BSD-TR-67-143

DIFFRACTION BY THE CONCAVE SURFACE OF THE PARABOLOID OF REVOLUTION

Technical Report No. 3

F 04-694-67 C-0055
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by

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February 1967

8525-3-T = RL-2184

Prepared for

Ballistic Systems Division, AFSC
Deputy for Ballistic Missile Re-entry Systems
Norton Air Force Base, California 92409

FOREWORD

This report was prepared by the Radiation Laboratory of the Department of Electrical Engineering of The University of Michigan under the direction of Dr. Raymond F. Goodrich, Principal Investigator, and Burton A. Harrison, Contract Manager. The work was performed under Contract F 04-694-67 C-0055 "Investigation of Re-entry Vehicle Surface Fields (SURF)". This work was administered under the direction of the Air Force Ballistic Systems Division, Norton Air Force Base, California 92409, by Lieutenant J. Wheatley BSYDF and was monitored by Mr. H.J. Katzman of the Aerospace Corporation.

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ABSTRACT

Let η_o denote the focal length of a paraboloid of revolution, and let D be the closure of the domain bounded by its concave surface. Then for a point source, with wave number k, located in D and on the axis of the paraboloid, the diffraction by the boundary of D is considered not only if $k\eta_o \gg 1$ but also if $k\eta_o \ll 1$. If $k\eta_o \gg 1$, an asymptotic representation of the total field on the boundary of D is derived for the Neumann boundary condition provided the source is far (with respect to wavelength) from the focus and the field point is far from the tip of the paraboloid. This representation is interpreted in terms of geometric optics.

If $k\eta_o \ll 1$, an asymptotic representation of the total field anywhere in D is derived for both Dirichlet and Neumann boundary conditions and for the source (field) point in the near field and field (source) point in the far field as well as for both source and field points in the near field. The near field result is compared with the solution of the corresponding potential problem. A necessary and sufficient condition for the existence of a solution to the corresponding Neumann potential problem is also derived.

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I
INTRODUCTION

1.1 Preliminary Discussion

For the most part, the solutions of diffraction problems have been confined to convex surfaces; relatively little has been done in the case of concave surfaces. For high frequency waves, diffraction by convex surfaces gives rise to single reflections and creeping waves while diffraction by concave surfaces gives rise to multiple reflections, caustic surfaces which are the envelopes of a family of multiply reflected waves and, in addition, whispering gallery waves (a form of traveling waves) if the source of radiation is near (with respect to wavelength) the concave surface. Some of the early considerations of these short wavelength effects are found in a paper by Rayleigh (1910) and also in his book (1896, Chapter 14). A recent investigation of the short wavelength diffraction by concave surfaces is illustrated by the papers of Kinber (1961a, b), which treat the circular cylinder and sphere, respectively. In this report, we shall study the diffraction by the concave surface of the paraboloid of revolution not only of high frequency waves, but also of low frequency waves and in the long wavelength limit. In the high frequency case, we shall consider a point source located on the axis of the paraboloid and, in addition, far (with respect to wavelength) from the focus. For waves of low frequency, the point source is considered to be anywhere on the axis of the paraboloid.

The restriction that the point source be far from the focus is the result of a fundamental difference between the short wavelength diffraction by the concave surface of the paraboloid of revolution and that of either the circular cylinder or sphere. This difference arises from the fact that while the paraboloid behaves as a typical concave surface for the point source far from the focus, the point source at the focus is a configuration whereby in the short wavelength limit, diffraction by a concave surface gives rise to a single reflection. The case of a dipole, with moment perpendicular to the axis, at the focal point of a perfectly conducting paraboloid of revolution has been investigated by Fock (1957) and Skalskaya (1955). Pinney (1946, 1947) considered the moment both perpendicular and parallel to the axis. Although there is a double reflection, it is natural to consider the plane wave problem in this category. The diffraction of a short wavelength plane wave by the concave surface of a parabolic cylinder (Dirichlet boundary condition) was studied by Lamb (1906), who indicated that the method could be extended to the paraboloid of revolution.

Perhaps the best starting point for a study of diffraction problems pertaining to the paraboloid of revolution is a book by Buchholz (1953) which includes a complete bibliography. Worthy of separate mention is an earlier written paper by Buchholz (1942/3) in which he considers a point source at the focus (Neumann boundary condition).

1.2 Mathematical Statement of the Problem

A paraboloid of revolution (with focal length η_0) divides three dimensional space into two domains. The domain whose boundary is the concave surface of the paraboloid shall be called the interior of the paraboloid of revolution. Let D be the closure of this domain and let $\rho(\underline{r})$ ($\underline{r}$ is the usual position vector) denote a point source distribution in D. The precise form of $\rho(\underline{r})$ depends on the definition of the point source; it will be specified later. Then the solution of the initial-boundary value problem for the inhomogeneous wave equation (with source distribution $\rho(\underline{r})$) in the domain D is a function $u(\underline{r}, t)$ defined in the space time domain $\left\{D\times(t>0)\right\}$, which is in $C^2(D)$ for each fixed positive t, which is in $C^2(t>0)$, and which satisfies

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \rho(\underline{r})\, e^{i\omega t} ,$$

u = 0 on the boundary of D (Dirichlet boundary condition),

or $\qquad\qquad (1.P_1)$

$\frac{\partial u}{\partial n} = 0$ on the boundary of D (Neumann boundary condition),

some prescribed initial conditions.

The solution of the time independent problem corresponding to $(1.P_1)$ is the function $v(\underline{r})$ which is in $C^2(D)$, and which satisfies

$$\nabla^2 v + k^2 v = \rho(\underline{r}) \qquad (k = \omega/c) \quad ,$$

$v = 0$ on the boundary of D,

or $\qquad\qquad\qquad\qquad (1.P_2)$

$\frac{\partial v}{\partial n} = 0$ on the boundary of D ,

as $|\underline{r}| \to \infty$, $v(\underline{r})$ corresponds to an outgoing wave and the flux across a confocal paraboloid is approaching a fixed value (radiation condition).

In this report, we shall find the following functions:

(i) $v(\underline{r})$, if $k\eta_0$ is sufficiently small, the point source is located anywhere on the axis of the paraboloid, and $\underline{r}$ is the position vector to any point in D (except the source) or on the boundary of D, and

(ii) $v(\underline{r})$, if $k\eta_0$ is sufficiently large, the point source is located on the axis of the paraboloid at a distance d from the focal point where $d = O(\eta_0)$, and $\underline{r}$ is the position vector to any point on the boundary of D.

In recent years, there has been considerable effort devoted to obtaining, for various domains D' and distributions $\rho'(\underline{r})$, a relation between the solution $u'(\underline{r}, t)$ of $(1.P_1)$ (with $D = D'$, $\rho(\underline{r}) = \rho'(\underline{r})$) and the solution $v'(\underline{r})$ of $(1.P_2)$ (with $D = D'$, $\rho(\underline{r}) = \rho'(\underline{r})$) in the form

$$u'(\underline{r}, t) = v'(\underline{r})\, e^{i\omega t} + u^*(\underline{r}, t) \quad ,$$

where $\lim_{t \to \infty} u^*(\underline{r}, t) = 0$. For a certain class of unbounded domains, such a result has been obtained by Zachmanoglou (1963). However, this class of unbounded domains does not contain the interior of the paraboloid of revolution, and with the exception of this observation we do not consider the question of existence of this type of relation for the interior of the paraboloid of revolution.

We conclude this section by considering, for a moment, problem $(1.P_2)$ defined on a domain D'' representing the closed, three dimensional exterior to a smooth, bounded, convex body. Then the radiation condition of $(1.P_2)$ becomes the three dimensional Sommerfeld radiation condition

$$\lim_{r \to \infty} r\left(\frac{\partial v}{\partial r} + ikv\right) = 0 \ .$$

Moreover, if $\rho''(\underline{r})$ is a distribution which has compact support in D'', then Ritt and Kazarinoff (1959, 1960) show that the solution $v''(\underline{r})$ to $(1.P_2)$ (with $D = D''$, $\rho = \rho''(\underline{r})$) is equal to the limit $\lim_{s \to 0^+} v''(\underline{r}, s)$, where $v''(\underline{r}, s)$ is a function defined in the domain $\{D'' \times (s > 0)\}$ and which satisfies

$$\nabla^2 v'' + \gamma^2 v'' = \rho''(\underline{r}) \qquad \left(\gamma = \frac{1}{c}(\omega - is)\right) ,$$

$v'' = 0$ on the boundary of D'' ,

or $\qquad (1.P_3)$

$\frac{\partial v''}{\partial n} = 0$ on the boundary of D'' ,

$$\int_{D''} \left|v''(\underline{r}, s)\right|^2 dV < \infty \quad (dV = \text{unit volume in } D'') \ .$$

We do not attempt to prove that this is true for a class of domains containing D and distributions containing $\rho(\underline{r})$. However, we show that it is true for D and $\rho(\underline{r})$ by deriving an integral representation from the formulation $(1.P_3)$ (with $D'' = D$, $\rho''(\underline{r}) = \rho(\underline{r})$), taking the limit as $s \to 0^+$, and then comparing the result with the corresponding integral representation for $v(\underline{r})$ derived from the formulation $(1.P_2)$. This is significant since for large $|\underline{r}|$, D is approaching a cylinder which implies that, for large $|\underline{r}|$, the behavior of $v(\underline{r})$ is no longer three but two dimensional in character (see Buchholz (1953, Chapter 18) for the form of $v(\underline{r})$ as $|\underline{r}| \to \infty$). The shape of D also affects the potential (see Section 2.4).

1.3 Coordinates of the Paraboloid of Revolution

A natural system of coordinates, i.e. a system for which the wave equation separates and the boundary of D is a level surface, may be defined in the following manner: two families of confocal paraboloids of revolution ($\xi = \xi_a$, $\eta = \eta_a$), with focal point at the origin, given by the equations

$$\rho^2 = 4\xi(\xi - z) \quad (\rho^2 = x^2 + y^2) , \qquad \text{and} \qquad \rho^2 = 4\eta(\eta + z) ,$$

together with the usual azimuth angle ϕ, where $0 \leqslant \xi$, $\eta < \infty$ and $0 \leqslant \phi < 2\pi$. This system of coordinates is called the coordinates of the paraboloid of revolution and is illustrated in Fig. (1-1). If the point source is considered to be located as shown in Fig. (1-2) (either possibility), then the domain D defined above is described by $0 \leqslant \eta \leqslant \eta_0$, $0 \leqslant \xi < \infty$, and $0 \leqslant \phi < 2\pi$, while the boundary of D is given by $\eta = \eta_0$. Moreover, since ρ is the radial coordinate of the usual cylindrical coordinate system, the coordinates of the paraboloid of revolution are related to the rectangular (x, y, z), cylindrical (ρ, ϕ, z), and spherical (r, θ, ϕ) coordinates by the following equations:

$$x = \rho \cos\phi = r \sin\theta \cos\phi = 2\sqrt{\xi\eta}\, \cos\phi ,$$

$$y = \rho \sin\phi = r \sin\theta \sin\phi = 2\sqrt{\xi\eta}\, \sin\phi ,$$

and

$$z = z = r\cos\theta = \xi - \eta .$$

Throughout this report, we shall perform calculations in the coordinates of the paraboloid of revolution. These calculations are greatly facilitated by the use of the general relations of curvilinear coordinates developed in Chapters 1 and 5 of Morse and Feshbach (1953). To use their relations, we observe that if we set

$$\xi_1 = \sqrt{2\xi} , \qquad \xi_2 = \sqrt{2\eta} , \quad \text{and} \quad \xi_3 = \cos\phi ,$$

their generalized curvilinear coordinates (ξ_1, ξ_2, ξ_3) will be defined in terms of the coordinates of the paraboloid of revolution given above. Hence, the scale factors h_1, h_2, and h_3 are:

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial \xi_1}\right)^2 + \left(\frac{\partial y}{\partial \xi_1}\right)^2 + \left(\frac{\partial z}{\partial \xi_1}\right)^2} = \sqrt{2(\xi + \eta)} ,$$

$$h_2 = \sqrt{\left(\frac{\partial x}{\partial \xi_2}\right)^2 + \left(\frac{\partial y}{\partial \xi_2}\right)^2 + \left(\frac{\partial z}{\partial \xi_2}\right)^2} = \sqrt{2(\xi + \eta)} ,$$

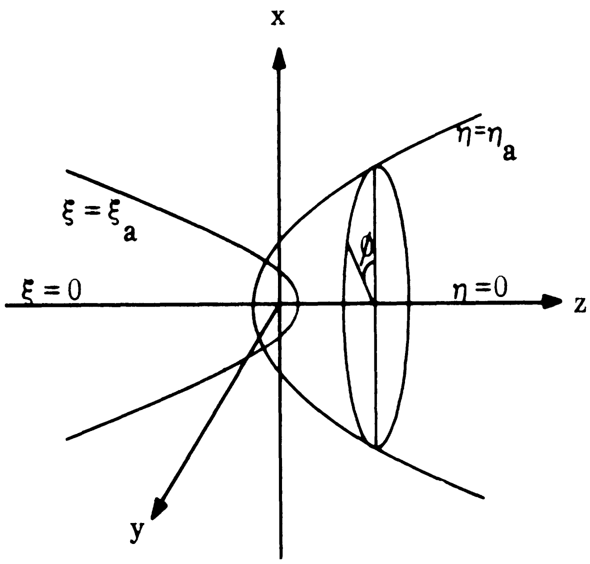

FIG. 1-1: COORDINATES OF THE PARABOLOID OF REVOLUTION

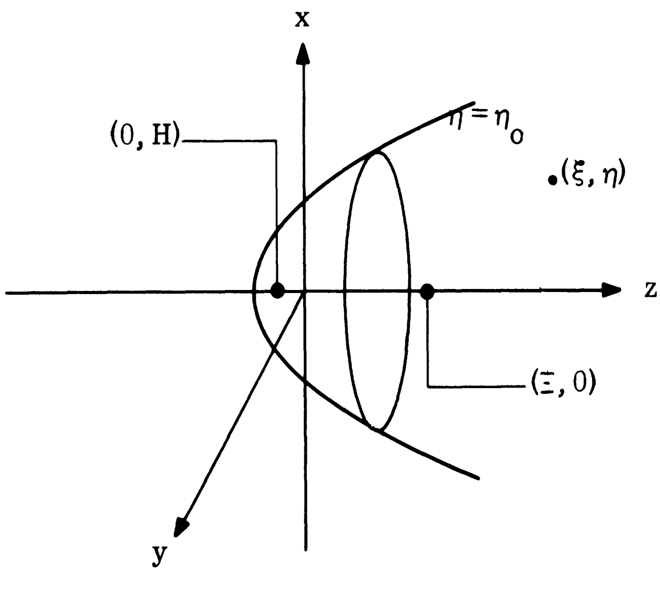

FIG. 1-2: LOCATION OF POINT SOURCE

and

$$h_3 = \sqrt{\left(\frac{\partial x}{\partial \xi_3}\right)^2 + \left(\frac{\partial y}{\partial \xi_3}\right)^2 + \left(\frac{\partial z}{\partial \xi_3}\right)^2} = \frac{2\sqrt{\xi\eta}}{\sin\phi} \quad .$$

Using the above two sets of equations, we can directly apply the relations of Chapters 1 and 5 of Morse and Feshbach (1953).

1.4 Integral Representations of the Solution

An integral representation of the solutions can be derived from both formulation $(1.P_2)$ and $(1.P_3)$ of Section 1.2. We first show how the method in Ritt and Kazarinoff (1959, 1960) can be applied to derive an integral representation from the formulation $(1.P_3)$. Thus, we begin with the inhomogeneous wave equation in which the wave number has an imaginary part:

$$\nabla^2 v + \gamma^2 v = \rho(\underline{r}) \qquad \left(\gamma = \frac{1}{c}(\omega - is)\right) \quad . \tag{1.1}$$

In the coordinates of the paraboloid of revolution, $\nabla^2 v$ has the representation given by (Morse and Feshbach, 1953, Chapter 5, together with Section 1.3)

$$\nabla^2 v = \frac{1}{2(\xi+\eta)} \left\{ \left[\frac{\partial}{\partial \xi}\left(2\xi \frac{\partial v}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(2\eta \frac{\partial v}{\partial \eta}\right) + \frac{\xi+\eta}{2\xi\eta} \frac{\partial^2 v}{\partial \phi^2} \right] \right\} .$$

But since $\rho(\underline{r})$ represents a point source on the axis of the paraboloid of revolution, the problems have axial symmetry; hence, the ϕ dependence can be removed. Therefore, equation (1.1) becomes

$$\frac{\partial}{\partial \xi}\left(\xi \frac{\partial v}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\eta \frac{\partial v}{\partial \eta}\right) + \gamma^2(\xi+\eta) = (\xi+\eta)\rho(\xi,\eta) \quad . \tag{1.2}$$

We consider first the point source at $(\Xi, 0)$. In this case, $(\xi+\eta)\rho(\xi,\eta) = C\,\delta(\xi-\Xi)\delta(\eta)$, where C is a constant which depends on the precise form of $\rho(\underline{r})$, i.e. on the definition of a point source. We now make the stipulation (or normalization) that our definition of a point source is such that $C = 1$. (We show in Appendix A that this implies that $\rho(\underline{r}) = 4\pi\delta(\underline{r}-\underline{r}_0)$, where $\underline{r}_0$ is the position vector of the point source.) Substitution of this choice in (1.2) yields

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial v}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\eta\frac{\partial v}{\partial\eta}\right)+\gamma^2(\xi+\eta)=\delta(\xi-\Xi)\delta(\eta) \;,$$

which can be written as

$$-L_\eta v - L_\xi v = \delta(\xi-\Xi)\delta(\eta) \;, \tag{1.3}$$

where

$$L_x y = \frac{-d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y \;, \qquad \text{with } p(x) = x,\; q(x) = -\gamma^2 x \;,$$

In order to proceed, we need to make a study of the operators L_η and L_ξ defined by

$$L_\eta y = -\frac{d}{d\eta}\left(\eta\frac{dy}{d\eta}\right) - \gamma^2\eta y \;, \qquad 0 \leqslant \eta \leqslant \eta_0 \;,$$

and

$$L_\xi y = -\frac{d}{d\xi}\left(\xi\frac{dy}{d\xi}\right) - \gamma^2\xi y \;, \qquad 0 \leqslant \xi < \infty \;,$$

since these operators do not correspond exactly with the ones studied in Ritt and Kazarinoff (1959, 1960). As they do, we shall use the papers by Sims (1957) and Phillips (1952) as the basis of this study. Before beginning, we note

$$p(x) = x, \qquad q(x) = -\gamma^2 x \quad \text{which implies} \quad \operatorname{Im} q(x) = \frac{2\omega s}{c^2}\,x \geqslant \text{some } q_0 \;.$$

Thus, although the operators L_η and L_ξ do not correspond exactly with the ones studied in Ritt and Kazarinoff (1959, 1960), the conditions on p and Im q do.

For L_η, the homogeneous differential equation to be studied is $L_\eta y - \lambda y = 0$. It can be written as

$$\frac{d^2y}{d\eta^2} + \frac{1}{\eta}\frac{dy}{d\eta} + \left(\gamma^2 + \frac{\lambda}{\eta}\right)y = 0. \tag{1.4}$$

The substitution $y = u\eta^{-1/2}$ results in the equation

$$\frac{d^2u}{d\eta^2} + \left(\gamma^2 + \frac{\lambda}{\eta} + \frac{1}{4\eta^2}\right)u = 0 \ ,$$

or

$$\frac{d^2u}{d(\pm 2i\gamma\eta)^2} + \left(-\frac{1}{4} + \frac{\lambda}{\pm 2i\gamma(\pm 2i\gamma\eta)} + \frac{1}{4(\pm 2i\gamma\eta)^2}\right) u = 0 \ , \qquad (1.4a)$$

which is Whittaker's equation. It has the two Whittaker functions $M_{\pm\lambda/2i\gamma,0}(\pm 2i\gamma\eta)$, $W_{\pm\lambda/2i\gamma,0}(\pm 2i\gamma\eta)$ as linearly independent solutions. A complete discussion of this equation together with these functions is found in Buchholz (1953). The solutions $M_{\pm\lambda/2i\gamma,0}(\pm 2i\gamma\eta)$ are regular at $\eta = 0$ and lie in $\mathcal{L}_2(0, \eta_0)$, where $\mathcal{L}_2(a, b)$ is the class of all square integrable functions on (a, b). Except for certain values of λ, the solutions $W_{\pm\lambda/2i\gamma,0}(\pm 2i\gamma\eta)$ are not regular at zero, but are in $\mathcal{L}_2(0, \eta_0)$ for all values of λ.

To see which pair of linearly independent solutions are most natural to use in the definition of the re solvent Green's function, we make use of the fact that the solutions $M_{\pm\lambda/2i\gamma,0}(\pm 2i\gamma\eta)$ are linearly dependent while the solutions $W_{\pm\lambda/2i\gamma,0}(\pm 2i\gamma\eta)$ are linearly independent. Then since $y_1(\eta,\lambda) = \eta^{-1/2} M_{\lambda/2i\gamma,0}(2i\gamma\eta)$ and $y_2(\eta,\lambda) = \eta^{-1/2} W_{\lambda/2i\gamma,0}(2i\gamma\eta)$ are two linearly independent solutions of (1.4) such that

(1) $y_1(\eta,\lambda)$ is regular at $\eta = 0$,

(2) $y_2(\eta,\lambda)$ is not regular at $\eta = 0$, except for the value $\frac{\lambda}{2i\gamma} = n + \frac{1}{2}$ or $\lambda = i\gamma(2n+1)$,

they are the natural solutions to use in the definition of the resolvent Green's function. Furthermore, the properties of L_η, $0 \leqslant \eta \leqslant \eta_0$, are:

(1) $\eta = 0$ is a regular singular point of $L_\eta y - \lambda y = 0$, $p(0) = 0$,

(2) $p(\eta_0) \neq 0$,

(3) for $\mathrm{Im}\,\lambda < q_o = 0$ (hence λ cannot equal $i\gamma(2n+1)$), the homogeneous equation $L_\eta y - \lambda y = 0$ has exactly one linearly independent solution regular at $\eta = 0$.

