THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING

DEPARTMENT OF ELECTRICAL ENGINEERING

Radiation Laboratory

ON THE SOLUTION SPACES FOR CERTAIN EQUATIONS OF ACOUSTICAL SCATTERING THEORY

by

ERGUN AR
RALPH E. KLEINMAN

January 1967

Grant GP-6140

8136-1-T = RL-2174

Contract With : National Science Foundation
Applied Mathematics and Statistics
Washington, D. C. 20550

Administered through:

OFFICE OF RESEARCH ADMINISTRATION • ANN ARBOR

ABSTRACT

The certain equations arising in the theory of acoustical scattering theory are discussed. The Neumann problem is rigorously solved.

TABLE OF CONTENTS

I

INTRODUCTION

We consider a spherical coordinate system $p = (r, \theta, \phi)$ erected with origin interior to a smooth, closed and bounded surface B, and let V denote the exterior volume. Then if G_o is the static Dirichlet Green's function for the surface B, and if G_k is the Dirichlet Green's function for the Helmholtz equation then it was shown by Kleinman [3] that

$$\tilde{u}(p,p_o) = -2ik\int_V dv_1 \frac{G_o(p,p_1)}{r_1}\frac{\partial}{\partial r_1}\left[r_1 e^{-ikr_1} u_k(p_1,p_o)\right]$$

$$+\frac{1}{4\pi}\int_B d\sigma_B \frac{e^{-ikr_B + ikR(p_B,p_o)}}{R(p_B,p_o)}\frac{\partial}{\partial n} G_o(p,p_B).$$

Here u_k is the regular part of G_k, and $\tilde{u} = e^{-ikr}u_k$, $R(p,p_1)$ denotes the distance between points p and p_1, dv_1 the volume element, $d\sigma$ the surface element, $\partial/\partial n$ the normal derivative directed out of V, and k is the complex wave number.

An integral equation for the corresponding Neumann problem was given by Ar and Kleinman [2] , that is, if G_o is the potential Neumann Green's function for surface B and if G_k (with the regular part u_k) is the Neumann Green's function for the Helmholtz equation then

$$\tilde{u}_p = -2ik \int_V dv_1 \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 \, \tilde{u}(p_1) \right]$$

$$+ ik \int_B d\sigma_B \, G_o(p,p_B) \, \hat{n} \cdot \hat{r}_B \, \tilde{u}(p_B)$$

$$- \int_B d\sigma_B \, G_o(p,p_B) \, e^{-ikr_B} \frac{\partial u(p_B)}{\partial n} .$$

with the same notation used above.

We write the above equation for the Neumann problem in the operator form

$$\tilde{u}(p) = L \circ \tilde{u} + u^{(o)} ,$$

with

$$L = kL_1 = kO + kO_1 ,$$

where

$$\omega \longrightarrow O \circ \omega = -2i \int_V dv_1 \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 \omega(p_1) \right] ,$$

$$\omega \longrightarrow O_1 \circ \omega = i \int_B d\sigma_B \, G_o(p,p_B) \, \hat{n} \cdot \hat{r}_B \, \omega(p_B) ,$$

and

$$u^{(o)} = - \int_B d\sigma_B \, G_o(p,p_B) \, e^{-ikr_B} \frac{\partial u(p_B)}{\partial n} .$$

With this notation it was shown by Ar [1] that above equation can be solved iteratively in the following function space W consisting of functions $\omega: V \longrightarrow E^1$ such that

(a) $\omega \in C^2(V)$, $\omega \in C^1(\bar{V})$,

(b) ω is analytic on the closed unit disc, in the complex $z = 1/r$ plane, having the expansion

$$\omega = \sum_{n=0}^{\infty} f_n(\theta, \phi)\, z^{n+1}, \quad |z| \leq 1,$$

(c) $f_n(\theta,\phi) = \sum_{m=n}^{\infty} Y_m(\theta,\phi)$, where Y_m is an m^{th} order spherical harmonic, i.e.

$$Y_m(\theta,\phi) = \sum_{\ell=-m}^{m} A_{\ell m} P_m^{\ell}(\cos\theta)\, e^{i\ell\phi},$$

with the norm

$$\|\omega\| = \max_{p\in\bar{V}} |\omega(p)| + \max_{|z|\leq 1} |\omega(\theta,\phi, 1/z)|,$$

for $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

In what follows we shall define another norm on the space W. Then we shall show that the operator L is bounded in this norm. The rest of the analysis for solving the main problem is the same as given in [1]. We shall only do the analysis for the more complicated Neumann problem. The analysis for the Dirichlet problem is essentially the same. Applications and the consequences of the method have been indicated in [1], [2], and [3] and therefore will not be repeated here.

