difficulties would arise if the Mie theory were used to derive the extinction cross section of the fog droplets. Instead, we are attempting to adopt a suggestion of van de Hulst in starting with an empirically fitted expression for the forward scattering amplitude valid for real index of refraction. A process of analytic continuation to complex indices is being performed; the resulting expression will be introduced into the forward scattering-extinction theorem to yield a form for the extinction cross section valid for larger droplets, and for the complex indices of refraction corresponding to the infrared wavelengths in which we are interested. This form should be amenable to computation of the extinction coefficient of the fog by integrating the cross section over the size distribution of the droplets. Results of either the derivation or the final computation will be reported when available.

III. SECOND PROBLEM AREA: INFORMATION EFFICIENCY AND CHOICE OF DETECTION SYSTEM

3.1 Introduction

The operation of an optical communication channel with an average signal power at the receiving end which is smaller than one energy quantum per sample of the modulated light wave depends, as far as the maximum rate of transmission of information is concerned, on the choice of the system of coding and modulation at the transmitter and on the successful detection and decoding at the receiver. In previous reports on this problem area we have primarily been concerned with the sample by sample detection and with the information and uncertainty, respectively, connected with each output sample after detection. The question of whether or not practically realizable coding-decoding schemes which can cope with the very large frequency of errors at this signal level are likely to be found within a reasonable time, has so far been left open.

In the next section we shall consider an approach to the coding problem in two successive steps. Essential in all coding is the introduction of redundancy into the signal; redundancy structured in such a way that the errors can be filtered out by a decoding process. When the errors are very numerous, say more than 50% of the digits, a very large redundancy is required. It is suggested that as a first step redundancy is added in a rather unsophisticated manner in the signal detection process itself, reducing the original number of independent samples to a smaller number of output digits with a lower percentage of errors. This string of digits will then be a more tractable input for an error correcting decoding process. The first step is approached as a conventional problem in statistical decision theory, modified for the quantum detector. A closed-form solution for the new error frequency has not yet been found, but a discussion of the parameters on which the solution depends leads to some interesting general conclusions.

In the light of these conclusions and the practical difficulties of designing narrow-band optical filters, as previously reported, the high efficiency of a simple binary photon counter channel suggested by Gordon (1962) looks more and more like an illusion. The simplest way of reducing the relative error rate is to reduce the bandwidth (or increase the transmitter power); then the number of photons per sample increases to the point where Gordon's analysis indicated that other types of channels promise higher efficiency.

For reasons of this nature we are continuing the theoretical study of an alternative type of channel, where a laser amplifier serves as the first component in the receiver, serving both as a narrow noise filter and for the purpose of raising the signal power to a level that can be more efficiently detected. We feel that a literal interpretation of Gordon's graphs of the channel efficiencies is biased in favor of the photon-counter channel, because the noise factor of the subsequent amplifier is not accounted for, while the laser amplifier noise is included. Because of the difference in signal levels, the noise in the post detection amplifier will necessarily affect the photon-counter channel, which will at least tend to even out the relative efficiencies of the two alternatives.

In Section 3.3 we present an additional contribution to the theoretical evaluation of the performance of a laser amplifier, giving general relationships between input and output signals as well as the noise fluctuations added by the amplifier.

3.2 Relations Between Detection and Coding in Optical Channels

In this section we shall examine the basic problems connected with the operation of an optical communication channel with a very low received signal level, on the average, less than one quantum per pair of samples.

In order to state the problem in simple idealized terms we shall assume a photo-electric detector functioning as a quantum counter. Its quantum efficiency may be taken as unity by a compensating increase in the predetection attenuation of the total incident radiation.

As a first alternative the following straightforward approach may be considered. Each radiation mode or pair of samples is detected consecutively; the result is a sequence of binary digits with a large number of 'errors' caused by the quantum-counter statistics. The optimization of the rate of transmission of information for given average transmitter power and the elimination of errors requires first, operation in a highly unsymmetric way, the 'on' digits being much less frequent than the 'off' digits. Secondly, the use of a highly redundant, very sophisticated code which permits the recovery of the information content without errors.

This approach meets with at least two important practical difficulties. In a physical transmitter not only the average power, but also the peak power is restricted. The realization of the optimum pulse energy for given average power may thus not be possible. The second consideration is that the design of error correcting codes for highly unsymmetric channels with very high error frequency is a largely unexplored field of considerable complexity.

Another alternative approach is to reduce the coding difficulties at the expense of a more sophisticated detection process. Each digit or unit at the output of the receiver and the input of the decoder may be obtained by processing collectively a group of samples rather than a single pair in the receiver. In this way the relative error frequency can be reduced, at the cost of a considerably reduced number of independent code digits. Theoretically, this simple way of introducing redundancy is wasteful of channel space and thus inferior to the first alternative, but this sacrifice may be worth taking in order to make the coding problem more reasonable.

The question may then be asked whether or not the equivalent result could be obtained simply by reducing the bandwidth of the signal and thus increasing the energy per sample. In a corresponding classical case, the detection of a completely specified signal in white Gaussian noise, the answer is affirmative. The statistical detection theory shows that the performance, in terms of error probabilities, of the ideal receiver depends on only one parameter - the energy of the signal divided by the spectral density of the noise power. It is consequently independent of whether the energy of the signal is concentrated in a narrow frequency band or has a wide spread in frequency.

We shall in this section try to answer this question for a simple model of an optical channel. The conventional procedure of statistical detection theory (Wald, 1950; Peterson, et al 1954; Van Meter et al, 1954) will be followed as closely as possible in the analysis of the detection of a completely specified signal (except for phase) by a receiver using a photon counter. The elementary task of the receiver is to make a decision whether or not a signal has been reaching the input terminals during a specified time interval T. The signal, a single pulse or code group of samples, is known to be an attenuated replica of one out of an ensemble of completely specified classical signals emitted by a distant transmitter. In order to simplify the first formulation of the problem we may temporarily consider this ensemble as having one single member. Later on, the results obtained may be averaged over a more realistic ensemble, taking into account fluctuations in the carrier wave generated at the transmitter, variation in propagation characteristics, etc.

The signals from the distant transmitter with added background radiation produce certain expectation values of the field variables at the receiver. These time-varying functions in the time interval T form an ensemble which we shall refer to as pre-observation space. This space is transformed into observation space by the photon counter, which yields a classically observable stream of electrons. An optimum decision rule built into the receiver projects the observation space into decision space, which has only two elements; 'yes' and 'no' . (See Fig. 3-1.) There are at least four sources of noise and consequently output errors: 1) the background radiation, represented by the noise ensemble $\mathcal{E}_N$ in the figure, 2) the uncertainty connected with the observation of the field according to quantum mechanics, i.e. the random projection of pre-observation space $\mathcal{E}_P$ into observation space $\mathcal{E}_O$, 3) dark current in the photo-cell, 4) amplifier noise. In the idealized model considered here we shall include only the first two, although the different statistics suggest that accounting for (3) and (4) by an equivalent increase in (1) is not justified.

First the points P in $\mathcal{E}_P$ and O in $\mathcal{E}_O$ must be assigned appropriate probabilities. If a point P is produced by thermal background radiation alone, each sample in the time interval T may be assigned a Gaussian probability density (or a Gaussian P-representation in Glauber's analogous quantum formulation) with zero mean and the variance appropriate to the temperature of the background. In the presence of the signal, the mean of the distribution of each sample is displaced by the expected value of the signal sample. In the projection of a point in $\mathcal{E}_P$ to a point in $\mathcal{E}_O$ each

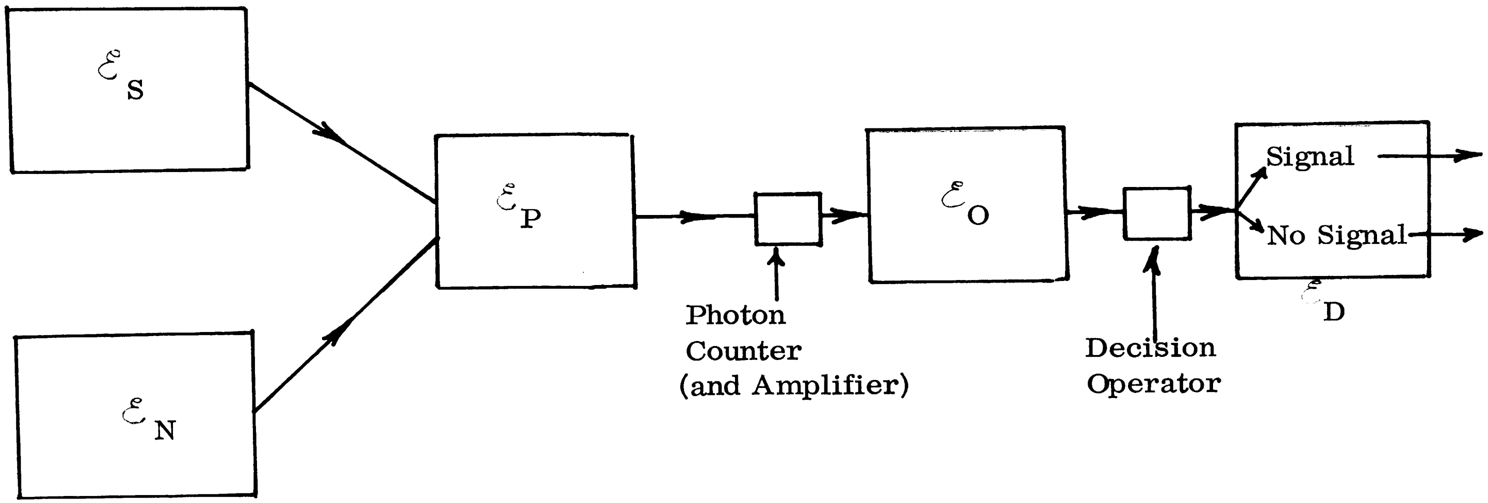

FIG. 3-1: $\mathcal{E}_S$ = Signal Ensemble (Transmitted Radiation); $\mathcal{E}_N$ = Noise Ensemble (Thermal Background Radiation), $\mathcal{E}_P$ = Pre-observation Space, $\mathcal{E}_O$ = Observation Space, and $\mathcal{E}_D$ = Decision Space.

pair of time samples of the former function changes into a time sample of the latter according to a Poisson probability law.