To find the resolvent Green's function of the operator L_η, $0 \leqslant \eta \leqslant \eta_o$, we need a solution $\phi_1(\eta,\lambda)$ of $L_\eta y - \lambda y = 0$ which satisfies the boundary condition at η_o, together with a solution $\phi_2(\eta,\lambda)$ of $L_\eta y - \lambda y = 0$ which is regular at $\eta = 0$. If we consider the Neumann problem, this is accomplished by the choice

$$\phi_1(\eta,\lambda) = y_2(\eta,\lambda)\left(\frac{dy_1(\eta,\lambda)}{d\eta}\right)_{\eta=\eta_o} - y_1(\eta,\lambda)\left(\frac{dy_2(\eta,\lambda)}{d\eta}\right)_{\eta=\eta_o} ,$$

and

$$\phi_2(\eta,\lambda) = y_1(\eta,\lambda) .$$

(Hereafter, we shall denote the derivative $\left(\frac{dF(\eta,\lambda)}{d\eta}\right)_{\eta=\eta_o}$ by $F'(\eta_o,\lambda)$.) With these definitions, the Wronskian $W\left[\phi_1(\eta,\lambda),\phi_2(\eta,\lambda)\right]$ of $\phi_1(\eta,\lambda)$ and $\phi_2(\eta,\lambda)$ becomes

$$W\left[\phi_1(\eta,\lambda),\phi_2(\eta,\lambda)\right] = (2i\gamma)y_1'(\eta_o,\lambda)\,W\left[y_2(\eta,\lambda),y_1(\eta,\lambda)\right] ,$$

which reduces to (Buchholz, 1953)

$$W\left[\phi_1(\eta,\lambda),\phi_2(\eta,\lambda)\right] = \frac{(2i\gamma)y_1'(\eta_o,\lambda)}{\eta\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)} .$$

It should be noted that

$$\phi_1(\eta_o,\lambda) = 2i\gamma\,W\left[y_2(\eta_o,\lambda),y_1(\eta_o,\lambda)\right] = \frac{2i\gamma}{\eta_o\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)} .$$

Thus, the resolvent Green's function $G_N(\eta,\eta',\lambda)$ can be written as

$$G_N(\eta,\eta',\lambda) = \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)}{(2i\gamma)y_1'(\eta_o,\lambda)}\begin{cases}\phi_1(\eta,\lambda)\phi_2(\eta',\lambda) & \eta > \eta' \\ \phi_1(\eta',\lambda)\phi_2(\eta,\lambda) & \eta < \eta'\end{cases} ,$$

while the resolvent operator has the representation

$$R_\lambda^{(N)} y = \int_0^{\eta_o} G_N(\eta, \eta', \lambda) y(\eta') d\eta' \ .$$

For the Dirichlet problem, a solution $\psi_1(\eta, \lambda)$ of $L_\eta y - \lambda y = 0$ which satisfies the boundary condition at η_o, together with a solution $\psi_2(\eta, \lambda)$ which is regular at $\eta = 0$ is given by

$$\psi_1(\eta, \lambda) = y_2(\eta, \lambda) y_1(\eta_o, \lambda) - y_1(\eta, \lambda) y_2(\eta_o, \lambda) \ ,$$

and

$$\psi_2(\eta, \lambda) = y_1(\eta, \lambda) \ .$$

Then

$$W\left[\psi_1(\eta, \lambda), \psi_2(\eta, \lambda)\right] = (2i\gamma) y_1(\eta_o, \lambda)\, W\left[y_2(\eta, \lambda), y_1(\eta, \lambda)\right] \ ,$$

which reduces to (Buchholz, 1953)

$$W\left[\psi_1(\eta, \lambda), \psi_2(\eta, \lambda)\right] = \frac{(2i\gamma) y_1(\eta_o, \lambda)}{\eta \Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)} \ .$$

Thus, the resolvent Green's function can be written as

$$G_D(\eta, \eta', \lambda) = \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)}{(2i\gamma) y_1(\eta_o, \lambda)} \begin{cases} \psi_1(\eta, \lambda)\psi_2(\eta', \lambda) & \eta > \eta' \\ \psi_1(\eta', \lambda)\psi_2(\eta, \lambda) & \eta < \eta' \end{cases}$$

while the resolvent operator has the same representation as above (with N replaced by D).

The properties of the operator L_ξ can be written down immediately. They are:

(1) $\xi = 0$ is a regular singular point for $L_\xi y - \lambda y = 0$, $p(0) = 0$,

(2) for $\operatorname{Im}\lambda < q_o = 0$, the homogeneous equation $L_\xi y - \lambda y = 0$ has exactly one linearly independent solution

$$y_1(\xi,\lambda) = \xi^{-1/2} M_{\lambda/2i\gamma,0}(2i\gamma\xi) \quad ,$$

which is regular at $\xi = 0$ and lies in $\mathcal{L}_2(0,\xi_o)$ $(0 < \xi_o < \infty)$, plus exactly one linearly independent solution

$$v_2(\xi,\lambda) = \xi^{-1/2} W_{\lambda/2i\gamma,0}(2i\gamma\xi) \ ,$$

which is regular at infinity and lies in $\mathcal{L}_2(\xi_o,\infty)$.

Hence, the resolvent Green's function $\tilde{G}(\xi,\xi',\lambda)$ can be written as

$$\tilde{G}(\xi,\xi',\lambda) = \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)}{2i\gamma} \begin{cases} y_1(\xi,\lambda)y_2(\xi',\lambda) & \xi < \xi' \\ y_1(\xi',\lambda)y_2(\xi,\lambda) & \xi > \xi' \end{cases} ,$$

while the resolvent operator $\tilde{R}_\lambda$ has the representation

$$\tilde{R}_\lambda y = \int_0^\infty G(\xi,\xi',\lambda)y(\xi')d\xi' \quad .$$

It should be observed that in this case $\tilde{R}_\lambda$ is not only analytic in $\operatorname{Im}\lambda < q_o = 0$, but also in the larger half-plane $\operatorname{Im}\lambda < k$.*

We can now proceed with the method in Ritt and Kazarinoff to find the integral representations for the solutions. Consider the Neumann problem, and let Γ be a path in the complex λ-plane defined by the straight line running from $-\infty - i\sigma$ to $\infty - i\sigma$, $0 < \sigma < k$. (We shall always assume that the λ-plane is cut at $\lambda = -3\pi/4$ and thus, $\arg\lambda$ satisfies $-3\pi/4 < \arg\lambda \lesssim 5\pi/4$.) Then, applying the

*See Appendix B.

resolvents $R_\lambda^{(N)}$, $\tilde{R}_{-\lambda}$ successively to equation (1.3), using the resolvent relation $R_\lambda^{(N)}(L_x y - \lambda y) = y$, and integrating along Γ, noting that the singular points of $R_\lambda^{(N)}$ lie above Γ while those of $\tilde{R}_{-\lambda}$ lie below Γ, we find the integral representation

$$v_N(\xi, \eta, \Xi, 0, s) = \frac{1}{2\pi i} \int_\Gamma \tilde{G}(\xi, \Xi, -\lambda) G_N(\eta, 0, \lambda) d\lambda \ ,$$

where $v_N(\xi, \eta, \Xi, 0, s)$ denotes the solution of the Neumann problem. Substituting for $\tilde{G}(\xi, \Xi, -\lambda)$ and $G_N(\eta, 0, \lambda)$, we obtain

$$v_N^{(0)}(\xi, \eta, \Xi, 0) = \lim_{s \to 0^+} \frac{1}{2\pi i} \int_\Gamma d\lambda \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2} + \frac{\lambda}{2i\gamma}\right)}{(2i\gamma)^2 y_1'(\eta_o, \lambda)} y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda) \cdot$$
$$\cdot\, y_1(0, \lambda) \phi_1(\eta, \lambda) \ ,$$

where $v_N^{(0)}(\xi, \eta, \Xi, 0) = \lim_{s \to 0^+} v_N(\xi, \eta, \Xi, 0, s)$, $\xi_1 = \min(\xi, \Xi)$, $\xi_2 = \max(\xi, \Xi)$, $\xi_1 < \xi_2$ and $\eta > 0$. But

$$y_1(0, \lambda) = (2i\gamma)^{1/2} \ ,$$

and

$$\phi_1(\eta \lambda) = y_2(\eta, \lambda) y_1'(\eta_o, \lambda) - y_1(\eta, \lambda) y_2'(\eta_o, \lambda) \ .$$

Therefore

$$v_N^{(0)}(\xi, \eta, \Xi, 0) = \lim_{s \to 0^+} \frac{(2i\gamma)^{-3/2}}{2\pi i} \int_\Gamma d\lambda \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2} + \frac{\lambda}{2i\gamma}\right)}{y_1'(\eta_o, \lambda)} y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda) \cdot$$
$$\cdot \left[y_2(\eta, \lambda) y_1'(\eta_o, \lambda) - y_1(\eta, \lambda) y_2'(\eta_o, \lambda) \right] . \tag{1.5}$$

If $\eta = \eta_o$,

$$\phi_1(\eta_o, \lambda) = \frac{2i\gamma}{\eta_o \Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)} \ .$$

Thus, the field on the surface is given by the simpler formula;

$$v_N^{(0)}(\xi,\eta_o,\Xi,0) = \lim_{s\to 0^+} \frac{(2i\gamma)^{-1/2}}{\eta_o(2\pi i)} \int_\Gamma d\lambda \frac{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1'(\eta_o,\lambda)} y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda) \ . \tag{1.6}$$

The above procedure may be repeated to obtain $v_D^{(0)}(\xi,\eta,\Xi,0)$, the solution to the Dirichlet problem for the point source at $(\Xi,0)$. Let Γ be the path in the complex λ-plane defined above; then

$$v_D^{(0)}(\xi,\eta,\Xi,0) = \lim_{s\to 0^+} \frac{(2i\gamma)^{-3/2}}{2\pi i} \int_\Gamma d\lambda \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1(\eta_o,\lambda)} y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)\cdot$$

$$\cdot\left[y_2(\eta,\lambda)y_1(\eta_o,\lambda)-y_1(\eta,\lambda)y_2(\eta_o,\lambda)\right] \ . \tag{1.7}$$

The field on the surface for the Dirichlet problem is given by the normal derivative

$$\left(\frac{\partial v_D^{(0)}(\xi,\eta,\Xi,0)}{\partial n}\right)_{\eta=\eta_o} \ .$$

Using the relation (Morse and Feshbach, 1953, Chapter 1, together with Section 1.3)

$$\frac{\partial v_D^{(0)}}{\partial n} = \frac{\eta^{1/2}}{(\xi+\eta)^{1/2}} \frac{\partial v_D^{(0)}}{\partial \eta} \ ,$$

we observe that the equation for the field on the surface is

$$\left.\frac{\partial v_D^{(0)}(\xi,\eta,\Xi,0)}{\partial n}\right|_{\eta=\eta_o} = \frac{-\lim_{s\to 0^+}}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}} \frac{(2i\gamma)^{-1/2}}{2\pi i} \int_\Gamma d\lambda \frac{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1(\eta_o,\lambda)} \cdot$$

$$\cdot\, y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda) \ , \tag{1.8}$$

where the condition $\xi_1 < \xi_2$ is understood not only for (1.8) but also for (1.5) through (1.7). In addition, we note the condition $\eta > 0$ for (1.5) and (1.7).

In the case of the point source at (0, H), $(\xi+\eta)\rho(\xi,\eta) = C\,\delta(\xi)\delta(\eta-H)$. Then the choice of $C = 1$ implies that equation (1.3) is replaced by

$$-L_\eta v - L_\xi v = \delta(\xi)\,\delta(\eta-H) \ .$$

Repeating the above procedure for the Neumann problem, we find

$$v_N(\xi,\eta,0,H,s) = \frac{1}{2\pi i}\int_\Gamma \tilde{G}(\xi,0,-\lambda)G_N(\eta,H,\lambda)d\lambda \ ,$$

while the solution of the Dirichlet problem has the same representation with N replaced by D. Substituting for the resolvent Green's functions, we obtain

$$v_N^{(0)}(\xi,\eta,0,H) = \lim_{s\to 0^+}\frac{(2i\gamma)^{-3/2}}{2\pi i}\int_\Gamma d\lambda\,\frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1'(\eta_0,\lambda)}\,y_2(\xi,-\lambda)y_1(\eta_1,\lambda)\cdot$$

$$\cdot\left[y_2(\eta_2,\lambda)y_1'(\eta_0,\lambda)-y_1(\eta_2,\lambda)y_2'(\eta_0,\lambda)\right] , \qquad (1.9)$$

$$v_N^{(0)}(\xi,\eta_0,0,H) = \lim_{s\to 0^+}\frac{(2i\gamma)^{-1/2}}{\eta_0(2\pi i)}\int_\Gamma d\lambda\,\frac{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1'(\eta_0,\lambda)}\,y_2(\xi,-\lambda)y_1(H,\lambda) \ , \qquad (1.10)$$

$$v_D^{(0)}(\xi,\eta,0,H) = \lim_{s\to 0^+}\frac{(2i\gamma)^{-3/2}}{2\pi i}\int_\Gamma d\lambda\,\frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1(\eta_0,\lambda)}\,y_2(\xi,-\lambda)y_1(\eta_1,\lambda)\cdot$$

$$\cdot\left[y_2(\eta_2,\lambda)y_1(\eta_0,-\lambda)-y_1(\eta_2,\lambda)y_2(\eta_0,\lambda)\right] , \qquad (1.11)$$

$$\left(\frac{\partial v_D^{(0)}(\xi,\eta,0,H)}{\partial n}\right)_{\eta=\eta_0} = \frac{-\lim_{s\to 0^+}}{\left[\eta_0(\xi+\eta_0)\right]^{1/2}}\,\frac{(2i\gamma)^{-1/2}}{2\pi i}\int_{\Gamma} d\lambda\,\frac{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{y_1(\eta_0,\lambda)}\,y_2(\xi,-\lambda)y_1(H,\lambda)\,, \tag{1.12}$$

where the definitions $\eta_1 = \min(\eta, H)$, $\eta_2 = \max(\eta, H)$, and the condition $\xi > 0$ are understood for (1.9) through (1.12), while the condition $\eta_1 < \eta_2$ is also understood for (1.9) and (1.11).

Consider now equations (1.5) through (1.12). We note that there is an essential difference between these equations and the similar equations in Ritt and Kazarinoff (1959, 1960). For these equations Γ is independent of s; thus, it may be possible to take the limit as $s\to 0^+$ inside the integral. According to Buchholz (1953), the functions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are entire functions of $\lambda/2i\gamma$ while the path Γ is defined so that the functions $\Gamma\left(\frac{\lambda}{2i\gamma}+\frac{1}{2}\right)$ and $\Gamma\left(-\frac{\lambda}{2i\gamma}+\frac{1}{2}\right)$ are analytic functions of $\lambda/2i\gamma$ on Γ. Therefore, we can take the limit as $s\to 0^+$ inside the integral and equations (1.5) through (1.12) are valid without the $\lim_{s\to 0^+}$ condition if the parameter γ is replaced by the paramter k.

Buchholz (1942/3, 1953) derives an integral representation for $v(\underline{r})$ from the formulation (1.P$_2$). For completeness we reproduce this derivation. We first consider the point source at $(\Xi, 0)$. For $\rho(\underline{r}) = 4\pi\delta(\underline{r}-\underline{r}_0)$ ($\underline{r}_0$ is the vector to the point source at $(\Xi,0)$), the free space Green's function has the form $-e^{-ikR}/R$ ($R = |r - r_0|$, time dependence $e^{i\omega t}$). Thus, he first derives an integral representation for $-e^{-ikR}/R$. It is

$$-\frac{e^{-ikR}}{R} = -\frac{2ik}{2\pi i}\int_{-\sigma'-i\infty}^{-\sigma'+i\infty} ds\,\Gamma(s+\tfrac{1}{2})\,\Gamma(-s+\tfrac{1}{2})m_s^{(0)}(2ik\xi_1)w_s^{(0)}(2ik\xi_2)w_{-s}^{(0)}(2ik\eta)\,,$$

where $|\sigma'| < 1/2$, $\xi \neq \Xi$, $\eta > 0$, $m_s^{(0)}(2ikx) = (2ikx)^{-1/2}M_{s,0}(2ikx)$ and $w_s^{(0)}(2ikx) = (2ikx)^{-1/2}W_{s,0}(2ikx)$. Then he assumes $v(\xi,\eta,\Xi,0)$ has the form

$$v(\xi,\eta,\Xi,0) = v'(\xi,\eta,\Xi,0) + \left(-\frac{e^{-ikR}}{R}\right)$$

with

$$v'(\xi,\eta,\Xi,0) = \frac{1}{2\pi i}\int_{-\sigma'-i\infty}^{-\sigma'+i\infty} ds\, \Gamma(s+\tfrac{1}{2})\,\Gamma(-s+\tfrac{1}{2})\, m_s^{(0)}(2ik\xi_1) w_s^{(0)}(2ik\xi_2) A_s m_{-s}^{(0)}(2ik\eta) ,$$

where again $|\sigma'| < 1/2$ and A_s is an unknown function of s. Hence, $v(\xi,\eta,\Xi,0)$ formally satisfies the inhomogeneous wave equation while the boundary condition may be satisfied by a suitable choice of A_s. For

$$\frac{\partial v}{\partial n} = 0 \quad \text{on the boundary,} \qquad A_s^{(N)} = 2ik\,\frac{w_{-s}^{(0)'}(2ik\eta_o)}{m_{-s}^{(0)'}(2ik\eta_o)} ,$$

while for

$$v = 0 \quad \text{on the boundary,} \qquad A_s^{(D)} = 2ik\,\frac{w_{-s}^{(0)}(2ik\eta_o)}{m_{-s}^{(0)}(2ik\eta_o)} ,$$

where again

$$F'(2ik\eta_o) = \left(\frac{d}{d\eta} F(2ik\eta)\right)_{\eta=\eta_o} .$$

We continue by considering explicitly the Neumann problem. The obvious modifications can be made for the Dirichlet problem. Substituting for $A_s^{(N)}$, we obtain

$$v_N(\xi,\eta,\Xi,0) = \frac{2ik}{2\pi i}\int_{-\sigma'-i\infty}^{-\sigma'+i\infty} ds\, \Gamma(s+\tfrac{1}{2})\,\Gamma(-s+\tfrac{1}{2})\,\frac{m_s^{(0)}(2ik\xi_1) w_s^{(0)}(2ik\xi_2)}{m_{-s}^{(0)'}(2ik\eta_o)} \cdot$$

$$\cdot \left[w_{-s}^{(0)'}(2ik\eta_o) m_{-s}^{(0)}(2ik\eta) - m_{-s}^{(0)'}(2ik\eta_o) w_{-s}^{(0)}(2ik\eta)\right] \qquad (1.13)$$

as a formal solution to the inhomogeneous wave equation which satisfies the boundary condition. If as $\xi \to \infty$ we compare the behavior of the integrand of (1.13) with the one in the representation for $-e^{-ikR}/R$, we note that $v_N(\xi, \eta, \Xi, 0)$ satisfies the radiation condition. Thus, it remains to show that the integral exists. Using the previous $y_1(x, \lambda)$, $y_2(x, \lambda)$ notation in equation (1.13), and substituting $-s$ for s, we find

$$v_N(\xi, \eta, \Xi, 0) = -\frac{(2ik)^{-1/2}}{2\pi i} \int_{\sigma'+i\infty}^{\sigma'-i\infty} ds\,\Gamma(s+\tfrac{1}{2})\Gamma(-s+\tfrac{1}{2}) \frac{y_1(\xi_1, -2iks)y_2(\xi_2, -2iks)}{y_1'(\eta_0, 2iks)} \cdot$$
$$\cdot \Big[y_1(\eta, 2iks)y_2'(\eta_0, 2iks) - y_2(\eta, 2iks)y_1'(\eta_0, 2iks)\Big]. \qquad (1.14)$$

It is shown in Appendix C or Buchholz (1942/3, 1953) that

$$\Gamma(-s+\tfrac{1}{2})\Big[y_1(\eta, 2iks)y_2'(\eta_0, 2iks) - y_2(\eta, 2iks)y_1'(\eta_0, 2iks)\Big]$$

is analytic in the complex s-plane, $y_1'(\eta_0, 2iks)$ has a countable infinity of simple zeros which lie along the imaginary axis, and the integrand of equation (1.14) is exponentially decreasing on a large semi-circle in the right half s-plane. Hence for $0 < \sigma' < 1/2$, the integral represents the zero solution which can be omitted by the further restriction $-1/2 < \sigma' \leqslant 0$. Moreover, for $\sigma' = 0$, the integral is not defined, and so we finally obtain the restriction $-1/2 < \sigma' < 0$. In addition, it is seen in Appendix C that along this path the integral converges. Thus, $v_N(\xi, \eta, \Xi, 0)$ given by equation (1.14) together with the restriction $-1/2 < \sigma' < 0$ is a solution of the Neumann problem defined by the formulation $(1.P_2)$. We write it as

$$v_N(\xi, \eta, \Xi, 0) = \frac{(2ik)^{-1/2}}{2\pi i} \int_{\substack{\sigma'+i\infty \\ (-1/2<\sigma'<0)}}^{\sigma'-i\infty} ds\,\Gamma(s+\tfrac{1}{2})\Gamma(-s+\tfrac{1}{2}) \frac{y_1(\xi_1, -2iks)y_2(\xi_2, -2iks)}{y_1'(\eta_0, 2iks)} \cdot$$
$$\cdot \Big[y_2(\eta, 2iks)y_1'(\eta_0, 2iks) - y_1(\eta, 2iks)y_2'(\eta_0, 2iks)\Big] \quad . \qquad (1.15)$$

If in (1.15) the substitution $s = \lambda/2ik$ is made, this equation becomes

$$v_N(\xi,\eta,\Xi,0) = \frac{(2ik)^{-3/2}}{2\pi i}\int_{-\infty+2ik\sigma'}^{\infty+2ik\sigma'} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_0,\lambda)}\cdot$$

$$\cdot\left[y_2(\eta,\lambda)y_1'(\eta_0,\lambda)-y_1(\eta,\lambda)y_2'(\eta_0,\lambda)\right],$$

with $-1/2 < \sigma' < 0$ which implies

$$v_N(\xi,\eta,\Xi,0) = \frac{(2ik)^{-3/2}}{2\pi i}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_0,\lambda)}\cdot$$

$$\cdot\left[y_2(\eta,\lambda)y_1'(\eta_0,\lambda)-y_1(\eta,\lambda)y_2'(\eta_0,\lambda)\right]. \tag{1.16}$$

As asserted, the integral in equation (1.16) agrees with the one in equation (1.5). Substituting $\eta=\eta_0$ in (1.16) yields the equation for the field on the surface

$$v_N(\xi,\eta_0,\Xi,0) = \frac{(2ik)^{-1/2}}{\eta_0(2\pi i)}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1'(\eta_0,\lambda)}, \tag{1.17}$$

where the integral in (1.17) agrees with the one in (1.6). Similarly, for the Dirichlet problem

$$v_D(\xi,\eta,\Xi,0) = \frac{(2ik)^{-3/2}}{2\pi i}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1(\eta_0,\lambda)}\cdot$$

$$\cdot\left[y_2(\eta,\lambda)y_1(\eta_0,\lambda)-y_1(\eta,\lambda)y_2(\eta_0,\lambda)\right], \tag{1.18}$$

while for the field on the surface

$$\left(\frac{\partial v_D(\xi,\eta,\Xi,0)}{\partial n}\right)_{\eta=\eta_o} = \frac{-1}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}} \frac{(2ik)^{-1/2}}{2\pi i} \int_{\substack{-\infty-i\sigma \\ (0<\sigma<k)}}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) \cdot$$

$$\cdot \frac{y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)}{y_1(\eta_o,\lambda)} , \qquad (1.19)$$

where the definitions $\xi_1 = \min(\xi,\Xi)$, $\xi_2 = \max(\xi,\Xi)$, and the condition $\xi \neq \Xi$ are understood not only for (1.19), but also (1.16) through (1.18). In addition, the condition $\eta > 0$ is understood for (1.16) and (1.18).