II

AN ALTERNATIVE SPACE FOR ITERATION

We recall that if ω is in our space W, then

$$\omega = \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^{n+1}} \quad , \quad r \geq \delta = 1$$

where

$$f_n = \sum_{m=n}^{\infty} Y_m^{(n)}(\theta,\phi) \quad , \quad Y_m^{(n)} = \sum_{\ell=-m}^{m} A_{\ell m}^{n} P_m^{\ell}(\cos\theta)e^{i\ell\phi} \quad .$$

Implicit in this definition is the fact that the series converge absolutely and uniformly and derivatives with respect to θ and ϕ also converge absolutely and uniformly. Thus, there exists a constant M such that

$$\sum_{m=n}^{\infty} \left| Y_m^{(n)} \right| < M \quad \text{for all } \theta \text{ and } \phi \ .$$

Note, here and all that follows $Y_m^{(n)}$ denote a m^{th} order spherical harmonic (depending on n). With this in mind, we define the following function mapping W into E^1.

$$u \rightarrow \| u \| = \max \left\{ \max_{p\in\bar{V}} |u(p)| , \max_{\substack{0\leq\theta\leq\pi \\ 0\leq\phi\leq 2\pi \\ r\geq b}} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left| Y_m^{(n)}(\theta,\phi) \right|}{r^{n+1}} \right\}$$

$$= \max \left\{ \max_{p\in\bar{V}} |u| , \max_{\theta,\phi} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left| Y_m^{(n)} \right|}{b^{n+1}} \right\} , \qquad (1.1)$$

where b is a constant such that $b > \delta = 1$.

Lemma 1 Function defined by (1) is a norm.

Proof

Since $\max\left(\max_{p\in\overline{V}}|u|, \chi\right) \geq \max_{p\in\overline{V}}|u|$ for all χ, and since $\max_{p\in\overline{V}}|u| > 0$ unless $u \equiv 0$, it follows that

$$\|u\| > 0 \quad \text{unless} \quad u \equiv 0. \tag{2}$$

We have $|\alpha u| = |\alpha|\,|u|$ for any complex number α. Therefore,

$$\max_{p\in\overline{V}} |\alpha u| = |\alpha| \max_{p\in\overline{V}} |u| \tag{3}$$

we also have

$$\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{\left|\alpha Y_m^{(n)}\right|}{r^{n+1}} = |\alpha|\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{\left|Y_m^{(n)}\right|}{r^{n+1}} .$$

Therefore,

$$\max_{\theta,\phi}\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{\left|\alpha Y_m^{(n)}\right|}{b^{n+1}} = |\alpha| \max_{\theta,\phi}\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{\left|Y_m^{(n)}\right|}{b^{n+1}} . \tag{4}$$

From (3) and (4), it follows that

$$\max\left\{\max_{p\in\overline{V}}|\alpha u|,\ \max_{\theta,\phi}\sum\sum\frac{\left|\alpha Y_m^{(n)}\right|}{b^{n+1}}\right\} =$$

$$= |\alpha| \max\left\{\max_{p\in\overline{V}}|u|,\ \max_{\theta,\phi}\sum\sum\frac{\left|Y_m^{(n)}\right|}{b^{n+1}}\right\} ,$$

or

$$\|\alpha u\| = |\alpha|\ \|u\| . \tag{5}$$

For u and v in W let

$$u = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{Y_m^{(n)}(\theta,\phi)}{r^{n+1}} \quad , \quad r \geq \delta = 1$$

$$v = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{Z_m^{(n)}(\theta,\phi)}{r^{n+1}} \quad , \quad r \geq \delta = 1 \quad .$$

Since $|u + v| \leq |u| + |v|$,

$$\max_{p\in\overline{V}} |u+v| \leq \max_{p\in\overline{V}} |u| + \max_{p\in\overline{V}} |v| \; . \tag{6}$$

Also,

$$\sum_{m=n}^{\infty} \left|Y_m^{(n)} + Z_m^{(n)}\right| \leq \sum_{m=n}^{\infty} \left|Y_m^{(n)}\right| + \sum_{m=n}^{\infty} \left|Z_m^{(n)}\right| \; . \tag{7}$$

From (7) it follows that

$$\max_{\theta,\phi} \left[\sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left|Y_m^{(n)} + Z_m^{(n)}\right|}{b^{n+1}} \right] \leq \max_{\theta,\phi} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left|Y_m^{(n)}\right|}{b^{n+1}} + \max_{\theta,\phi} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left|Z_m^{(n)}\right|}{b^{n+1}} \; . \tag{8}$$

From (6) and (8) it follows that

$$\|u + v\| \leq \|u\| + \|v\| \; . \tag{9}$$

To justify the last step we must show that if A, B, C, D, are non-negative constants then

$$\max(A+B,\ C+D) \leq \max(A,\ C) + \max(B,\ D)\ . \tag{10}$$

Let $\max(A+B,\ C+D) = A+B$.

Then since

$$A \leq \max(A,\ C)$$

and

$$B \leq \max(B,\ D)\ ,$$

we have

$$A + B \leq \max(A,\ C) + \max(B,\ D)\ .$$

This is sufficient to prove (10). The validity of (10) justifies the last step (9). With (2), (5) and (9) it follows that (1) defines a norm, proving the lemma.