Let p_i be the value of the field variable at the ith sample and the number of samples

$$2n = 2\,(f_2 - f_1)T \tag{3.1}$$

i.e., twice the bandwidth times the length of the observation interval. The corresponding samples of the signal s_i are all known. The variables p_i and s_i are so normalized that p_i^2 and s_i^2 represent the instantaneous power of the respective radiation incident on the photon counter. It is convenient for the following analysis to reduce the number of sampling instants by one half and specify envelope and phase at each time

$$p_i = P_{ic}\cos\omega t + P_{is}\sin\omega t = P_i\cos(\omega t - \Phi_i) \qquad 0 < i \leqslant n \tag{3.2}$$

$$s_i = S_{ic}\cos\omega t + S_{is}\sin\omega t = S_i\cos(\omega t - \theta_i) \qquad 0 < i < n \ . \tag{3.3}$$

This change of variables for the Gaussian distribution leads to the following distribution in pre-observation space in the absence of a signal

$$dF_{S_0}(P, \Phi) = \prod_{i=0}^{n} \frac{d\Phi_i}{2\pi} \cdot \frac{P_i dP_i}{N} \exp\left\{-\frac{P_i^2}{2N}\right\} \tag{3.4}$$

where N is the noise variance.

In the presence of a signal, the Guassian variable of mean zero and variance N is p-s; the corresponding distribution is, after the above transformation of variables:

$$dF_{S_1}(P, \Phi) = \prod_i \cdot \frac{d\Phi_i}{2\pi} \cdot \frac{P_i dP_i}{N} \exp\left\{-\frac{1}{2N}\left[P_i^2 + S_i^2 - 2P_iS_i\cos(\Phi_i - \theta_i)\right]\right\} \tag{3.5}$$

The subscripts S_0 and S_1 refer to subsets in the signal space $\mathcal{S}_S$, the former being the null set, no signal, and the latter containing the completely specified single signal.

The operation of the photon counter depends only on the power, not the phase of the incident radiation. The last two expressions may consequently be integrated with respect to Φ_i before the observation space distribution functions are evaluated.

$$dF_{S_0}(P) = \prod_{i=0}^{n} \exp\left\{-\frac{P_i^2}{2N}\right\} \frac{P_i dP_i}{N} \tag{3.8}$$

$$dF_{S_1}(P) = \exp\left\{-\frac{L(S)}{N_o}\right\} \prod_{i=0}^{n} \exp\left\{-\frac{P_i^2}{2N}\right\} \cdot I_0\left(\frac{P_i S_i}{N}\right) \frac{P_i dP_i}{N} \tag{3.9}$$

where N_o is the noise power per cycle bandwidth, i. e. the spectral density of the noise. L(S) is the total energy of the signal, evaluated from the average of the signal power

$$\bar{S} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} S_i^2 = \frac{1}{T} \int_0^T \left[S(t)\cos(\omega t - \Phi_i)\right]^2 dt = \frac{L(S)}{T} \tag{3.10}$$

$$\sum_{i=1}^{n} \frac{1}{2} S_i^2 = (f_2 - f_1) \cdot L(S) \quad . \tag{3.11}$$

The symbol $I_o(x)$ denotes as usual the zero order Bessel function with imaginary argument, i. e., $J_o(jx)$.

Each sample pair (i) may be considered as an independent travelling wave mode of the radiation field. The expected value of the number of quanta in the mode is

$$m_i = \frac{1}{2} \frac{P_i^2 \tau}{\hbar\omega} = \frac{P_i^2}{2N} \cdot \frac{N_o}{\hbar\omega} = \frac{P_i^2}{2N} \cdot \mu \quad , \tag{3.12}$$

defining the new parameters μ, average number of thermal photons per mode, and τ, the time between t_i and t_{i+1}, or the reciprocal of the bandwidth.

Let r_i be the number of photons counted by the detector in the ith interval. With the Poisson distribution for r_i, given m_i, the observation space distributions are obtained as follows:

$$P_{S_0}(r) = \int_0^{\infty} dF_{S_0}(m_i) \cdot P_{m_i}(r) = \prod_i \int_0^{\infty} \frac{dm_i}{\mu} \exp\left(-\frac{m_i}{\mu}\right) \cdot \frac{m_i^{r_i}}{r_i!} \cdot e^{-m_i} = \prod_i \frac{\mu^{r_i}}{(1+\mu)^{r_i+1}} \qquad (3.13)$$

a well known result for thermal radiation.

In the presence of a signal

$$P_{S_1}(r) = \int_0^{\infty} dF_{S_1}(m) P_m(r) = \prod_i \exp\left\{-\frac{S_i^2}{2N}\right\} \int_0^{\infty} \frac{dm_i}{\mu} \cdot I_o\left(\frac{P_i S_i}{N}\right) \exp\left\{-m_i \frac{1+\mu}{\mu}\right\} \frac{m_i^{r_i}}{r_i!} \quad . \qquad (3.14)$$

After the substitutions

$$x^2 = \frac{m_i(1+\mu)}{\mu} = \frac{P_i^2}{2N}(1+\mu) \qquad (3.15)$$

$$\beta^2 = \gamma = \frac{S_i^2}{2N(1+\mu)} \qquad (3.16)$$

each integral in (3.14) can be written

$$\mathcal{J} = \frac{2}{r!} \frac{\mu^r}{(1+\mu)^{r+1}} \int_0^{\infty} e^{-x^2} I_o(2\beta x) x^{2r+1} dx = \frac{\mu^r}{(1+\mu)^{r+1}} F(r+1;1;\beta^2) \quad , \qquad (3.17)$$

where the function F(a, b, z) is the confluent hypergeometric function (see Watson, 1952, p. 395). For b = 1 this function may be expressed in elementary functions and finite polynomials

$$F(r+1;1;\gamma)=\frac{1}{r!}\frac{d^r}{d\gamma^r}(\gamma^r e^{\gamma})=Z_r(\gamma)e^{\gamma} \quad . \tag{3.18}$$

Consequently we may write the distribution in the presence of a signal in the form

$$P_{S_1}(r)=\prod_i e^{-\mu\gamma_i}\cdot Z_{r_i}(\gamma_i)\frac{\mu^{r_i}}{(1+\mu)^{r_i+1}} \quad . \tag{3.19}$$

The likelihood ratio for any point in observation space represented by a radius vector $\vec{r}$ is accordingly

$$\ell(\vec{r})=\frac{P_{S_1}(r)}{P_{S_0}(r)}=\exp\left\{-\mu\sum_i\gamma_i\right\}\prod_i Z_{r_i}(\gamma_i) \quad . \tag{3.20}$$

Here γ_i and μ are known characteristics of signal and noise normalized with respect to photon energy as specified by (3.16) and (3.12). The components of the random vector occur only in the polynomial $Z_{r_i}(\gamma_i)$.

For the purpose of changing the product into a sum we calculate the logarithm of the likelihood ratio

$$\log\left[\ell(r)\right]=-\mu\sum_i\gamma_i+\sum_i\log Z_{r_i}(\gamma_i)=-A+\alpha \tag{3.21}$$

The first term on the right is independent of r; it may also be written in previous notation

$$A=\frac{L(S)}{N_o}\frac{\mu}{1+\mu} \tag{3.22}$$

The second term

$$\alpha = \sum_i \log Z_{r_i}(\gamma_i) \qquad (3.23)$$

is clearly a monotone function of the likelihood ratio and can be used as a basis for a decision whether or not the observed point in observation space indicates the arrival of a signal. Unfortunately this function is rather difficult to manipulate in mathematical operations. Its most important properties, however, are easily stated. It is real, positive and monotone increasing for all finite, real and positive values of r and γ. This follows from the fact that only positive exponents occur in the polynomial, which has only positive terms; the first term is $\gamma^r/r!$ and the last two $r\gamma+1$. It has the form of the Laguerre rth order polynomial of $-\gamma/2$, divided by $\sqrt{2}$. In the limit of small signals α is simply the sum of $r_i\gamma_i$ and represents the first order correlation between these two variables. In general, α is evidently a certain nonlinear measure of such correlation between the vector $\vec{r}$ and the a priori known signal envelope represented by the samples S_i.

A very large class of decision rules including Bayes and Minimx rules can be given in the form

$$D\left[\ell(r) \geqslant \lambda_c\right] = 1$$
$$D\left[\ell(r) < \lambda_c\right] = 0 \qquad (3.24)$$

where the threshold value of λ_c of the likelihood ratio may be selected by minimizing a risk function of some kind. Since the last term of (3.21) is a monotone function of $\ell(r)$, an equivalent statement of the decision rule is

$$D\left[\alpha \geqslant \alpha_c\right] = 1 \quad , \qquad (3.25)$$

if

$$\log \lambda_c = -\frac{L(S)}{N_o} \cdot \frac{\mu}{\mu+1} + \alpha_c \qquad (3.26)$$

In the simple case of a classical detector of a signal in the white Gaussian noise the variable corresponding to α is the linear correlation between the receiver input and the known signal; then a matched filter constitutes a perfect analog computer, producing at its output a current or voltage proportional to the time integral of the product of the two. In the case of the quantum detector considered here, the analog computation of an acceptable variable on which to base the decision is considerably more complicated.

By restricting the class of signals to be considered, the problem can be simplified. Let us limit the choice to binary signals; each sample S_i has only two possible values; zero and S_k, and let the duty factor

$$\eta = n_k/n \tag{3.27}$$

be the ratio of the number of non-zero samples n_k to the total number of n samples in the observation interval T. Disregarding the obvious practical obstacles we can in principle obtain α from $\vec{r}$ in the following way. The receiver input ($\vec{r}$) is first transformed by a nonlinear transducer without energy storage, designed to give an output proportional to log $Z_r(\gamma_k)$. Since all nonzero values of the signal are the same, there is no need to compensate for the nonlinearity in S. The transducer output is fed to a filter matched to the signal, in principle a tapped delay line, giving a result proportional to α. The final component of an automatic receiver is a trigger circuit that checks the output of the filter at the end of the observation interval T and produces a unit digit, if this output exceeds the value calibrated to correspond to the threshold value α_c, a zero digit otherwise.

The performance of a receiver operating with the decision rule (3.25) is evaluated in terms of the relative frequencies of erroneous decisions, 'misses' as well as 'false alarms'. To determine these quantities it is convenient to derive the distribution functions of $\ell(r)$ or its counterpart $\alpha(r)$, from which the 'receiver operating characteristic', the ROC-diagram can be plotted (Fig. 3-2). From this diagram the error probabilities are easily obtained for any chosen value of the threshold ratio λ_c.