For the point source at (0, H), we begin with the following integral representation for $-e^{-ikR}/R$:

$$-\frac{e^{-ikR}}{R} = -\frac{2ik}{2\pi i}\int_{-\sigma'-i\infty}^{\sigma'+i\infty} ds\, \Gamma(s+\tfrac{1}{2})\,\Gamma(-s+\tfrac{1}{2})\,w_s^{(0)}(2ik\xi)\,m_{-s}^{(0)}(2ik\eta_1)\,w_{-s}^{(0)}(2ik\eta_2) ,$$

where $|\sigma'| < 1/2$, $\xi > 0$, $\eta_1 = \min(\eta, H)$, $\eta_2 = \max(\eta, H)$, and $\eta \neq H$. Proceeding as above, we would then obtain

$$v_N(\xi,\eta,0,H) = \frac{(2ik)^{-3/2}}{2\pi i} \int_{\substack{-\infty-i\sigma \\ (0<\sigma<k)}}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) \frac{y_2(\xi,-\lambda)y_1(\eta_1,\lambda)}{y_1'(\eta_o,\lambda)} \cdot$$

$$\cdot \left[y_2(\eta_2,\lambda)y_1'(\eta_o,\lambda) - y_1(\eta_2,\lambda)y_2'(\eta_o,\lambda)\right], \qquad (1.20)$$

$$v_N(\xi,\eta_o,0,H) = \frac{(2ik)^{-1/2}}{\eta_o(2\pi i)} \int_{\substack{-\infty-i\sigma \\ (0<\sigma<k)}}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) \frac{y_2(\xi,-\lambda)y_1(H,\lambda)}{y_1'(\eta_o,\lambda)} , \qquad (1.21)$$

$$v_D(\xi,\eta,0,H) = \frac{(2ik)^{-3/2}}{2\pi i}\int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{y_2(\xi,-\lambda)y_1(\eta_1,\lambda)}{y_1(\eta_0,\lambda)} \cdot$$
$$(0<\sigma<k)$$
$$\cdot\left[y_2(\eta_2,\lambda)y_1(\eta_0,\lambda) - y_1(\eta_2,\lambda)y_2(\eta_0,\lambda)\right], \quad (1.22)$$

$$\left(\frac{\partial v_D(\xi,\eta,0,H)}{\partial n}\right)_{\eta=\eta_0} = \frac{-1}{\left[\eta_0(\xi+\eta_0)\right]^{1/2}}\,\frac{(2ik)^{-1/2}}{2\pi i}\int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) \cdot$$
$$(0<\sigma<k)$$
$$\cdot\ \frac{y_2(\xi,-\lambda)y_1(H,\lambda)}{y_1(\eta_0,\lambda)}, \quad (1.23)$$

with the above conditions on η_1, η_2, and ξ.

1.5 Results of the Asymptotic Analysis of the Integral Representations

In Section 1.4 we derived two equivalent sets of integral representations, each containing one representation for the solution of the various problems formulated. We first discuss some general results pertaining to each representation. We do this by considering explicitly equation (1.16), and noting that the results pertaining to (1.16) either pertain directly or with a slight modification to the remaining integral representations. In Appendix C, we prove that the integral representation (1.16) may, for all positions of the source and field points, be replaced by a convergent residue series expansion obtained by summing the residues at the zeros of the function $y_1'(\eta_0,\lambda)$ (poles of $G_N(\eta,\eta',\lambda)$). In Section 3.2, we prove that this series converges quite slowly for short wavelengths or high frequencies, i.e. for wave numbers k ($k = \omega/c = 2\pi/\lambda$) and focal lengths η_0 such that $k\eta_0 \gg 1$, Hence if $k\eta_0 \gg 1$, this residue series behaves like the Mie type series found in the diffraction by smooth, bounded, convex bodies (Ritt and Kazarinoff, 1959; 1960).

The above behavior for $k\eta_0 \gg 1$ leads us to investigate the residue series for long wavelengths or low frequencies, i.e. for wave numbers k and focal lengths

η_o such that $k\eta_o \ll 1$. If this series is to be truly analogous to the Mie type series found in the diffraction by smooth, bounded, convex bodies, it should readily yield the first term in the asymptotic expansion for $v_N(\xi, \eta, \Xi, 0)$ if $k\eta_o \ll 1$. We show this to be true in Sections 2.2 and 2.5. The explicit results established in Chapter 2 are summarized below in Theorems 1 through 4.

The possibility of further analogy with the diffraction by smooth, bounded, convex bodies suggests that in order to find an asymptotic representation for $v_N(\xi, \eta, \Xi, 0)$ if $k\eta_o \gg 1$, we should examine the residue series obtained by summing the residues at the poles of the Γ-function $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ (poles of $G(\xi, \xi', -\lambda)$). We show in Appendix C, that this residue series is convergent if and only if the source and field points are subject to the restriction $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$. Thus, any asymptotic representation of $v_N(\xi, \eta, \Xi, 0)$ derived from this residue series would be subject to the same restriction and could not be used to describe $v_N(\xi, \eta, \Xi, 0)$ throughout the interior of the paraboloid. We show in Sections 3.3 and 3.4 that if $k\eta_o \gg 1$, an asymptotic representation of the integral in equation (1.17) which is valid for all positions of the source and field points can be derived by considering the saddle point contributions to the integral. The explicit results obtained in Chapter 3 are summarized below in Theorems 5 through 8.

The results of this report differ from those of Buchholz (1942/3, 1953) in that his analysis is not directly concerned with large or small values of $k\eta_o$. He is primarily concerned with analyzing the total field by considering the separate terms (or modes) in each of the two residue series, particularly the one obtained by summing the residues at the zeros of $y_1'(\eta_o, \lambda)$. However he does obtain, by different methods, an asymptotic representation of the zeros of $y_1(\eta_o, \lambda)$ (1953, Chapter 17, pg. 189, eq. 14) and $y_1'(\eta_o, \lambda)$ (1942/3, pg. 432, eq. 3.10) if $k\eta_o \ll 1$. Comparing his results with those of Section 2.1, we see that the first terms agree and the error term in these equations (Section 211, equations (2.15a) and (2.13a), respectively) has the same order as his explicit second term.

1.5.1 Low Frequency Theorems

For $k\eta_0 \ll 1$ we study two distinct cases. The first considers both the source and field points to be near (with respect to wavelength) the focal points; it is called the near field and is defined by the mathematical conditions $k\xi \ll 1$, $k\Xi \ll 1$. One reason for interest in this case is the investigation of a relationship between the first term in the asymptotic expansion of the total near field and the solution of the corresponding potential problem. For the Dirichlet boundary condition, we obtain (Sections 2.2 and 2.3) the following result:

Theorem 1: If $k\eta_0 \ll 1$, then for the Dirichlet boundary condition the first term in the asymptotic expansion of the total near field (of a point source on the axis of the interior of a paraboloid of revolution) is equal to the solution of the corresponding Dirichlet potential problem.

In the attempt to prove a theorem for the Neumann boundary condition corresponding to Theorem 1, we find two essential differences between the Neumann and Dirichlet problems. The first is that the asymptotic representation of the total near field contains a term which has a logarithmic dependence on $k\xi$. The second is that the Neumann potential cannot be regular at infinity in the usual sense. Instead, we find (Section 2.4) the following condition at infinity:

Theorem 2: A necessary and sufficient condition that ϕ be a solution to the Neumann potential problem (for a point charge on the axis of the interior of the paraboloid of revolution) is that

$$\lim_{\xi \to \infty} \xi \int_0^{\eta_0} \frac{\partial \phi}{\partial \xi} \, d\eta = 1 \ .$$

This result is most likely a characteristic of the domain defined by the interior of a paraboloid of revolution, since the corresponding configuration for a cone does not contradict the usual regularity condition. In addition, it implies that the solution of the Neumann potential problem contains a term which has a logarithmic dependence on ξ. Furthermore, if we call the other terms in this solution the regular part of the Neumann potential, the results of Sections 2.2 and 2.4 also yield:

Theorem 3: If $k\eta_0 \ll 1$, then for the Neumann boundary condition the term, which is independent of k, in the asymptotic representation of the total near field (of a point source on the axis of the interior of a paraboloid of revolution) is equal (to within some constant) to the regular part of the solution of the corresponding Neumann potential problem.

The second case studied if $k\eta_0 \ll 1$ is the point source (field point) in the near field and field point(point source) in the far field. This case is defined by the mathematical conditions $\xi_1/\eta_0 = O(1)$, $k\xi_2 \gg 1$, and corresponds to a configuration of some physical interest. We obtain (Section 2.5) the following result:

Theorem 4: If $k\eta_0 \ll 1$, then for a point source (on the axis of the interior of a paraboloid of revolution) in the near field and field point in the far field, the solution to the Dirichlet problem is exponentially small, while the solution to the Neumann problem contains a single propagating term.

This result can be interpreted in terms of waveguide theory. At a great distance from the source, we can assume that the field is approximately governed by the homogeneous wave equation. In addition, the paraboloid locally resembles a cylinder. Thus the local far field satisfies the boundary value problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial z^2}+k^2 v=0 \quad \text{within an infinite circular cylinder,}$$

$$v=0 \quad \text{or} \quad \frac{\partial v}{\partial n}=0 \quad \text{on the boundary of the cylinder.}$$

The solutions of this axially symmetric cylindrical waveguide problem for the Neumann boundary condition have a propagating term for all values of k (η_0 fixed), while for k small enough the solutions for the Dirichlet boundary condition have no propagating terms.

1.5.2 High Frequency Theorems

If $k\eta_0 \gg 1$, we again investigate two distinct cases. The first considers the source to be far (with respect to wavelength) from the focus and far (with respect to the transition region of $y_1'(\eta_0, \lambda)$) from the tip of the paraboloid. In this case we explicitly study the field on the surface if the point source is at $(\Xi, 0)$ and the Neumann boundary condition holds (equation (1.17)). We obtain (Sections 3.3 and 3.4) the following results:

Theorem 5: If $k\eta_0 \gg 1$, $k\Xi \gg 1$, $k\xi \gg 1$, then $v_N(\xi, \eta_0, \Xi, 0)$ can be asymptotically represented as a finite sum of integrals, where the sum always has at least one term and the total number of terms depends only on η_0, Ξ and ξ. This first term is approximately equal to $-2e^{-ikR}/R$, while the remaining terms (if they exist) can be estimated by the saddle point method.

Theorem 6: Let η_0, Ξ and ξ be such that there exist N terms $(N > 1)$ in the sum of Theorem 5, and let n be an integer such that $1 \leqslant n \leqslant N-1$. Then a sufficient condition that the saddle point equations of the (n+1)th integral have an approximate solution is

$$(\xi_1/\xi_2)^{\frac{1}{2(2n+1)}} \ll 1 \ .$$

Moreover, if $\xi_1 = \Xi$, $\xi_2 = \xi$, and

$$(\Xi/\xi)^{\frac{1}{2(2n+1)}} \ll 1 \ ,$$

the (n+1)th integral is approximately equal to

$$\frac{-2}{\sqrt{\Xi} \cdot \sqrt{\xi}} (\Xi/\xi)^{\frac{1}{2(2n+1)}} e^{-ik\psi(\Xi, \xi, \eta_0, n)} e^{\pi i \frac{n}{2}} \ ,$$

where $\psi(\Xi, \xi, \eta_0, n) = \xi - \Xi + \eta_0 + 2n\eta_0$.

Theorem 7: Let η_o, Ξ and ξ be such that there exists N terms $(N > 1)$ in the sum of Theorem 5, and let Ξ and ξ be such that $(\Xi/\xi)^{1/6} \ll 1$. Then the geometric path traveled by a ray that leaves the source at $(\Xi, 0)$ and arrives at (ξ, η_o) via one reflection is approximately equal to

$$\xi - \Xi + \eta_o + 2\eta_o = \psi(\Xi, \xi, \eta_o, 1) \ .$$

We do not extend Theorem 7 to values of $n > 1$. However, in Section 3.4.3 we do show that if a ray leaves the source at $(\Xi, 0)$ and arrives at (ξ, η_o) via n reflections where the path length of the initial ray and each subsequent reflected ray is a priori large, then the total path length traveled by such a ray is approximately equal to $\psi(\Xi, \xi, \eta_o, n)$. This relation together with Theorems 5, 6 and 7 gives strong impetus to interpret the results obtained in terms of the theory of geometric optics. This interpretation is discussed fully in Section 3.4.3. We also show how similar results can be obtained if the source is at (0, H) and $-k^2H$ is outside the transition region of $y_1'(\eta_o, \lambda)$.

The second case studied if $k\eta_o \gg 1$ is the point source at a point (0, H) near the tip of the paraboloid ($-k^2H$ is inside the transition region of $y_1'(\eta_o, \lambda)$). In this case the sum in Theorem 5 depends on $k\eta_o$. Hence, we consider directly the residue series of equation (1.21) obtained from summing the residues at the zeros of the function $y_1'(\eta_o, \lambda)$. We express our results (Section 3.5) in the following theorem:

Theorem 8: If $k\eta_o \gg 1$, $k\xi \gg 1$, and $-k^2H$ is in the transition region of $y_1'(\eta_o, \lambda)$, then the residue series for $v_N(\xi, \eta_o, 0, H)$ (residues at the zeros of $y_1'(\eta_o, \lambda)$) has two distinct sets of nonexponentially small residue terms. Let $\sum_w$ denote the set whose terms become exponentially small if $-k^2H$ is allowed outside the transition region of $y_1'(\eta_o, \lambda)$. Then if $-k^2H$ lies inside the transition region of $y_1'(\eta_o, \lambda)$, the approximate distance-dependent phase of each of the terms of $\sum_w$ is equal to the arc length from the tip of the paraboloid to the field point (ξ, η_o) under consideration. Furthermore, for field points (ξ, η_o) such that $\xi/\eta_o \gg 1$, the approximate distance-dependent amplitude of each of the terms of $\sum_w$ is equal to $1/\rho$, where ρ is the radius of the cylindrical cross section of the paraboloid of revolution.

Therefore, in this case we observe the existence of a set of residues (previously exponentially small) that can be interpreted as cylindrical waves traveling along the surface of the paraboloid of revolution. These waves are called whispering gallery waves.

II
LOW FREQUENCY (THIN PARABOLOID) DIFFRACTION

As indicated in Section 1.3 a paraboloid of revolution may be characterized by a focal length η_o. For a given wave number k, the mathematical condition $k\eta_o \ll 1$ corresponds either to small values of k (low frequency diffraction) or small values of η_o (thin paraboloid). We wish to investigate the integral representations in this case. For convenience these representations are now given a slightly different form. Let us define (the $m_\chi^{(0)}(z)$, $w_\chi^{(0)}(z)$ notation being used by Buchholz, 1953)

$$v_1(x,\lambda) = m^{(0)}_{\lambda/2ik}(2ikx) = (2ikx)^{-1/2} M_{\lambda/2ik,\,0}(2ikx) = (2ik)^{-1/2}\, y_1(x,\lambda)\,,$$

$$v_2(x,\lambda) = w^{(0)}_{\lambda/2ik}(2ikx) = (2ikx)^{-1/2} W_{\lambda/2ik,\,0}(2ikx) = (2ik)^{-1/2}\, y_2(x,\lambda)\,,$$

and

$$v_1'(x_o,\lambda) = \left(\frac{d}{d(2ikx)}\, v_1(x,\lambda)\right)_{x=x_o}.$$

(This notation for the derivative is different from that of Chapter 1 and Appendix C where the primes simply refer to differentiation with respect to x.) Then equations (1.18) through (1.19) become

$$v_N(\xi,\eta,\Xi,0) = \frac{1}{2\pi i}\int_{-\infty-i\sigma}^{\infty-i\sigma} d\lambda\, \Gamma\!\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\!\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)\,v_2(\xi_2,-\lambda)}{v_1'(\eta_o,\lambda)}\,\cdot$$
$$(0<\sigma<k) \qquad \cdot\left[v_2(\eta,\lambda)v_1'(\eta_o,\lambda) - v_1(\eta,\lambda)v_2'(\eta_o,\lambda)\right], \tag{2.1}$$

$$v_N(\xi,\eta_o,\Xi,0) = \frac{1}{2\pi i(2ik\eta_o)}\int_{-\infty-i\sigma}^{\infty-i\sigma} d\lambda\, \Gamma\!\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{v_1'(\eta_o,\lambda)}\,, \qquad (0<\sigma<k) \tag{2.2}$$

$$v_D(\xi,\eta,\Xi,0) = \frac{1}{2\pi i}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)\,v_2(\xi_2,-\lambda)}{v_1(\eta_o,\lambda)}\cdot \tag{2.3}$$

$$\cdot\left[v_2(\eta,\lambda)\,v_1(\eta_o,\lambda)-v_1(\eta,\lambda)\,v_2(\eta_o,\lambda)\right]$$

$$\left(\frac{\partial v_D(\xi,\eta,\Xi,0)}{\partial n}\right)_{\eta=\eta_o} = \frac{-1}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}}\cdot\frac{1}{2\pi i}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda)\,v_2(\xi_2,-\lambda)}{v_1(\eta_o,\lambda)}, \tag{2.4}$$

where again $\xi_1 = \min(\xi,\Xi)$, $\xi_2 = \max(\xi,\Xi)$ and $\xi \neq \Xi$, $\eta > 0$. Similarly, equations (1.20) through (1.23) become

$$v_N(\xi,\eta,0,H) = \frac{1}{2\pi i}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_2(\xi,-\lambda)\,v_1(\eta_1,\lambda)}{v_1(\eta_o,\lambda)}\cdot \tag{2.5}$$

$$\cdot\left[v_2(\eta_2,\lambda)\,v_1'(\eta_o,\lambda)-v_1(\eta_2,\lambda)\,v_2'(\eta_o,\lambda)\right],$$

$$v_N(\xi,\eta_o,0,H) = \frac{1}{2\pi i(2ik\eta_o)}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_2(\xi,-\lambda)\,v_1(H,\lambda)}{v_1'(\eta_o,\lambda)}, \tag{2.6}$$

$$v_D(\xi,\eta,0,H) = \frac{1}{2\pi i}\int_{\substack{-\infty-i\sigma\\(0<\sigma<k)}}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_2(\xi,-\lambda)\,v_1(\eta_1,\lambda)}{v_1(\eta_o,\lambda)}\cdot$$

$$\cdot\left[v_2(\eta_2,\lambda)\,v_1(\eta_o,\lambda)-v_1(\eta_2,\lambda)\,v_2(\eta_o,\lambda)\right], \tag{2.7}$$

$$\left(\frac{\partial v_D(\xi,\eta,0,H)}{\partial n}\right)_{\eta=\eta_o} = \frac{-1}{\left[\eta_o(\xi+\eta_o)\right]^{1/2}}\cdot\frac{1}{2\pi i}\int_{-\infty-i\sigma}^{\infty-i\sigma} d\lambda\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\frac{v_2(\xi,-\lambda)v_1(H,\lambda)}{v_1(\eta_o,\lambda)}\,, \quad (0<\sigma<k) \tag{2.8}$$

where as previously, $\eta_1 = \min(\eta, H)$, $\eta_2 = \max(\eta, H)$ and $n \neq H$, $\xi > 0$.