Lemma 2

If $u \in W$ then $\left| u - \frac{f_o}{r} \right| \leq \frac{C}{r^2} \|u\|$, for all $p \in \bar{V}$, where f_o is the first coefficient in the expansion for u, and C a positive constant.

Proof

We have for $u \in W$

$$u = \sum_{n=0}^{\infty} \frac{f_n}{r^{n+1}}\ , \qquad r \geq \delta = 1\ ;$$

also,

$$\max_{\substack{\theta,\phi \\ r \geq b}} \sum_{n=0}^{\infty} \frac{|f_n|}{r^{n+1}} = \max_{\theta,\phi} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \left| \sum_{m=n}^{\infty} Y_m^{(n)} \right| \leq$$

$$\leq \max_{\theta,\phi} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left| Y_m^{(n)} \right|}{b^{n+1}}\ .$$

Hence, with the definition of our norm (1) it follows that

$$\max_{\theta,\phi} \sum_{n=0}^{\infty} \frac{|f_n|}{b^{n+1}} \leq \|u\| \quad . \tag{11}$$

If $r \leq b$, then since $|f_o| \leq \|u\|$ by (11)

$$\begin{aligned} \left|u - \frac{f_o}{r}\right| &\leq |u| + \frac{|f_o|}{r} \leq \|u\| \frac{\|u\|}{r} \leq \frac{b}{r} \|u\| + \frac{1}{r} \|u\| = \\ &= \frac{b}{r^2} (1 + b) \|u\| \quad . \end{aligned} \tag{12}$$

If $r \geq b$, then

$$\begin{aligned} \left|u - \frac{f_o}{r}\right| &= \left| \sum_{n=1}^{\infty} \frac{|f_n|}{r^{n+1}} \right| \leq \sum_{n=1}^{\infty} \frac{|f_n|}{r^{n+1}} = \\ &= \frac{1}{r^2} \sum_{n=1}^{\infty} \frac{|f_n|}{r^{n-1}} \leq \frac{1}{r^2} \sum_{n=1}^{\infty} \frac{|f_n|}{b^{n-1}} \\ &= \frac{b^2}{r^2} \sum_{n=1}^{\infty} \frac{|f_n|}{b^{n+1}} \leq \frac{b^2}{r^2} \sum_{n=0}^{\infty} \frac{|f_n|}{b^{n+1}} \leq \frac{b^2}{r^2} \|u\|, \text{ by (11)} \end{aligned} \tag{13}$$

From (12) and (13) we conclude that

$$\left|u - \frac{f_o}{r}\right| \leq \frac{C}{r^2} \|u\| \quad , \quad \text{for all } p \in \overline{V} \quad , \tag{14}$$

where C is a positive constant. Thus proving the lemma.

Bounding the Operator L.

We recall that

$$L = kL_1 = kO + kO_1 \tag{15}$$

with

$$\omega \longrightarrow O\bullet\omega = -2i \int_V dv_1 \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 \omega(p_1) \right] \tag{16}$$

and

$$\omega \longrightarrow O_1 \bullet \omega = i \int_B d\sigma_B G_o(p,p_B)\, \hat{n}\bullet\hat{r}_B \omega(p_B) \; . \tag{17}$$

From (16) integrating by parts with respect to r_1 once we obtain

$$O\bullet\omega = 2i \int_B d\sigma_B G_o(p,p_B) \left[\omega(p_B) - \frac{f_o}{r_B} \right] + $$
$$+ 2i \int_V dv_1 \frac{1}{r_1} \frac{\partial}{\partial r_1} \left[r_1 G_o(p,p_1) \right] \left[\omega(p_1) - \frac{f_o}{r_1} \right] = O_2 \bullet \omega + O_3 \bullet \omega, \tag{18}$$

thus splitting the operator O into two parts with

$$\omega \longrightarrow O_2 \bullet \omega = 2i \int_B d\sigma_B G_o(p,p_B) \left[\omega(p_B) - \frac{f_o}{r_B} \right] \tag{19}$$

and

$$\omega \longrightarrow O_3 \bullet \omega = 2i \int_V dv_1 \frac{1}{r_1} \frac{\partial}{\partial r_1} \left[r_1 G_o(p,p_1) \right] \left[\omega(p_1) - \frac{f_o}{r_1} \right] \quad . \tag{20}$$

We now proceed to show in the following lemmas that the operators O_1, O_2 and O_3 ~~and therefore~~ $L_1 = O + O_1 = O_1 + O_2 + O_3$ is bounded in norm (1).

Lemma 3

The operator O_3 (Eq. (20)) is bounded.