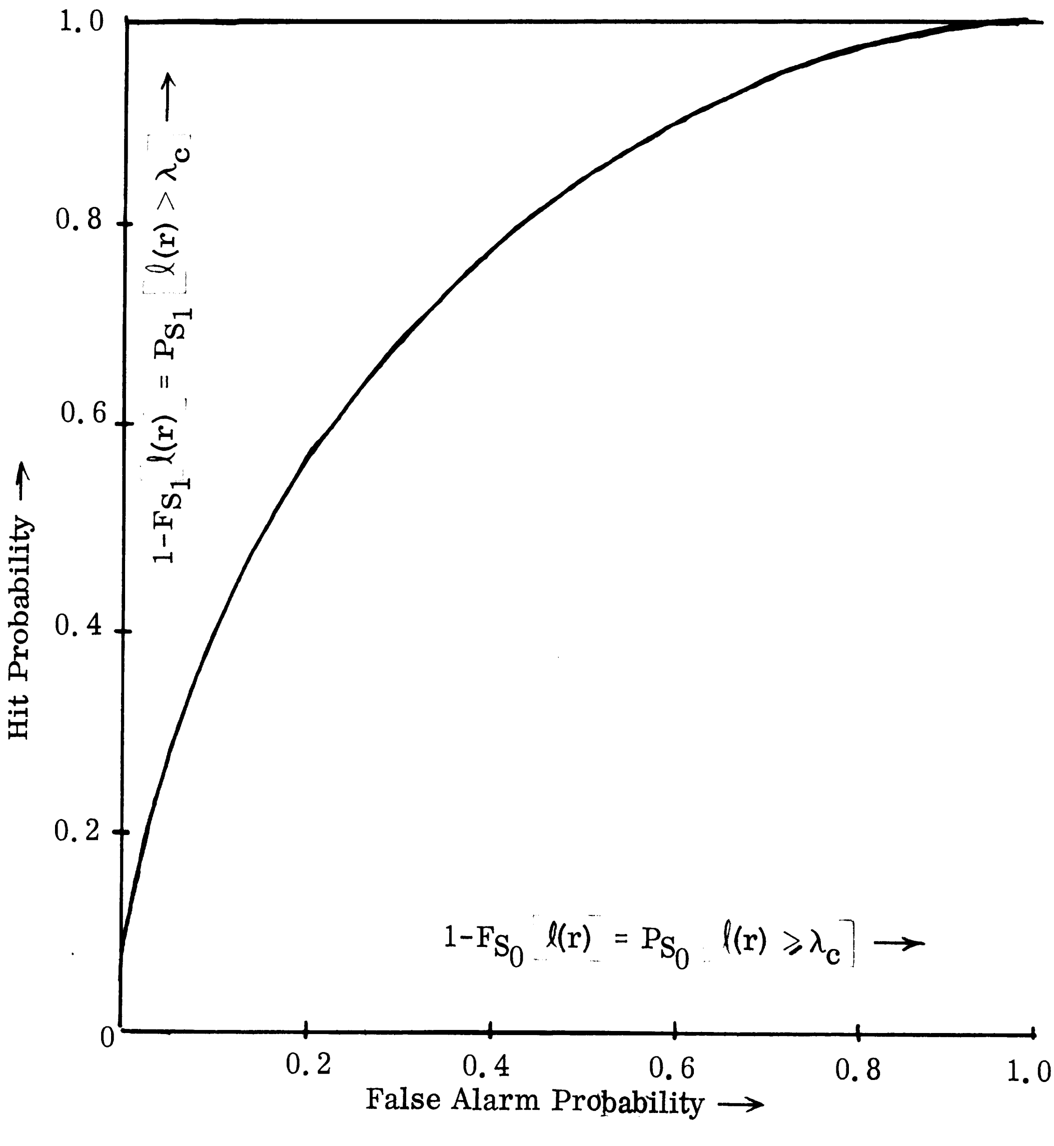

FIG. 3-2: RECEIVER OPERATING CHARACTERISTIC

The task of deriving the distribution functions of α in the absence and presence of a signal, respectively, has not been completed at the time of writing. Since α is the sum of a number (n_k) of independent random variables with identical distributions, the method of characteristic functions offers a reasonable approach to this problem.

$$\phi^{\alpha}_{S_0}(\tau) = \left[\sum_{r=1}^{\infty} \exp\left\{ j\,\tau \log Z_r(\gamma_k) \right\} P_{S_0}(r) \right]^{n_k} \tag{3.28}$$

is the characteristic function in the absence of a signal; the corresponding expression with $P_{S_1}(r)$ as weighting function yields the characteristic function when a signal is present. The inverse transformations give the distribution in respective cases.

For reasonably large n_k another possible approach is to calculate the mean and variance of α by elementary methods and then to approximate the distributions by Gaussian functions according to the central limit theorem.

The Gaussian approximations permit some general conclusions. Let the two distributions have means m_1 and m_2 variances σ_1^2 and σ_2^2 respectively. Then

$$\log \ell(r) = \frac{(\alpha-m_1)^2}{2\sigma_1^2} - \frac{(\alpha-m_2)^2}{2\sigma_2^2} + \log\frac{\sigma_1}{\sigma_2} = \alpha - A \quad . \tag{3.29}$$

Since the equality must hold for all values of α, we see immediately that

$$\frac{\alpha^2}{2\sigma_1^2} - \frac{\alpha^2}{2\sigma_2^2} = 0 \tag{3.30}$$

$$\sigma_1^2 = \sigma_2^2 \quad . \tag{3.31}$$

Consequently, as long as α is chosen to be a linear function of $\log \ell(r)$ and n_k is large enough to make the distributions approximately Gaussian, the latter two must have the same variance. Writing $\sigma = \sigma_1 = \sigma_2$ we obtain the additional equations

$$2\alpha(m_2-m_1) = 2\sigma^2\alpha \tag{3.32}$$

$$m_2^2 - m_1^2 = 2\sigma^2 A \tag{3.33}$$

or

$$m_1 = A - \frac{1}{2}\sigma^2 \tag{3.34}$$

$$m_2 = A + \frac{1}{2}\sigma^2 \tag{3.35}$$

One of the three quantities m_1, m_2 and σ^2 must be calculated, before these equations can determine the distributions. For instance,

$$m_1 = E_{S_0}(\alpha) = n_k \sum_{r=1}^{\infty} \log Z_r(\gamma_k) \cdot \frac{\mu^r}{(1+\mu)^{r+1}} \quad . \tag{3.36}$$

In the small-signal limit, where

$$\log Z_r(\gamma) \cong r\gamma \quad , \tag{3.37}$$

$$m_1 \cong n_k \mu \gamma = A \; . \tag{3.38}$$

and both distributions must degenerate to a δ-function at unity likelihood ratio in order to satisfy the equations (3. 34) and (3. 35). The distribution of α is under these circumstances independent of whether a signal was present or not. Evidently the nonlinear character of the correlation operation indicated by the right-hand side of (3. 23) is essential for the detection. Because the logarithm increases more slowly than linearly with its argument, m_1 is at larger signal values depressed below the value A and the difference between the abscissae of the peaks of the probability densities m_2-m_1 permits detection by a decision function such as (3. 25).

Since the statistics of the likelihood ratio, represented by the characteristic functions (3.28), means and variances (3. 34), (3. 35) and (3. 36), have not been evaluated in closed form or numerically at the time of writing, no complete discussion of the error probabilities under various conditions can be given. It is only possible to point out the independent parameters on which the solution of the detection process depends and make comparisons with the classical detection problem.

After the choice of binary signals only, the independent parameters reduce to three, which may be taken as μ, A and n_k, as can be seen from (3.20) to (3.22) and the later definition of n_k (3.27)

$$\mu = \frac{N_o}{\hbar\omega} \tag{3.39}$$

$$A = \mu\ \gamma_k \cdot n_k = \frac{L(S)}{N_o + \hbar\omega} \tag{3.40}$$

$$n_k = \eta(f_2 - f_1) \cdot T \quad . \tag{3.41}$$

In the classical case there is only one parameter, A, with $\hbar\omega$ omitted in the denominator. Only the total signal energy and the spectral density of the noise matter. Whether or not the signal energy is spread over a long time and wide frequency band is of no consequence for the error frequency in the optimum receiver.

The photo detector does not support corresponding statements. The spreading of L(S) over a large number of samples affects the sample power S_k (i.e. γ_k) which is one of the parameters in the nonlinear correlation operator (3.23). When the available signal energy is low, it is advantageous to concentrate it in a few samples in order to reach a more favorable operating range of the nonlinear correlator.

Suppose that the maximum pulse energy L(S) as well as the average power L(S)/T are given, so that T cannot be independently varied under optimum conditions. Then it is clear from the analysis that the ratio of duty factor to bandwidth which gives a certain n_k is immaterial; it will not affect the error frequency.

In other words, in order to reduce the relative error frequency, it is equally favorable to reduce the bandwidth as to use a wide-band system with a large number of independent samples per pulse or unit code group.

These conclusions are arrived at under the assumption that a strictly binary pulse code is used; for each consecutive non-overlapping interval T a yes-no decision is made, giving a binary digit with a certain reasonably small probability of error. The output of the correlator, however, contains potentially more information than one bit per pulse, minus the equivocation, because it is continuous in time, and in a broadband case the times when its peaks occur may be well enough defined to accommodate a pulse-position modulation rather than the simple binary pulse-code modulation. Since a pulse-position-modulation would have higher error frequencies again, this defeats the purpose under discussion: reduction of an inherently too large error frequency. The higher error frequency follows of course from the fact that the same data must serve as a basis for a decision between an increased number of alternatives.

Returning to the binary channel, let us consider the choice of the threshold condition λ_c. If the transmitted signal is symmetric, $P(1) = P(0) = 1/2$, also the error probabilities should optimally be symmetric, i.e. the rate of misses equal to the rate of false alarms. In the Gaussian approximation ($n_k \gg 1$) the probability density curves are symmetrically placed with respect to the line $\alpha_c = \frac{1}{2}(m_1+m_2)$; this threshold value for the decision consequently leads to equal error rates (Fig. 3-3).

