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INTEGRAL EQUATIONS APPROPRIATE TO CURRENT'S INDUCED ON A PERTURBATION OF A CONDUCTING SURFACE

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ABSTRACT

An integral equation useful for computations of currents induced on a perturbed conducting surface by an incident plane wave is developed, where the perturbation is of stub or protuberance type. However unlike the usual approach, the free space Green's function is not used. Instead a dyadic type Green's function whose tangential components vanish everwhere on the unperturbed surface is employed. In this way the resulting integral equation involves an integral taken only over the perturbed surface. An explicit expression is developed for the kernal of the integral equation for the case of the unperturbed surface being a sphere. A great simplication is achieved in the integral equation approach for a special class of perturbations on a sphere. For these cases, the vector field can be represented in terms of two scalars; and the resulting vector integral equation reduced to two sets of coupled scalar integral equations, taken only over the surface of the perturbation.

I

INTEGRAL EQUATION FOR A GENERAL PERTURBED SURFACE

The integral equation for the currents on the perturbed portion of a general perturbed surface will be developed in this section. The surfaces under consideration will be perfectly conducting, and the perturbations that are considered, will be limited to protuberance types. Define the following:

S_o : surface of unperturbed perfectly conducting shape

S_1 : surface of perturbation or protuberance

S : surface composed of S_1 and the unperturbed portion of S_o .

Let $(\underline{E}, \underline{H})$ be the total Maxwellian field (harmonic time dependence $\exp(-i\omega t)$ assumed), which are generated by a plane wave $(\underline{E}^i, \underline{H}^i)$ incident upon the perfectly conducting shape S. The incident field will be given explicitly by the relation

$$\underline{E}^i = \hat{\underline{x}} e^{+ikz} \quad . \tag{1}$$

In the far field $(R \rightarrow \infty)$, the total field $\underline{E} = \underline{E}^i + \underline{E}^s$ will have the form

$$\left.\begin{aligned} &\underline{E} \sim \hat{\underline{x}} e^{+ikz} + \frac{e^{ikR}}{R} \underline{f}(\theta, \phi) \\ &\text{and} \\ &\sqrt{\frac{\mu_o}{\epsilon_o}}\, \underline{H} \sim \hat{\underline{y}} e^{+ikz} + \frac{e^{ikR}}{R} \hat{\underline{R}} \times \underline{f}(\theta, \phi) \quad . \end{aligned}\right\} \tag{2}$$

Let $(\underline{\tilde{E}}, \underline{\tilde{H}})$ be the total Maxwellian fields which are generated by a magnetic dipole at $\underline{R}_o$, in the presence of the unperturbed surface S_o. The orientation of the dipole will be given by the unit vector $\underline{m}$. The component of this field, due to the source only, has the form

$$\underline{\tilde{E}}^i = \underline{\nabla} \times \frac{e^{ikr}}{r}\, \underline{m}, \quad \text{where } \underline{r} = \left|\underline{R} - \underline{R}_o\right| \tag{3}$$

and in the far field the total field $(\underline{\tilde{E}}, \underline{\tilde{H}})$ has the form

$$\left.\begin{aligned} \underline{\tilde{E}} &\sim \frac{e^{ikR}}{R}\, \underline{\tilde{f}}(\theta, \phi; \underline{R}_o) \\ \sqrt{\frac{\mu_o}{\epsilon_o}}\, \underline{\tilde{H}} &\sim \frac{e^{ikR}}{R}\, \underline{\hat{R}} \times \underline{\hat{f}}(\theta, \phi; R_o) \; . \end{aligned}\right\} \tag{4}$$

Employing the well-known Lorentz lemma, the following integral relation connecting the two fields, is obtained

$$\int_{S+S_\infty+\Sigma} \underline{n} \cdot (\underline{E} \times \underline{\tilde{H}} - \underline{\hat{E}} \times \underline{H})\, ds = 0 \tag{5}$$

where the surface comprises three separate surfaces, S, S_∞ which is a sphere of infinite radius, and Σ a small sphere enclosing the dipole source at $\underline{r}_o$. The vector $\underline{n}$ is the unit outward normal to the surfaces. (See figure 1).

Since $\underline{n} \times \underline{\tilde{E}}$ vanishes on S_o, and $\underline{n} \times \underline{E}$ vanishes on S, the above integral reduces to

$$\int_{S_1} \underline{n} \cdot \underline{\tilde{E}} \times \underline{H}\, ds = \int_{\Sigma + S_\infty} \underline{n} \cdot \left[\underline{E} \times \underline{\tilde{H}} - \underline{\tilde{E}} \times \underline{H}\right] ds \tag{6}$$

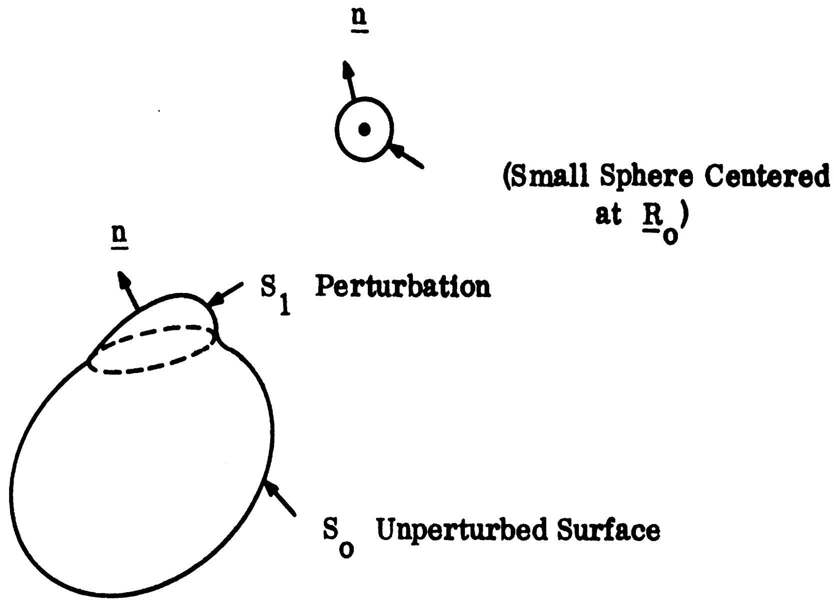

FIG. 1: GEOMETRY OF SURFACE

The integral taken over the sphere of infinite radius can be evaluated as follows. Explicitly it is given by the relation

$$\sqrt{\frac{\mu_0}{\epsilon_0}} \lim_{R\to\infty} \int_0^{2\pi} \int_0^{\pi} R^2 \sin\theta \left[-\tilde{E}_\phi H_\theta + E_\phi \tilde{H}_\theta - E_\theta \tilde{H}_\phi + \tilde{E}_\theta H_\phi\right] d\theta\, d\phi$$

$$= - \lim_{R\to\infty} \int_0^{2\pi} \int_0^{\pi} d\theta\, d\phi \left[\cos\phi \tilde{f}_\theta - \tilde{f}_\phi \sin\phi\right] R \sin\theta (1-\cos\theta) \exp\left[ikR(1+\cos\theta)\right]$$

$$= - \lim_{R\to\infty} \frac{-i}{k} \int_0^{2\pi} d\phi \int_0^{\pi} (1-\cos\theta) \left[\cos\phi \tilde{f}_\theta + \tilde{f}_\phi \sin\phi\right] d \exp\left[ikR(1+\cos\theta)\right]$$

$$= -\frac{i2}{k} \int_0^{2\pi} d\phi \left[\cos\phi\, \tilde{f}_\theta - \tilde{f}_\phi \sin\phi\right]_{\theta=\pi}$$

$$= -\frac{4\pi i}{k} \hat{\underline{x}} \cdot \tilde{\underline{f}}(\theta=\pi;\, \underline{R}_0) \tag{7}$$