Proof

From the Eq. (20) and the estimate of the Lemma 2, it follows that for some constant $C > 0$,

$$\left|O_3 \cdot \omega\right| \leq C \frac{\|\omega\|}{r} \int_V dv_1 \frac{1}{r_1^2} \left| \frac{\partial}{\partial r_1} \left(r_1 G_o (p, p_1)\right)\right| . \tag{21}$$

The integrand is of $O\left(\frac{1}{R(p, p_1)}\right)$ as $R \longrightarrow 0$, and of $\left(\frac{r}{r_1^4}\right)$ as $r_1 \longrightarrow \infty$, hence for some constant $C_1 > 0$.

$$\max_{p \epsilon \bar{V}} \left|O_3 \cdot \omega\right| \leq C_1 \|\omega\| . \tag{22}$$

Next, in order to show that $\|O_3 \cdot \omega\| < C \|\omega\|$ for some constant $C > 0$ we must show (see definition of norm (1)) that if

$$O_3 \cdot \omega = \sum_{n=0}^{\infty} \frac{g_n(\theta, \phi)}{r^{n+1}} , \quad r \geq \delta = 1 \tag{23}$$

and

$$g_n = \sum_{m=n}^{\infty} z_m^{(n)} , \tag{24}$$

then

$$\max_{\theta,\phi} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left|Z_m^{(n)}\right|}{b^{n+1}} \leq C_2 \|\omega\| \quad , \tag{25}$$

for some constant $C_2 > 0$ $(b > \delta = 1)$. Note that $O_3 \circ \omega$ has the expansion indicated by (23), (24) and (25) because we have already shown that L and all the other operators involved map the space W into itself.

Now we proceed to carry out the quite tedious task of showing (25). First, neglecting the constants

$$O_3 \cdot \omega = \int_V dv_1 \frac{1}{r_1} \frac{\partial}{\partial r_1} \left[r_1 G_o\right] \left[\omega - \frac{f_o}{r_1}\right]$$

$$= \int_V dv_1 \frac{1}{r_1} (\omega - \frac{f_o}{r_1}) \frac{\partial}{\partial r_1} \left[r_1 u_o\right] - \frac{1}{4\pi} \int_V dv_1 \frac{1}{r_1} (\omega - \frac{f_o}{r_1}) \frac{\partial}{\partial r_1} (\frac{r_1}{R}) \tag{26}$$

where u_o is the regular part of the static Green's function. Consider the first integral. For $r \geq \delta = 1$

$$\int_V dv_1 \frac{1}{r_1}(\omega - \frac{f_o}{r_1}) \frac{\partial}{\partial r_1} (r_1 u_o) =$$

$$= \int_{V_{int}} dv_1 \frac{1}{r_1} (\omega - \frac{f_o}{r_1}) \frac{\partial}{\partial r_1} \left[r_1 \sum_{n=0}^{\infty} \frac{Y_n(\theta,\phi;p_1)}{r^{n+1}}\right] + \tag{27}$$

$$+ \int_{V_{ext}} dv_1 \frac{1}{r_1} (\omega - \frac{f_o}{r_1}) \frac{\partial}{\partial r_1} \left[r_1 \sum_{n=0}^{\infty} \frac{Y_n(\theta,\phi\, ; \,\theta_1,\phi_1)}{(rr_1)^{n+1}}\right]$$

where V_{int} is the volume exterior to the surface B and interior to the sphere of radius $\delta = 1$, V_{ext} is the volume exterior to that sphere. The expansions in spherical harmonics for the Static Green's function are given previously.

Since the integration with respect to p_1 does not change the fact that the integrals are still n^{th} order spherical harmonics, we must show that

$$\max_{\theta,\phi} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \left| \int_{V_{int}} dv_1 \frac{1}{r_1} \left(\omega - \frac{f_o}{r_1}\right) \frac{\partial}{\partial r_1} \left[r_1 Y_n(\theta,\phi;p_1)\right] - \int_{V_{ext}} dv_1 \frac{1}{r_1} \left(\omega - \frac{f_o}{r_1}\right) \frac{n}{r_1^{n+1}} Y_n(\theta,\phi;\theta_1,\phi_1) \right| \tag{28}$$

$$= \max_{\theta,\phi} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \sum_{m=n}^{\infty} \left| Z_m^{(n)} \right| \leq K_1 \|\omega\| , \text{ some constant } K_1 > 0.$$

Clearly,

$$\sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \left| \int_{V_{int}} dv_1 \frac{1}{r_1} \left(\omega - \frac{f_o}{r_1}\right) \frac{\partial}{\partial r_1} \left[r_1 Y_n(\theta,\phi;p_1)\right] - \int_{V_{ext}} dv_1 \left(\omega - \frac{f_o}{r_1}\right) \frac{n}{r_1} \frac{Y_n(\theta,\phi;\theta_1,\phi_1)}{r_1^{n+1}} \right| \leq \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \cdot \tag{29}$$

$$\cdot \left\{ \int_{V_{int}} dv_1 \frac{1}{r_1} \left| \omega - \frac{f_o}{r_1} \right| \left| \frac{\partial}{\partial r_1} (r_1 Y_n) \right| + \int_{V_{ext}} dv_1 \frac{n}{r_1} \left| \omega - \frac{f_o}{r_1} \right| \frac{|Y_n|}{r_1^{n+1}} \right\} .$$