2.1 Low Frequency (Thin Paraboloid) Poles

The "low frequency poles" are those poles of the integrands of equations (2.1) through (2.8) corresponding to the zeros of the functions $v_1(\eta_o, \lambda)$ and $v_1'(\eta_o, \lambda)$ if $k\eta_o \ll 1$. In order to analyze these zeros we consider $v_1(\eta, \lambda)$ for $k\eta \ll 1$. We first assume $0 \leqslant |\lambda/2k| \leqslant O(1)$. Then the power series expansion for $v_1(\eta, \lambda)$ (or $m_\chi^{(0)}(z)$ to keep the equations brief) is (Buchholz, 1953, Chapter 2, equation 7)

$$m_\chi^{(0)}(z) = e^{-z/2}\sum_{r=0}^{\infty}\frac{(\frac{1}{2}-\chi)_r\, z^r}{(1)_r\, r!}\,,$$

which for $z \to 0$ can be written as

$$m_\chi^{(0)}(z) = \left[1-\frac{z}{2}+\frac{z^2}{4}+O(z^3)\right]\left[1+(\frac{1}{2}-\chi)z+\frac{(\frac{1}{2}-\chi)(\frac{1}{2}-\chi+1)}{4}z^2+O(z^3)\right]$$

or

$$m_\chi^{(0)}(z) = 1-\frac{z}{2}+(\frac{1}{2}-\chi)z+\frac{z^2}{4}-(\frac{1}{2}-\chi)\frac{z^2}{2}+\frac{(\frac{1}{2}-\chi)(\frac{1}{2}-\chi+1)}{4}z^2+O(z^3)\,.$$

We set $\chi = \lambda/2ik$, $z = 2ik\eta$ to obtain

$$v_1(\eta,\lambda) = 1-\lambda\eta+O\left((k\eta)^2\right),$$

implying that $v_1(\eta_o, \lambda)$ has no zeros for $0 \leqslant |\lambda/2k| \leqslant O(1)$. We also see that for $z \to 0$

$$\frac{d}{dz} m_\chi^{(0)}(z) = -\chi + \frac{3}{8} z + \chi^2 \frac{z}{2} + O(z^2) \quad .$$

This implies

$$v_1'(\eta, 0) = O(k\eta) \ ,$$

$$v_1'(\eta, \lambda) = -\frac{\lambda}{2ik} + O(k\eta) \qquad (\lambda \neq 0) \ ,$$

or that $\lambda = 0$ is asymptotically a zero of $v_1'(\eta_0, \lambda)$, and there are no other zeros of $v_1'(\eta_0, \lambda)$ for $0 \leqslant |\lambda/2k| \leqslant O(1)$.

To investigate the zeros for $|\lambda/2k| \gg 1$ or $|\lambda/k| \gg 1$ we can use the theory developed in Appendix D. In the Whittaker equation

$$\frac{d^2u}{d(2ik\eta)^2} + \left(-\frac{1}{4} + \frac{\lambda}{2ik(2ik\eta)} + \frac{1}{4(2ik\eta)^2}\right) u = 0 \, ,$$

we identify from Appendix D the parameters $\ell = \lambda/2ik$, $z = 2ik\eta$ and thus

$$s = \frac{2ik\eta}{4\frac{\lambda}{2ik}} \quad .$$

This implies that for $-3\pi/4 < \arg\lambda \leqslant 5\pi/4$ (recall that this is the convention adopted for the λ-plane), $-\pi/4 \leqslant \arg s < 7\pi/4$. In particular $|s| = k\eta \left|k/\lambda\right| \ll 1$, $\left|\frac{\lambda}{k} s\right| = k\eta \ll 1$, and we can apply equations (D.35) to $M_{\lambda/2ik,\, 0}(2ik\eta)$ for any value of $\arg\lambda$. Therefore

$$M_{\lambda/2ik,\,0}(2ik\eta) = (2ik\eta)^{1/2} J_0(\zeta) \left[1 + O\left((k\eta)\frac{k}{\lambda}\right)\right] \quad \text{if} \quad |\zeta| \leqslant N,$$

$$M_{\lambda/2ik,\,0}(2ik\eta) = (2ik\eta)^{1/2} J_0(\zeta) \left[1 + O(k/\lambda)\right] \qquad \text{if} \quad |\zeta| > N,$$

where ζ is given by equation (D.28a). Then we can write

$$\zeta = 2\lambda^{1/2} \eta^{1/2} \left[1 + \frac{(k\eta)}{6} \frac{k}{\lambda} + O\left((k\eta)^2 \frac{k^2}{\lambda^2}\right)\right] \quad . \qquad (2.9)$$

It immediately follows that

$$v_1(\eta,\lambda) = J_o(\zeta)\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] \qquad \left(|\zeta|\leqslant N\right), \tag{2.10a}$$

$$v_1(\eta,\lambda) = J_o(\zeta)\left[1+O(k/\lambda)\right] \qquad \left(|\zeta|> N\right), \tag{2.10b}$$

from which

$$v_1'(\eta,\lambda) = J_o'(\zeta)\frac{d\zeta}{d(2ik\eta)}\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right]+J_o(\zeta)\,O(k/\lambda) \quad \left(|\zeta|\leqslant N\right),$$

$$v_1'(\eta,\lambda) = J_o'(\zeta)\frac{d\zeta}{d(2ik\eta)}\left[1+O(k/\lambda)\right]+J_o(\zeta)\,O(k/\lambda) \quad \left(|\zeta|> N\right).$$

The second term for $|\zeta|>N$ is present since the order term $O\left((k\eta)\frac{k}{\lambda}\right)$ also occurs in equation (2.10b) but is not written explicitly since it is of lower order than $O(k/\lambda)$. Using equation (2.9) we obtain

$$v_1'(\eta,\lambda) = \frac{\lambda^{1/2}}{2ik\eta^{1/2}}J_o'(\zeta)\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right]+J_o(\zeta)\,O(k/\lambda) \quad \left(|\zeta|\leqslant N\right), \tag{2.11a}$$

$$v_1'(\eta,\lambda) = \frac{\lambda^{1/2}}{2ik\eta^{1/2}}J_o'(\zeta)\left[1+O(k/\lambda)\right]+J_o(\zeta)\,O(k/\lambda) \quad \left(|\zeta|> N\right). \tag{2.11b}$$

Let ζ_r, $r=1,2,3\ldots$, $r=-1,-2,-3,\ldots$ denote respectively the positive and negative zeros of $J_o'(\zeta)$. Then the zeros of the function $v_1'(\eta,\lambda)$ can be represented by $\zeta=\zeta_r+\Delta$. In order to find Δ, we use the Taylor expansion of $J_o'(\zeta)$ about a zero ζ_r. Thus both (2.11a) and (2.11b) imply that for a zero of $v_1'(\eta,\lambda)$

$$\frac{\lambda^{1/2}}{2ik\eta^{1/2}}\Delta\,O(1) = O(k/\lambda),$$

and hence the zeros of the function $v_1'(\eta,\lambda)$ are given by the equation

$$\zeta = \zeta_r+O\left((k\eta)^{1/2}\frac{k^{3/2}}{\lambda^{3/2}}\right) \tag{2.12}$$

for arbitrary ζ. Since we are explicitly interested in the zeros λ_r in the complex λ-plane, we must solve equation (2.12) for λ_r. Substituting for ζ we obtain the equation

$$2\lambda^{1/2}\eta^{1/2}\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] = \zeta_r + O\left((k\eta)^{1/2}\frac{k^{3/2}}{\lambda^{3/2}}\right) \quad ,$$

which has the solution

$$\lambda_r = \frac{\zeta_r^2}{4\eta}\left[1+O\left(\frac{(k\eta)^2}{\zeta_r^2}\right)\right]. \tag{2.13}$$

But since $\zeta_r = -\zeta_{-r}$, we need only consider $r = 1, 2, 3, \ldots$. Then $k\eta \ll 1$ implies $\lambda_r/k \gg 1$, since $\zeta_1 \simeq 3.832$ implies

$$\frac{\lambda_r}{k} \sim \frac{\zeta_r^2}{4k\eta} \geqslant \frac{\zeta_1^2}{4k\eta} \gg 1 .$$

This demonstrates consistency with the assumption of $|\lambda/k| \gg 1$ and therefore $v_1'(\eta_o, \lambda)$ has real positive zeros given by the equation

$$\lambda_r = \frac{\zeta_r^2}{4\eta_o}\left[1+O\left(\frac{(k\eta_o)^2}{\zeta_r^2}\right)\right] \quad (r = 1, 2, 3, \ldots) \ . \tag{2.13a}$$

To find the zeros of $v_1(\eta_o, \lambda)$ we need, in addition to equation (2.10), the estimate

$$O\left((k\eta)^{1/2}(k\eta)^2\frac{k^2}{\lambda^2}\right)$$

of $v_1(\eta, \lambda)$ at a zero of $J_o(\zeta)$ (Appendix D). Thus if β_r $(r = 1, 2, 3, \ldots, r = -1, -2, -3, \ldots)$ denote respectively the positive and negative zeros of $J_o(\zeta)$, the zeros of $v_1(\eta, \lambda)$ are given by the equation

$$\zeta = \beta_r + O\left((k\eta)^{1/2}(k\eta)^2\frac{k^2}{\lambda^2}\right) \tag{2.14}$$

for arbitrary $\bar{\zeta}$. Substituting for ζ we obtain the equation

$$2\lambda^{1/2}\eta^{1/2}\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] = \beta_r + O\left((k\eta)^{1/2}(k\eta)^2\frac{k^2}{\lambda^2}\right) ,$$

which has the solution

$$\lambda_r = \frac{\beta_r^2}{4\eta}\left[1+O\left(\frac{(k\eta)^2}{\beta_r^2}\right)\right] . \tag{2.15}$$

Again since $\beta_r = -\beta_{-r}$, we need only consider $r = 1, 2, 3, \ldots$. Also $k\eta \ll 1$ implies $\lambda_r/k \gg 1$, since $\beta_1 \simeq 2.405$ implies

$$\frac{\lambda_r}{k} \sim \frac{\beta_r^2}{4k\eta} \geqslant \frac{\beta_1^2}{4k\eta} \gg 1 .$$

Therefore we again have consistency with the assumption of $|\lambda/k| \gg 1$ and $v_1(\eta_o, \lambda)$ has real positive zeros given by

$$\lambda_r = \frac{\beta_r^2}{4\eta_o}\left[1+O\left(\frac{(k\eta_o)^2}{\beta_r^2}\right)\right] \quad (r = 1, 2, 3, \ldots). \tag{2.15a}$$

We conclude this section with an observation about the zeros given by equation (2.13a). Since $\cos(\zeta - \frac{\pi}{4})$ governs the behavior of $J_o(\zeta)$ for large ζ, we see that the limiting behavior of these zeros for large ζ_r is described by equation (C.4a). That this limiting relationship exists follows from the fact that for fixed $k\eta$ the value of

$$s = -\frac{k^2\eta}{\lambda_r} = -k\eta(k/\lambda_r)$$

can be made small by taking k/λ_r small enough (λ_r/k large enough). Then although

$$\left|\frac{\lambda_r}{k} s\right| = k\eta$$

is not small, the asymptotic representations given by equations (D.34) and expansions given by equations (D.28) through (D.30) still apply. A similar observation may be made for the zeros given by equation (2.15a)

2.2 Residue Series for the Near Field

Let $\lambda_o = 0$ and λ_r, $r = 1, 2, 3, \ldots$ denote the positive zeros of $v_1'(\eta_o, \lambda)$. Then we prove in Appendix C that the integral representation (2.1) can be replaced by a convergent residue series expansion. The result obtained is

$$v_N(\xi, \eta, \Xi, 0) = \sum_{r=0}^{\infty} \Gamma\left(\frac{\lambda_r}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda_r}{2ik} + \frac{1}{2}\right) \frac{v_1(\xi_1, -\lambda_r) v_2(\xi_2, -\lambda_r)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\lambda_r}} \cdot \left[v_2(\eta, \lambda_r) v_1'(\eta_o, \lambda_r) - v_1(\eta, \lambda_r) v_2'(\eta_o, \lambda_r)\right] ,$$

which upon separating the $r = 0$ term reduces to

$$v_N(\xi, \eta, \Xi, 0) = \frac{\Gamma(1/2)\Gamma(1/2) v_1(\xi_1, 0) v_2(\xi_2, 0)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=0}} \left[v_2(\eta, 0) v_1'(\eta_o, 0) - v_1(\eta, 0) v_2'(\eta_o, 0)\right] +$$

$$+ \frac{1}{2ik\eta_o} \sum_{r=1}^{\infty} \Gamma\left(\frac{\lambda_r}{2ik} + \frac{1}{2}\right) \frac{v_1(\xi_1, -\lambda_r) v_2(\xi_2, -\lambda_r)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\lambda_r}} \frac{v_1(\eta, \lambda_r)}{v_1(\eta_o, \lambda_r)} . \qquad (2.16)$$

We make this separation since although $v_1'(\eta_o, 0) \sim 0$, $v_2(\eta, 0)$ and $v_2'(\eta_o, 0) \sim \infty$ and thus the ratio

$$\frac{1}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=0}} \left[v_2(\eta, 0) v_1'(\eta_o, 0) - v_1(\eta, 0) v_2'(\eta_o, 0)\right]$$

must be carefully evaluated.

In order to estimate the derivative in the $r = 0$ term, we employ the power series expansion for $m_\chi^{(0)}(z)$ given in Section 2.1. From the expansion of $\frac{d}{dz} m_\chi^{(0)}(z)$ we obtain

$$\frac{d}{d\chi}\frac{d}{dz}\,m_\chi^{(0)}(z) = -1+\chi z+O(z^2) \quad .$$

Using this form with $\chi = \frac{\lambda}{2ik}$ and $z = 2ik\eta$, we find

$$\frac{d}{d\lambda}\,v_1'(\eta,\lambda) = -\frac{1}{2ik}+\lambda\eta+O\left((k\eta)^2\right)$$

and

$$\left(\frac{d}{d\lambda}\,v_1'(\eta_o,\lambda)\right)_{\lambda=0} = -\frac{1}{2ik}+O\left((k\eta_o)^2\right) \quad .$$

The other members of the $r=0$ term are given by (Buchholz, 1953, Appendix I)

$$v_1(\xi_1,0) = J_o(k\xi_1) \ ,$$

$$v_2(\xi_2,0) = -\frac{i\pi^{1/2}}{2}\,H_o^{(2)}(k\xi_2)$$

$$v_1(\eta,0) = J_o(k\eta) \ ,$$

$$v_1'(\eta_o,0) = -\frac{k\eta_o}{4i}\left[1+O\left((k\eta_o)^2\right)\right] \ ,$$

$$v_2(\eta,0) = -\frac{i\pi^{1/2}}{2}\,H_o^{(2)}(k\eta) = -\frac{1}{\pi^{1/2}}\ln\frac{k\eta}{2}+O(1)+\frac{(k\eta)^2}{4\pi^{1/2}}\ln\frac{k\eta}{2}+O\left((k\eta)^2\right) \quad .$$

$$v_2'(\eta_o,0) = -\frac{1}{2ik\eta_o\pi^{1/2}}+\frac{k\eta_o}{4i\pi^{1/2}}\ln\frac{k\eta_o}{2}+O(k\eta_o) \ .$$

Therefore equation (2.16) becomes

$$v_N(\xi,\eta,\Xi,0) = \frac{i\pi}{2\eta_o} J_o(k\xi_1) H_o^{(2)}(k\xi_2)\Big[1+O(k\eta_o)\Big] +$$

$$+ \frac{1}{2ik\eta_o} \sum_{r=1}^{\infty} \Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right) \frac{v_1(\xi_1,-\lambda_r)\, v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r}} \; \frac{v_1(\eta,\lambda_r)}{v_1(\eta_o,\lambda_r)} \quad . \qquad (2.17)$$

To estimate the functions depending on η in the r^{th} $(r \geqslant 1)$ term, we use the asymptotic representations obtained in Section 2.1. From equations (2.10) and (2.11) we find

$$v_1(\eta,\lambda_r) = J_o(\zeta_\eta^{(r)}) \left[1+O\left(\frac{(k\eta)^2}{\zeta_r^2}\right)\right] \qquad \left(\left|\zeta_\eta^{(r)}\right| \leqslant N\right),$$

$$v_1(\eta,\lambda_r) = J_o(\zeta_\eta^{(r)}) \left[1+O\left(\frac{k\eta}{\zeta_r^2}\right)\right] \qquad \left(\left|\zeta_\eta^{(r)}\right| > N\right),$$

$$v_1'(\eta_o,\lambda) = \frac{\lambda^{1/2}}{2ik\eta_o^{1/2}} J_o'(\zeta_{\eta_o})\left[1+O\left((k\eta_o)\frac{k}{\lambda}\right)\right] + J_o(\zeta_{\eta_o}) O(k/\lambda) \qquad \left(\left|\zeta_{\eta_o}\right| \leqslant N\right),$$

$$v_1'(\eta_o,\lambda) = \frac{\lambda^{1/2}}{2ik\eta_o^{1/2}} J_o'(\zeta_{\eta_o})\Big[1+O(k/\lambda)\Big] + J_o(\zeta_{\eta_o}) O(k/\lambda) \qquad \left(\left|\zeta_{\eta_o}\right| > N\right),$$

where

$$\zeta_{\eta_o} = 2\lambda^{1/2}\eta_o^{1/2}\left[1+O\left((k\eta_o)\frac{k}{\lambda}\right)\right],$$

$$\zeta_\eta^{(r)} = 2\lambda_r^{1/2}\eta^{1/2}\left[1+O\left((k\eta)\frac{k}{\lambda_r}\right)\right].$$

Moreover, by substituting for λ_r the value given in equation (2.13a), we observe that

$$\zeta^{(r)}_{\eta_o} = \zeta_r\left[1+O\left(\frac{(k\eta_o)^2}{\zeta_r^2}\right)\right].$$

Therefore, taking the derivatives with respect to λ, setting $\lambda = \lambda_r$, and expanding the functions

$$J_o'\left(\zeta_r + O\left(\frac{(k\eta_o)^2}{\zeta_r}\right)\right) \quad , \qquad J_o''\left(\zeta_r + O\left(\frac{(k\eta_o)^2}{\zeta_r}\right)\right)$$

in a Taylor's series, we obtain

$$\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = \frac{1}{2ik}\, J_o''(\zeta_r)\left[1+O\left(\frac{(k\eta_o)^2}{\zeta_r}\right)\right] \qquad (\zeta_r \leqslant N) \ ,$$

$$\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = \frac{1}{2ik}\, J_o''(\zeta_r)\left[1+O\left(\frac{k\eta_o}{\zeta_r^2}\right)+O\left(\frac{(k\eta_o)^2}{\zeta_r}\right)\right] \qquad (\zeta_r > N).$$

Since ζ_r is a positive zero of $J_o'(\zeta)$, Bessel's differential equation,

$$J_o''(\zeta) + \frac{1}{\zeta}\, J_o'(\zeta) + J_o(\zeta) = 0 \ ,$$

implies that

$$J_o''(\zeta_r) = -J_o(\zeta_r) \ .$$

Hence, the above derivatives become

$$\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = -\frac{1}{2ik}\, J_o(\zeta_r)\left[1+O\left(\frac{(k\eta_o)^2}{\zeta_r}\right)\right] \qquad (\zeta_r \leqslant N), \quad (2.18a)$$

$$\left(\frac{d}{d\lambda}\, v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r} = -\frac{1}{2ik}\, J_o(\zeta_r)\left[1+O\left(\frac{k\eta_o}{\zeta_r^2}\right)+O\left(\frac{(k\eta_o)^2}{\zeta_r}\right)\right] \quad (\zeta_r > N), \quad (2.18b)$$