The integral taken over the small sphere of vanishing radius $r = \left|\underline{R} - \underline{R}_0\right|$ centered at $\underline{R}_0$ is given by

$$\lim_{r\to 0} \int \hat{\underline{r}} \cdot \left[\underline{E} \times \tilde{\underline{H}}^i - \tilde{\underline{E}}^i \times \underline{H}\right] r^2\, d\Omega \tag{8}$$

where $\underline{E}^i$ is given by Eq. (3). For r small it can be shown that

$$\underline{E}^i \sim - \frac{(\hat{\underline{r}} \times \underline{m})}{r^2} \quad ,$$

$$i\omega\mu_o\, \underline{H}^i \sim \frac{[3\hat{\underline{r}}(\underline{m} \cdot \hat{\underline{r}}) - \underline{m}]}{r^3} \quad ,$$

in which case expression (8) reduces to the form

$$\lim_{r \to 0} \int \hat{\underline{r}} \cdot \left[(\hat{\underline{r}} \times m) \times \underline{H} + i \, \frac{\underline{E} \times \underline{m}}{r\omega\mu_o} \right] d\Omega \, .$$

To evaluate the integral local spherical polar coordinates (r, θ', ϕ') are used, with the z-axis being in the direction of the vector $\underline{m}$. The fields at a point (r, θ', ϕ') on the surface are expanded out in a Taylor type series about the origin. It follows that $\hat{\underline{r}} \cdot (\hat{\underline{r}} \times \underline{m}) \times \underline{H} = -(\underline{H} \cdot \underline{m}) + (\underline{r} \cdot \underline{m})(\underline{H} \cdot \hat{\underline{r}})$

$$= -(\underline{H} \cdot \underline{m}) \sin^2\theta' + \sin\theta' \cos\theta' \, (H_x \cos\phi' + H_y \sin\phi')$$

where the components of $\underline{H}$ are those given at the origin ie: $\underline{H} = \underline{H}(\underline{R}_o)$.
Similarly

$$\hat{\underline{r}} \cdot \frac{\underline{E} \times \underline{m}}{r} = \frac{\sin\theta'}{r} (\cos\phi' \, E_y - \sin\phi' \, E_x)$$

$$= \frac{\sin\theta'}{r} \left[\cos\phi' \, E_y(\underline{R}_o) - \sin\phi' \, E_x(\underline{R}_o) \right]$$

$$\sin\theta' \cos\phi' \left[\sin\theta' \cos\phi' \frac{\partial E_y}{\partial x} + \sin\theta' \sin\phi' \frac{\partial E_y}{\partial y} + \cos\theta' \frac{\partial E_y}{\partial z} \right]$$

$$- \sin\theta' \sin\phi' \left[\sin\theta' \cos\phi' \frac{\partial E_x}{\partial x} + \sin\theta' \sin\phi' \frac{\partial E_x}{\partial y} + \cos\theta' \frac{\partial E_x}{\partial z} \right] \quad .$$

The above integral reduces to

$$\int_0^{2\pi}\int_0^{\pi} (\sin\theta')^3 \, d\theta' d\phi' \left\{ -(\underline{H}\cdot\underline{m}) + \frac{i}{2\omega\mu_o}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\right\} = -4\pi \, \underline{H}(\underline{R}_o)\cdot\underline{m} \tag{9}$$

With these results the integral relation (6) reduces to

$$\underline{H}(\underline{R}_o)\cdot\underline{m} = -\frac{1}{4\pi}\int_{S_1} \underline{n}'\cdot\underline{\tilde{E}}(\underline{R}';\, \underline{R}_o)\times\underline{H}(\underline{R}')\,ds' - \frac{i}{k}\,\tilde{f}_x(\theta=\pi;\, \underline{R}_o)\sqrt{\frac{\epsilon_o}{\mu_o}} \,. \tag{10}$$

It is evident that the above can be expressed in the form

$$\underline{H}(\underline{R}_o)\cdot\underline{m} = \underline{H}^o(\underline{R}_o)\cdot\underline{m} + \frac{1}{4\pi}\int_{S_1} \underline{\tilde{E}}(\underline{R}';\, \underline{R}_o)\cdot\left[\underline{n}'\times\underline{H}(\underline{R}')\right]\, ds' \tag{11}$$

for $\underline{R}_o$ outside surface S. $\underline{H}^o(\underline{R}_o)$ is the field generated at the point $\underline{R}_o$ by the plane wave [given by Eq. (1)], in the presence of the unperturbed perfectly conducting surface S_o. The above equation indicates that if the solutions are available for the unperturbed surface (ie: $\underline{H}^o$ and $\underline{\tilde{E}}$ can be prescribed), then the field $\underline{H}$ at a point $\underline{R}_o$ generated by a plane wave incident upon the perturbed surfaces is given in terms of the current on the perturbed section.

An integral equation for this current is obtained by letting $\underline{R}_o$ approach the surface of the perturbation for tangential orientations of the dipole $\underline{m}$. It can be shown that when $\underline{R}_o$ is on S_1, the following equation is obtained

$$\underline{H}(\underline{R}_o)\cdot\underline{m}_t = 2\,\underline{H}^o(\underline{R}_o)\cdot\underline{m}_t + \frac{1}{2\pi}\int_{S_1} \underline{\tilde{E}}(\underline{R}';\, \underline{R}_o;\, \underline{m}_t)\cdot\left[\underline{n}'\times\underline{H}(\underline{R}')\right]\, ds' \tag{12}$$

where $\underline{m}_t$ is tangential to the surface.

II

APPLICATION TO A PERTURBATION ON A SPHERE

The special case will be considered, where the unperturbed surface S_o is a sphere of radius a,

From Stratton (p.565), the value of $\underline{H}^o(\underline{R}_o)$ corresponding to the total field generated at the point $\underline{R}_o$ by a plane wave incident on the perfectly conducting sphere can be obtained immediately as follows.

$$\sqrt{\frac{\mu_o}{\epsilon_o}}\;\underline{H}^o(\underline{R}) = \sum_{n=1} i^n \frac{2n+1}{n(n+1)}\left(b_n\,\underline{M}^{(3)}_{e1n} + i\,a_n\,\underline{N}^{(3)}_{o1n}\right) + \hat{\underline{y}}\;e^{ikz} \tag{13}$$

where

$$a_n = \frac{j_n(\rho)}{h^{(1)}_n(\rho)} \tag{14}$$

$$b_n = \frac{[\rho\, j_n(\rho)]'}{[\rho\, h^{(1)}_n(\rho)]'} \tag{15}$$

with $\rho = ka$.

To obtain an expression for the kernel of the integral Eq. (12), the electric field generated by the dipole at $\underline{R}_o$ is decomposed into two parts

$$\underline{\tilde{E}} = \underline{\tilde{E}}^i + \underline{\tilde{E}}^s \tag{16}$$

where $\underline{\tilde{E}}^i$ is the field of the dipole in free space, given by

$$\underline{\tilde{E}}^i = \underline{\nabla} \times \frac{e^{ikr}}{r}\;\underline{m}$$

and $\underline{\tilde{E}}^s$ is the scattered field due to the sphere. From Appendix A it is shown that $\underline{\tilde{E}}^i$ can be expressed in the form

$$\tilde{\underline{E}}^{i} = \underline{\nabla} \times \underline{\nabla} \times \underline{R}\, \tilde{\psi}^{i} + \underline{\nabla} \times \underline{R}\, \tilde{\chi}^{i} \tag{17}$$

where

$$\left.\begin{aligned} \tilde{\psi}^{i} &= \underline{m} \cdot \underline{\nabla}_{o} \times \underline{R}_{o}\, \pi \\ \tilde{\chi}^{i} &= \underline{m} \cdot \underline{\nabla}_{o} \times \underline{\nabla}_{o} \times \underline{R}_{o}\, \pi \end{aligned}\right\} \tag{18}$$

and

$$\pi = ik \sum_{n=1} \frac{(2n+1)}{n(n+1)} \begin{Bmatrix} j_n(kR)\, h_n^{(1)}(kR_o) \\ h_n^{(1)}(kR)\, j_n(kR_o) \end{Bmatrix} P_n(\cos\gamma) \qquad \begin{matrix} R < R_o \\ R > R_o \end{matrix} \tag{19}$$

with $\underline{R} \cdot \underline{R}_o = RR\cos\gamma$.