Note the Y_n in both integrals arise from the regular part of the static Green's function. That is

$$u_o = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi; p_1)}{r^{n+1}} \qquad \text{for} \quad r \geq \delta = 1$$

$$= \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi; \theta_1 \phi_1)}{(r r_1)^{n+1}} \qquad \text{for } r, r_1 \geq \delta = 1 \,. \tag{30}$$

The fact that the series converges absolutely and remains absolutely convergent after multiplying by r_1 and differentiating with respect to r_1 means that

$$\left| \frac{\partial}{\partial r_1} \left[r_1 Y_n(\theta, \phi; p_1) \right] \right| < A\, \delta^{n+1} = A \tag{31}$$

for some constant A independent of n, θ, ϕ, and p_1. Similarly, there is a constant B such that

$$\left| Y_n(\theta, \phi; \theta_1, \phi_1) \right| < B\, \delta^{2n} = B \quad . \tag{32}$$

Now using the estimate in the Lemma 2, Eqs. (31) and (32), we obtain

$$\int_{V_{int}} \frac{1}{r_1} \left| \omega - \frac{f_o}{r_1} \right| \left| \frac{\partial}{\partial r_1} (r_1 Y_n) \right| dv_1 \leq C \|\omega\| \cdot$$

$$\cdot\, A \int_{V_{int}} \frac{1}{r_1} \sin\theta_1\, dr_1\, d\theta_1\, d\phi_1 \leq D \|\omega\| \,, \tag{33}$$

for some constant $D > 0$. And, similarly

$$\int_{V_{ext}} \frac{n}{r_1} \left| \omega - \frac{f_o}{r_1} \right| \frac{\left| Y_n(\theta,\phi;\theta_1,\phi_1) \right|}{r_1^{n+1}} \, dv_1 \le C \, \|\omega\| \cdot$$

$$\cdot \, B \int_{V_{ext}} \frac{n}{r_1^{n+4}} dv_1 \le E \, \|\omega\| \, , \tag{34}$$

for some constant $E > 0$. With the help of (33) and (34) we obtain

$$\max_{\theta,\phi} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \left| \int_{V_{int}} dv_1 \frac{1}{r_1} (\omega - \frac{f_o}{r_1}) \frac{\partial}{\partial r_1} \left[r_1 Y_n(\theta,\phi;p_1) \right] - \right.$$

$$\left. - \int_{V_{ext}} dv_1 \frac{1}{r_1} (\omega - \frac{f_o}{r_1}) \frac{n}{r_1^{n+1}} \cdot Y_n(\theta,\phi;\theta_1,\phi_1) \right| \le \sum_{n=0}^{\infty} \frac{C_3 \|\omega\|}{b^{n+1}} \tag{35}$$

$$= \frac{\cancel{C_3 \|\omega\|}}{b-1} = C_4 \, \|\omega\| \, ,$$

C_3, C_4 are appropriate positive constants; we also recall that $b > \delta = 1$.

This completes the first part (for the regular part of static Green's function).

Next we go on to the singular part of the static Green's fucntion (second integral in (26)).

$$\frac{1}{4\pi}\int_V \frac{1}{r_1}\left(\omega-\frac{f_o}{r_1}\right)\frac{\partial}{\partial r_1}\left(\frac{r_1}{R}\right) dv_1 =$$

$$= \frac{1}{4\pi}\int_V \frac{1}{r_1}\left(\omega-\frac{f_o}{r_1}\right)\frac{\partial}{\partial r_1}\left\{ r_1 \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\gamma)\right\} dv_1 \quad , \tag{36}$$

where $\cos\gamma = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)$. For $r \geq \delta = 1$ this becomes

$$\frac{1}{4\pi}\int_{V_{int}} \frac{1}{r_1}\left(\omega-\frac{f_o}{r_1}\right)\frac{\partial}{\partial r_1}\sum_{n=0}^{\infty}\frac{r_1^{n+1}}{r^{n+1}} P_n(\cos\gamma) \; +$$

$$+ \frac{1}{4\pi}\int_1^r dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \, r_1 \left(\omega-\frac{f_o}{r_1}\right) \cdot$$

$$\cdot \sin\theta_1 \frac{\partial}{\partial r_1}\left\{\sum_{n=0}^{\infty}\frac{r_1^{n+1}}{r^{n+1}} P_n(\cos\gamma)\right\} + \tag{37}$$

$$+ \frac{1}{4\pi}\int_r^{\infty} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 r_1 \sin\theta_1 \left(\omega-\frac{f_o}{r_1}\right)\frac{\partial}{\partial r_1}\left\{\sum_{n=0}^{\infty}\frac{r^n}{r_1^n} P_n(\cos\gamma)\right\} .$$

For $r \geq \delta = 1$, however,

$$\omega - \frac{f_o}{r_1} = \sum_{n=1}^{\infty} \frac{f_n(\theta_1, \phi_1)}{r_1^{n+1}} \quad .$$