In order to further estimate (2.17) we use the mathematical conditions defining the near field. Since the near field corresponds to the physical problem of both the source and field points near (with respect to wavelength) the origin, these conditions are $k\bar{\xi} \ll 1$, $k\xi \ll 1$. Then for these values of k, ξ_1, ξ_2

$$J_o(k\xi_1) = 1 + O\left((k\xi_1)^2\right), \tag{2.19a}$$

$$H_o^{(2)}(k\xi_2) = \frac{2}{i\pi} \ln \frac{k\xi_2}{2} + \frac{2}{i\pi} \gamma + 1 + O\left((k\xi_2)^2 \ln \frac{k\xi_2}{2}\right), \tag{2.19b}$$

where γ is the Euler constant. We now determine the behavior of $v_1(\xi_1, -\lambda_r)$ and $v_2(\xi_2, -\lambda_r)$ for $r \geq 1$, $k\xi_j \ll 1$, $j = 1, 2$. It suffices to obtain $M_{-\lambda_r/2ik, 0}(2ik\xi_1)$ and $W_{-\lambda_r/2ik, 0}(2ik\xi_2)$. These functions are solutions of the equation

$$\frac{d^2u}{d(2ik\xi_j)^2} + \left(-\frac{1}{4} - \frac{\lambda_r}{2ik(2ik\xi_j)} + \frac{1}{4(2ik\xi_j)^2}\right)u = 0 \quad (j = 1, 2),$$

and upon identifying from Appendix D the parameters $\ell = -\lambda_r/2ik$, $z = 2ik\xi_j$, we obtain

$$s_{\xi_j} = \frac{2ik\xi_j}{4\left(-\frac{\lambda_r}{2ik}\right)}.$$

Then we see from equation (2.13a) that s_{ξ_j} is real valued and positive. We also note that for $-3\pi/4 < \arg\lambda \leq 5\pi/4$, the s-plane defined by

$$s = \frac{2ik\xi}{4\left(\frac{-\lambda}{2ik}\right)}$$

is described by $-\pi/4 \leq \arg s < 7\pi/4$. In addition, since

$$s_{\xi_j} = \frac{k^2 \xi_j}{\lambda_r} \simeq \frac{4(k\xi_j)}{\zeta_r^2}(k\eta_o) \ll 1$$

as well as

$$\frac{\lambda_r}{k} s_{\xi_j} = k\xi_j \ll 1 \ ,$$

we can apply equations (D.35) and (D.38) to obtain

$$M_{-\lambda_r/2ik,\, 0}(2ik\xi_1) = (2ik\xi_1)^{1/2} J_o(\zeta_{\xi_1}^{(r)}) \left[1 + O\left(\frac{(k\xi_1)^2}{\zeta_r^2}\right)\right] \quad \left(\left|\zeta_{\xi_1}^{(r)}\right| \leqslant N\right) ,$$

$$M_{-\lambda_r/2ik,\, 0}(2ik\xi_1) = (2ik\xi_1)^{1/2} J_o(\zeta_{\xi_1}^{(r)}) \left[1 + O\left(\frac{k\xi_1}{\zeta_r^2}\right)\right] \quad \left(\left|\zeta_{\xi_1}^{(r)}\right| > N\right) ,$$

$$W_{-\lambda_r/2ik,\, 0}(2ik\xi_2) = \frac{i\pi^{1/2}}{2^{1/2}} F_{-\lambda_r/2ik}(2ik\xi_2)^{1/2} H_o^{(1)}(\zeta_{\xi_2}^{(r)}) \left[1 + O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right] ,$$

where

$$F_{-\lambda_r/2ik} = \exp\left\{-\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} \ ,$$

and from equation (D.28a)

$$\zeta_{\xi_j}^{(r)} = 2i\lambda_r^{1/2} \xi_j^{1/2} \left[1 + O\left(\frac{(k\xi_j)(k\eta_o)}{\zeta_r^2}\right)\right] .$$

Hence

$$v_1(\xi_1, -\lambda_r) = J_o(\zeta_{\xi_1}^{(r)}) \left[1 + O\left(\frac{(k\xi_1)^2}{\zeta_r^2}\right)\right] \quad \left(\left|\zeta_{\xi_1}^{(r)}\right| \leqslant N\right) , \qquad (2.20a)$$

$$v_1(\xi_1, -\lambda_r) = J_o(\zeta^{(r)}_{\xi_1})\left[1+O\left(\frac{k\xi_1}{\zeta_r^2}\right)\right] \qquad \left(\left|\zeta^{(r)}_{\xi_1}\right| > N\right), \tag{2.20b}$$

$$v_2(\xi_2, -\lambda_r) = \frac{i\pi^{1/2}}{2^{1/2}} F_{-\lambda_r/2ik} H_o^{(1)}(\zeta^{(r)}_{\xi_2})\left[1+O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right]. \tag{2.21}$$

Now

$$\left|\frac{\lambda_r}{2ik}\right| \simeq \zeta_r^2/8k\eta_o \gg 1 ,$$

and since $\arg \lambda_r/2ik = -\pi/2$, Stirling's formula (Erdélyi et al, 1953) gives

$$\Gamma\left(\frac{\lambda_r}{2ik} + \frac{1}{2}\right) = \sqrt{2\pi}\, \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\}\left[1+O(k/\lambda_r)\right].$$

Substituting this and equations (2.10), (2.18), (2.19), (2.20) and (2.21) into (2.17), we obtain

$$v_N(\xi, \eta, \Xi, 0) = \frac{1}{\eta_o} \ln k\xi_2 + \frac{1}{\eta_o}\gamma + \frac{i\pi}{2\eta_o} - \frac{i\pi}{\eta_o}\sum_{r=1}^{\infty} \frac{J_o(\zeta^{(r)}_{\xi_1})H_o^{(1)}(\zeta^{(r)}_{\xi_2})J_o(\zeta^{(r)}_{\eta})}{\left[J_o(\zeta_r)\right]^2} \cdot \left[1+O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right], \tag{2.22}$$

where only the terms of greatest order have been retained. Expanding $J_o(\zeta^{(r)}_{\xi_1})$ and $H_o^{(1)}(\zeta^{(r)}_{\xi_2})$ about $\zeta^{(r)}_{\xi_j}(0) = 2i\lambda_r^{1/2}(0)\xi_j^{1/2}$, $j = 1, 2$, where $\lambda_r(0) = \zeta_r^2/4\eta_o$, expanding $J_o(\zeta^{(r)}_{\eta})$ about $\zeta^{(r)}_{\eta}(0) = 2\lambda_r^{1/2}(0)\eta^{1/2}$, and substituting the results into equation (2.22), we find

$$v_N(\xi,\eta,\Xi,0) = \frac{1}{\eta_o}\ell n\, k\xi_2 + \frac{1}{\eta_o}\gamma + \frac{i\pi}{2\eta_o} - \frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)H_o^{(1)}\left(\zeta_{\xi_2}^{(r)}(0)\right)J_o\left(\zeta_{\eta}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \cdot$$

$$\cdot\left[1+O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right] \quad , \tag{2.23}$$

where again only the term of greatest order has been retained.

The residue series for the point source at (0, H) may be similarly obtained. Summing the residues of the integral representation (2.5) yields

$$v_N(\xi,\eta,0,H) = \frac{\Gamma(1/2)\Gamma(1/2)\, v_2(\xi,0)\, v_1(\eta_1,0)}{\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=0}}\left[v_2(\eta_2,0)v_1'(\eta_o,0) - v_1(\eta_2,0)v_2'(\eta_o,0)\right] +$$

$$+\frac{1}{2ik\eta_o}\sum_{r=1}^{\infty}\frac{\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)v_2(\xi,-\lambda_r)\,v_1(\eta_1,\lambda_r)}{\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\cdot\frac{v_1(\eta_2,\lambda_r)}{v_1(\eta_o,\lambda_r)} \quad , \tag{2.24}$$

which is analogous to equation (2.16). This can be written as

$$v_N(\xi,\eta,0,H) = \frac{i\pi}{2\eta_o}H_o^{(2)}(k\xi)\left[1+O(k\eta_o)\right] +$$

$$+\frac{1}{2ik\eta_o}\sum_{r=1}^{\infty}\frac{\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)v_2(\xi,-\lambda_r)v_1(\eta_1,\lambda_r)}{\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\cdot\frac{v_1(\eta_2,\lambda_r)}{v_1(\eta_o,\lambda_r)} \quad . \tag{2.25}$$

Then, since the near field is defined by $k\xi \ll 1$, substituting the corresponding asymptotic representations gives

$$v_N(\xi,\eta,0,H) = \frac{1}{\eta_o}\,\ln k\xi + \frac{1}{\eta_o}\gamma + \frac{i\pi}{2\eta_o} - \frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{H_o^{(1)}(\zeta_\xi^{(r)})J_o(\zeta_{\eta_1}^{(r)})J_o(\zeta_{\eta_2}^{(r)})}{\left[J_o(\zeta_r)\right]^2}\cdot$$

$$\cdot\left[1+O\left(\frac{k\xi}{\zeta_r^2}\right)\right], \qquad (2.26)$$

where

$$\zeta_{\eta_j}^{(r)} = 2\lambda_r^{1/2}\eta_j^{1/2}\left[1+O\left((k\eta_j)\frac{(k\eta_o)}{\zeta_r^2}\right)\right],$$

$$\zeta_\xi^{(r)} = 2i\lambda_r^{1/2}\xi^{1/2}\left[1+O\left((k\xi)\frac{(k\eta_o)}{\zeta_r^2}\right)\right],$$

and only the term of greatest order has been retained. Expanding about $\zeta_{\eta_j}^{(r)}(0) = 2\lambda_r^{1/2}(0)\eta_j^{1/2}$, $j = 1, 2$, $\zeta_\xi^{(r)}(0) = 2i\lambda_r^{1/2}(0)\xi^{1/2}$, we find

$$v_N(\xi,\eta,0,H) = \frac{1}{\eta_o}\,\ln k\xi + \frac{1}{\eta_o}\gamma + \frac{i\pi}{2\eta_o} -$$

$$-\frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{H_o^{(1)}\left(\zeta_\xi^{(r)}(0)\right)J_o\left(\zeta_{\eta_1}^{(r)}(0)\right)J_o\left(\zeta_{\eta_2}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\left[1+O\left(\frac{k\xi}{\zeta_r^2}\right)\right], \qquad (2.27)$$

where again only the term of greatest order has been retained.

The solutions for the Dirichlet problem are found in the same manner as above. Consider the point source at $(\Xi, 0)$. If $\lambda_1, \lambda_2, \ldots \lambda_r, \ldots$ now denote the positive zeros of $v_1(\eta_o, \lambda)$ (given by equation (2.15a)), then the integral representation (2.3) becomes

$$v_D(\xi,\eta,\Xi,0)=\sum_{r=1}^{\infty}\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\cdot\left[-v_1(\eta,\lambda_r)v_2(\eta_o,\lambda_r)\right],$$

and on using the Wronskian relation this reduces to

$$v_D(\xi,\eta,\Xi,0)=\frac{1}{2ik\eta_o}\sum_{r=1}^{\infty}\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\frac{v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r)}{\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\left[-\frac{v_1(\eta,\lambda_r)}{v_1'(\eta_o,\lambda_r)}\right]. \tag{2.28}$$

Then in order to evaluate (2.28) asymptotically, it remains only to find $\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}$. From equations (2.10)

$$\frac{d}{d\lambda}v_1(\eta_o,\lambda)=J_o'(\zeta_{\eta_o})\frac{\eta_o^{1/2}}{\lambda^{1/2}}\left[1+O\left((k\eta)\frac{k}{\lambda}\right)\right] \qquad \text{if } |\zeta_{\eta_o}|\leqslant N,$$

$$\frac{d}{d\lambda}v_1(\eta_o,\lambda)=J_o'(\zeta_{\eta_o})\frac{\eta_o^{1/2}}{\lambda^{1/2}}\left[1+O(k/\lambda)\right] \qquad \text{if } |\zeta_{\eta_o}|>N.$$

Thus

$$\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}=\frac{2\eta_o}{\beta_r}J_o'(\beta_r)\left[1+O\left(\frac{(k\eta_o)^2}{\beta_r^2}\right)\right] \quad \left(|\beta_r|\leqslant N\right), \tag{2.29a}$$

$$\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}=\frac{2\eta_o}{\beta_r}J_o'(\beta_r)\left[1+O\left(\frac{k\eta_o}{\beta_r^2}\right)\right] \quad \left(|\beta_r|>N\right). \tag{2.29b}$$

Substituting equations (2.29) and the other corresponding asymptotic representations into (2.28), we obtain, upon expanding as before and retaining only the terms of greatest order,

$$v_D(\xi,\eta,\Xi,0) = -\frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)H_o^{(1)}\left(\zeta_{\xi_2}^{(r)}(0)\right)J_o\left(\zeta_{\eta}^{(r)}(0)\right)}{\left[J_o'(\beta_r)\right]^2}\left[1+O\left(\frac{k\xi_2}{\beta_r^2}\right)\right]. \tag{2.30}$$

For the point source at (0, H) the integral representation (2.7) gives

$$v_D(\xi,\eta,0,H) = \frac{1}{2ik\eta_o}\sum_{r=1}^{\infty}\Gamma\left(\frac{\lambda_r}{2ik}+\frac{1}{2}\right)\frac{v_2(\xi,-\lambda_r)v_1(\eta_1,\lambda_r)}{\left(\frac{d}{d\lambda}v_1(\eta_o,\lambda)\right)_{\lambda=\lambda_r}}\left[-\frac{v_1(\eta_2,\lambda_r)}{v_1'(\eta_o,\lambda_r)}\right]. \tag{2.31}$$

Then substituting the corresponding asymptotic representations into (2.31), expanding as before and retaining only the terms of greatest order, we find

$$v_D(\xi,\eta,0,H) = -\frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{H_o^{(1)}\left(\zeta_{\xi}^{(r)}(0)\right)J_o\left(\zeta_{\eta_1}^{(r)}(0)\right)J_o\left(\zeta_{\eta_2}^{(r)}(0)\right)}{\left[J_o'(\beta_r)\right]^2}\left[1+O\left(\frac{k\xi_2}{\zeta_r^2}\right)\right]. \tag{2.32}$$

2.3 Dirichlet Potential Problem

We consider first the Dirichlet potential problem corresponding to the wave problem for a point source at $(\Xi, 0)$. In order to represent the solution to this potential problem we use an integral representation of $1/R$ given by Morse and Feshbach (1953, Chapter 10). They define a paraboloid of revolution coordinate system by

$$x = \lambda\omega\cos\phi, \quad y = \lambda\omega\sin\phi, \quad z = \frac{1}{2}(\lambda^2-\omega^2) \qquad \left(r = \frac{1}{2}(\lambda^2+\omega^2)\right),$$

in which, for a point source at $(\lambda = \lambda_o,\ \omega = 0)$, the integral representation of $1/R$ is

$$\frac{1}{R} = i\pi\int_0^{\infty}J_o(t\lambda)J_o(t\lambda_o)H_o^{(1)}(it\omega)t\,dt \qquad (\omega > 0).$$

Then if the substitutions $\lambda = \sqrt{2\xi}$, $\omega = \sqrt{2\eta}$, $\lambda_o = \sqrt{2\Xi}$ are made, the above coordinate system reduces to the one which we are considering, and the above integral representation for $1/R$ becomes

$$\frac{1}{R} = i\pi \int_0^\infty J_o(t\sqrt{2\xi_1})J_o(t\sqrt{2\xi_2})H_o^{(1)}(it\sqrt{2\eta})t\,dt \quad (\eta > 0) , \tag{2.33}$$

where again $\xi_1 = \min(\xi, \Xi)$, $\xi_2 = \max(\xi, \Xi)$. Therefore we can assume that the potential $\phi_D(\xi, \eta, \Xi, 0)$ has the form

$$\phi_D(\xi, \eta, \Xi, 0) = -\frac{1}{R} + \phi_S(\xi, \eta, \Xi, 0),$$

where

$$\phi_S(\xi, \eta, \Xi, 0) = i\pi \int_0^\infty J_o(t\sqrt{2\xi_1})J_o(t\sqrt{2\xi_2})A_t J_o(it\sqrt{2\eta})t\,dt ,$$

with A_t an unknown function of t. The boundary condition implies

$$A_t = \frac{H_o^{(1)}(it\sqrt{2\eta_o})}{J_o(it\sqrt{2\eta_o})} ,$$

and hence

$$\phi_D(\xi, \eta, \Xi, 0) = -i\pi \int_0^\infty t\,dt \, \frac{J_o(t\sqrt{2\xi_1})J_o(t\sqrt{2\xi_2})}{J_o(it\sqrt{2\eta_o})} \cdot$$

$$\cdot \left[H_o^{(1)}(it\sqrt{2\eta})J_o(it\sqrt{2\eta_o}) - H_o^{(1)}(it\sqrt{2\eta_o})J_o(it\sqrt{2\eta})\right]$$

is an integral representation of the potential.

In order to compare the potential with the term in equation (2.30) which is independent of k, we obtain a series representation for the potential. In order that this series representation be more readily comparable, we first substitute $t = \sqrt{2v}$ which yields

$$\phi_D(\xi,\eta,\Xi,0) = -2i\pi \int_0^\infty v\,dv \, \frac{J_o(2v\sqrt{\xi_1})J_o(2v\sqrt{\xi_2})}{J_o(2iv\sqrt{\eta_o})} \cdot$$

$$\cdot \left[H_o^{(1)}(2iv\sqrt{\eta})\,J_o(2iv\sqrt{\eta_o}) - H_o^{(1)}(2iv\sqrt{\eta_o})J_o(2iv\sqrt{\eta})\right] . \qquad (2.34)$$

Then we consider the function

$$\psi(v) = \frac{1}{J_o(2iv\sqrt{\eta_o})}\left[H_o^{(1)}(2iv\sqrt{\eta})J_o(2iv\sqrt{\eta_o}) - H_o^{(1)}(2iv\sqrt{\eta_o})J_o(2iv\sqrt{\eta})\right],$$

whereupon we note

$$\psi(v\,e^{-\pi i}) = \frac{1}{J_o(2iv\sqrt{\eta_o})}\left[\left\{2H_o^{(1)}(2iv\sqrt{\eta}) + H_o^{(2)}(2iv\sqrt{\eta})\right\} J_o(2iv\sqrt{\eta_o}) - \right.$$

$$\left. - \left\{2H_o^{(1)}(2iv\sqrt{\eta_o}) + H_o^{(2)}(2iv\sqrt{\eta_o})\right\} J_o(2iv\sqrt{\eta})\right] ,$$

which reduces to

$$\psi(v\,e^{-\pi i}) = \frac{1}{J_o(2iv\sqrt{\eta_o})}\left[H_o^{(1)}(2iv\sqrt{\eta})J_o(2iv\sqrt{\eta_o}) - H_o^{(1)}(2iv\sqrt{\eta_o})J_o(2iv\sqrt{\eta})\right]$$

or

$$\psi(v\,e^{-\pi i}) = \psi(v) \; .$$

Therefore if we write equation (2.34) as

$$\phi_D(\xi,\eta,\Xi,0) = -i\pi \int_0^\infty v\,dv\, J_o(2v\sqrt{\xi_1})H_o^{(1)}(2v\sqrt{\xi_2})\psi(v) -$$

$$- i\pi \int_0^\infty v\,dv\, J_o(2v\sqrt{\xi_1})H_o^{(2)}(2v\sqrt{\xi_2})\psi(v) \; ,$$

we can consider the substitution $v = we^{-\pi i}$ in the second integrand to obtain

$$\phi_D(\xi,\eta,\Xi,0) = -i\pi \int_0^{\infty} v dv \, J_0(2v\sqrt{\xi_1}) H_0^{(1)}(2v\sqrt{\xi_2}) \psi(v) +$$

$$+ i\pi \int_0^{-\infty} w dw \, J_0(2w\sqrt{\xi_1}) H_0^{(1)}(2w\sqrt{\xi_2}) \psi(w) \quad .$$

This reduces to

$$\phi_D(\xi,\eta,\Xi,0) = -i\pi \int_{-\infty}^{\infty} v dv \, J_0(2v\sqrt{\xi_1}) H_0^{(1)}(2v\sqrt{\xi_2}) \psi(v) \ ,$$

and thus the equation for the potential becomes

$$\phi_D(\xi,\eta,\Xi,0) = -i\pi \int_{-\infty}^{\infty} v dv \, \frac{J_0(2v\sqrt{\xi_1}) H_0^{(1)}(2v\sqrt{\xi_2})}{J_0(2iv\sqrt{\eta_0})} \Big[H_0^{(1)}(2iv\sqrt{\eta}) J_0(2iv\sqrt{\eta_0}) -$$

$$- H_0^{(1)}(2iv\sqrt{\eta_0}) J_0(2iv\sqrt{\eta}) \Big] \ . \tag{2.35}$$

Let β_r $(r = 1, 2, 3, \ldots, -1, -2, -3, \ldots)$ again denote the zeros of $J_0(\beta)$. Thus $v_r = i\beta_r / 2\sqrt{\eta_0}$ $(r = 1, 2, 3, \ldots)$ denote the zeros of $J_0(2iv\sqrt{\eta_0})$ along the positive imaginary axis and hence also the poles of the integrand of equation (2.35) in the upper half plane. Now for large $|v|$ in the upper half plane, we can use the asymptotic representations (Erdélyi et al, 1953) of $J_0(2v\sqrt{\xi_1})$, $H_0^{(1)}(2v\sqrt{\xi_2})$, $H_0^{(1)}(2iv\sqrt{\eta})$ and $J_0(2iv\sqrt{\eta})$ to show that the integrand of (2.35) is exponentially small provided $\eta > 0$, $\xi_1 < \xi_2$. Therefore the residue theorem implies

$$\phi_D(\xi,\eta,\Xi,0) = 2\pi i(-i\pi) \sum_{r=1}^{\infty} \frac{v_r J_0(2v_r\sqrt{\xi_1}) H_0^{(1)}(2v_r\sqrt{\xi_2})}{\left(\frac{d}{dv} J_0(2iv\sqrt{\eta_0})\right)_{v=v_r}} \Big[-H_0^{(1)}(2iv_r\sqrt{\eta_0}) J_0(2iv_r\sqrt{\eta}) \Big] \ ,$$

which upon using the Wronskian relation for the Bessel functions $J_o(2iv\sqrt{\eta_o})$, $H_o^{(1)}(2iv\sqrt{\eta_o})$, and substituting $v_r = i\beta_r/2\sqrt{\eta_o}$, reduces to

$$\phi_D(\xi,\eta,\Xi,0) = -\frac{i\pi}{\eta_o}\sum_{r=1}^{\infty} \frac{J_o(i\beta_r\sqrt{\xi_1/\eta_o})H_o^{(1)}(i\beta_r\sqrt{\xi_2/\eta_o})}{\left[J_o'(\beta_r)\right]^2}\; J_o(\beta_r\sqrt{\eta/\eta_o}) \;. \tag{2.36}$$

This series is identical to the one in equation (2.30) representing the term independent of k.