The scattered field can be written in a similar manner as follows:

$$\tilde{\underline{E}}^{s} = \underline{\nabla} \times \underline{\nabla} \times \underline{R}\, \tilde{\psi}^{s} + \underline{\nabla} \times \underline{R}\, \tilde{\chi}^{s} \quad . \tag{20}$$

The boundary condition on the total field

$$\hat{\underline{R}} \times \left[\tilde{\underline{E}}^{s} + \tilde{\underline{E}}^{i}\right] = 0, \qquad R = a .$$

can be expressed in the form

$$\tilde{\chi}^{i} + \tilde{\chi}^{s} = 0 \qquad R = a \tag{21}$$

$$\frac{\partial}{\partial R}\left\{R\left[\tilde{\psi}^{i} + \tilde{\psi}^{s}\right]\right\} = 0 \qquad R = a . \tag{22}$$

It is immediately evident that

$$\tilde{\chi}^{s} = ik\, \underline{m} \cdot \underline{\nabla}_{o} \times \underline{\nabla}_{o} \times \underline{R}_{o} \sum_{n=1} \frac{2n+1}{n(n+1)}\, a_n\, h_n^{(1)}(kR)\, h_n^{(1)}(kR_o) P_n(\cos\gamma) \tag{23}$$

$$\tilde{\psi}^s = -ik\,\underline{m}\cdot\underline{\nabla}_o \times \underline{R}_o \sum_{n=1} \frac{2n+1}{n(n+1)}\, b_n h_n^{(1)}(kR)\, h_n^{(1)}(kR_o) P_n(\cos\gamma) \tag{24}$$

where the constants a_n and b_n are given by Eqs. (14) and (15) respectively. The kernel of the integral Eq. (12) has the explicit form

$$\tilde{\underline{E}}(\underline{R}, \underline{R}_o) = \underline{\nabla}\times\underline{\nabla}\times\underline{R}(\tilde{\psi}^s + \tilde{\psi}^i) + \underline{\nabla}\times\underline{R}(\tilde{\chi}^s + \tilde{\chi}^i) \tag{25}$$

where the scalar functions are given by Eqs. (18), (19), (23) and (24).

Explicit forms have now been found for $\underline{H}^o(\underline{R})$ and $\tilde{\underline{E}}(\underline{R}; \underline{R}_o)$ associated with the integral equation for the currents $\underline{n} \times \underline{H}$ on the perturbation.

To complete the analysis the appropriate explicit expression for the far field should be obtained from Eq. (11). In this case the value of $\tilde{\underline{E}}(\underline{R}, \underline{R}_o)$ for $R_o \to \infty$ is needed. It can be shown that

$$\tilde{\underline{E}}^i(\underline{R}, \underline{R}_o) \underset{R_o\to\infty}{\sim} -\frac{e^{ikR_o - ikR\cos\gamma}}{R_o}\, ik\hat{\underline{R}}_o \times \underline{m}, \tag{26}$$

$$\tilde{\chi}^s \underset{R_o\to\infty}{\sim} (\underline{m}\cdot\underline{\alpha})\, k\frac{e^{ikR_o}}{R_o} \sum_{n=1} \frac{(2n+1)}{n(n+1)}\, a_n (-i)^{n+1} h_n^{(1)}(kR) P_n^{(1)}(\cos\gamma) \tag{27}$$

$$\tilde{\psi}^s \underset{R_o\to\infty}{\sim} -i\,(\underline{m}\cdot\underline{\beta})\frac{e^{ikR_o}}{R_o} \sum_{n=1} \frac{(2n+1)}{n(n+1)}\, b_n (-i)^{n+1} h_n^{(1)}(kR) P_n^{(1)}(\cos\gamma) \tag{28}$$

where

$$\underline{\alpha} = \frac{\left[\hat{\underline{\phi}}_o \dfrac{\partial\cos\gamma}{\sin\theta_o\,\partial\phi_o} + \hat{\underline{\theta}}_o \dfrac{\partial\cos\gamma}{\partial\theta_o}\right]}{\sin\gamma} \tag{29}$$

$$\underline{\beta} = \frac{\left[\underline{\hat{\theta}}_o \dfrac{\partial \cos\gamma}{\sin\theta_o \, \partial\phi_o} - \underline{\hat{\phi}}_o \dfrac{\partial \cos\gamma}{\partial\theta_o}\right]}{\sin\gamma} \tag{30}$$

with

$$\cos\gamma = \cos\theta \cos\theta_o + \sin\theta \sin\theta_o \cos(\phi - \phi_o) \quad .$$

When $R_o \longrightarrow \infty$, $\underline{\tilde{E}}(\underline{R}, R_o)$ is essentially the total field produced at $\underline{R}$ by a plane wave incident on the conducting sphere $R = a$.

To obtain the far scattered field expansion (11) is used with $R_o \longrightarrow \infty$ and the vector $\underline{m}$ taken to be in the $\underline{\hat{\theta}}_o$ or $\underline{\hat{\phi}}_o$ directions. The far scattered field is comprised of two parts, the scattered field produced by a plane wave on perfectly conducting sphere given by the term $\underline{H}^o(\underline{R}_o) \cdot \underline{m}$, and the scattered field due to the perturbation $\underline{H}_p(\underline{R}_o) \cdot \underline{m}$ where

$$\underline{H}_p(\underline{R}_o) \cdot \underline{m} = \frac{1}{4\pi} \int_{S_1} \underline{\tilde{E}}(\underline{R}', \underline{R}_o ; \underline{m}) \cdot \left[\underline{n} \times \underline{H}(R')\right] dS'$$

where $\underline{\tilde{E}}(R', R_o; \underline{m})$ for $R_o \longrightarrow \infty$ is found from Eqs. (25), (26), (27) and (28).

III

SPECIAL CLASSES OF PERTURBATION ON A SPHERE

Great simplication of the integral equation approach is achieved when the perturbation on the sphere belongs to a special class. This special class of perturbation is characterized by surfaces which are comprised of spherical caps R = constant and conical sections $f(\theta, \phi)$ = constant. Two examples of such surfaces are

(1) the conical stub: with cap $R = b$, and conical side $\theta = \theta_0$

(2) the flat plate: with cap $R = b$, and sides $\theta = \theta_0$, $\theta = \pi - \theta_0$, $\phi = \phi_0$ and $\phi = -\phi_0$.

For these types of perturbations, the total field ($\underline{E}$, $\underline{H}$) generated by the plane wave ($\underline{E}^i$, $\underline{H}^i$) incident on the perturbed sphere, can be represented in terms of two independent scalars as follows:

$$\underline{H} = \underline{\nabla} \times \underline{R}\psi + \frac{1}{k} \underline{\nabla} \times \underline{\nabla} \times \underline{R}\chi \tag{31}$$

$$-i\sqrt{\epsilon_0/\mu_0}\,\underline{E} = \frac{1}{k} \underline{\nabla} \times \underline{\nabla} \times R\psi + \underline{\nabla} \times \underline{R}\chi \quad . \tag{32}$$

Using the relations

$$\underline{\nabla} \times \underline{\nabla} \times \underline{R}\psi = \hat{\underline{R}}k^2 R\psi + \underline{\nabla}\left(\frac{\partial R\psi}{\partial R}\right),$$

$$\underline{\nabla} \times \underline{R}\chi = \underline{\nabla}\chi \times \underline{R} \ ,$$

it follows the boundary condition $\underline{n} \times \underline{E} = 0$, applied to a spherical cap R = constant, yields the conditions

$$\underline{\hat{R}} \times \underline{\nabla} \frac{\partial R\psi}{\partial R} = 0 \quad \text{and} \quad \underline{\nabla}\chi - \underline{\hat{R}}(\underline{\hat{R}} \cdot \underline{\nabla}\chi) = 0 \ .$$

Since both the operators $\underline{\hat{R}} \times \underline{\nabla}$ and $\left[\underline{\nabla} - \hat{R}(\underline{\hat{R}} \cdot \underline{\nabla})\right]$ imply differentiation along the surface, the above two conditions can be expressed in reduced form as follows

$$\frac{\partial\, R\psi}{\partial R} = 0 \ , \quad \text{and} \quad \chi = 0 \quad \text{for } R = \text{constant} \ .$$

In a similar manner it can be shown that the boundary condition $\underline{n} \times \underline{E} = 0$ applied to a conical portion $f(\theta, \phi) = c$ reduces to the conditions

$$\psi = 0 \quad \text{and} \quad \frac{\partial \chi}{\partial n} = 0$$

where $\frac{\partial}{\partial n}$ implies the normal derivative.