So (37) may be written as

$$\frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{n+1}{r^{n+1}} \int_{V_{int}} \frac{1}{r_1}(\omega - \frac{f_o}{r_1}) r_1^n P_n(\cos\gamma) + \frac{1}{4\pi}\int_1^r dr_1 \int_0^{2\pi}\int_0^{\pi} \times$$

$$\times \sin\theta_1 d\theta_1 d\phi_1 \sum_{m=1}^{\infty} \frac{f_m(\theta_1, \phi_1)}{r_1^m} \sum_{n=0}^{\infty} \frac{r_1^n}{r^{n+1}} (n+1) P_n(\cos\gamma) + \tag{38}$$

$$+ \frac{1}{4\pi}\int_r^{\infty} dr_1 \int_0^{2\pi}\int_0^{\pi} \sin\theta_1 d\theta_1 d\phi_1 \sum_{m=1}^{\infty} \frac{f_m(\theta_1, \phi_1)}{r_1^m} \sum_{n=0}^{\infty} \frac{(-n) r^n}{r_1^{n+1}} P_n(\cos\gamma).$$

Recall that

$$f_m(\theta_1, \phi_1) = \sum_{\ell=m}^{\infty} Y_\ell^{(m)}(\theta_1, \phi_1) \quad ,$$

so that

$$\int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \sin\theta_1 f_m(\theta_1, \phi_1) P_n(\cos\gamma) = 0, \quad n < m.$$

$$= \frac{4\pi}{2n+1} Y_n^{(m)}(\theta, \phi), \quad n \geq m.$$

Hence from (38) we have

$$
\begin{aligned}
&\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(n+1)}{r^{n+1}}\int_{V_{int}} dv_1\,\frac{1}{r_1}\left(\omega-\frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma)+\int_1^r dr_1\sum_{m=1}^{\infty}\,\times\\
&\times\sum_{n=m}^{\infty}\frac{Y_n^{(m)}(\theta,\phi)}{(2n+1)}\,\frac{r_1^{n+1}}{r^{n+1}}\,(n+1)-\int_r^{\infty} dr_1\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{Y_n^{(m)}(\theta,\phi)}{2n+1}\,\times\\
&\times\frac{nr^n}{r_1^{n+m+1}}=\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{n+1}{r^{n+1}}\int_{V_{int}} dv_1\left(\omega-\frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma)\;+\\
&+\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{Y_n^{(m)}(\theta,\phi)}{2n+1}\,\frac{n+1}{r^{n+1}}\,\frac{(r^{n-m+1}-1)}{(n-m+1)}-\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{Y_n^{(m)}(\theta,\phi)nr^n}{(2n+1)(n+m)^{n+m}}\;.
\end{aligned}
\tag{39}
$$

Now we reorder the last two sums to put them into the "right" form. These sums may be written as

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{Y_n^{(m)}(\theta,\phi)}{r^m(2n+1)}\left(\frac{n+1}{n-m+1}-\frac{m}{m+n}\right)=\\
&-\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{Y_n^{(m)}(\theta,\phi)(n+1)}{r^{n+1}(2n+1)(n-m+1)}\;.
\end{aligned}
\tag{40}
$$

The first term can be written, changing the roles of m and n and starting the first sum at zero as

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} \frac{Y_m^{(n+1)}(\theta,\phi)}{(2m+1)r^{n+1}} \left(\frac{m+1}{m-1} - \frac{m}{n+m+1}\right) = \\
&= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} \frac{Y_m^{(n+1)}(\theta,\phi)(n+1)}{r^{n+1}(m-n)(n+m+1)} \, .
\end{aligned}
\tag{41}
$$

Next, the second term in (40) can be handled similarly; thus

$$
\sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \frac{Y_n^{(m)}(\theta,\phi)(n+1)}{r^{n+1}(2n+1)(n-m+1)} = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \frac{Y_n^{(n-m)}(\theta,\phi)(n+1)}{r^{n+1}(2n+1)(m+1)} \, . \tag{42}
$$

Thus, we can write the integral (36) involving the singular part of the static Green's function, with the help of (41) and (42) as

$$
\begin{aligned}
&\frac{1}{4\pi}\int_V dv_1 \frac{1}{r_1}\left(\omega - \frac{f_o}{r_1}\right)\frac{\partial}{\partial r_1}\left(\frac{r_1}{R}\right) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \left\{ \frac{n+1}{4\pi}\int_{V_{int}} dv_1 \; x \right. \\
&\quad x \; \frac{1}{r_1}\left(\omega - \frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma) + \sum_{m=n+1}^{\infty} \frac{Y_m^{(n+1)}(\theta,\phi)(n+1)}{(m-n)(n+m+1)} \; - \\
&\quad \left. - \sum_{m=0}^{n-1} \frac{Y_n^{(n-m)}(\theta,\phi)(n+1)}{(2n+1)(m+1)} \right\} ,
\end{aligned}
\tag{43}
$$

where the last term is identically zero for n = 0.