A similar argument can be employed for the Dirichlet potential problem corresponding to the wave problem for a point source at (0, H). In this case the integral representation for 1/R becomes

$$\frac{1}{R} = i\pi \int_0^{\infty} J_o(t\sqrt{2\xi})J_o(it\sqrt{2\eta_1})H_o^{(1)}(it\sqrt{2\eta_2})t\,dt \;,$$

where as previously $\eta_1 = \min(\eta, H)$, $\eta_2 = \max(\eta, H)$. Then the potential has the integral representation

$$\phi_D(\xi,\eta,0,H) = -i\pi \int_0^{\infty} t\,dt\, \frac{J_o(t\sqrt{2\xi})J_o(it\sqrt{2\eta_1})}{J_o(it\sqrt{2\eta_o})}\left[H_o^{(1)}(it\sqrt{2\eta_2})J_o(it\sqrt{2\eta_o}) - H_o^{(1)}(it\sqrt{2\eta_o})J_o(it\sqrt{2\eta_2})\right] ,$$

which by the same analysis as before reduces to

$$\phi_D(\xi,\eta,0,H) = -i\pi \int_{-\infty}^{\infty} v\,dv\, \frac{H_o^{(1)}(2v\sqrt{\xi})J_o(2iv\sqrt{\eta_1})}{J_o(2iv\sqrt{\eta_o})}\left[H_o^{(1)}(2iv\sqrt{\eta_2})J_o(2iv\sqrt{\eta_o}) - H_o^{(1)}(2iv\sqrt{\eta_o})J_o(2iv\sqrt{\eta_2})\right]. \tag{2.37}$$

For $\xi > 0$, $\eta_1 < \eta_2$ the integrand of (2.37) decreases exponentially as $|v| \to \infty$ in the upper half plane. Thus as above

$$\phi_D(\xi, \eta, 0, H) = -\frac{i\pi}{\eta_o} \sum_{r=1}^{\infty} \frac{H_o^{(1)}(i\beta_r\sqrt{\xi/\eta_o})J_o(\beta_r\sqrt{\eta_1/\eta_o})J_o(\beta_r\sqrt{\eta_2/\eta_o})}{\left[J_o'(\beta_r)\right]^2} \quad , \quad (2.38)$$

the series being identical to the one in equation (2.32) for the term which is independent of k.

2.4 Neumann Potential Problem

We consider first the Neumann potential problem corresponding to the wave problem for a point source at $(\Xi, 0)$. If we attempt to represent the solution to this problem in the same manner as we did for the Dirichlet potential, we find that substituting the value of A_t implied by the boundary condition leads to a divergent integral. Therefore we seek another method of representing the solution to the Neumann potential problem together with some insight into why the previous technique fails.

We recall that for interior Neumann potential problems the condition that a solution exist implies that the boundary condition cannot be specified arbitrarily (Koshlyakov et al, 1964, Chapter 18). Consequently we seek an analogous condition for the domain defined by the interior of a paraboloid of revolution. Let us consider the bounded volume $V_{\xi'}$ defined by the intersection of the paraboloid of revolution $\eta = \eta_o$ with the paraboloid of revolution $\xi = \xi'$ (Fig. 2-1). If S represents the boundary of $V_{\xi'}$, we define the surface areas S_1 and S_2 by

$$S_1 = S \cap (\eta = \eta_o) \, ,$$

$$S_2 = S \cap (\xi = \xi') \, .$$

Then we can apply Gauss' theorem (Morse and Feshbach, 1953, Chapter 1) to the volume $V_{\xi'}$ and any suitably defined vector $\bar{A}$ to obtain

$$\iint_{S_1} \bar{A} \cdot \hat{n} \, dS_1 + \iint_{S_2} \bar{A} \cdot \hat{n} \, dS_2 = \iiint_{V_{\xi'}} \nabla \cdot \bar{A} \, dV_{\xi'} \quad ,$$

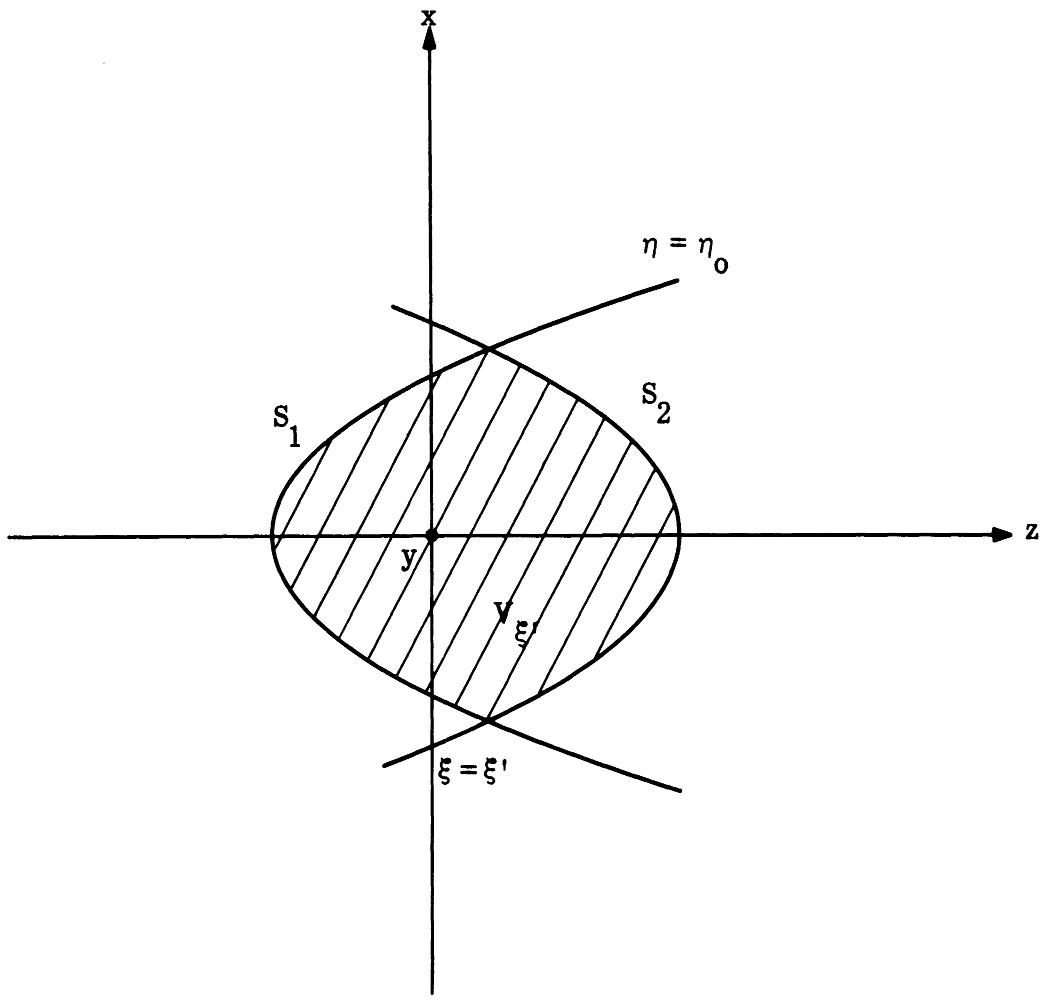

FIG. 2-1: VOLUME AND SURFACE AREAS FOR GAUSS' THEOREM.

where $\hat{n}$ is the unit outward normal to $V_{\xi'}$. We substitute for dS_1, dS_2, $dV_{\xi'}$ (Morse and Feshbach, 1953, Chapter 1 together with Section 1.3), and taking the limit as $\xi' \rightarrow \infty$ of both sides of the equation, we find

$$\int_0^{2\pi}\int_0^{\infty} \bar{A}\cdot\hat{n}\left[2\sqrt{\eta_o(\eta_o+\xi)}\,d\xi d\phi\right] + \lim_{\xi'\rightarrow\infty}\int_0^{2\pi}\int_0^{\eta_o} \bar{A}\cdot\hat{n}\left[2\sqrt{\xi'(\xi'+\eta)}\,d\eta d\phi\right]$$

$$= \int_0^{2\pi}\int_0^{\infty}\int_0^{\eta_o} \nabla\cdot\bar{A}\left[2(\xi+\eta)\,d\xi d\eta d\phi\right]. \qquad (2.39)$$

Let ϕ represent a possible solution to the Neumann potential problem under consideration. Then ϕ must satisfy the boundary condition $\left(\frac{\partial\phi}{\partial n}\right)_{\eta=\eta_o} = 0$ as well as the condition

$$\int_0^{2\pi}\int_0^{\infty}\int_0^{\eta_o} \nabla^2\phi\left[2(\xi+\eta)\,d\xi d\eta d\phi\right] = 4\pi \quad ,$$

which follows from the Poisson equation. But an allowable choice of $\bar{A}$ in equation (2.39) is $\bar{A} = \nabla\phi$ (Morse and Feshbach, 1953, Chapter 1). Therefore (2.39) becomes

$$\lim_{\xi'\rightarrow\infty}\int_0^{2\pi}\int_0^{\eta_o} \nabla\phi\cdot\hat{n}\left[2\sqrt{\xi'(\xi'+\eta)}\,d\eta d\phi\right] = 4\pi \ .$$

Now since

$$\bar{A}\cdot\hat{n} = \nabla\phi\cdot\hat{n} = \frac{\partial\phi}{\partial n} \ ,$$

we find, upon using (Morse and Feshbach, 1953, Chapter 1, together with Section 1.3)

$$\left(\frac{\partial\phi}{\partial n}\right)_{\xi=\xi'} = \frac{\partial\phi}{\partial\xi'}\,\frac{(\xi')^{1/2}}{(\xi'+\eta)^{1/2}} \quad ,$$

changing the dummy variable from ξ' to ξ, and carrying out the ϕ-integration,

$$\lim_{\xi \to \infty} \xi \int_0^{\eta_0} \frac{\partial \phi}{\partial \xi} d\eta = 1 \ . \tag{2.40}$$

This equation thus represents a necessary condition at infinity for the existence of a solution to the Neumann potential problem being considered. It is also the reason why we cannot represent the solution to the Neumann potential problem as we did for the Dirichlet potential problem; assuming a solution in this form implies a behavior at infinity which is contrary to the validity of equation (2.40). This result is characteristic of the domain defined by the interior of the paraboloid of revolution. The same is not true for the domain defined by the interior of a cone (half cone angle less than $\pi/2$) where the application of Gauss' theorem does not lead to any contradiction. This is shown in Appendix F.

We can show that condition (2.40) is also sufficient for existence by producing a solution to the potential problem

$$\nabla^2 \phi = 4\pi\delta(\bar{r} - \bar{r}_0) \qquad (\bar{r}_0 \text{ the vector to the point } (\Xi, 0)),$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on the boundary } \eta = \eta_0 \ , \tag{2.P}$$

$$\lim_{\xi \to \infty} \xi \int_0^{\eta_0} \frac{\partial \phi}{\partial \xi} d\eta = 1 \ ,$$

by the method of separation of variables (Morse and Feshbach, 1953, Chapter 7). We write the inhomogeneous potential equation in the coordinates of the paraboloid of revolution

$$\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \phi}{\partial \eta}\right) = \delta(\xi - \Xi)\xi(\eta) \ . \tag{2.41}$$

Then if ζ_r, $r = 1, 2, 3, \ldots$ again denotes the positive zeros of $J_0'(\zeta) = 0$, we observe that the functions $J_0(\gamma_\eta^{(r)})$, $\gamma_\eta^{(r)} = \zeta_r\sqrt{\eta/\eta_0}$, and the function 1 are solutions of the homogeneous potential equation satisfying the boundary condition. Thus we assume a solution of the inhomogeneous equation to be of the form

$$\phi_N(\xi, \eta, \Xi, 0) = A_o(\xi) + \sum_{r=1}^{\infty} A_r(\xi) J_o(\gamma_\eta^{(r)}) \ . \tag{2.42}$$

In addition, since the functions 1, $J_o(\gamma_\eta^{(r)})$ form a complete set in the space of C^∞ functions, by the theory of distributions, we can find the expansion

$$\delta(\eta) = \frac{1}{\eta_o} + \frac{1}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o(\gamma_\eta^{(r)})}{\left[J_o(\zeta_r)\right]^2} \ . \tag{2.43}$$

Substituting (2.42) and (2.43) into (2.41), we obtain the following ordinary differential equations

$$\frac{d}{d\xi}\left(\xi \frac{dA_o(\xi)}{d\xi}\right) = \frac{1}{\eta_o} \delta(\xi - \Xi), \tag{2.44}$$

$$\frac{d}{d\xi}\left(\xi \frac{dA_r(\xi)}{d\xi}\right) - \lambda_r(0) A_r(\xi) = \frac{1}{\eta_o} \frac{\delta(\xi - \Xi)}{\left[J_o(\zeta_r)\right]^2}, \quad r = 1, 2, 3, \ldots, \tag{2.45}$$

where again $\lambda_r(0) = \zeta_r^2/4\eta_o$. We solve equation (2.45) by constructing the Green's function from linearly independent solutions of the homogeneous equation (Friedman, 1956, Chapter 3). The pertinent solutions are $J_o(\gamma_\xi^{(r)})$ $(\gamma_\xi^{(r)} = i\zeta_r\sqrt{\xi/\eta_o})$, which are regular at $\xi = 0$ and lie in $\mathcal{L}_2(0, \xi_o)$ $(0 < \xi_o < \infty)$, and $H_o^{(1)}(\gamma_\xi^{(r)})$, which are regular at infinity and lie in $\mathcal{L}_2(\xi_o, \infty)$. The $H_o^{(1)}(\gamma_\xi^{(r)})$ solutions are required in order that condition (2.40) not be violated. Thus since

$$J_o(\gamma_\xi^{(r)}) \frac{d}{d\xi} H_o^{(1)}(\gamma_\xi^{(r)}) - \frac{d}{d\xi} J_o(\gamma_\xi^{(r)}) H_o^{(1)}(\gamma_\xi^{(r)}) = \frac{i}{\pi\xi} \ ,$$

the Green's function for equation (2.45), written as

$$-\frac{d}{d\xi}\left(\xi \frac{dA_r(\xi)}{d\xi}\right) + \lambda_r(0) A_r(\xi) = -\frac{1}{\eta_o} \frac{\delta(\xi - \Xi)}{\left[J_o(\zeta_r)\right]^2} \ ,$$

is

$$G(t,\bar{s},-\lambda_r(0)) = -\pi/i \begin{cases} J_o(\gamma_t^{(r)})H_o^{(1)}(\gamma_{\bar{s}}^{(r)}) & t<\bar{s} \\ J_o(\gamma_{\bar{s}}^{(r)})H_o^{(1)}(\gamma_t^{(r)}) & \bar{s}<t . \end{cases}$$

Hence the solution to equation (2.45) is given by

$$A_r(\xi) = \int_0^\infty G(\xi,\bar{s},-\lambda_r(0)) \left[-\frac{1}{\eta_o} \frac{\delta(\bar{s}-\Xi)}{\left[J_o(\zeta_r)\right]^2} \right] d\bar{s}$$

or

$$A_r(\xi) = \pi/i \, \frac{J_o(\gamma_{\xi_1}^{(r)})H_o^{(1)}(\gamma_{\xi_2}^{(r)})}{\eta_o\left[J_o(\zeta_r)\right]^2} , \quad r = 1,2,3,\ldots, \tag{2.46}$$

where

$$\gamma_{\xi_1}^{(r)} = i\zeta_r\sqrt{\xi_1/\eta_o} , \quad \gamma_{\xi_2}^{(r)} = i\zeta_r\sqrt{\xi_2/\eta_o} , \qquad \xi_1 < \xi_2 .$$

We now show that condition (2.40) is sufficient also for the existence of a solution to equation (2.44). To construct the Green's function we use 1 for the solution of the homogeneous equation which is regular at $\xi = 0$ and lies in $\mathcal{L}_2(0,\xi_o)$, and $\log\xi$ for the linearly independent solution which has the ξ dependence at infinity required by (2.40). Since the Wronskian $W(\log\xi, 1) = -1/\xi$, the solution to equation (2.44), written as

$$-\frac{d}{d\xi}\left(\xi \frac{dA_o(\xi)}{d\xi}\right) = -\frac{1}{\eta_o}\delta(\xi-\Xi) ,$$

is given by

$$A_o(\xi) = \frac{1}{\eta_o}\begin{cases} 1 & \xi < \Xi \\ \log\xi & \xi > \Xi \end{cases}$$
$$\equiv \frac{1}{\eta_o} H(\xi,\Xi) . \tag{2.47}$$

Substituting (2.47) and (2.46) into (2.42), we find

$$\phi_N(\xi, \eta, \Xi, 0) = \frac{1}{\eta_o} H(\xi, \Xi) - \frac{i\pi}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o(\gamma_{\xi_1}^{(r)}) H_o^{(1)}(\gamma_{\xi_2}^{(r)}) J_o(\gamma_\eta^{(r)})}{\left[J_o(\zeta_r)\right]^2} \tag{2.48}$$

as a solution to the potential problem (2.P). It should be noted that this solution plus any constant is also a solution to (2.P). Examining the series in equation (2.48), we see that it agrees (to within some constant) with the term, which is independent of k, in the field (equation (2.23)) for the corresponding Neumann problem. This comparison completes our investigation of potential problems. We do not consider any relation between uniqueness and the behavior at infinity.

There is little difference for the Neumann potential problem corresponding to the wave problem for the point source at (0, H). The first part of the previous discussion does not change at all; equation (2.40) remains the same. The second part of the discussion changes only in that we are now considering the equation

$$\nabla^2 \phi = 4\pi\, \delta(\bar{r} - \bar{r}_o) \quad (\bar{r}_o \text{ the vector to the point } (0, H)),$$

which when written in the coordinates of the paraboloid of revolution becomes

$$\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \phi}{\partial \eta}\right) = \delta(\xi)\, \delta(\eta - H) \ . \tag{2.49}$$

The assumption of a solution to (2.49) is then of the form

$$\phi_N(\xi, \eta, 0, H) = B_o(\xi, H) + \sum_{r=1}^{\infty} B_r(\xi, H) J_o(\gamma_\eta^{(r)}) \ , \tag{2.50}$$

while the expansion of the δ-function is now

$$\delta(\eta - H) = \frac{1}{\eta_o} + \frac{1}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o\left(\gamma_H^{(r)}\right) J_o\left(\gamma_\eta^{(r)}\right)}{\left[J_o(\zeta_r)\right]^2} \quad , \tag{2.51}$$

with $\gamma_H^{(r)} = \zeta_r\sqrt{H/\eta_o}$. This leads to the ordinary differential equations

$$\frac{d}{d\xi}\left(\xi\frac{dB_o(\xi)}{d\xi}\right) = \frac{1}{\eta_o}\delta(\xi) , \tag{2.52}$$

$$\frac{d}{d\xi}\left(\xi\frac{dB_r(\xi)}{d\xi}\right) - \lambda_r(0)B_r(\xi) = \frac{1}{\eta_o}\delta(\xi)\frac{J_o(\gamma_H^{(r)})}{\left[J_o(\zeta_r)\right]^2} , \quad r = 1, 2, \ldots, \tag{2.53}$$

with solutions

$$B_o(\xi) = \frac{1}{\eta_o}\log\xi \qquad (\xi > 0) ,$$

$$B_r(\xi) = \pi/i\,\frac{J_o(\gamma_H^{(r)})H_o^{(1)}(\gamma_\xi^{(r)})}{\eta_o\left[J_o(\zeta_r)\right]^2} \qquad (\xi > 0) , \quad r = 1, 2, \ldots, \tag{2.55}$$

and thus

$$\phi_N(\xi, \eta, 0, H) = \frac{1}{\eta_o}\log\xi - \frac{i\pi}{\eta_o}\sum_{r=1}^{\infty}\frac{H_o^{(1)}(\gamma_\xi^{(r)})J_o(\gamma_H^{(r)})J_o(\gamma_\eta^{(r)})}{\left[J_o(\zeta_r)\right]^2} \tag{2.56}$$

plus any constant. Examining the series in equation (2.56), we find that it agrees (to within some constant) with the term, which is independent of k, in the field (equation (2.27)) for the corresponding Neumann wave problem.

2.5 Residue Series for the Far Field

We consider first equation (2.17) without any assumptions on $k\xi_j$, $j = 1, 2$. Then using the formulas developed in Section 2.2 we can write

$$v_N(\xi, \eta, \Xi, 0) \sim \frac{i\pi}{2\eta_o}J_o(k\xi_1)H_o^{(2)}(k\xi_2) -$$

$$- \frac{\sqrt{2\pi}}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_1(\xi_1, -\lambda_r)v_2(\xi_2, -\lambda_r) , \tag{2.57}$$

where the error terms have been omitted for the sake of simplicity. We will continue to omit them throughout most of the section. The far field is defined by the mathematical condition $k\xi_2 \gg 1$; the case of physical interest is $\xi_1/\eta_o = O(1)$. For these conditions the behavior of $v_1(\xi_1, -\lambda_r)$ is at once determined by equations (2.20). It also immediately follows that

$$J_o(k\xi_1) \sim 1 ,$$

$$H_o^{(2)}(k\xi_2) \sim \left(\frac{2}{\pi k\xi_2}\right)^{1/2} e^{-ik\xi_2} e^{i\pi/4} .$$

Substituting for $H_o^{(2)}(k\xi_2)$ and $v_1(\xi_1, -\lambda_r)$, we write equation (2.57) as

$$v_N(\xi, \eta, \Xi, 0) \sim J_o(k\xi_1)\left[-\frac{1}{\eta_o}\left(\frac{\pi}{2ik\xi_2}\right)^{1/2} e^{-ik\xi_2}\right]$$

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2, -\lambda_r) , \quad (2.58)$$

so as to retain the dependence on $k\xi_1$ in the first term.