Maximization of the rate of flow of information for given average transmitted power, however, leads to an asymmetric signal, $P(1) \ll P(0)$, unless the channel is operating at nearly classical power levels. Under this condition also the receiver threshold value should be simultaneously optimized, since otherwise the false alarm errors would swamp the transmitted pulses. This optimization requires both a complete quantitative solution of the detection problem and a working knowledge of practically realizable coding procedures for asymmetric binary channels. In the absence of both, we have to postpone further discussion of this subject.

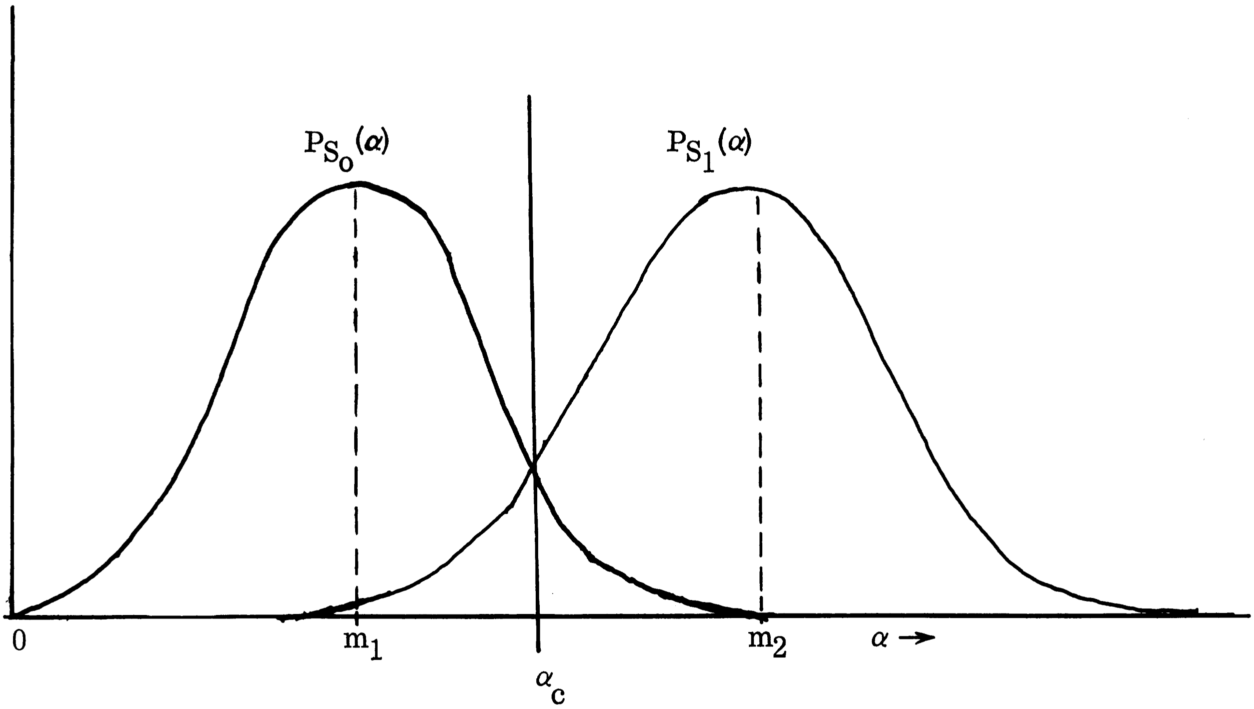

FIG. 3-3: PROBABILITY DENSITIES OF CORRELATION OUTPUT

Other essential steps in making the statistical detection analysis provide practically valuable guide lines for the design of optical communications involve the removal or reduction of some of the idealization and simplification of the theoretical model analyzed. The first consideration should be to account for the addition of noise in the processing of the output of the photocell, primarily amplifier noise. Whether the photo current is amplified by an electron multiplier or by any other type of amplifier, noise will necessarily be added, which will affect the likelihood ratio and its distribution functions as seen from the amplifier output. The performance of the receiver will consequently be poorer than indicated by the idealized theory presented above. The resulting increase in error frequency may necessitate operation at so much higher pulse energy that other types of receivers such as superheterodynes, laser amplifier systems, etc. are preferable, since theory indicates that they are more effective at a higher energy per sample (Gordon, 1962).

3.3 Quantum Statistics of Laser Amplifiers

3.3.1 Introduction

Quantum amplifiers have been studied by several authors (Shimoda, 1957; Louisell, 1961, Gordon 1963a, 1963b). Shimoda et al (1957) have presented an analysis of the amplification and fluctuations of the number of photons. Louisell et al (1961) have studied the amplification and fluctuations of the field amplitude, as well as the number of photons, and have derived expressions for the probability distribution of the field amplitudes at the output, for various forms of input fields. The model used by Louisell (1961) and Gordon (1963a) is constructed in analogy to the classical model for parametric amplifiers. In their analyses of the statistics of the field amplitude, the amplifier is characterized by a single parameter, namely the gain.

The present analysis is aimed particularly at the travelling-wave quantum amplifier although the model can be easily adapted to other types of amplifiers. For the sake of manipulative ease, we assume that the atoms of the amplifying medium (active material) have sharply defined energy levels. This restriction will be relaxed in subsequent work. Both the radiation field and the active material are described in terms of density operators. Thus, we are able to determine the field density operator at the output for an arbitrary input. The amplifier is characterized by two parameters; the gain and the population inversion. For quantum amplifiers at infrared and optical frequencies, the lower level of the active material is not empty. A ratio of the population of the upper level to that of the lower of the order of 0.9 may not be atypical. It is of importance therefore, to take explicit account of this fact. In addition, the case of an amplifier with time-dependent populations of the levels of the active material can be studied by a straightforward generalization of the present treatment.

In previous studies of a binary information channel (Hok et al 1964, 1965) with a laser preamplifier, we have used the results of Shimoda (1957). In this respect the present work constitutes a justification of those results which are also obtained as a special case of this analysis. As long as an information channel is based on detecting the number of quanta, the aforementioned considerations are justified. However, for a channel based on measuring the field amplitude and its phase, one would need to know the effect of the amplification process on these quantities. Due to the quantum nature of the processes involved, the statistics of the output and the noise performance of the device will depend on the particular variable to be measured, the state of the input signal and the characteristics of the amplifier. By way of contrast let it be mentioned that when quantum effects are negligible, the characteristics of the device alone should determine the performance of the channel independently of what is measured. Lastly, in the light of Glauber's work (1963a, 1963b) it is of importance to examine the question of coherence in connection with information channels and amplifiers.

3.3.2 Formulation of Problem

The physical problem we wish to study can be represented schematically as in Fig. 3-4. Electromagnetic energy carrying information enters the input of a

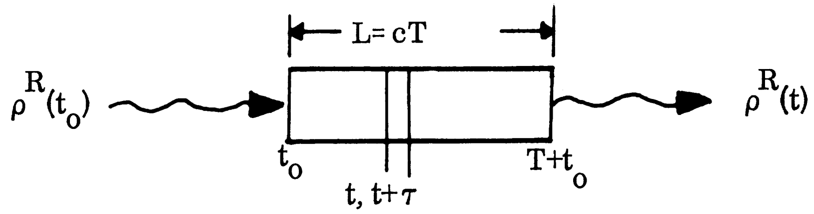

FIG. 3-4

laser amplifier. The state of the input field is represented by a density operator $\rho^R(t_0)$ and the state of the output by $\rho^R(T+t_0)$. The length of the amplifier is L. The signal enters at time t_0 and leaves the amplifier at time $T+t_0$. Neglecting dispersion, we shall have

$$L = cT \quad , \tag{3.41}$$

where c is the speed of light. Knowing the density operator of the signal constitutes as complete a knowledge of the state of the signal as is allowed by quantum theory. Thus, the problem we wish to solve is to determine $\rho^R(T+t_0)$ in terms of $\rho^R(t_0)$ and the characteristics of the amplifier. The latter is an assemblage of material particles grouped in atoms (or molecules). It is assumed that the atoms do not interact with each other although they may interact with external fields such as pumping fields. Let H^R be the hamiltonian of the radiation field, H^A the hamiltonian of the particles and V the interaction between the two. The total non-relativistic hamiltonian then is

$$H=H^R+H^A-\sum_\sigma \frac{e_\sigma}{m_\sigma c}\,\underline{p}_\sigma \cdot \underline{A}(\underline{r}^\sigma)+\sum_\sigma \frac{e_\sigma^2}{2m_\sigma c^2}\,\underline{A}^2\,(\underline{r}^\sigma) \ , \tag{3.42}$$

where the index σ refers to the σth particle, e_σ and m_σ are its charge and mass, and

$\underline{r}^{\sigma}$, $\underline{p}_{\sigma}$ its position and momentum operator respectively. $\underline{A}(\underline{r})$ is the vector potential. (Here we need consider only transverse fields.) Let $\{ \underline{u}_k(\underline{r}) \}$ be the orthonormal, complete set of eigenvectors appropriate to the problem. These **vectors** satisfy the equations

$$\underline{\nabla} \cdot \underline{u}_k(\underline{r}) = 0, \tag{3.43a}$$

$$\nabla^2 \underline{u}_k(\underline{r}) + \frac{\omega_k^2}{c^2} \underline{u}_k(\underline{r}) = 0 , \tag{3.43b}$$

where

$$\omega_k = ck \ . \tag{3.43c}$$

For the applicational environment pertaining to the present study, the appropriate set of eigenvectors will probably the the free space eigenvectors . These are

$$\underline{u}_k(\underline{r}) = (2\pi)^{-3/2} \underline{\epsilon}_k e^{i\underline{k}\cdot\underline{r}} \ , \tag{3.44}$$

where the index k is assumed to contain the polarization index as well. In terms of these eigenvectors, $A(\underline{r})$ can be written in the form

$$\underline{A}(\underline{r}) = c \sum_k \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \left\{ a_k \underline{u}_k(\underline{r}) + a_k^+ \underline{u}_k(\underline{r}) \right\} \ , \tag{3.45}$$

where a_k and a_k^+ are the usual annihilation and creation operators, obeying the commutation relations

$$\left[a_k, a_{k'}^+ \right] = \delta_{kk'} \ , \tag{3.46a}$$

and

$$\left[a_k, a_{k'} \right] = \left[a_k^+, a_{k'}^+ \right] = 0 \ . \tag{3.46b}$$

Processes in which more than one photon are simultaneously emitted or absorbed, or processes in which photons are scattered do not have any importance in laser amplifiers. At least one wishes to construct an amplifier satisfying this requirement. Thus, we may dispense with the term $\underline{A}^2$. The total hamiltonian now is

$$H = H^R + H^A + \sum_k d_k (a_k + a_k^+) , \tag{3.47a}$$

where

$$H^R = \sum_k \hbar\omega_k (a_k^+ a_k + \frac{1}{2}) \tag{3.47b}$$

and d_k is a particle operator defined by

$$d_k \equiv - \sum_\sigma \frac{e_\sigma}{m_\sigma} \left(\frac{\hbar}{2\omega_k}\right)^{1/2} (\underline{p}_\sigma \cdot \underline{\epsilon}_k) . \tag{3.47c}$$

In defining d_k, we have used the dipole approximation and have replaced $e^{i\underline{k}\cdot\underline{r}}$ by 1. Hence, d_k is the dynamic electric dipole moment operator. Since the atoms are uncorrelated, the operator d_k (which is the collective dipole moment) can be written in the form

$$d_k = \sum_{j=1}^{N} d_k^{(j)} , \tag{3.48}$$

where the summation is over all atoms. Strictly speaking, d_k is the projection of the dipole operator on the polarization vector $\underline{\epsilon}_k$.