Thus, since the incident field can be decomposed in a similar form as Eq. (31), namely

$$\underline{H}^i = \underline{\nabla} \times \underline{R}\psi^i + \frac{1}{k} \underline{\nabla} \times \underline{\nabla} \times \underline{R}\chi^i \tag{33}$$

it follows that the vector problem can be reduced to the following two scalar problems:

(1) find $\psi = \psi^s + \psi^i$ such that

(a) ψ^i is the given component of the incident field,

(b) ψ^s satisfies the scalar Helmoltz equation and represents, an outgoing wave

(c) with boundary conditions

$$\left.\begin{aligned} &\frac{\partial\, R\psi}{\partial R} = 0 \quad \text{for } R = \text{constant} \ , \\ &\text{and} \\ &\psi = 0 \quad \text{for } f(\theta, \phi) = \text{constant} \ . \end{aligned}\right\} \tag{34}$$

(2) to find $\chi = \chi^i + \chi^s$ such that

(a) χ^i is the given component of the incident field,

(b) χ^s satisfies the scalar Helmoltz equation and the outgoing radiation condition,

(c) with the boundary conditions

$$\left.\begin{aligned} &\chi = 0 \quad \text{for } R = \text{constant}, \\ &\frac{\partial \chi}{\partial n} = 0 \quad \text{for} \quad f(\theta, \phi) = \text{constant} \quad . \end{aligned}\right\} \tag{35}$$

The above decomposition of the vector field in two scalar quantities will result in a simplication of the appropriate integral equation, the development of which follows. Define the following:

S_o : the surface of the unperturbed perfectly conducting sphere $R = a$

S_1 : the surface of the perturbation whcih will be comprised of two parts,

(a) C the sides given by the cone $f(\theta, \phi) = \text{constant}$, $a \leq R \leq b$, and

(b) T the cap given by $R = b$,

S : the surface comprised of S_1 and the unperturbed portion of S.

The scalar quantities χ and ψ associated with the total field $(\underline{E}; \underline{H})$ generated by the plane wave incident on S, must satisfy the following boundary conditions

$$\begin{aligned} &\left(\frac{\partial R\psi}{\partial R}\right)_a = 0 \ , \quad \chi = 0 \quad \text{for surface } S_o \\ &\left(\frac{\partial R\psi}{\partial R}\right)_b = 0 \ , \quad \chi = 0 \quad \text{for cap } T \\ &\psi = 0 \ , \quad \frac{\partial \chi}{\partial n} = 0 \quad \text{for sides } C \quad . \end{aligned} \tag{37}$$

In addition, the field $(\underline{E}^o, \underline{H}^o)$ (which is the total field generated by the plane incident on the unperturbed surface $S_o(R = a)$) and associated scalar quantitites χ^o and ψ^o will be introduced. It then follows that

$$\left.\begin{array}{l} \dfrac{\partial R\psi^o}{\partial R} = 0 \\ \chi^o = 0 \end{array}\right\} \text{for } R = a \quad . \tag{38}$$

To derive the appropriate integral equation for ψ, the Green's function $G_1(\underline{R}, \underline{R}_o)$ must be introduced, where

$$\nabla^2 G_1 + k^2 G_1 = -4\pi\,\delta\,(R - R_o) \tag{39}$$

and which satisfies the boundary condition

$$\frac{\partial R G_1}{\partial R} = 0 \quad \text{for} \quad R = a \ . \tag{40}$$

The Green's function has the explicit form

$$G_1(\underline{R}, \underline{R}_o) = \frac{e^{ikr}}{r} - ik \sum_{n=0} (2n+1)\, b_n h_n^{(1)}(kR)\, h_n^{(1)}(kR_o)\, P_n(\cos\gamma) \tag{41}$$

where $r = \left|\underline{R} - \underline{R}_o\right|$, $RR_o \cos\gamma = \underline{R} \cdot \underline{R}_o$, and the constant b_n is given by Eq. (15).

Integrating the relationship

$$\psi\,\nabla^2 G_1 - G_1\,\nabla^2\psi = -4\pi\,\psi\,\delta\,(\underline{R} - \underline{R}_o)$$

one obtains

$$\psi(\underline{R}_o) = \frac{1}{4\pi}\int_{S + S_\infty} \underline{n} \cdot (\psi\,\underline{\nabla} G_1 - G_1\,\underline{\nabla}\psi)\,dS \tag{42}$$

where S_∞ is an infinite sphere, and $\underline{n}$ is the inward normal to the enclosed volume.

Similarly it can be shown that

$$\psi^{o}(R_o) = \frac{1}{4\pi}\int_{S_o+S_\infty} \underline{n}\cdot(\psi^{o}\,\underline{\nabla}G_1 - G_1\,\underline{\nabla}\psi^{o})\,dS \;. \tag{43}$$

The integral over the surface S_o vanishes since G_1 and ψ^o satisfy the same boundary conditions on this surface. On combining Eqs. (42) and (43) one obtains

$$\psi(\underline{R}_o) = \psi^{o}(\underline{R}_o) + \frac{1}{4\pi}\int_{S_1}\left(\psi\frac{\partial G_1}{\partial n} - G_1\frac{\partial\psi}{\partial n}\right)dS$$

$$+\frac{1}{4\pi}\int_{S_\infty}\underline{n}\cdot\left[(\psi-\psi^{o})\,\underline{\nabla}G_1 - G_1\underline{\nabla}(\psi-\psi^{o})\right]dS \;.$$

The integral over the infinite sphere vanishes, since $\psi - \psi^o$ does not contain the incident field, inwhich case both G and $\psi - \psi^o$ satisfy the outgoing radiation condition. Hence the above reduces to

$$\psi(\underline{R}_o) = \psi^{o}(\underline{R}_o) + \frac{1}{4\pi}\int_{S_1}\left(\psi\frac{\partial G_1}{\partial n} - G_1\frac{\partial\psi}{\partial n}\right)dS \;. \tag{44}$$

In a similar manner it can be shown that

$$\chi(\underline{R}_o) = \chi^{o}(\underline{R}_o) + \frac{1}{4\pi}\int_{S_1}\left(\chi\frac{\partial G_2}{\partial n} - G_2\frac{\partial\chi}{\partial n}\right)dS \tag{45}$$

where the Green's function $G_2(\underline{R}, \underline{R}_o)$ is given by the relation

$$G_2(\underline{R}, \underline{R}_o) = \frac{e^{ikr}}{r} - ik\sum_{n=0}(2n+1)\,a_n\,h_n^{(1)}(kR_o)\,h_n^{(1)}(kR)\,P_n(\cos\gamma) \tag{46}$$

with the constant a_n given by Eq. (14). The above integral representation can be written as follows where use is made of boundary conditions (37)

$$\psi(\underline{R}_o) = \psi^o(\underline{R}_o) - \frac{1}{4\pi}\int_C G_1 \frac{\partial \psi}{\partial n} dS + \frac{1}{4\pi}\int_T \psi \left[\frac{\partial G_1}{\partial n} + \frac{G_1}{b}\right] dS \tag{46}$$

$$\chi(\underline{R}_o) = \chi^o(\underline{R}_o) + \frac{1}{4\pi}\int_C \chi \frac{\partial G_2}{\partial n} dS - \frac{1}{4\pi}\int_T G_2 \frac{\partial \chi}{\partial n} dS \tag{47}$$

where $\underline{n}$ is the outward normal. A description of the surfaces are given in Fig. 2. The fundamental problem remains to find $\frac{\partial \psi}{\partial n}$ and χ on the side C and ψ and $\frac{\partial \chi}{\partial n}$ on the cap T, for when these are known, the total fields χ and ψ can be determined. The integral equations for those quantities can be developed by letting $\underline{R}_o$ approach the surface, and taking the appropriate limit. It thus follows that for $\underline{R}_o$ on the surface T