To estimate $v_2(\xi_2, -\lambda_r)$ we need to determine the order of magnitude of

$$s_{\xi_2} = \frac{k^2\xi_2}{\lambda_r} = \frac{k}{\lambda_r}(k\xi_2) \sim \frac{4k\eta_o}{\zeta_r^2}(k\xi_2) .$$

This depends on the relation between $k\eta_o$ and $k\xi_2$ as well as the value of ζ_r^2; the latter dependence is a function of r in addition. Since $k\xi_2 \gg 1$ there are three different possibilities for its order of magnitude; they are $k\xi_2 \gg 1/k\eta_o$, $k\xi_2 = O(1/k\eta_o)$ and $k\xi_2 \ll 1/k\eta_o$. For $k\xi_2 \gg 1/k\eta_o$ we can write the series in equation (2.58) as

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=1}^{M}\frac{J_o\left(\zeta_\eta^{(r)}(\rho)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2,-\lambda_r)$$

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=M+1}^{N}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2,-\lambda_r)$$

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=N+1}^{\infty}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2,-\lambda_r)\ , \tag{2.59}$$

where M is such that $r \leqslant M$ implies $\zeta_r^2 = O(1)$ $(s_{\xi_2} >> 1)$, N is such that $r > N$ implies $\zeta_r^2 \geqslant O(k\xi_2)$ $(s_{\xi_2} << 1)$, and $r \epsilon [M+1,\ N]$ implies $\zeta_r^2 = O\left((k\eta_o)(k\xi_2)\right)$ $\left(s_{\xi_2} = O(1)\right)$. We will refer to the three sums as $\sum_1$, $\sum_2$, $\sum_3$. Thus the problem of estimating $v_2(\xi_2, -\lambda_r)$ is reduced to estimating it in each of the three sums $\sum_1$, $\sum_2$, $\sum_3$.

In order to evaluate $v_2(\xi_2, -\lambda_r)$ in $\sum_1$ and $\sum_2$ we use the results of Appendix D. We can write

$$\bar{\rho} = -2i\ell = -2i\left(-\frac{\lambda_r}{2ik}\right) = \lambda_r/k \simeq \frac{\zeta_r^2}{4k\eta_o}\ .$$

Then in $\sum_1$, $\bar{\zeta}_{\xi_2}$ is large and s_{ξ_2} is large; thus expansions (D.9) and (D.10) inserted in equations (D.15) yield

$$v_2(\xi_2,-\lambda_r) \sim (2ik\xi_2)^{-1/2}\exp\left\{-\frac{\lambda_r}{2ik}\log(2ik\xi_2)\right\}e^{-ik\xi_2}\ . \tag{2.60}$$

Hence $\sum_1$ becomes

$$\sum_1 \sim -\frac{1}{\eta_o}\sqrt{\frac{\pi}{ik\xi_2}}\sum_{r=1}^{M}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}\cdot$$

$$\cdot\exp\left\{-\frac{\lambda_r}{2ik}\log(2ik\xi_2)\right\}e^{-ik\xi_2}.$$

Evaluating the two exponentials we obtain

$$\sum_1 \sim -\frac{1}{\eta_o}\sqrt{\frac{\pi}{ik\xi_2}}\sum_{r=1}^{M}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{-\frac{\zeta_r^2\pi}{8k\eta_o}\right\}e^{-ik\xi_2}.$$

$$\cdot\exp\left\{-\frac{i\zeta_r^2}{8k\eta_o}\log\frac{\zeta_r^2}{16(k\xi_2)(k\eta_o)e}\right\}. \tag{2.61}$$

For $\sum_2$, $s_{\xi_2} = O(1)$, and equations (D.15) give

$$v_2(\xi_2,-\lambda_r) \sim (2ik\xi_2)^{-1/2}(\lambda_r/k)^{1/6}\exp\left\{-\frac{\lambda_r}{2ik}\log-\frac{\lambda_r}{2ike}\right\}\bar{\Psi}\left(s_{\xi_2}(0)\right)\bar{\zeta}_{\xi_2}^{(r)}(0)H_{1/3}^{(2)}\left(\bar{\zeta}_{\xi_2}^{(r)}(0)\right), \tag{2.62}$$

with

$$s_{\xi_2}(0) = \frac{4k\eta_o}{\zeta_r^2}(k\xi_2)$$

and

$$\bar{\zeta}_{\xi_2}^{(r)}(0) = \frac{\lambda_r}{k}\bar{\Phi}\left(s_{\xi_2}(0)\right).$$

Therefore $\sum_2$ becomes

$$\sum_2 \sim -\frac{1}{\eta_o}\sqrt{\frac{\pi}{ik\xi_2}}\sum_{r=M+1}^{N}\left(\frac{\zeta_r^2}{4k\eta_o}\right)^{7/6}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{-\frac{\zeta_r^2\pi}{8k\eta_o}\right\}\cdot$$

$$\cdot\bar{\Psi}\left(s_{\xi_2}(0)\right)\bar{\zeta}_{\xi_2}^{(r)}(0)H_{1/3}^{(2)}\left(\bar{\zeta}_{\xi_2}^{(r)}(0)\right). \tag{2.63}$$

From equations (2.61) and (2.63) we see that both $\sum_1$ and $\sum_2$ are exponentially small.

In order to evaluate $v_2(\xi_2, -\lambda_r)$ for $\sum_3$ we need only use equation (2.21). This gives

$$\sum_3 \sim -\frac{i\pi}{\eta_o} \sum_{r=N+1}^{\infty} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_{\xi_1}^{(r)}(0)\right) H_o^{(1)}\left(\zeta_{\xi_2}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} .$$

Since $\zeta_{\xi_j}^{(r)}(0) = i\zeta_r(\xi_j/\eta_o)^{1/2}$, we may write $\sum_3$ as

$$\sum_3 \sim -\frac{i\pi}{\eta_o} \sum_{r=N+1}^{\infty} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\zeta_r(\xi_1/\eta_o)^{1/2}\right\} \exp\left\{-\zeta_r(\xi_2/\eta_o)^{1/2}\right\}. \tag{2.64}$$

We write the product of the exponentials as

$$\exp\left\{\zeta_r\left[(\xi_1/\eta_o)^{1/2} - (\xi_2/\eta_o)^{1/2}\right]\right\} .$$

But since $\xi_1/\eta_o = O(1)$, and $k\xi_2 \gg 1/k\eta_o$ implies $\xi_2/\eta_o \gg 1/k^2\eta_o^2$, this product is much less than

$$\exp\left\{-\zeta_r\left[\frac{1}{k\eta_o} - 1\right]\right\} ,$$

and $\sum_3$ is also exponentially small. This implies that the contribution from the residue series in equation (2.57) is exponentially small; thus it is much smaller than the error term of the $r = 0$ term. Therefore we may write equation (2.57) as

$$v_N(\xi, \eta, \Xi, 0) = \frac{i\pi}{2\eta_o} J_o(k\xi_1) H_o^{(2)}(k\xi_2) \left[1 + O(k\eta_o)\right] , \tag{2.65}$$

while upon using $k\xi_1 \ll 1$ equation (2.58) becomes

$$v_N(\xi,\eta,\Xi,0) = -\frac{1}{\eta_o}\left(\frac{\pi}{2ik\xi_2}\right)^{1/2} e^{-ik\xi_2}\left[1+O(k\xi_1)+O(k\eta_o)\right], \tag{2.66}$$

since $k\eta_o \gg 1/k\xi_2$.

In the case of $k\xi_2 = O(1/k\eta_o)$ we can write the series in equation (2.58) as

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=1}^{M}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2,-\lambda_r) -$$

$$-\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=M+1}^{\infty}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)J_o\left(\zeta_{\xi_1}^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_2(\xi_2,-\lambda_r),$$

where M is such that $r \lesssim M$ implies $\zeta_r^2 = O(1)$ $\left(s_{\xi_2} = O(1)\right)$, and for $r > M$ $\zeta_r^2 \gg 1$ ($s_{\xi_2} \ll 1$). Then the discussion above for $\sum_2$ applies to the first sum here, a similar discussion to the one above for $\sum_3$ applies to the second sum. The results are the same as above; equation (2.65) is valid, while we add the error term $O(1/k\xi_2)$ to equation (2.66). Finally, for the case $k\xi_2 \ll 1/k\eta_o$ we see that the series in equation (2.58) does not need to be divided into separate sums. The condition $s_{\xi_2} \ll 1$ is true for all terms; a similar argument to the one for the third sum above shows the series is exponentially small. Equation (2.65) is valid and in equation (2.66) we replace $O(k\eta_o)$ by $O(1/k\xi_2)$.

For the corresponding Dirichlet problem we first consider equation (2.28) without any restrictions on $k\xi_j$, $j = 1, 2$. Using the formulas developed in Section 2.2, we obtain

$$v_D(\xi,\eta,\Xi,0) \sim -\frac{\sqrt{2\pi}}{\eta_o}\sum_{r=1}^{\infty}\frac{J_o\left(\zeta_\eta^{(r)}(0)\right)}{\left[J_o'(\beta_r)\right]^2}\exp\left\{\frac{\lambda_r}{2ik}\log\frac{\lambda_r}{2ike}\right\}v_1(\xi_1,-\lambda_r)v_2(\xi_2,-\lambda_r). \tag{2.67}$$

However, the preceding arguments for the Neumann problem show that this residue series is exponentially small; thus we have the result that

$$v_D(\xi, \eta, \Xi, 0) \sim 0.$$

If we now consider the point source at (0, H), it is easily seen that the arguments do not differ from those above. We begin with equations (2.25) and (2.31) without any restriction on $k\xi$. Substituting the formulas of Section 2.2, we find

$$v_N(\xi, \eta, 0, H) = \frac{i\pi}{2\eta_o} H_o^{(2)}(k\xi) -$$

$$- \frac{\sqrt{2\pi}}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_H^{(r)}(0)\right)}{\left[J_o(\zeta_r)\right]^2} \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} v_2(\xi, -\lambda_r) , \quad (2.68)$$

$$v_D(\xi, \eta, 0, H) = - \frac{\sqrt{2\pi}}{\eta_o} \sum_{r=1}^{\infty} \frac{J_o\left(\zeta_\eta^{(r)}(0)\right) J_o\left(\zeta_H^{(r)}(0)\right)}{\left[J_o'(\beta_r)\right]^2} \exp\left\{\frac{\lambda_r}{2ik} \log \frac{\lambda_r}{2ike}\right\} v_2(\xi, -\lambda_r). \quad (2.69)$$

Then, since the far field is defined by $k\xi \gg 1$, comparison of these equation with equation (2.58) shows that we can refer to the previous arguments.

2.6 Interpretation of Far Field Results

The results of the previous section can be simply interpreted. Let us consider a source in the near field (at $(\Xi, 0)$ or $(0, H)$, it makes no difference), and investigate the field $u_N(\xi, \eta)$ or $u_D(\xi, \eta)$ in a neighborhood of the point (ξ_f, η_f) in the far field. We make the following two observations:

(i) The field far away from the source is approximately governed by the homogeneous wave equation

$$\nabla^2 u + k^2 u = 0.$$

(ii) The radius of the circular cross section of the paraboloid is $\rho = 2\sqrt{\xi\eta_o}$. Hence at (ξ_f, η_f)

$$\frac{d\rho}{d\xi} = \sqrt{\frac{\eta_o}{\xi_f}} \ll 1 ;$$

thus the paraboloid looks like a cylinder in a neighborhood of (ξ_f, η_f).

Therefore the local behavior of the field at the point (ξ_f, η_f) is governed by the axially symmetric cylindrical waveguide problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial^2 z^2} + k^2 u = 0 ,$$

$$u = 0 \text{ or } \frac{\partial u}{\partial n} = 0 \text{ at } \rho = \rho_f = 2\sqrt{\xi_f \eta_o} .$$

This problem can be solved by the method of separation of variables (Morse and Feshbach, 1953, Chapter 11). For the Dirichlet boundary condition we write the solution as

$$u_D(\rho, z) = \sum_{r=1}^{\infty} J_o\left(\rho\sqrt{k^2 - \lambda_r}\right)\left[A_r e^{i\sqrt{\lambda_r} z} + B_r e^{-i\sqrt{\lambda_r} z}\right], \tag{2.70}$$

where ρ, z are the associated cylindrical coordinates (Section 1.3), A_r and B_r depend only on r, λ_r is a solution of the equation

$$\frac{k^2\rho_f^2 - \zeta_r^2}{\rho_f^2} = \lambda_r , \tag{2.71}$$

and ζ_r $(r = 1, 2, 3, \ldots)$ are again the positive zeros of $J_o(\zeta) = 0$. From (2.71) we see that there are values of k (considering η_o fixed) for which all λ_r are negative $(k^2\rho_f^2 - \zeta_1^2 < 0)$. For these values of k the solution becomes, upon defining $i\Lambda_r = \sqrt{\lambda_r}$, $\Lambda_r > 0$,

$$u_D(\rho, z) = \sum_{r=1}^{\infty} J_o\left(\rho\sqrt{k^2 - \lambda_r}\right) A_r e^{-\Lambda_r z} ,$$

which is exponentially small. Therefore as $k \rightarrow 0$ we find that for k less than some k_o there are no more propagating solutions ("low frequency cutoff") and the field is exponentially small. This corresponds to the result observed in Section 2.5.

The solution to the Neumann problem can be written as

$$u_N(\rho, z) = \sum_{r=0}^{\infty} J_o\left(\rho\sqrt{k^2 - \lambda_r}\right)\left[C_r e^{i\sqrt{\lambda_r} z} + D_r e^{-i\sqrt{\lambda_r} z}\right], \tag{2.72}$$

where C_r and D_r depend only on r, λ_r is a solution of the equation

$$\frac{k^2 \rho_f^2 - \beta_r^2}{\rho_f^2} = \lambda_r , \tag{2.73}$$

where $\beta_o = 0$ and the β_r $(r = 1, 2, 3, \ldots)$ are again the positive zeros of $J_o'(\beta) = 0$. Since $\lambda_o = k^2$, $J_o\left(\rho\sqrt{k^2 - \lambda_o}\right) = 1$, and choosing the solution corresponding to a time dependence of $e^{i\omega t}$ we obtain

$$u_N(\rho, z) = D_o e^{-ikz} + \sum_{r=1}^{\infty} J_o\left(\rho\sqrt{k^2 - \lambda_r^2}\right)\left[C_r e^{i\sqrt{\lambda_r} z} + D_r e^{-i\sqrt{\lambda_r} z}\right] . \tag{2.74}$$

The series behaves as in the Dirichlet problem. Thus as $k \rightarrow 0$ it exhibits "low frequency cutoff". However, the $r = 0$ term remains; there exists a propagating term for the Neumann problem. This again corresponds to the results in Section 2.5.

CHAPTER III
HIGH FREQUENCY (FAT PARABOLOID) DIFFRACTION

In this chapter we investigate the integral representations for the total field if $k\eta_o \gg 1$. This condition corresponds either to large values of k (high frequency diffraction) or large values of η_o (a fat paraboloid).

We noted previously (Sections 1.4, 1.5, Appendix C) that Buchholz (1953) shows that $v_1(\eta_o, \lambda)$, $v_1'(\eta_o, \lambda)$ have a countable infinity of simple zeros which lie along the real axis. Moreover, the proof that all the zeros are real valued does not use any asymptotic representations (Buchholz, 1953, Chapter 17). Let $\lambda_1, \lambda_2, \lambda_3 \ldots; \Lambda_1, \Lambda_2, \Lambda_3, \ldots$ denote the zeros of $v_1(\eta_o, \lambda)$ and $v_1'(\eta_o, \lambda)$ respectively. It is proved in Appendix C that the integral representations (2.1) through (2.8) can be replaced by convergent residue series expansions. For equations (2.1) and (2.5) we find

$$v_N(\xi, \eta, \Xi, 0) = \frac{1}{2ik\eta_o} \sum_{r=1}^{\infty} \Gamma\left(\frac{\Lambda_r}{2ik} + \frac{1}{2}\right) \frac{v_1(\xi_1, -\Lambda_r) v_2(\xi_2, -\Lambda_r)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_r}} \cdot \frac{v_1(\eta, \Lambda_r)}{v_1(\eta_o, \Lambda_r)} , \tag{3.1}$$

$$v_N(\xi, \eta, 0, H) = \frac{1}{2ik\eta_o} \sum_{r=1}^{\infty} \Gamma\left(\frac{\Lambda_r}{2ik} + \frac{1}{2}\right) \frac{v_2(\xi, -\Lambda_r) v_1(H, \Lambda_r)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_r}} \cdot \frac{v_1(\eta, \Lambda_r)}{v_1(\eta_o, \Lambda_r)} ; \tag{3.2}$$

the other representations yield similarly behaving residue series.

We first want to determine whether equations (3.1) and (3.2) can be used to obtain an asymptotic representation of the field. That is to say we seek to learn whether or not the residue series are converging rapidly. It will be shown in Section 3.2 that they are not rapidly convergent. This demonstration is based upon the detailed discussion of the zeros of $v_1(\eta_o, \lambda)$ and $v_1'(\eta_o, \lambda)$ that is given in Section 3.1. The positive results we have obtained are given in Sections 3.3, 3.4, and 3.5. These results stem from the application of the saddle point method to derive an asymptotic representation for $v_N(\xi, \eta_o, \Xi, 0)$ if $k\eta_o \gg 1$, $k\Xi \gg 1$, and $k\xi \gg 1$. In either case (convergence or saddle point method), the discussion of the Dirichlet problem is quite similar to that of the Neumann problem and is not explicitly considered.

3.1 High Frequency Zeros

We first note that we need only to consider $v_1(\eta, \lambda)$ for $k\eta \gg 1$ and λ real (see above). We shall study the zeros of v_1 in three pairs of regions of λ, namely the regions corresponding to the conditions $0 \lesssim |\lambda/2k| \ll k\eta$, $|\lambda/k| = O(k\eta)$, and $|\lambda/k| \gg k\eta$. These conditions correspond to regions of validity of the various asymptotic representations of v_1 derived in Appendix D. If $k\eta$ is large, these regions of validity overlap and cover the real line. However, we do not use this fact and do not attempt to prove it. Roughly speaking, what we do show is that there exist sets of zeros in the range of λ where $|\lambda/k| = O(k\eta)$, sets of large cardinality for which the corresponding residue terms have a nonnegligible sum.

If we make the same identification of parameters as in Section 2.1, we observe that the condition $|\lambda/k| \gg k\eta$ implies that $|s_\eta| = k\eta|k/\lambda| \ll 1$ (recall that $s_\eta = \frac{2ik\eta}{4\lambda/2ik}$). Therefore, this case has already been considered in Chapter 2 (as indicated in the discussion following equation (2.15a); see also Appendix C). The zeros in this range (see equation (C.4a)) do not affect the rapidity of convergence of the residue series. The residues they contribute are exponentially small, involving negative exponents that monotonically decrease. Therefore, these zeros are not considered further in this section.

The situation as regards the zeros in the range where $0 \lesssim |\lambda/2k| \ll k\eta$ is rather more complicated. If in addition to the condition $k\eta \gg 1$ ($k\xi \gg 1$), we stipulate that $k\xi_1 \gg 1$ and $k\xi_2 \gg 1$ ($k\eta_1 \gg 1$ and $k\eta_2 \gg 1$), the slow convergence of the residue series (3.1) ((3.2)) may be demonstrated without considering these zeros. However, if $k\xi_1 \ll 1$ ($k\eta_1 \ll 1$), these zeros generate the dominant residue contribution and so are discussed further. Since $|2ik\eta| \gg |\lambda/2k|$, then (Buchholz, 1953, Chapter 7)

$$M_{\lambda/2ik,\,0}(2ik\eta) = \frac{(2ik\eta)^{-\lambda/2ik}\, e^{ik\eta}}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} \left[1 + O(1/k\eta)\right] +$$

$$+ \frac{(2ik\eta)^{\lambda/2ik}\, e^{-ik\eta}}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)}\, e^{-\pi\lambda/2k}\, e^{\pi i/2} \left[1 + O(1/k\eta)\right] ,$$

from which we obtain

$$v_1(\eta, \lambda) = (2ik\eta)^{-1/2} \left\{ \frac{(2ik\eta)^{-\lambda/2ik}\, e^{ik\eta}}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} + \right.$$

$$\left. + \frac{(2ik\eta)^{\lambda/2ik}\, e^{-ik\eta}}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)}\, e^{-\pi\lambda/2k}\, e^{\pi i/2} \right\} \left[1 + O(1/k\eta)\right]. \tag{3.3}$$

Hence, the zeros of $v_1(\eta_o, \lambda)$ for $k\eta_o \gg 1$ and $0 \lesssim |\lambda/2k| \ll k\eta$ are asymptotic to the solutions of the equation

$$\frac{(2ik\eta_o)^{-\lambda/2ik}\, e^{ik\eta_o}}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} = \frac{(2ik\eta_o)^{\lambda/2ik}\, e^{-ik\eta_o}}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)}\, e^{-\pi\lambda/2k}\, e^{-\pi i/2} . \tag{3.4}$$

That such zeros exist for some values of $k\eta_o$ can be demonstrated by setting $\lambda = 0$. Then (3.4) becomes

$$e^{i(k\eta_o - \frac{\pi}{4})} + e^{-i(k\eta_o - \frac{\pi}{4})} = 0 ;$$

thus if $k\eta_o$ is a solution of $\cos(k\eta - \frac{\pi}{4}) = 0$, a corresponding zero λ^* of $v_1(\eta_o, \lambda)$ exists and is asymptotic to 0.