Our task is to show that for some constant $C_5 > 0$,

$$\max_{\theta,\phi} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \sum_{m=n}^{\infty} \left| Z_m^{(n)}(\theta,\phi) \right| \leq C_5 \, \|\omega\| \ , \tag{44}$$

where

$$Z_n^{(n)} = \frac{n+1}{4\pi} \int_{V_{int}} dv_1 \frac{1}{r_1} \left(\omega - \frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma) - \frac{n+1}{2n+1} \sum_{m=0}^{n-1} \frac{Y_n^{(n-m)}(\theta,\phi)}{m+1}$$

and

$$Z_m^{(n)} = \frac{Y_m^{(n+1)}(\theta,\phi)(n+1)}{(m-n)(m+n+1)} \ , \qquad m \geq n+1 \ .$$

Clearly

$$\sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \sum_{m=n}^{\infty} \left| Z_m^{(n)} \right| = \sum_{n=0}^{\infty} \frac{\left| Z_n^{(n)} \right|}{b^{n+1}} + \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \frac{\left| Z_m^{(n)} \right|}{b^{n+1}} \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{n+1}{4\pi b^{n+1}} \left| \int_{V_{int}} dv_1 \left(\omega - \frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma) \right| +$$

$$+ \sum_{n=0}^{\infty} \frac{n+1}{(2n+1)b^{n+1}} \left| \sum_{m=0}^{n-1} \frac{Y_n^{(n-m)}(\theta,\phi)}{m+1} \right| + \tag{45}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \sum_{m=n+1}^{\infty} \frac{Y_m^{(n+1)}(\theta,\phi)(n+1)}{(m-n)(m+n+1)} \ .$$

Consider the three terms on the right hand side of the inequality (45) separately. Firstly,

$$\left| \int_{V_{int}} \frac{1}{r_1}\left(\omega - \frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma) dv_1 \right| \leq \int_{V_{int}} \frac{1}{r_1} \left| \omega - \frac{f_o}{r_1} \right| r_1^n \left| P_n(\cos\gamma) \right| dv_1 .$$

But

$$\left| \omega - \frac{f_o}{r_1} \right| \leq \frac{C}{r_1^2} \, \|\omega\| , \qquad \left| P_n(\cos\gamma) \right| \leq 1 , \qquad 0 < r_1 \leq \delta = 1.$$

So

$$\left| \int_{V_{int}} dv_1 \frac{1}{r_1}\left(\omega - \frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma) \right| \leq Q_1 \, \|\omega\| , \tag{46}$$

where Q_1 is a positive constant independent of ω.

Notice that, with $b > \delta = 1$,

$$\sum_{n=0}^{\infty} \frac{n+1}{4\pi b^{n+1}} Q_1 \|\omega\| = \frac{Q_1 \|\omega\|}{4\pi b} \sum_{n=0}^{\infty} \frac{n+1}{b^n} =$$

$$= \frac{Q_1 \|\omega\|}{4\pi b} \frac{d}{d(\frac{1}{b})} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} = \frac{Q_1 \|\omega\|}{4\pi b} \frac{b^2}{(1-b)^2} \leq Q_2 \|\omega\| , \quad \text{some } Q_2 > 0. \tag{47}$$

Thus, with (47), (46) we have

$$\sum_{n=0}^{\infty} \frac{n+1}{4\pi b^{n+1}} \left| \int_{V_{int}} dv_1 \frac{1}{r_1}\left(\omega - \frac{f_o}{r_1}\right) r_1^n P_n(\cos\gamma) \right| \leq Q_2 \|\omega\| . \tag{48}$$

Secondly,

$$\sum_{n=1}^{\infty} \frac{(n+1)}{(2n+1)b^{n+1}} \left| \sum_{m=0}^{n-1} \frac{Y_n^{(n-m)}}{m+1} \right| \leq \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{(n+1)\left|Y_n^{(n-m)}\right|}{(2n+1)b^{n+1}(m+1)} \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{b^n} \sum_{m=n}^{\infty} \left|Y_m^{(n)}\right| \qquad \text{(replacing n by n - 1)} \tag{49}$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{b^n} \sum_{m=n}^{\infty} \left|Y_m^{(n)}\right| \leq b\,\|\omega\| \quad \text{(by definition of our norm).}$$

Thirdly,

$$\sum_{n=0}^{\infty} \frac{1}{b^{n+1}} \sum_{m=n+1}^{\infty} \frac{\left|Y_m^{(n+1)}\right|(n+1)}{(m-n)(m+n+1)} \leq \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{b^{n+1}} \left|Y_m^{(n+1)}\right| \leq$$

$$\leq b\,\|\omega\| \quad , \quad \left(\text{since } \frac{n+1}{(m-n)(m+n+1)} \leq 1 \right) \text{ as above.} \tag{50}$$

Combining (47), (49) and (50) we obtain the inequality (44) which completes the analysis for the singular part. Equation (44) with the inequality for the regular part, (35), yield (25). Finally, combining (25) with (22) we obtain

$$\|Q_3 \circ \omega\| \leq C\,\|\omega\| \quad , \tag{51}$$

where C is an appropriate constant, proving the Lemma 3.