$$\psi(\underline{R}_o) = 2\psi^o(\underline{R}_o) - \frac{1}{2\pi}\int_C G_1 \frac{\partial \psi}{\partial n} dS + \frac{1}{2\pi}\int_T \psi \left(\frac{\partial G_1}{\partial n} + \frac{G_1}{b}\right) dS \tag{48}$$

$$\frac{\partial \chi(\underline{R}_o)}{\partial n_o} = \frac{2\partial \chi^o(R_o)}{\partial n_o} + \frac{1}{2\pi}\int_C \chi \frac{\partial^2 G_2}{\partial n \partial n_o} dS - \frac{1}{2\pi}\int_T \frac{\partial G_2}{\partial n_o}\frac{\partial \chi}{\partial n} dS \tag{49}$$

and for $\underline{R}_o$ on the surface C

$$\frac{\partial \psi(R_o)}{\partial n_o} = \frac{2\partial \psi^o(R_o)}{\partial n_o} - \frac{1}{2\pi}\int_C \frac{\partial G_1}{\partial n_o}\frac{\partial \psi}{\partial n} dS + \frac{1}{2\pi}\int_T \psi \frac{\partial}{\partial n_o}\left[\frac{\partial G_1}{\partial n} + \frac{G_1}{b}\right] dS \tag{50}$$

$$\chi(\underline{R}_o) = 2\chi^o(\underline{R}_o) + \frac{1}{2\pi}\int_C \chi \frac{\partial G_2}{\partial n} dS - \frac{1}{2\pi}\int_T G_2 \frac{\partial \chi}{\partial n} dS\ . \tag{51}$$

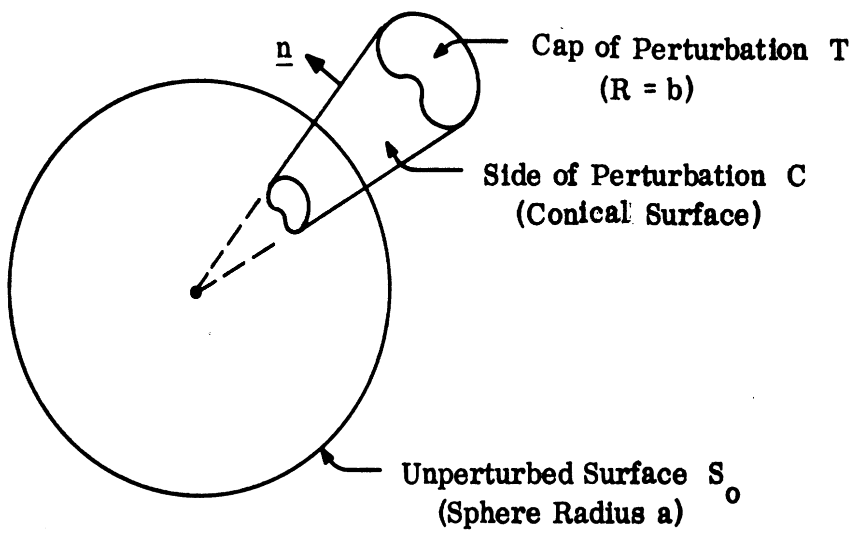

FIG. 2: GEOMETRY OF SURFACE PERTURBATION.

It is of particular importance to briefly investigate the behavior of the scalar functions and their normal derivatives in the immediate vicinity of the edge. The top edge formed by the intersection of the conical side C and cap T, has an exterior wedge angle of $3\pi/2$ radians. From Maue (1949) it is seen that the tangential component of the magnetic field which is parallel to the edge, $H_{\parallel}$, is finite whereas the other tangential component which is perpendicular to the edge, $H_{\perp}$, must behave like $s^{-1/3}$ where s is the distance from the edge. In the local region of the wedge the leading term of χ and ψ are solutions of the two-dimensional Laplace's equation, with solutions of the form

$$s^{\pm\nu} \cos \nu\phi' \quad , \quad s^{\pm\nu} \sin \nu\phi' \ , \log s$$

where ϕ' is the angle measured from the top face T. Applying the boundary conditions Eq. (37) on the wedge faces $\phi' = 0$ and $\phi' = 3\pi/2$, it follows that

$$\nu = 1/3, \ 2/3, \ \ldots .$$

However using relation (31) and the conditions that $H_{\parallel}$ is finite and $H_{\perp} \sim s^{-1/3}$, it can be shown that the lowest admissable solution is given by $\nu = 5/3$. This implies that χ, ψ, $\frac{\partial \chi}{\partial n}$, and $\frac{\partial \psi}{\partial n}$ are finite at the edge.

The far field expression (for $R_o \to \infty$) can be easily obtained. Using the relations,

$$G_1(\underline{R}, \underline{R}_o) \sim \frac{e^{ikR_o}}{R_o} F_1(\theta_o, \phi_o; \underline{R}) , \tag{52}$$

and

$$G_2(\underline{R}, \underline{R}_o) \sim \frac{e^{ikR_o}}{R_o} F_2(\theta_o, \phi_o; \underline{R}) , \tag{53}$$

where

$$F_1 = e^{-ikR\cos\gamma} - \sum_{n=0} (2n+1)\, b_n (-i)^n h_n^{(1)}(kR)\, P_n(\cos\gamma) \quad , \tag{54}$$

$$F_2 = e^{-ikR\cos\gamma} - \sum_{n=0} (2n+1)\, a_n (-i)^n h_n^{(1)}(kR)\, P_n(\cos\gamma) \quad , \tag{55}$$

it follows that

$$\psi(\underline{R}_o) - \psi^o(\underline{R}_o) \sim \frac{e^{ikR_o}}{R_o}\, f_1(\theta_o, \phi_o)$$

$$\psi(\underline{R}_o) - \psi^o(\underline{R}_o) \sim \frac{e^{ikR_o}}{R_o}\, f_2(\theta_o, \phi_o)$$

where

$$f_1(\theta_o, \phi_o) = -\frac{1}{4\pi}\int_C F_1 \frac{\partial\psi}{\partial n}\, dS + \frac{1}{4\pi}\int_T \frac{\psi}{b}\left[\frac{\partial R F_1}{\partial R}\right]_{R=b} dS \tag{56}$$

$$f_2(\theta_o, \phi_o) = \frac{1}{4\pi}\int_C \chi \frac{\partial F_2}{\partial n}\, dS - \frac{1}{4\pi}\int_T F_2 \frac{\partial\chi}{\partial n}\, dS \; . \tag{57}$$

The scattered far field due to the perturbation is given

$$\underline{H}_p^s(\underline{R}_o) \sim \frac{e^{ikR_o}}{R_o}\left[\hat{\theta}_o \frac{1}{\sin\theta_o}\frac{\partial}{\partial\phi_o} - \hat{\phi}_o \frac{\partial}{\partial\theta_o}\right] f_1(\theta_o, \phi_o) \tag{58}$$

$$ik\, \frac{e^{ikR_o}}{R_o}\left[\hat{\theta}_o \frac{\partial}{\partial\theta_o} + \hat{\phi}_o \frac{1}{\sin\theta_o}\frac{\partial}{\partial\phi_o}\right] f_2(\theta_o, \phi_o) \quad .$$

The total scattered field is obtained by adding to expression (58) the scattered field produced by the plane wave incident upon the unperturbed conducting sphere R = a.

IV

SUMMARY

4.1 Geometry: The perfectly-conducting surface consists of a stub on a sphere of radius a, with the surface of the stub described by a conical side C and cap T, where

$$C: \quad \theta = F(\phi)\ , \qquad a \leqslant R \leqslant b$$

$$T: \quad R = b\ , \qquad \theta \text{ and } \phi \text{ contained in the cone C}$$

Examples (1) conical stub: $C;\ \theta = \theta_o,\ a \leqslant R \leqslant b,\ 0 \leqslant \phi \leqslant 2\pi$

$T;\ R = b,\ 0 \leqslant \theta \leqslant \theta_o,\ 0 \leqslant \phi \leqslant 2\pi$

(2) flat plate: $C;$ surfaces $\theta = \theta_o,\ \theta = \pi - \theta_o;\ \phi = \phi_o,\ \phi = -\phi_o$

$T;\ R = b\ .$

4.2 Incident Field: The incident field is a plane wave (harmonic time dependence), of polarization $\underline{e}$, and direction of incident given by $-\underline{k}_1$, where

$$\underline{k}_1 = k_1 (\sin\theta_1 \cos\phi_1,\ \sin\theta_1 \sin\phi_1,\ \cos\theta_1)\ .$$

In particular, the incident field is described by the relation

$$\underline{E}^i = \underline{e} \exp(-i\underline{k}_1 \cdot \underline{R})$$

where $\underline{e} = e_1 \hat{\underline{\theta}}_1 + e_2 \hat{\underline{\phi}}_1$.