The problem has thus been reduced, as usual, to the interaction of an assemblage of harmonic oscillator with an assemblage of atoms. To keep bookkeeping complexity to a minimum we shall consider a single mode of the field and study its interaction with the atoms. The generalization to more than one mode is straightforward . Dropping the index k and the summation over k, we now have

$$H = H^R + H^A + d(a^+ + a), \tag{3.49a}$$

and

$$H^R = \hbar\omega \, (a^+ a + \frac{1}{2}) \quad , \tag{3.49b}$$

where ω is the frequency of the oscillator.

3.3.3. Time Evolution of the Density Operator

Let $\rho(t)$ be the density operator of the compound system (harmonic oscillator plus atoms) at time t. The density operator at a later time $(t+\tau)$ is given by (Margenau and Murphy, 1963)

$$\rho(t+\tau) = U(\tau)\rho(t)U^{+}(\tau) , \tag{3.50}$$

where $U(\tau)$ is the time evolution operator for the system and is given by

$$U(\tau) = e^{-\frac{i}{\hbar}H\tau} . \tag{3.51}$$

Introducing

$$H^{o} \equiv H^{R}+H^{A} , \tag{3.52a}$$

and

$$V \equiv d(a^{+}+a) \tag{3.52b}$$

we have

$$H = H^{o}+V . \tag{3.52c}$$

From usual perturbation theory (Messiah, 1964) we have

$$U(\tau)=U^{(o)}(\tau)-\frac{i}{h}\int_{0}^{\tau} ds\, U^{(o)}(\tau-s)V\, U^{(o)}(s)- -\frac{1}{h^2}\int_{0}^{\tau} ds \int_{0}^{s} ds'\, U^{(o)}(\tau-s)VU^{(o)}(s-s')VU^{(o)}(s')+\cdots \tag{3.53}$$

Retaining terms up to and including the second order in V will be sufficient for our purposes. The operator $U^{(o)}(\tau)$ is defined by

$$U^{(o)}(\tau) \equiv e^{-\frac{i}{\hbar}H^{o}\tau} \tag{3.54}$$

For the sake of brevity let us also introduce the operators

$$W^{(1)}(\tau) \equiv -\frac{i}{h}\int_0^\tau ds U^{(o)}(\tau-s) V U^{(o)}(s) \ , \tag{3.55a}$$

and

$$W^{(2)}(\tau) \equiv -\frac{i}{h^2}\int_0^\tau ds \int_0^s ds' U^{(o)}(\tau-s) V U^{(o)}(s-s') V U^{(o)}(s') \ , \tag{3.55b}$$

in terms of which we have

$$U(\tau) = U^{(o)}(\tau) + W^{(1)}(\tau) + W^{(2)}(\tau) \ , \tag{3.56}$$

where terms of order higher than the second in V have been neglected. Combining now (3.50) and (3.56), we obtain

$$\rho(t+\tau) = U^{(o)}(\tau)\rho(t)U^{(o)+}(\tau) + U^{(o)}(\tau)\rho(t)W^{(1)+}(\tau) + W^{(1)}(\tau)\rho(t)U^{(o)+}(\tau)$$
$$+ W^{(1)}(\tau)\rho(t)W^{(1)+}(\tau) + U^{(o)}(\tau)\rho(t)W^{(2)+}(\tau) + W^{(2)}(\tau)\rho(t)U^{(o)+}(\tau), \tag{3.57}$$

where we have again neglected terms of order higher than the second in V.

The density operator $\rho(t)$ represents the whole system. Now, as the wave progresses from left to right in Fig. 3-4, it finds itself in an environment of new atoms each time. The populations of the various levels of the atoms are controlled from outside by means of pumping. Of course, this is true only if the field is not so strong as to saturate the amplifier. With this understanding the level populations will be assumed constant throughout the amplifier. Then, we may write

$$\rho(t) = \rho^{\pi}(t)\rho^{A}(t) \ . \tag{3.58}$$

This does not mean that field and atoms are constantly uncoupled, In fact (3.57) describes precisely the coupling between the two. The operator $\rho^R(t)$ describes the field at time t, while $\rho^A(t)$ describes the state of that part of the amplifier that has not interacted yet with the field.

Our ultimate goal is to describe the field after a certain amount of amplification. Thus, let us define the reduced density operator (Fano, 1957)

$$\rho^R(t) \equiv Tr_A \rho(t) \quad , \tag{3.59}$$

where Tr_A means that the trace with respect to atomic variables is taken. This definition is consistent with Eq. (3.58) since $Tr_A \rho^A(t)=1$. To calculate the operator $\rho^R(t+\tau)=Tr_A \rho(t+\tau)$ we shall use (3.57). To this end, we consider the representation $\{|n\rangle|i\rangle\}$, where

$$a^+ a|n\rangle = n|n\rangle \quad , \quad n=0,1,2,\ldots \tag{3.60a}$$

and

$$H^A|i\rangle = E_i|i\rangle \ . \tag{3.60b}$$

In a laser amplifier, the amplification is due to transitions between two particular levels of the atoms. Let us assume therefore, that H^A has only two eigenstates $|1\rangle$ and $|2\rangle$ where $E_2 > E_1$. The frequency of the transition $|2\rangle \rightarrow |1\rangle$ will be denoted by ω_o. That is

$$\omega_o = \frac{E_2 - E_1}{\hbar} \ . \tag{3.61}$$

Let, furthermore, N_1, N_2, N be the number of atoms in the lower state, the number of atoms in the upper state and the total number of atoms, respectively. Then, we shall have $N=N_1+N_2$. It will be convenient to introduce the probabilities for an atom to be found in the lower or upper state. These are given by the diagonal matrix elements of the operator ρ^A in the representation $\{|i\rangle\}$. Thus we have

$$\rho^A_{11} = \frac{N_1}{N} \qquad \text{and} \qquad \rho^A_{22} = \frac{N_2}{N} \quad . \tag{3.62}$$

In the calculation that will follow, we shall neglect the off-diagonal matrix elements of ρ^A. This is the random phase approximation and will be satisfactory as long as we to not have any appreciable correlation between atoms. Moreover, we assume that the atoms do not exhibit a permanent electric dipole moment in either of the two energy eigenstates. This means that d, and therefore V, will not have diagonal non-vanishing matrix elements in the representation $\{|i\rangle\}$. When this fact is combined with the neglect of the off-diagonal matrix elements of ρ^A, one finds that the Tr_A of the second and third terms in Eq. (3.57) vanishes. Thus we have

$$\rho^R(t+\tau)=Tr_A\,\rho(t+\tau)=Tr_A\Big\{U^{(o)}(\tau)\rho^R(t)\rho^A(t)U^{(o)+}(\tau)+W^{(1)}(\tau)\rho^R(t)\rho^A(t)W^{(1)+}(\tau)$$

$$+U^{(o)}(\tau)\rho^R(t)\rho^A(t)W^{(2)+}(\tau)+W^{(2)}(\tau)\rho^R(t)\rho^A(t)U^{(o)+}(\tau)\Big\}, \qquad (3.63)$$

where we have used (3.58). The operator ρ^R is an operator in radiation-field space. Considering an arbitrary matrix element in the representation $\{|n\rangle\}$ we obtain

$$\rho^R_{mn}(t+\tau)=\langle m|Tr_A\,\rho(t+\tau)|n\rangle=\langle m|Tr_A\Big\{U^{(o)}(\tau)\rho^R(t)\rho^A(t)U^{(o)+}(\tau)+W^{(1)}(\tau)\rho^R(t)\rho^A(t)W^{(1)+}(\tau)+$$

$$+U^{(o)}(\tau)\rho^R(t)\rho^A(t)W^{(2)+}(\tau)+W^{(2)}(\tau)\rho^R(t)\rho^A(t)U^{(o)+}(\tau)\Big\}|n\rangle. \qquad (3.64)$$

The calculation of the right-hand side of this equation is rather straightforward although somewhat lengthy. It will be presented in a future report. The result is

$$\rho^R_{mn}(t+\tau)=e^{-i(m-n)\omega\tau}\Big\{\rho^R_{mn}(t)-\tau\Big[c_2\{(m+1)K^*+(n+1)K\}+c_1(mK+nK^*)\Big]$$

$$\rho^R_{mn}(t)+\tau b_2\sqrt{mn}\,\rho^R_{(m-1)(n-1)}(t)+\tau b_1\sqrt{(m+1)(n+1)}\,\rho^R_{(m+1)(n+1)}\Big\}, \qquad (3.65)$$

where

$$c_j=\rho^A_{jj}\,|d_{21}|^2\,\hbar^{-2}, \qquad j = 1, 2, \qquad (3.66a)$$

$$b_j=2\pi c_j\,\delta(\omega-\omega_o)=2\pi\hbar^{-2}\rho^A_{jj}\,|d_{21}|^2\,\delta(\omega-\omega_o), \qquad (3.66b)$$

and

$$K = i \frac{P}{\omega-\omega_o} + \pi\ \delta(\omega-\omega_o)\ . \qquad (3.66c)$$

The symbol P denotes the Cauchy principal pact and K^* is the complex conjugate of K The following relations, which will be useful to us, are obtained from (3. 66).

$$K+K^* = 2\pi\,\delta(\omega-\omega_o) \qquad (3.67a)$$

$$c_j \mathrm{Re}\ K = \frac{b_j}{2}\ ,\ j = 1, 2 \qquad (3.67b)$$

$$c_j(K+K^*) = b_j\ . \qquad (3.67c)$$

The appearance of δ-function in (3. 65) is due to our having considered a single mode of the radiation field and an atomic system with sharp energy levels. When one considers broadened energy levels, the δ-functions will be replaced by line shape or density of states functions. Also when more than one mode is considered one will have to sum (or integrate) over all modes. In any event Eq. (3. 65) will not be modified except for the fact that the quantities b_j, c_j and K which characterize the coupling between field and atoms, will have different expressions. In this sense (3. 65) contains essentially all the physics we are interested in the for time being.