Lemma 4

The operator O_2, Eq. (19), is bounded.

Proof

From the Eq. (19) it follows, with the estimate of Lemma 2, that for some constant $C > 0$,

$$\left|O_2 \circ \omega\right| \leq C\|\omega\| \int_B d\sigma_B \frac{\left|G_o(p,p_B)\right|}{r_B} \leqq C\|\omega\| \quad x$$

$$x \left\{ \int_B \frac{1}{4\pi r_B R(p,p_B)} d\sigma_B + \int_B \left|u_o(p,p_B)\right| \frac{1}{r_B} d\sigma_B \right\} . \tag{52}$$

The first term on the right is the potential of a single layer distribution of density $1/4\pi r_B$. Since $r_B \neq 0$ (the origin was taken with B) and the surface is smooth, closed and finite, this density is uniformly Hölder continuous which means that the potential is continuously differentiable for all points $p \epsilon \bar{V}$. The second term on the right hand side of (52) is the integral of a bounded function over a finite surface and, hence, is bounded. Thus, there exists a constant $N_1 > 0$ such that

$$\max_{p \epsilon \bar{V}} \left|O_2 \circ \omega\right| \leq N_1 \|\omega\| . \tag{53}$$

If

$$O_2 \circ \omega = \sum_{n=0}^{\infty} \frac{g_n(\theta,\phi)}{r^{n+1}} \quad , \quad r \geq \delta = 1$$

and

$$g_n = \sum_{m=n}^{\infty} Z_m^{(n)}$$

then by exactly the same long procedure of Lemma 3 it can be shown that

$$\max_{p \in \bar{V}} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left| Z_m^{(n)} \right|}{b^{n+1}} \leq N_2 \|\omega\| \tag{54}$$

for some constant $N_2 > 0$.

From (53), (54) and the definition of our norm, it now follows that

$$\left\| O_2 \circ \omega \right\| < N \|\omega\| \quad , \tag{55}$$

for some constant $N > 0$, proving the lemma.

Lemma 5

The operator O_1, Eq. (17), is bounded.

Proof

We have with Eq. (17)

$$\left| O_1 \circ \omega \right| \leq \int_B d\sigma_B \left| G_o(p, p_B) \right| \left| \hat{n} \cdot \hat{r}_B \right| \left| \omega(p_B) \right| \quad . \tag{56}$$

By definition

$$\left| \omega(p_B) \right| \leq \|\omega\| \quad . \tag{57}$$

Also, $\hat{n}$ and $\hat{r}_B$ are unit vectors,

$$\left| \hat{n} \cdot \hat{r}_B \right| \leq 1 \quad , \tag{58}$$

thus,

$$\left| O_1 \circ \omega \right| \leq \|\omega\| \int_B \left| G_o(p, p_o) \right| d\sigma_B \quad . \tag{59}$$

The surface integral is bounded by the same argument given in the first part of Lemma 4 (Eq. (52)). Hence, for some constant $A_1 > 0$

$$\max_{p \in \bar{V}} \left| O_1 \cdot \omega \right| \leq A_1 \|\omega\| \quad . \tag{60}$$

Furthermore, if

$$O_1 \cdot \omega = \sum_{n=0}^{\infty} \frac{g_n(\theta, \phi)}{r^{n+1}} \quad , \qquad r \geq \delta = 1 \tag{61}$$

and

$$g_n(\theta, \phi) = \sum_{m=n}^{\infty} Z_m^{(n)} \tag{62}$$

then, again, by exactly the same argument of Lemma 3, it follows that

$$\max_{\theta, \phi} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left| Z_m^{(n)} \right|}{b^{n+1}} \leq A_2 \|\omega\| \tag{63}$$

for some constant $A_2 > 0$.

From (60), (63) and the definition of the norm, it follows that

$$\left\| O_1 \cdot \omega \right\| \leq A \|\omega\| \tag{64}$$

for some constant $A > 0$, proving the lemma.

It now follows immediately from the Lemmas 3, 4 and 5 that:

Corollary. The operator $L = kL_1$ (Eq. (15)) is bounded; and there exists a complex number $k_o > 0$ such that for $|k| < |k_o|$, $\|L\| < 1$.

III

REFERENCES

[1] Ar, Ergun (1966), "On the Helmholtz Equation for an Acoustically Rigid Scatterer," The University of Michigan Radiation Laboratory Report 7359-1-T, NSF Grant GP-4581

[2] Ar, Ergun and R. E. Kleinman (1966), "The Exterior Neumann Problem for the Three-dimensional Helmholtz Equation," <u>Arch. Rat. Mech. Anal.</u> <u>23</u>, pp. 218-236.

[3] Kleinman, R. E. (1965), "The Dirichlet Problem for the Helmholtz Equation," <u>Arch. Rat. Mech. Anal.</u> <u>18</u>, No. 3, pp. 205-229.