4.3 Fundamental Integral Equations: There are four unknown quantities to be found, ψ and $\frac{\partial \chi}{\partial n}$ which are associated with the surface T, and $\frac{\partial \psi}{\partial n}$ and χ which are associated with the surface C The resultant four integral equations decouple into two sets of two The two integral equations involving ψ and $\frac{\partial \psi}{\partial n}$ are as follows, for $\underline{R}_o$ on T,

$$\psi(\underline{R}_o) = 2\psi^o(\underline{R}_o) - \frac{1}{2\pi}\int_C G_1(\underline{R}, \underline{R}_o)\frac{\partial \psi}{\partial n} + \frac{1}{2\pi}\int_T \psi\left(\frac{\partial G_1}{\partial R} + \frac{G_1}{R}\right) ds$$

and $\underline{R}_o$ on C,

$$\frac{\partial \psi(R_o)}{\partial n_o} = \frac{2\partial\psi^o(R_o)}{\partial n_o} - \frac{1}{2\pi}\int_C \frac{\partial G_1}{\partial n_o}\frac{\partial \psi}{\partial n} ds + \frac{1}{2\pi}\int_T \psi \frac{\partial}{\partial n_o}\left(\frac{\partial G_1}{\partial R} + \frac{G_1}{R}\right) ds .$$

The Green's function $G_1(\underline{R}, \underline{R}_o)$ has the precise form

$$G_1(\underline{R}, \underline{R}_o) = \frac{e^{ikr}}{r} - ik \sum_{n=0} (2n+1) b_n h_n^{(1)}(kR) h_n^{(1)}(kR_o) P_n(\cos\gamma)$$

and its derivative is given by

$$\frac{\partial G_1}{\partial n_o} = \underline{n}_o \cdot \underline{\nabla}_o G_1(\underline{R}, \underline{R}_o)$$

where $\underline{n}_o$ is the unit outward normal to the surface (in this case the conical side C). In the above expression for the Green's function, the following relations are used,

$$r = \left| \underline{R} - \underline{R}_o \right| ,$$

$$\cos\gamma = \cos\theta\cos\theta_o + \sin\theta\sin\theta_o\cos(\phi-\phi_o) ,$$

$$b_n = \frac{[\rho j_n(\rho)]'}{[\rho h_n^{(1)}(\rho)]'} \qquad \text{with } \rho = ka .$$

The function $\psi^o(\underline{R}_o)$ is given by the relation (see Appendix B)

$$\psi^o(\underline{R}_o) = \sqrt{\frac{\epsilon_o}{\mu_o}} (\underline{\alpha} \cdot \underline{e}) \sum_{n=1} \frac{(-i)^n (2n+1)}{n(n+1)} P_n^{(1)}(\cos\gamma_1)\left[j_n(kR_o) - b_n h_n^{(1)}(kR_o)\right]$$

where $\cos\gamma_1 = \cos\theta_1\cos\theta_o + \sin\theta_1\sin\theta_o\cos(\phi_o - \phi_1)$

and

$$\underline{\alpha} = \hat{\underline{\theta}}_1 \left[-\cos\theta_o\sin\theta_1 + \sin\theta_o\cos\theta_1\cos(\phi_o-\phi_1)\right] \Big/ \sin\gamma_1$$
$$+ \hat{\underline{\phi}}_1 \left[\sin\theta_o\sin(\phi_1-\phi_o)\right] \Big/ \sin\gamma_1 .$$

The two integral equations involving χ and $\frac{\partial\chi}{\partial n}$ are as follows for $\underline{R}_o$ on T

$$\frac{\partial\chi(R_o)}{\partial n_o} = \frac{2\partial\chi^o(\underline{R}_o)}{\partial R_o} + \frac{1}{2\pi}\int_C \chi \frac{\partial}{\partial n}\left(\frac{\partial G_2}{\partial R_o}\right) ds - \frac{1}{2\pi}\int_T \frac{\partial G_2}{\partial R_o}\frac{\partial\chi}{\partial n} ds$$

and for $\underline{R}_o$ on C

$$\chi(\underline{R}_o) = 2\chi^o(\underline{R}_o) + \frac{1}{2\pi}\int_C \chi\frac{\partial G_2}{\partial n} ds - \frac{1}{2\pi}\int_T G_2 \frac{\partial\chi}{\partial n} ds .$$

The Green's function $G_2(\underline{R}, \underline{R}_o)$ is given explicitly by the relation

$$G_2(\underline{R}, \underline{R}_o) = \frac{e^{ikr}}{r} - ik \sum_{n=0} (2n+1) a_n h_n^{(1)}(kR) h_n^{(1)}(kR_o) P_n(\cos\gamma)$$

where

$$a_n = \frac{j_n(\rho)}{h_n^{(1)}(\rho)} \qquad \text{with } \rho = ka .$$

The known function $\chi^o(\underline{R}_o)$ is given by the relation (from Appendix B)

$$\chi^o(\underline{R}_o) = i\sqrt{\frac{\epsilon_o}{\mu_o}} \, (\underline{\alpha} \cdot \underline{e} \times \hat{\underline{k}}_1) \sum_{n=1} \frac{(-i)^n (2n+1) P_n^{(1)}(\cos\gamma_1)}{n(n+1)} \left[j_n(kR_o) - a_n h_n^{(1)}(kR_o) \right].$$

4.4 <u>The Far Field:</u> The far scattered field is composed of two parts

$$\underline{H}^s = \underline{H}^{os} + \underline{H}_p^s$$

where $\underline{H}^{os}$ is the scattered field due to the perfectly-conducting sphere R=a, and $\underline{H}^{ps}$ is the scattered field due to the stub. Since the expression for $\underline{H}^{os}$ is well-known, only the addition term $\underline{H}^{ps}$ will be given here.

$$\underline{H}_p^s(\underline{R}_o) = \frac{e^{ikR_o}}{R_o} \left[\hat{\underline{\theta}}_o \frac{1}{\sin\theta_o} \frac{\partial}{\partial\phi_o} - \hat{\underline{\phi}}_o \frac{\partial}{\partial\theta_o} \right] f_1(\theta_o, \phi_o)$$

$$+ ik \frac{e^{ikR_o}}{R_o} \left[\hat{\underline{\theta}}_o \frac{\partial}{\partial\theta_o} + \hat{\underline{\phi}}_o \frac{1}{\sin\theta_o} \frac{\partial}{\partial\phi_o} \right] f_2(\theta_o, \phi_o)$$

where

$$f_1(\theta_o, \phi_o) = -\frac{1}{4\pi}\int_C F_1 \frac{\partial\psi}{\partial n}\, ds + \frac{1}{4\pi}\int_T \psi\left[\frac{\partial F_1}{\partial R} + \frac{F_1}{R}\right]_{r=b} ds$$

$$f_2(\theta_o, \phi_o) = \frac{1}{4\pi}\int_C \chi \frac{\partial F_2}{\partial n}\, ds - \frac{1}{4\pi}\int_T F_2 \frac{\partial \chi}{\partial R}\, ds$$

with

$$F_1 = e^{-ikR\cos\gamma} - \sum_{n=0} (2n+1)\, b_n (-i)^n\, h_n^{(1)}(kR)\, P_n(\cos\gamma)$$

$$F_2 = e^{-ikR\cos\gamma} - \sum_{n=0} (2n+1)\, a_n (-i)^n\, h_n^{(1)}(kR)\, P_n(\cos\gamma) \quad .$$

APPENDIX A

It is the purpose of this appendix to prove the relation

$$\underline{\nabla} \times \left(\frac{e^{ikr}}{r} \underline{t} \right) = \underline{\nabla} \times \underline{\nabla} \times \underline{R} (\underline{t} \cdot \underline{\nabla}_o \times \underline{R}_o \pi) + \underline{\nabla} \times \underline{R} (\underline{t} \cdot \underline{\nabla}_o \times \underline{\nabla}_o \times \underline{R}_o \pi) \tag{A.1}$$

where

$$\pi = ik \sum_{n=1} \frac{(2n+1)}{n(n+1)} j_n(kR)\, h_n^{(1)}(kR_o)\, P_n(\cos\gamma) \qquad R < R_o \; .$$

In the above relation, $\underline{t}$ is a unit constant vector, $\underline{r} = \underline{R} - \underline{R}_o$, γ is the angle between the two vectors $\underline{R}$ and $\underline{R}_o$, and the subscript on the operator $\underline{\nabla}_o$ indicates differentiation with respect to the variables (R_o, θ_o, ϕ_o).