We define now the derivative of $\rho^R(t)$ as follows:

$$\dot{\rho}^R(t) = \frac{1}{\tau}\left[\rho^R(t+\tau) - \rho^R(t)\right]\ . \qquad (3.68)$$

The time increment τ is understood as macroscopically small, but large compared to characterist times of the atomic transitions in consideration. With the above definition in mind, we subtract $\rho^R_{mn}(t)$ from both sides of (3. 65) and divide by τ . Then, the following differential equation is obtained.

$$\dot{\rho}^R_{mn}(t) = -\left[c_2\left((m+1)K^* + (n+1)K\right) + c_1\left(mK + nK^*\right) + i(m-n)\omega\right]$$

$$\rho^R_{mn}(t) + b_2\sqrt{mn}\,\rho^R_{(m-1)(n-1)}(t) + b_1\sqrt{(m+1)(n+1)}\,\rho^R_{(m+1)(n+1)}(t)\ . \qquad (3.69)$$

The matrix elements of ρ^R form a square infinite-dimensional matrix. Let us consider that subset of matrix elements for which $m-n=\ell$ where ℓ is a fixed integer. The above equation shows that the matrix elements belonging to a subset corresponding to a given ℓ are coupled with each other and are not coupled to matrix elements of any other subset. This introduces considerable simplification, and in order to make use of this simplification we let

$$m = n + \ell\ . \qquad (3.70a)$$

Substituting into (3. 69), rearranging somewhat, and introducing the symbols

$$\nu \equiv \omega - (c_2 - c_1)\ mK\ , \qquad (3.70b)$$

$$b \equiv b_2 + b_1 \qquad (3.70c)$$

$$\gamma_\ell \equiv \left(\frac{b}{2} + i\nu\right)\ell\ , \qquad (3.70d)$$

and

$$c \equiv b_2 + \gamma_\ell\ , \qquad (3.70e)$$

we obtain

$$\dot{\rho}_{(m+\ell)m}(t) = -(bm+c)\rho_{(m+\ell)m}(t) + b_2\sqrt{(m+\ell)m}\,\rho_{(m+\ell-1)(m-1)}(t)$$

$$+ b_1\sqrt{(m+\ell+1)(m+1)}\ \rho_{(m+\ell+1)(m+1)}(t)\ . \qquad (3.71)$$

For a given ℓ, we have a set of infinitely many, coupled differential equations. The set corresponding to $\ell = 0$ contains the diagonal matrix elements and by letting $\ell=0$ in (3. 71) we obtain the equation that Shimoda et al (1957) ahve obtained. By considering all the sets for $\ell=0, 1, 2, \ldots,$ we have the complete density matrix. The solution of (3. 71) therefore, will provide the solution for the complete density matrix.

3.3.4 Expectation Values of the Field Operators

Before discussing the solution of the differential equation for the matrix elements of the density operator, we study the effect of the amplification process on the expectation values and fluctuations of certain field operators. It turns out that, in order to calculate such quantities, one does not need the solutions of Eq.(3.71). The reason is that one can, with the help of Eq.(3.71), write and solve differential equations for the expectation values and fluctuations themselves. This can probably be done for moments higher than the second as well. However, the calculation of probability distributions of field amplitudes, for example, will require the solution for the complete density operator.

The energy operator is one of the operators of interest. Its expectation value and fluctuations have been studied by Shimoda et al (1957). There is nothing to be added here. We have used their results in previous work (Hok, 1964). Three other quantities that are of particular interest to us here are the field coordinates (position and momentum of the harmonic oscillator or equivalently, electric and magnetic field) and their phase. We now turn to the sutdy of these quantities.

Expectation Values of Field Coordinates. The oscillator coordinate operators are given by

$$q = \sqrt{\frac{\hbar}{2\omega}}\ (a^{\dagger} + a)\ , \tag{3.72a}$$

and

$$p = i\sqrt{\frac{\hbar\omega}{2}}\ (a^{\dagger} - a)\ . \tag{3.72b}$$

It is convenient to introduce the dimensionless hermitian operators

$$Q \equiv a^{\dagger} + a\ , \tag{3.73a}$$

and

$$P \equiv i(a^{\dagger} - a)\ , \tag{3.73b}$$

which are related to q and p through the equation

$$q = Q\sqrt{\frac{\hbar}{2\omega}} \quad \text{and} \quad p = P\sqrt{\frac{\hbar\omega}{2}} \quad . \tag{3.74}$$

Let now $\langle a^\dagger(t)\rangle \equiv T_r\left\{\rho^R(t)a^\dagger\right\}$ be the expectation value of $a^\dagger$. The matrix elements of $a^\dagger$ are given by

$$\langle n|a^\dagger|m\rangle = \sqrt{m+1}\;\delta_{n,m+1} \quad . \tag{3.75}$$

Using this relation we obtain

$$\frac{\partial}{\partial t}\langle a^\dagger(t)\rangle = T_r\left\{\dot{\rho}^R(t)a^\dagger\right\} = \sum_{m=0}^{\infty}\dot{\rho}^R_{m(m+1)}(t)\sqrt{m+1} \quad . \tag{3.76}$$

From Eq. (3.22) we find that

$$\dot{\rho}^R_{m(m+1)}(t) = i\nu\rho^R_{m(m+1)}(t) - (b_2 + \frac{b}{2})\rho^R_{m(m+1)}(t) - bm\rho^R_{m(m+1)}(t) +$$

$$+ b_2\sqrt{m(m+1)}\,\rho^R_{(m-1)m}(t) + b_1\sqrt{(m+1)(m+2)}\;\rho^R_{(m+1)(m+2)}(t) \; . \tag{3.77a}$$

Combining this with Eq. (3.76) and after some rearrangement and simplification, we arrive at the equation

$$\frac{\partial}{\partial t}\langle a^\dagger(t)\rangle = (K + i\nu)\langle a^\dagger(t)\rangle \; , \tag{3.77b}$$

where K is defined by

$$K = \frac{b_2 - b_1}{2} \; . \tag{3.77c}$$

Eq. (3.76) has the obvious solution

$$\langle a^\dagger(t)\rangle = \langle a^\dagger(o)\rangle\, e^{(K+i\nu)t} \tag{3.78}$$

where $\langle a^{\dagger}(o)\rangle$ is the expectation value at the initial time which, for our purpose, is the time zero at which the signal enters the input. The time that the signal spends inside the amplifier is related to the length of the amplifier through Eq. (3.41).

Taking now the complex conjugate of Eq. (3.78) we obtain the solution for $\langle a(t)\rangle$, namely,

$$\langle a(t)\rangle = \langle a(o)\rangle e^{(K-i\nu)t} . \tag{3.79}$$

Combining Eqs. (3.78) and (3.79) we obtain

$$\langle Q(t)\rangle = \left[\langle Q(o)\rangle \; \mathrm{Cos}\,\nu t + \langle P(o)\rangle \, \mathrm{Sin}\,\nu t\right] e^{Kt} , \tag{3.80a}$$

$$\langle P(t)\rangle = \left[-\langle Q(o)\rangle \, \mathrm{Sin}\,\nu t + \langle P(o)\rangle \, \mathrm{Cos}\,\nu t\right] e^{Kt} . \tag{3.80b}$$

The corresponding equations for the oscillator coordinates q and p are:

$$\langle q(t)\rangle = \left[\langle q(o)\rangle \; \mathrm{Cos}\,\nu t + \frac{1}{\omega}\langle p(o)\rangle \; \mathrm{Sin}\,\nu t\right] e^{Kt} , \tag{3.81a}$$

$$\langle p(t)\rangle = \left[-\omega \langle q(o)\rangle \; \mathrm{Sin}\,\nu t + \langle p(o)\rangle \; \mathrm{Cos}\,\nu t\right] e^{Kt} . \tag{3.81b}$$

These equations are similar to those obtained by Louisell et al (1961). They differ slightly because, here, we have accounted for a shift in the oscillator frequency which will be discussed later. The equations of Louisell (1961) are obtained if we replace ν by ω, which is equivalent to neglecting the shift represented by $(c_2-c_1)\mathrm{Im}K$ (see Eq. (3.70b)). The exponential represents the gain which increases exponentially with the difference (b_2-b_1) and the time that the signal travels inside the amplifier. For future use we shall introduce the quantity

$$G \equiv e^{2Kt} \tag{3.82}$$

The field coordinates increase as the square root of G. Of course, the gain is larger than unity only if the upper level is more populated than the lower i.e. if $b_2 > b_1$.

This is understood to be the case throughout this treatment, since we are interested in an amplifier and not an attenuator. It is interesting to note nevertheless, that the case of the attenuator is implicit in this study.