The vector field can be expressed in terms of the orthogonal set of spherical vector wave functions

$$\underline{M}_{\substack{e\\o}mn} = \underline{\nabla} \times \underline{R}\, \psi_{\substack{e\\o}mn} \; ,$$

$$k\, \underline{N}_{\substack{e\\o}mn} = \underline{\nabla} \times \underline{\nabla} \times \underline{R}\, \psi_{\substack{e\\o}mn} \; ,$$

where

$$\psi_{\substack{e\\o}mn} = j_n(kR)\, P_n^m(\cos\theta) \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \; ,$$

as follows

$$\underline{\nabla} \times \left(\frac{e^{ikr}}{r} \underline{t} \right) = \underline{\nabla} \times \left(\underline{R} \sum a_\nu \psi_\nu \right) + \frac{1}{k} \underline{\nabla} \times \underline{\nabla} \times \left(\underline{R} \sum b_\nu \psi_\nu \right)$$

$$\underline{\nabla} \times \underline{\nabla} \times \left(\frac{e^{ikr}}{r} \underline{t} \right) = \underline{\nabla} \times \underline{\nabla} \times \left(\underline{R} \sum a_\nu \psi_\nu \right) + k\, \underline{\nabla} \times \left(\underline{R} \sum b_\nu \psi_\nu \right) \; .$$

The summation over ν indicates summation over $n = 1, 2, 3, \ldots,$ $m = 0, 1, 2, \ldots, n,$ and for both even e and odd o functions.

From the orthogonal properties of the vector wave functions, integrated over the surface of a sphere, it follows that

$$\int_S (\nabla \times \underline{R}\psi_v) \cdot (\nabla \times \frac{e^{ikr}}{r} \underline{t})\, dS = a_v \int_S (\nabla \times \underline{R}\psi_v \cdot \nabla \times \underline{R}\psi_v)\, dS \tag{A.2}$$

$$\int_S (\nabla \times \underline{R}\psi_v) \cdot (\nabla \times \nabla \times \frac{e^{ikr}}{r} \underline{t})\, dS = b_v k \int_S (\nabla \times \underline{R}\psi_v \cdot \nabla \times \underline{R}\psi_v)\, dS \tag{A.3}$$

From Stratton (p. 418) it is seen that

$$\beta_v = \int_S \nabla \times \underline{R}\psi_v \cdot \nabla \times \underline{R}\psi_v\, dS = (1+\delta)\, \frac{2\pi R^2}{(2n+1)}\, \frac{(n+m)!}{(n-m)!}\, n(n+1)\left[j_n(kR)\right]^2 \tag{A.4}$$

where $\delta = 0$ if $m>0$, and $\delta = 1$ if $m = 0$.

In order to compute the left-hand side of Eqs. (A.1) and (A.2) the following lemmas are needed.

Lemma I: if S is a closed spherical surface of radius R, then

$$\int_S \nabla\psi \times \underline{R} \frac{e^{ikr}}{r}\, dS = \underline{\nabla}_o \times \underline{R}_o \int_S \frac{e^{ikr}}{r}\, \psi\, dS \tag{A.5}$$

Proof: The left-hand side of (A.5) can be written in the form

$$\int_S \nabla\left(\frac{e^{ikr}}{r}\psi\right) \times \underline{R}\, dS - \int_S \psi \nabla \frac{e^{ikr}}{r} \times \underline{R}\, dS \quad .$$

Since S is a closed surface and $\underline{R}$ is the normal to the surface, then the first integral vanishes leaving the second integral

$$\int \psi \underline{\nabla}_o \frac{e^{ikr}}{r} \times \underline{R}\, dS = \int \psi \underline{\nabla}_o \frac{e^{ikr}}{r} \times \underline{R}_o\, dS + \int \psi \underline{\nabla}_o \frac{e^{ikr}}{r} \times \underline{r}\, dS \quad .$$

The second integral vanishes since $\underline{\nabla}_o\left(\frac{e^{ikr}}{r}\right)$ is parallel to $\underline{r}$. This leaves

$$\int \underline{\nabla}_o\left(\frac{e^{ikr}}{r}\psi\right) \times \underline{R}_o \, dS = \underline{\nabla}_o \times \left[\underline{R}_o \int \frac{e^{ikr}}{r}\psi \, dS\right]$$

namely, the right-hand side of Eq. (A.5)

Lemma II:

$$\int_S \underline{\nabla} \times (\underline{R}\psi) \cdot \underline{\nabla} \times \frac{e^{ikr}}{r}\underline{t} \, dS = \underline{t} \cdot \underline{\nabla}_o \times \underline{\nabla}_o \times \left\{\underline{R}_o \int_S \psi \frac{e^{ikr}}{r} dS\right\} \qquad (A.6)$$

Proof: The left-hand side can be written in the form

$$\underline{t} \cdot \int (\underline{\nabla}\psi \times \underline{R}) \times \nabla \frac{e^{ikr}}{r} \, dS$$

$$= \underline{t} \cdot \int \underline{\nabla}_o \frac{e^{ikr}}{r} \times (\underline{\nabla}\psi \times \underline{R}) = \underline{t} \cdot \underline{\nabla}_o \times \int_S \frac{e^{ikr}}{r} \underline{\nabla}\psi \times \underline{R} \, dS$$

and using lemma I, this reduces to

$$\underline{t} \cdot \underline{\nabla}_o \times \underline{\nabla}_o \times \left\{\underline{R}_o \int \frac{e^{ikr}}{r} \psi \, dS\right\}$$

Lemma III:

$$\int_S (\underline{\nabla} \times \underline{R}\psi) \cdot (\underline{\nabla} \times \underline{\nabla} \times \frac{e^{ikr}}{r} t) \, dS = k^2 \underline{t} \cdot \underline{\nabla}_o \times \left\{\underline{R}_o \int \psi \frac{e^{ikr}}{r} dS\right\} \qquad (A.7)$$

Proof: The left-hand side can be written in the form

$$k^2 \int_S \underline{\nabla}\psi \times \underline{R} \cdot \underline{t} \frac{e^{ikr}}{r} \, dS + \int_S (\underline{\nabla}\psi \times \underline{R}) \cdot (\underline{t} \cdot \underline{\nabla}_o)\underline{\nabla}_o \frac{e^{ikr}}{r} \, dS$$

$$= \left[k^2 \underline{t} + (\underline{t} \cdot \underline{\nabla}_o)\underline{\nabla}_o\right] \cdot \int \underline{\nabla}\psi \times \underline{R} \frac{e^{ikr}}{r} \, dS$$

and in using lemma I, this reduces to

$$\left[k^2 \underline{t} + (\underline{t} \cdot \underline{\nabla}_o) \underline{\nabla}_o\right] \cdot \underline{\nabla}_o \times \left\{\underline{R}_o \int \psi \frac{e^{ikr}}{r} dS\right\}$$

$$= k^2 \underline{t} \cdot \underline{\nabla}_o \times \left\{\underline{R}_o \int \psi \frac{e^{ikr}}{r} dS\right\} .$$

The results of the above lemmas can now be used to evaluate the unknown quantities a_v and b_v. It follows that

$$\beta_v a_v = \underline{t} \cdot \underline{\nabla}_o \times \underline{\nabla}_o \times \left\{\underline{R}_o \int_S \psi_v \frac{e^{ikr}}{r} dS\right\}$$

$$\beta_v b_v = k \underline{t} \cdot \underline{\nabla}_o \times \left\{\underline{R}_o \int_S \psi_v \frac{e^{ikr}}{r} dS\right\}.$$

It follows that

$$\sum_v a_v \psi_v (\underline{R}) = \underline{t} \cdot \underline{\nabla}_o \times \underline{\nabla}_o \times (\underline{R}_o \pi) \tag{A.8}$$

$$\frac{1}{k} \sum b_v \psi_v (\underline{R}) = \underline{t} \cdot \underline{\nabla}_o \times (\underline{R}_o \pi) \tag{A.9}$$

where

$$\pi = \sum \frac{\psi_o(\underline{R})}{\beta_v} \int_S \psi_v (\underline{R}') \frac{e^{ikr}}{r} dS' \tag{A.10}$$

and $r = \left|\underline{R}' - \underline{R}_o\right|$.

Simplification is achieved when the coordinate system is rotated lining the z-axis up with the direction $\underline{R}_o$. In this case

$$\pi(R, R_o, \gamma) = \sum_{n=1} \frac{(2n+1)}{n(n+1)4\pi} \int_S \frac{\psi_{eon}(R,\gamma)\psi_{eon}(R',\gamma')}{\left[j_n(kR')\right]^2} \frac{e^{ikr}}{r} \sin\gamma' d\gamma' \, d\phi'$$

$$= ik \sum_{n=1} \frac{(2n+1)}{2n(n+1)} \psi_{eon}(R,\gamma) \; (2n+1) h_n^{(1)}(kR_o) \int \left[P_n(\cos\gamma')\right]^2 \sin\gamma' \, d\gamma'$$

$$= ik \sum_{n=1} \frac{(2n+1)}{n(n+1)} j_n(kR) \, h_n^{(1)}(kR_o) \, P_n(\cos\gamma) \; . \qquad \text{(A.11)}$$

It follows immediately from (A.8), (A.9) and (A.11) that (A.1 is proved.

APPENDIX

B

The problem is to find explicit expressions for ψ^o and χ^o associated with the field $(\underline{E}^o, \underline{H}^o)$ generated by a plane wave incident upon the perfectly-conducting sphere $R = a$, by the relations

$$\left.\begin{aligned} \underline{H}^o &= \underline{\nabla} \times \underline{R}\psi^o + \frac{1}{k}\underline{\nabla} \times \underline{\nabla} \times \underline{R}\chi^o \\ -i\sqrt{\epsilon_o/\mu_o}\;\underline{E}^o &= \frac{1}{k}\underline{\nabla} \times \underline{\nabla} \times \underline{R}\psi^o + \underline{\nabla} \times \underline{R}\chi^o \;. \end{aligned}\right\} \qquad (B.1)$$

From Section II, it was shown that that the total field generated by a magnetic dipole at $\underline{R}_o$,

$$\underline{\tilde{E}}^i = \underline{\nabla} \times \frac{e^{ikr}}{r}\,\underline{m}$$

in the presence of a perfectly conducting sphere of radius $R = a$, could be written in the form

$$-i\sqrt{\epsilon_o/\mu_o}\;\underline{\tilde{E}} = \frac{1}{k}\underline{\nabla} \times \underline{\nabla} \times \underline{R}\tilde{\psi} + \underline{\nabla} \times \underline{R}\tilde{\chi}$$

where

$$\tilde{\psi} = -i\sqrt{\epsilon_o/\mu_o}\;k \quad \underline{m} \cdot \underline{\nabla}_o \times \underline{R}_o\,\tilde{\pi}$$

$$\tilde{\chi} = -i\sqrt{\epsilon_o/\mu_o} \quad \underline{m} \cdot \underline{\nabla}_o \times \underline{\nabla}_o \times \underline{R}_o\,\pi$$

with

$$\left.\begin{matrix}\tilde{\pi}\\ \pi\end{matrix}\right\} = ik\sum_{n=1}\frac{(2n+1)}{n(n+1)}\;P_n(\cos\gamma)\,h_n^{(1)}(kR_o)\left[j_n(kR) - \left\{\begin{matrix}b_n\\ a_n\end{matrix}\right\}h_n^{(1)}(kR)\right].$$

Let $R_o \rightarrow \infty$ and $\underline{m} \perp \hat{\underline{R}}_o$ in which case $\underline{\tilde{E}}^i$ becomes a plane wave

$$\underline{\tilde{E}}^i \sim \left[ik \frac{e^{ikR_o}}{R_o} \right] e^{-ikR\cos\gamma} \, \underline{m} \times \hat{\underline{R}}_o \; .$$

In this case we have

$$\tilde{\chi} \sim k\sqrt{\frac{\epsilon_o}{\mu_o}} \, e^{ikR_o} \, \underline{m} \cdot \underline{\nabla}_o \sum_{n=1} (-i)^n \frac{(2n+1)}{n(n+1)} P_n(\cos\gamma) \left[j_n(kR) - a_n h_n^{(1)}(kR) \right] \tag{B.2}$$

$$\tilde{\psi} \sim -ik\sqrt{\frac{\epsilon_o}{\mu_o}} \, e^{ikR_o} \, \hat{\underline{R}}_o \times \underline{m} \cdot \underline{\nabla}_o \sum_{n=1} (-i)^n \frac{(2n+1)}{n(n+1)} P_n(\cos\gamma) \left[j_n(kR) - b_n h_n^{(1)}(kR) \right] . \tag{B.3}$$

The term $\underline{\nabla}_o P_n(\cos\gamma)$ can be written in the form

$$\underline{\nabla}_o P_n(\cos\gamma) = \left[\hat{\underline{\theta}}_o \frac{\partial \cos\gamma}{\partial \theta_o} + \hat{\underline{\phi}}_o \frac{\partial \cos\gamma}{\sin\theta_o \, \partial \phi_o} \right] \frac{\partial P_n(\cos\gamma)}{R_o \, \partial \cos\gamma}$$

$$= \frac{1}{R_o} \, \underline{\alpha} \, P_n^{(1)}(\cos\gamma) \tag{B.4}$$

where

$$\underline{\alpha} = \underline{\theta}_o \left[-\cos\theta \sin\theta_o + \sin\theta \cos\theta_o \cos(\phi - \phi_o) \right] / \sin\gamma \tag{B.5}$$

$$+ \hat{\underline{\phi}}_o \left[\sin\theta \sin(\phi_o - \phi) / \sin\gamma \right]$$

Normalize $\underline{E}^i$ so that it represents a plane wave of unit amplitude, i.e.

$$\underline{\tilde{E}}^i = \underline{e} \exp(-i\underline{k}_o \cdot \underline{R})$$

where

$$\underline{k}_o = k_o (\sin\theta_o \cos\phi_o, \sin\theta_o \sin\phi_o, \cos\theta_o)$$

and

$$\underline{e} = \hat{\underline{\theta}}_o e_1 + \hat{\underline{\phi}}_o e_2 \quad .$$

Then it follows that the total field generated by the plane wave incident on a conducting sphere of radius a, can be expressed in terms of the two scalars χ^o and ψ^o by relations (B.1), where

$$\chi_o = i\sqrt{\frac{\epsilon_o}{\mu_o}} (\underline{\alpha} \cdot \underline{e} \times \hat{\underline{R}}_o) \sum_{n=1} \frac{(-1)^n (2n+1)}{n(n+1)} P_n^{(1)}(\cos\gamma) \left[j_n(kR) - a_n h_n^{(1)}(kR) \right]$$

$$\psi_o = \sqrt{\frac{\epsilon_o}{\mu_o}} (\underline{\alpha} \cdot \underline{e}) \sum_{n=1} \frac{(-1)^n (2n+1)}{n(n+1)} P_n^{(1)} (\cos\gamma) \left[j_n(kR) - b_n h_n^{(1)}(kR) \right] .$$

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