Expectation Value of the Energy. As already mentioned, this question has been answered by Shimoda et all (1957). Here, we shall simply give the result since it will be needed in the following sections. It is convenient to define a photon number operator

$$f \equiv a^{\dagger} a \,. \tag{3.83}$$

The expected number of photons then is

$$\langle f(t) \rangle = \langle a^{\dagger}(t)\, a(t) \rangle , \tag{3.84}$$

and the expected energy $\hbar\omega \langle f(t) \rangle$. To find an equation for $\langle f(t) \rangle$, we proceed exactly as we did for $\langle a^{\dagger}(t) \rangle$. Thus, we obtain the equation

$$\frac{\partial}{\partial t} \langle f(t) \rangle = 2K \langle f(t) \rangle + b_2 \,, \tag{3.85}$$

whose solution is

$$\langle f(t) \rangle = \left(\langle f(o) \rangle + \frac{b_2}{b_2 - b_1} \right) e^{2Kt} - \frac{b_2}{b_2 - b_1} \,. \tag{3.86}$$

A more convenient form is

$$\langle f(t) \rangle = G \langle f(o) \rangle + (G-1)\lambda \,, \tag{3.87}$$

where we have introduced the quantity

$$\lambda \equiv \frac{b_2}{b_2 - b_1} \,, \tag{3.88}$$

which characterizes the population inversion. For active materials, that is for $b_2 > b_1$, we shall have $\lambda \geq 1$. The equality is attained under complete inversion. As far as amount of amplification is concerned, one would desire this extreme case. For very large gain, i.e. for $G \gg 1$, Eq.(3.87) can be approximated by

$$\langle f(t) \rangle = G\left[\langle f(o) \rangle + \lambda\right] . \tag{3.89}$$

This shows that there will be an output even if the number of input photons ($\langle f(o) \rangle$) is zero. Clearly, this is due to spontaneous emission.

3.3.5 Fluctuations

In the previous section, the expected values of the field amplitudes were expressed in terms of the expected values at the input and the gain. The outcomes of measurements of these amplitudes at the output will fluctuate about the expected values. It is desirable, therefore, to have a quantitative estimate of these fluctuations upon which one can base a criterion for the usefulness of the amplifier. A rather conventional measure of the fluctuation is given by the quantity

$$\mathcal{E}_Q^2 = \frac{\langle \Delta Q^2 \rangle}{\langle Q \rangle^2} , \tag{3.90}$$

where

$$\langle \Delta Q^2 \rangle = \langle Q^2 \rangle - \langle Q \rangle^2 . \tag{3.91}$$

A similar quantity $\mathcal{E}_P^2$ measures the fluctuations of P. We now calculate $\langle Q^2(t) \rangle$ and $\langle P^2(t) \rangle$.

From the definition of Q we have

$$Q^2 = a^{\dagger}a + a^{\dagger}a + 2a^{\dagger}a + 1 , \tag{3.92}$$

which, upon using Eq. (3.84) becomes

$$\langle Q^2(t)\rangle = \langle a^\dagger(t)\, a^\dagger(t)\rangle + \langle a(t)\, a(t)\rangle + 2\langle f(t)\rangle + 1. \tag{3.93}$$

The quantity $\langle f(t)\rangle$ has been calculated in the preceding section. Observing that

$$\langle a(t)\, a(t)\rangle = \langle a^\dagger(t)\, a^\dagger(t)\rangle^* \tag{3.94}$$

we conclude that we only need to calculate the quantity $\langle a^\dagger(t)\, a^\dagger(t)\rangle$. Using Eq. (3.75) we obtain

$$\langle a^\dagger(t)\, a^\dagger(t)\rangle = \mathrm{Tr}\left\{ a^\dagger a^\dagger \rho^R(t)\right\} = \sum_{m=0}^{\infty} \rho^R_{m(m+2)}(t) \sqrt{(m+1)(m+2)}\ , \tag{3.95}$$

and therefore

$$\frac{d}{dt}\langle a^\dagger(t)\, a^\dagger(t)\rangle = \sum_{m=0}^{\infty} \dot{\rho}^R_{m(m+2)}(t) \sqrt{(m+1)(m+2)}\ . \tag{3.96}$$

Combining this with Eq. (3.71) and performing some manipulations, one obtains the following simple differential equation:

$$\frac{d}{dt}\langle a^\dagger(t)\, a^\dagger(t)\rangle = 2(K + i\nu)\langle a^\dagger(t)\, a^\dagger(t)\rangle, \tag{3.97}$$

which has the solution

$$\langle a^\dagger(t)\, a^\dagger(t)\rangle = \langle a^\dagger(o)\, a^\dagger(o)\rangle e^{2(K+i\nu)t}\ . \tag{3.98}$$

From this and Eq. (3.94) we obtain

$$\langle a(t)\, a(t)\rangle = \langle a(o)\, a(o)\rangle e^{2(K-i\nu)t}\ . \tag{3.99}$$

Now, Eq. (3.93) becomes

$$\langle Q^2(t)\rangle = \langle a^\dagger(o)\, a^\dagger(o)\rangle e^{2(K+i\nu)t} + \langle a(o)\, a(o)\rangle e^{2(K-i\nu)t} + 2\langle f(t)\rangle + 1\,. \tag{3.100}$$

Writing the exponentials in terms of sines and cosines and using Eq. (3.87) we obtain

$$\langle Q^2(t)\rangle = 2G\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(r)} \mathrm{Cos}\, 2\nu t + 2G\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(i)} \mathrm{Sin}\, 2\nu t + \\ + 2G\langle f(o)\rangle + 2(G-1)\lambda + 1, \tag{3.101}$$

where the superscripts (r) and (i) indicate the real and imaginary parts, respectively, of the quantity they qualify.

From the definition of P, we have

$$P^2 = -a^\dagger a^\dagger - a a + 2a^\dagger a + 1\,, \tag{3.102}$$

and using the previous analysis we obtain

$$\langle P^2(t)\rangle = -2G\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(r)} \mathrm{Cos}\, 2\nu t - 2G\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(i)} \mathrm{Sin}\, 2\nu t + \\ + 2G\langle f(o)\rangle + 2(G-1)\lambda + 1\,. \tag{3.103}$$

Taking the squares of Eqs. (3.80), we find

$$\langle Q(t)\rangle^2 = G\langle Q(o)\rangle^2 \mathrm{Cos}^2 \nu t + G\langle P(o)\rangle^2 \mathrm{Sin}^2 \nu t + \\ + G\langle Q(o)\rangle\langle P(o)\rangle \mathrm{Sin}\, 2\nu t \tag{3.104a}$$

and

$$\langle P(t)\rangle^2 = G\langle Q(o)\rangle^2 \mathrm{Sin}^2 \nu t + G\langle P(o)\rangle^2 \mathrm{Cos}^2 \nu t - \\ - G\langle Q(o)\rangle\langle P(o)\rangle \mathrm{Sin}\, 2\nu t\,. \tag{3.104b}$$

To study the fluctuations of the field amplitudes, it suffices to consider the average values and deviations at a particular time. In fact, this time should be the time T at which the signal leaves the amplifier. Thereafter, the signal will be free and evolve like a free harmonic oscillator. The phase of the output depends on the phase of the input and the length of the amplifier. Since we are not considering phases at this point, we may assume that the time T that the signal spends inside the amplifier is given by $T=M\frac{2\pi}{\nu}$ where M is a large integer. This assumption simplifies the equations somewhat without affecting the conclusion concerning the statistics of the amplitudes. Thus, we have

$$\langle Q^2(t)\rangle = 2G\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(r)} + 2G\langle f(o)\rangle + 2(G-1)\lambda + 1\,, \quad (3.105a)$$

$$\langle P^2(t)\rangle = -2G\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(r)} + 2G\langle f(o)\rangle + 2(G-1)\lambda + 1\,, \quad (3.105b)$$

$$\langle Q(t)\rangle^2 = G\langle Q(o)\rangle^2\,, \quad (3.106a)$$

$$\langle P(t)\rangle^2 = G\langle P(o)\rangle^{(2)}\,. \quad (3.106b)$$

Using the definitions of Q and P one can easily show that

$$2\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(r)} = \frac{1}{2}\left\{\langle Q^2(o)\rangle - \langle P^2(o)\rangle\right\}, \quad (3.107a)$$

and

$$\frac{1}{2}\left\{\langle Q^2(o)\rangle + \langle P^2(o)\rangle\right\} = 2\langle f(o)\rangle + 1\,. \quad (3.107b)$$

By virtue of these relations, the quantitiy $2\langle a^\dagger(o)\, a^\dagger(o)\rangle^{(r)}$ can be eliminated from Eqs. (3.105) which become

$$\langle Q^2(t)\rangle = G\langle Q^2(o)\rangle + (2\lambda-1)(G-1)\,, \quad (3.108a)$$

$$\langle P^2(t)\rangle = G\langle P^2(o)\rangle + (2\lambda-1)(G-1)\,. \quad (3.108b)$$

Combining these equations with Eqs. (3.106) and (3.90), we obtain

$$\langle \Delta Q^2(t) \rangle = G \langle \Delta Q^2(o) \rangle + (2\lambda-1)(G-1) \ , \tag{3.109a}$$

$$\langle \Delta P^2(t) \rangle = G \langle \Delta P^2(o) \rangle + (2\lambda-1)(G-1) \ , \tag{3.109b}$$

and therefore

$$\mathcal{E}_Q^2 = \mathcal{E}_{Q_o}^2 + \frac{(2\lambda-1)}{\langle Q(o) \rangle^2} \left(\frac{G-1}{G}\right) , \tag{3.110a}$$

$$\mathcal{E}_P^2 = \mathcal{E}_{P_o}^2 + \frac{(2\lambda-1)}{\langle P(o) \rangle^2} \left(\frac{G-1}{G}\right) . \tag{3.110b}$$

Recalling the definition of λ (Eq. (3.88) and introducing the symbol μ for the quantity $(2\lambda-1)$, we have

$$\mu = 2\lambda - 1 = \frac{b_2 + b_1}{b_2 - b_1} \ . \tag{3.111a}$$

If we characterize the population inversion by a maser temperature T_m, then μ asssumes the form

$$\mu = \frac{e^{\hbar\omega_o/k|T_m|} + 1}{e^{\hbar\omega_o/k|T_m|} - 1} \ . \tag{3.111b}$$

For active materials, we shall have $\mu \geq 1$, the equality occuring when the lower level is empty. In terms of μ, we have

$$\mathcal{E}_Q^2 = \mathcal{E}_{Q_o}^2 + \frac{\mu}{\langle Q(o) \rangle^2} \left(\frac{G-1}{G}\right) , \tag{3.112a}$$

$$\mathcal{E}_P^2 = \mathcal{E}_{P_o}^2 + \frac{\mu}{\langle P(o) \rangle^2} \left(\frac{G-1}{G}\right) . \tag{3.112b}$$

The quantities $\mathcal{E}_Q^2$ and $\mathcal{E}_P^2$ characterize the amplifier noise referring to measurements of Q and P, respectively. As discussed earlier, they are different, in general. The quantities $\mathcal{E}^2_{\phi_o}$ and $\mathcal{E}^2_{P_o}$ refer to the input and depend on the state of the input signal. The other terms depend on the parameter of the device, and the input field amplitudes. They decrease as the field amplitudes at the input increase. These terms are due to spontaneous omission. They will be present even when the gain of the amplifier is unity that is when $b_2 - b_1 = 0$. To see this, one recalls that $G = e^{(b_2 - b_1)}$. It follows then that for $b_2 = b_1$ we have

$$\lim_{(b_2 - b_1) \to 0} \mu \frac{G-1}{G} = \lim_{(b_2 - b_1) \to 0} \left(\frac{b_2 + b_1}{b_2 - b_1} \right) \left(\frac{G-1}{G} \right) = 2b_2 t \, .$$

For large gain $(G \gg 1)$, Eqs. (3.112) can be approximated by

$$\mathcal{E}^2_Q = \mathcal{E}^2_{Q_o} + \frac{\mu}{\langle Q(o) \rangle^2} \, , \tag{3.113a}$$

$$\mathcal{E}^2_P = \mathcal{E}^2_{P_o} + \frac{\mu}{\langle P(o) \rangle^2} \, . \tag{3.113b}$$

It is important to note that large gain does not imply $b_2 \gg b_1$ and therefore even for large gain μ will not be equal to unity, in general. The relative values of b_2 and b_1 are determined by the physical properties of the active material, while the gain can be made as large as desired by increasing the length of the amplifier. From Eqs. (3.112), we conclude that the quantum mechanical fluctuations at the input (which depend on the state of the input signal) go through the amplification process unchanged. In addition, one has the spontaneous emission noise whose relative importance decreases as the field amplitude of the input signal increases. The terms $\mathcal{E}^2_{Q_o}$ and $\mathcal{E}^2_{P_o}$ are due to what one might call quantum noise. They are a manifestation of the fact that a quantum mechanical variable cannot be determined precisely unless the system is in an eigenstate of the variable in question. The second term, in Eqs. (3.112), are due to the internal noise of the amplifier which is spontaneous emission. One sees therefore, that the quantum amplifier does not change the quantum noise of the signal. It only adds to it the noise of spontaneous emission. Recalling the relationship between (Q, P) and (q, p) (see Eqs. (3.74)) we can write Eqs. (3.112) in the more familiar forms

$$\mathcal{E}_q^2 = \mathcal{E}_{q_o}^2 + \frac{\hbar\omega}{2\omega^2 \langle q(o) \rangle^2} \mu \left(\frac{G-1}{G}\right) , \qquad (3.114a)$$

and

$$\mathcal{E}_p^2 = \mathcal{E}_{p_o}^2 + \frac{\hbar\omega}{2 \langle p(o) \rangle^2} \mu \left(\frac{G-1}{G}\right) . \qquad (3.114b)$$

Again the quantities $\mathcal{E}_q^2$ and $\mathcal{E}_p^2$ are dimensionless. These equations indicate that the effect of internal noise on a measurement of q_2, for example, decreases as $\omega^2 \langle q(o) \rangle^2$ increases in comparison to $\hbar\omega/2$, and similarly for p. The significant conclusion is that the effect of the spontaneous emission noise is determined primarily by the state of the signal at the input. To see it more clearly, note that for $b_1 = 0$ and large gain one may replace the quantity $\mu(G-1/G)$ by 1. The weaker the signal, therefore, the more it will be affected by spontaneous emission. This is reminiscent of Gordon's (1962) conclusion about energy measurements.

3.3.6 Results for a Special Case

Let us consider now the case in which the input signal is in a pure coherent state in the Glauber (1963) sense. Such a state is represented by $|\alpha\rangle$ and is defined by $a|\alpha\rangle = \alpha|\alpha\rangle$. Then we shall have

$$\langle Q(o) \rangle = 2\,\mathrm{Re}\,\alpha, \quad \langle P(o) \rangle = 2\,\mathrm{Im}\,\alpha . \qquad (3.115)$$

It is straightforward to show that

$$\langle Q^2(o) \rangle = 4(\mathrm{Re}\,\alpha)^2 + 1$$

and

$$\langle P^2(o) \rangle = 4(\mathrm{Im}\,\alpha)^2 + 1.$$

Consequently

$$\langle \Delta Q^2(o) \rangle = \langle \Delta P^2(o) \rangle = 1 \; . \qquad (3.116)$$

Then, Eqs. (3.109) give

$$\langle \Delta Q^2(t) \rangle = \langle \Delta P^2(t) \rangle = G + (2\lambda - 1)(G-1) \; . \qquad (3.117)$$

It is not surprising that the uncertainties associated with P and Q are equal because this is a property of the coherent state. And the amplifier preserves this property. If we consider the limiting case in which $\lambda=1$ and $G \gg 1$, and transform to q and p, we find

$$\langle \Delta q^2(t) \rangle \cong \frac{\hbar}{\omega} G \; , \qquad (3.118a)$$

and

$$\langle \Delta p^2(t) \rangle \cong \hbar\omega\, G \; . \qquad (3.118)b$$

Thus, we recapture the results that Louisell et al (1961) have obtained, as a special case of ours.

IV DISCUSSION OF FUTURE WORK

4.1 First Problem Area

In this chapter we shall try to look ahead into the future, to estimate the work that can be completed under the present contract and covered by the Final Report, as well as to enumerate a number of unsolved problems and incompletely investigated phenomena which are related to the purpose of this project.

In the first problem area we shall collect and reorganize for the Final Report the material presented in the Interim Reports on absorption by normal constituents of the atmosphere as well as extinction by rain, clouds, fog, and haze. In order to obtain quantitative conclusions in time for the Final Report, it has been necessary to accept rather uncritically specifications published in the literature of such things as droplet size, distributions associated with various meteorological conditions, although the literature does not imply that these are very accurately determined. We plan to return to this question (from the electromagnetic rather than the meteorological point of view) if time permits, by investigating the sensitivity of calculations of attenuation to the existing degree of uncertainity in specification of weather conditions. The result will indicate to what extent further research on this particular subject is desirable.

As far as subjects for future investigations are concerned, it may be pointed out that most calculations of attenuation by weather conditions have been based on the theory of electromagnetic extinction by spheres, since it is comparitively tractable and it is assumed that water droplets are involved. However, the possibility that elongated ice crystals are present in high-altitude clouds.should be investigated. If so, a "feasibility study" related to the theoretical analysis of attenuation by an ensemble of such crystals should be carried out. In case of encouraging results of this study, such an analysis constitutes a worthwhile subject for future research.

Another related general area for future investigation is the source, generation process and theoretical boundaries of the "sky noise" discussed in the Introduction to Chapter III of this Interim Report.

4.2 Second Problem Area

In the final report on the second problem area we expect to begin with a critical review of the theory of communication by means of electromagnetic radiation. This will involve a discussion of the basic ideas of quantum field theory, coherent and incoherent states of the field, the entropy concept in quantum statistical mechanics, the basic uncertainties in field measurements and the maximum amount of measurable information carried by an electromagnetic radiation field of given frequency, bandwidth and average intensity.

Subsequently, we then present a theoretical investigation of an optical communication channel employing optical means for discrimination against background radiation and a photon counter for observing the incident radiation under the assumption of a perfect filter and a perfect counter, the limits of received information and error statistics will be estimated, the former from entropy calculations and the latter from statistical decision theory. The statistical detection analysis presented in Section 3.2 of this report will presumably be extended in several directions; if it is found feasible without lengthy computations, the distribution functions of the likelihood ratio will be found, and the effects of additional noise sources will be investigated. In order to approach the theoretical rate of transmission of information in actual operation it is also necessary to solve the problem of finding an efficient way of coding the signals. This problem will be analyzed and ways and means of reaching more and more efficient solutions will be discussed. Because of the large error frequencies and the peculiar statistical properties of the channel, it is not expected that more than very crude solutions can be suggested within the remaining duration of the contract. As we have pointed out before, coding for asymmetric binary channels is a largely unexplored field.

Since there are serious doubts that the photon-counter channel can be implemented to give satisfactory communication at extremely low signal level, we have devoted considerable attention in this and previous reports to an alternative type of channel, which uses a laser amplifier as the first receiver component. The fundamental study of an idealized model of a laser amplifier given in Section 3.3 of this report will be included in the final report and developed in more realistic direction to account for broadened energy levels in the active material resulting in an amplifier with nonzero bandwidth. This will make it possible to consider a complex input wave form of relativity broad spectrum and to evaluate the effectiveness of the amplifier as a filter for rejection of background noise. The use of a laser amplifier as a filter and preamplifier for a photon counter will be investigated; we hope that time will permit the inclusion of the result in the final report. In a receiver of this type the laser provides both selectivity and gain, at the cost of added noise from spontaneous emission. The study of such a combination falls very neatly in line with the topics already covered under this contract although not explicitly mentioned in the task statement.

This problem area offers many unsolved problems for future research. As mentioned above, the efficient coding of channels using detectors with Poisson statistics is a largely unexplored field, where a lot of work is required. Another rich territory of investigation is the application and detailed properties of lasers in communication. Both as oscillators and amplifiers lasers promise to make it possible to extend ultra-high-frequency and microwave techniques into the optical range as well as to create entirely new optical techniques. For this purpose, more detailed knowledge of the behavior of lasers is desirable, in the linear as well as in the nonlinear domain. The quantum-mechanical formalism presented in this report can readily be extended to account for nonlinear effects as well. By considering a reduced density operator for the active material (as was done for the field density operator in this Interim Report) one can develop differential equations for the population of the states of the active material, which are coupled to the field density operator. The resulting system of nonlinear equations can be used to study the behavior of a laser under a very wide range of conditions. The nonlinearity controls the stability of amplitude and phase of an oscillator and causes the "phase-lock" phenomenon which has been successfully explored for communication over extremely long distances at conventional frequencies. The performance of a laser with regard to added noise in various receiver applications such as heterodyne oscillator and "phase-lock" detector offers a wide variety of important problems for future research.

4.3. Third Problem Area

The survey of principles for narrow-band tunable optical filters presented in the first two interim reports will be included in the final report substantially as given there.

There is certainly room for extensive work in this area. However, progress in this area depends primarily on the following factors:

1. Discovery of new electro-optical effects and materials
2. Development of new materials and devices utilizing already known phenomena.

It thus appears that the most fruitful efforts in this area would be along the lines of developmental and inventive type of research rather than theoretical extension and extrapolation.

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