A similar result follows for $v_1'(\eta_o, \lambda)$. By differentiating in equation (3.3) we find

$$v_1'(\eta,\lambda) = (2ik\eta)^{-1/2}\left\{\frac{(2ik\eta)^{-\lambda/2ik}\, e^{ik\eta}}{2\Gamma\left(\frac{1}{2}-\frac{\lambda}{2ik}\right)} - \frac{(2ik\eta)^{\lambda/2ik}\, e^{-ik\eta}}{2\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)}\, e^{-\pi\lambda/2k}\, e^{\pi i/2}\right\} \cdot$$

$$\cdot\left[1+O(1/k\eta)\right] . \qquad (3.5)$$

Therefore the zeros of $v_1'(\eta_o, \lambda)$ for $k\eta_o \gg 1$ and $0 \leqslant |\lambda/2k| \ll k\eta$ are asymptotic to the solutions of the equation

$$\frac{(2ik\eta_o)^{-\lambda/2ik}\, e^{ik\eta_o}}{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2ik}\right)} = \frac{(2ik\eta_o)^{\lambda/2ik}\, e^{-ik\eta_o}}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)}\, e^{-\pi\lambda/2k}\, e^{\pi i/2} . \qquad (3.6)$$

This implies that if $k\eta_o$ is a solution of $\sin(k\eta - \frac{\pi}{4}) = 0$, a corresponding zero Λ^* of $v_1'(\eta_o, \lambda)$ exists and is asymptotic to 0. The determination of all the zeros in this range depends on an explicit solution of (3.6) (or (3.4) for the zeros of $v_1(\eta_o, \lambda)$). We have not obtained such a solution.

The important region is that where $|\lambda/k| = O(k\eta)$. It contains the zeros which govern the rapidity of convergence of the residue series (3.1) ((3.2)) if $k\eta \gg 1$, $k\xi_1 \gg 1$, $k\xi_2 \gg 1$ ($k\xi \gg 1$, $k\eta_1 \gg 1$, $k\eta_2 \gg 1$). We observe that $|\lambda/k| = O(k\eta)$ implies $|s_\eta| = O(1)$. In this range we can thus use the Airy function representation for $v_1(\eta,\lambda)$ if $\arg s_\eta = 0$ and $s_\eta > 1$, or if $s_\eta < 1$ but $1 - s_\eta = O\left(1/(k\eta)^{2/3}\right)$. If $\arg s_\eta = 0$, $s_\eta < 1$, $1 - s_\eta \gg 1/(k\eta)^{2/3}$, and $s_\eta \gg 1/(k\eta)^{2/3}$, we can use either the Airy or Bessel function representations for $v_1(\eta,\lambda)$; while if $\arg s_\eta = \pi$, we may use only the Bessel function representation. These cases will be presented in the order determined by beginning with negative values of λ satisfying the inequality $\lambda < -k^2\eta$, proceeding to negative values of λ satisfying the relation $\lambda \geqslant -k^2\eta$, and finally ending with positive values of λ.

Case I. $\lambda < -k^2\eta$; $|\lambda/k| = O(k\eta)$

For negative values of λ ($\arg\lambda = \pi$, $\arg s_\eta = 0$) satisfying the inequality $\lambda < -k^2\eta$ we observe that $s_\eta < 1$. Thus for values of λ such that

$$1 + \frac{k^2\eta}{\lambda} \gg \frac{1}{(k\eta)^{2/3}} \qquad \left(1 - s_\eta \gg \frac{1}{(k\eta)^{2/3}}\right) ,$$

equations (D. 34) may be used to obtain

$$M_{\lambda/2ik,\,0}(2ik\eta) = (\lambda/ik)^{1/2}\left[\frac{\Phi(s_\eta)}{\phi(s_\eta)}\right]^{1/2} J_0(\zeta) + \frac{\Psi(s_\eta)\zeta_\eta^5\, O(1)}{(k\eta)^{9/2}} \qquad \left(|\zeta_\eta| \leq N\right) \quad (3.7a)$$

$$M_{\lambda/2ik,\,0}(2ik\eta) = (\lambda/ik)^{1/2}\left[\frac{\Phi(s_\eta)}{\phi(s_\eta)}\right]^{1/2} J_0(\zeta_\eta) + \frac{\Psi(s_\eta)\zeta_\eta^{1/2}\left[e^{i\zeta_\eta}O(1) + e^{-i\zeta_\eta}O(1)\right]}{(k\eta)^{3/2}}$$

$$\left(|\zeta_\eta| > N\right), \quad (3.7b)$$

with

$$\phi^2(s_\eta) = \frac{1}{s_\eta}(1 - s_\eta) ,$$

where by (D. 23)

$$\Phi(s_\eta) = \int_0^{s_\eta} \left(\frac{1-s}{s}\right)^{1/2} ds , \quad (3.8)$$

$$\zeta_\eta = \frac{\lambda}{ik}\Phi(s_\eta) . \quad (3.9)$$

In order to avoid repetition of long equations, we observe that the representation of $v_1(\eta, \lambda) = (2ik\eta)^{-1/2} M_{\lambda/2ik,\,0}(2ik\eta)$ follows immediately from equations (3. 7). By differentiation we find

$$v_1'(\eta, \lambda) = (2ik\eta)^{-1/2}\left[M'_{\lambda/2ik,\,0}(2ik\eta) + O(1/k\eta)\right] , \tag{3.10}$$

where

$$M'_{\lambda/2ik,\,0}(2ik\eta) = \frac{d}{d(2ik\eta)}\, M_{\lambda/2ik,\,0}(2ik\eta) \quad .$$

Therefore, since

$$\frac{ds_\eta}{d(2ik\eta)} = \frac{ik}{2\lambda} = O(1/k\eta)$$

and

$$\frac{d\zeta_\eta}{d(2ik\eta)} = \frac{1}{2}\left(\frac{1-s_\eta}{s_\eta}\right)^{1/2} ,$$

equations (3.7) imply

$$M'_{\lambda/2ik,\,0}(2ik\eta) = \left(\frac{\lambda}{ik}\right)^{1/2}\left\{\left[\frac{\Phi(s_\eta)}{\phi(s_\eta)}\right]^{1/2} J_o'(\zeta_\eta)\cdot\frac{1}{2}\left(\frac{1-s_\eta}{s_\eta}\right)^{1/2} + O(1/k\eta)\right\} \tag{3.11}$$

for all allowed ζ_η. We can now assert that $v_1(\eta, \lambda)$ and $v_1'(\eta, \lambda)$ have no zeros in the range of λ where equations (3.7) through (3.11) are valid. This follows from the fact that equation (3.9) shows that $\arg \zeta_\eta = \pi/2$, and there are no complex zeros of $J_o(\zeta)$ or $J_o'(\zeta)$ (Erdélyi et al, 1953).

We continue to consider negative values of λ satisfying $\lambda < -k^2\eta$; in addition, we consider only those values of λ for which

$$\frac{-k^2\eta}{\lambda} \gg \frac{1}{(k\eta)^{2/3}} \qquad \left(s_\eta \gg \frac{1}{(k\eta)^{2/3}}\right) \quad .$$

We further allow values of λ in a neighborhood of $-k^2\eta$ defined by the relation

$$1 + \frac{k^2\eta}{\lambda} = O\left(\frac{1}{(k\eta)^{2/3}}\right) \qquad \left(1 - s_\eta = O\left(\frac{1}{(k\eta)^{2/3}}\right)\right) .$$

Then equations (D.21) yield

$$M_{\lambda/2ik,\,0}(2ik\eta) = \frac{C\left(\frac{\lambda}{2ik}\right)}{\sqrt{2\pi}}\, e^{\pi i/4}\, e^{-\pi\lambda/2k}\left[\bar{\Psi}(s_\eta)\,Ai(-\bar{\sigma}_\eta) + O(1/k\eta)\right] \quad \left(\left|\bar{\zeta}_\eta\right| \leqslant N\right) ,$$

(3.12a)

$$M_{\lambda/2ik,\,0}(2ik\eta) = \frac{C\left(\frac{\lambda}{2ik}\right)}{\sqrt{2\pi}}\, e^{\pi i/4}\, e^{-\pi\lambda/2k}\left[\bar{\Psi}(s_\eta)\,Ai(-\bar{\sigma}_\eta) + \right.$$

$$\left. + \frac{\bar{\Psi}(s_\eta)\bar{\zeta}_\eta^{-1/6}\left\{\bar{E}^1(\bar{\zeta}_\eta) + \bar{E}^2(\bar{\zeta}_\eta)\right\}}{k\eta}\right] \quad \left(\left|\bar{\zeta}_\eta\right| > N\right) ,$$

(3.12b)

with

$$\bar{\phi}^2(s_\eta) = \frac{1}{s_\eta}(s_\eta - 1) ,$$

$$C\left(\frac{\lambda}{2ik}\right) = 2(3)^{1/6}\pi^{1/2}\left(-\frac{\lambda}{2k}\right)^{1/6} ,$$

$$\bar{\sigma}_\eta = \left(\frac{3}{2}\,\bar{\zeta}_\eta\right)^{2/3} ,$$

and, from equations (D.3), (D.4) and (D.6),

$$\bar{\Phi}(s_\eta) = \int_1^{s_\eta}\left(\frac{s-1}{s}\right)^{1/2} ds , \tag{3.13}$$

$$\bar{\zeta}_\eta = \frac{i\lambda}{k}\int_{s_\eta}^{1}\left(\frac{1-s}{s}\right)^{1/2} ds , \tag{3.14}$$

$$\bar{\Psi}(s_\eta) = \left[\bar{\Phi}(s_\eta)\right]^{1/6}\left[\bar{\phi}(s_\eta)\right]^{-1/2} . \tag{3.15}$$

Representations for $v_1(\eta, \lambda)$ follow immediately from equations (3.12). In order to obtain a representation for $v_1'(\eta, \lambda)$, we find one for $M'_{\lambda/2ik, 0}(2ik\eta)$. Since

$$\frac{ds_\eta}{d(2ik\eta)} = \frac{ik}{2\lambda} = O(1/k\eta) \qquad \text{(recall that } |\lambda/k| = O(k\eta)),$$

$$\frac{d(-\bar{\sigma}_\eta)}{d(2ik\eta)} = -\left(\frac{3}{2}\bar{\zeta}\right)^{-1/3} \frac{d\bar{\zeta}_\eta}{d(2ik\eta)},$$

and
$$\frac{d\bar{\zeta}_\eta}{d(2ik\eta)} = \frac{1}{2}\left(\frac{1-s_\eta}{s_\eta}\right)^{1/2},$$

differentiating in equation (3.12) gives

$$M'_{\lambda/2ik, 0}(2ik\eta) = \frac{C\left(\frac{\lambda}{2ik}\right)}{\sqrt{2\pi}} e^{\pi i/4} e^{-\pi\lambda/2k} \left[\bar{\psi}(s_\eta)\left\{-\left(\frac{3}{2}\bar{\zeta}_\eta\right)^{-1/3}\right\} \cdot \right.$$
$$\left. \cdot Ai'(-\bar{\sigma}_\eta) \cdot \frac{1}{2}\left(\frac{1-s_\eta}{s_\eta}\right)^{1/2} + O(1/k\eta)\right] \tag{3.16}$$

for all allowed $\bar{\zeta}_\eta$. We can now assert that $v_1(\eta, \lambda)$ and $v_1'(\eta, \lambda)$ have no zeros if $\lambda < -k^2\eta$. This follows from what has been shown in the previous paragraph, together with equation (3.14) which implies that $\arg \bar{\zeta}_\eta = 3\pi/2$ and $\arg \bar{\sigma}_\eta = \pi$ ($-\bar{\sigma}_\eta$ is positive, and there are only negative zeros of $Ai(\sigma)$ or $Ai'(\sigma)$ (Abramowitz and Stegun, 1964, Chapter 10)).

Case II. $-k^2\eta \leq \lambda < 0$; $|\lambda/k| = O(k\eta)$

We proceed to consider negative values of λ satisfying the relation $\lambda \geq -k^2\eta$. In this case $s_\eta \geq 1$, thus equations (3.12) and (3.16) are valid for $M_{\lambda/2ik, 0}(2ik\eta)$ and $M'_{\lambda/2ik, 0}(2ik\eta)$, respectively. However, equations (D.3) now yield

$$\bar{\zeta}_\eta = -\lambda/k \int_1^{s_\eta} \left(\frac{s-1}{s}\right)^{1/2} ds . \tag{3.17}$$

For $s_\eta = 1$, this implies $\bar{\zeta}_\eta = 0$, which in turn implies both $\bar{\sigma}_\eta = 0$ and $-\bar{\sigma}_\eta = 0$. Consequently, the zeros $\lambda_n(\Lambda_n)$ of $v_1(\eta, \lambda)$ $(v_1'(\eta, \lambda))$ satisfy the inequality $\lambda_n > -k^2\eta$ $(\Lambda_n > -k^2\eta)$. We consider two possibilities for values of $\lambda > -k^2\eta$. The first assumes values of λ in a neighborhood of $-k^2\eta$ defined by the relation

$$-\frac{k^2\eta}{\lambda} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right) \qquad \left(s_\eta - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right)\right),$$

while the second treats those values of λ for which

$$-\frac{k^2\eta}{\lambda} - 1 \gg \frac{1}{(k\eta)^{2/3}} \qquad \left(s_\eta - 1 \gg \frac{1}{(k\eta)^{2/3}}\right) \quad .$$

In either case we note that $\arg \bar{\zeta}_\eta = 0$. This implies $\arg \bar{\sigma}_\eta = 0$, or that $-\bar{\sigma}_\eta$ is negative; thus $v_1(\eta, \lambda)$ or $v_1'(\eta, \lambda)$ may have zeros in these ranges of λ.

To see whether such zeros exist, we note that for the first possibility equations (D.7) and (D.8) give

$$\bar{\phi}(s_\eta) = (s_\eta - 1)^{1/2} \left[1 + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right] \quad , \tag{3.18}$$

$$\bar{\Phi}(s_\eta) = \frac{2}{3}(s_\eta - 1)^{3/2} \left[1 + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right] . \tag{3.19}$$

From these equations together with (3.13), (3.15) and (3.17), we find

$$\bar{\zeta}_\eta = -\frac{\lambda}{k} \cdot \frac{2}{3}(s_\eta - 1)^{3/2} \left[1 + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right], \tag{3.20}$$

$$\bar{\psi}(s_\eta) = (2/3)^{1/6} \left[1 + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right] \quad . \tag{3.21}$$

In addition,

$$\bar{\sigma}_\eta = (-\frac{\lambda}{k})^{2/3}(s_\eta - 1) \left[1 + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right] . \tag{3.22}$$

If we now substitute (3.21) into (3.12a) (in this range of λ $\bar{\zeta}_\eta = O(1)$) and use the definition of $v_1(\eta, \lambda)$, we obtain

$$v_1(\eta,\lambda) = \frac{(2ik\eta)^{-1/2}}{\sqrt{2\pi}} (2/3)^{1/6} C\left(\frac{\lambda}{2ik}\right) e^{\pi i/4} e^{-\pi\lambda/2k} \left[Ai(-\bar{\sigma}_\eta) + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right] , \tag{3.23}$$

with $\bar{\sigma}_\eta$ given by equation (3.22). Then if $-\sigma_p$ $(p = 1, 2, 3, \ldots)$ is a zero of $Ai(-\sigma) = 0$ $(\sigma_p > 0,\ \sigma_{p+1} > \sigma_p)$, we can perform a Taylor expansion of

$$Ai\left[-\sigma_p + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right]$$

about $-\sigma_p$ to show that in this range of λ there exist zeros λ_p of $v_1(\eta, \lambda)$ which are given by

$$\sigma_p + O\left(\frac{1}{(k\eta)^{2/3}}\right) = \left(-\frac{\lambda_p}{k}\right)^{2/3} (s_\eta^{(p)} - 1)\left[1 + O\left(\frac{1}{(k\eta)^{2/3}}\right)\right], \tag{3.24}$$

where $s_\eta^{(p)} = -k^2\eta/\lambda_p$. This equation may be solved by noting that

$$-\frac{k^2\eta}{\lambda} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right)$$

implies

$$\lambda = -k^2\eta\left[1 + \frac{A}{(k\eta)^{2/3}}\right] , \tag{3.25}$$

where A is independent of $k\eta$ and $|A| \ll (k\eta)^{2/3}$. Substituting equation (3.25) into (3.24) and solving for A yields

$$\lambda_p = -k^2\eta\left[1 - \frac{\sigma_p}{(k\eta)^{2/3}} + O\left(\frac{1}{(k\eta)^{4/3}}\right)\right] \quad \left(p = 1, 2, 3 \ldots,\ \sigma_p \ll (k\eta)^{2/3}\right) \tag{3.26}$$

as the equation governing the zeros of $v_1(\eta, \lambda)$ in the given range of λ.

A similar equation exists for the zeros of $v_1'(\eta, \lambda)$. Let $-\Sigma_p$ $(p = 1, 2, 3, \ldots)$ denote the zeros of $\mathrm{Ai}'(-\sigma)$ $\left(\Sigma_p > 0,\ \Sigma_{p+1} > \Sigma_p\right)$. Then using equation (3.16) for $M'_{\lambda/2ik,\,0}(2ik\eta)$, we can proceed as above to derive

$$\Lambda_p = -k^2\eta\left[1 - \frac{\Sigma_p}{(k\eta)^{2/3}} + O\left(\frac{1}{(k\eta)^{4/3}}\right)\right] \quad \left(p = 1, 2, 3\ldots,\ \Sigma_p \ll (k\eta)^{2/3}\right) \tag{3.27}$$

as the equation governing the zeros of $v_1'(\eta, \lambda)$ in the given range of λ.

The second possibility for $\lambda > -k^2\eta$

$$\left(-\frac{k^2\eta}{\lambda} - 1 \gg \frac{1}{(k\eta)^{2/3}}\right)$$

will be treated somewhat differently. We show that zeros of $v_1(\eta, \lambda)$ $(v_1'(\eta, \lambda))$ exist, but do not represent them explicitly. This is permissible since we never need such a representation in our discussion of the diffraction problem. From equation (3.17) we note that in this case $\bar{\zeta}_\eta = O(k\eta)$; thus we see from equation (3.12b) that there exist zeros of $v_1(\eta, \lambda)$ if there exist solutions λ_p of the equation

$$\sigma_p + O(1/k\eta) = \left[-\frac{3\lambda}{2k}\int_1^{s_\eta}\left(\frac{s-1}{s}\right)^{1/2} ds\right]^{2/3} \quad (p > \text{some } p_0). \tag{3.28}$$

Since σ_p are the zeros of $\mathrm{Ai}(-\sigma) = 0$, such λ_p exist if

$$f(\lambda) = -\frac{3\lambda}{2k}\int_1^{-k^2\eta/\lambda}\left(\frac{s-1}{s}\right)^{1/2} ds$$

is an increasing function of λ. But

$$f'(\lambda) = -3/2k\int_1^{s_\eta}\left(\frac{s-1}{s}\right)^{1/2} ds - \frac{3\lambda}{2k}\left(\frac{s_\eta - 1}{s_\eta}\right)^{1/2}\left(\frac{k^2\eta}{\lambda^2}\right),$$

and we can use equation (D.5) to show

$$f'(\lambda) = -3/2k\left[\sqrt{s_\eta(s_\eta - 1)} - \log\left(\sqrt{s_\eta - 1} + \sqrt{s_\eta}\right)\right] + \frac{3}{2k}\sqrt{s_\eta(s_\eta - 1)}$$

or

$$f'(\lambda) = \frac{3}{2k}\log\left(\sqrt{s_\eta - 1} + \sqrt{s_\eta}\right) .$$

Since $s_\eta > 1$, $f'(\lambda) > 0$, which implies that $f(\lambda)$ is an increasing function of λ. Therefore, there exist solutions λ_p of equation (3.28); these solutions are the zeros of $v_1(\eta, \lambda)$ in this range of λ.

A similar argument can be made for the zeros of $v_1'(\eta, \lambda)$. We can show as above that there exist zeros Λ_p of $v_1'(\eta, \lambda)$; these zeros are solutions of the equation

$$\sum_p + O(1/k\eta) = \left[-\frac{3\lambda}{2k}\int_1^{s_\eta}\left(\frac{s-1}{s}\right)^{1/2} ds\right]^{2/3} \quad (p > \text{some } p_0) . \qquad (3.29)$$

Case III. $\lambda > 0$; $|\lambda/k| = O(k\eta)$

Last we consider positive values of λ ($\arg\lambda = 0$, $\arg s_\eta = \pi$). In this case equations (3.7) and equation (3.11) are valid for $M_{\lambda/2ik, 0}(2ik\eta)$ and $M'_{\lambda/2ik, 0}(2ik\eta)$, respectively. Moreover, by (D.25),

$$\Phi(s_\eta) = -i\int_0^{s_\eta}\left(\frac{s-1}{s}\right)^{1/2} ds , \qquad (3.30)$$

and as a result

$$\zeta_\eta = -\lambda/k\int_0^{s_\eta}\left(\frac{s-1}{s}\right)^{1/2} ds . \qquad (3.31)$$

If we let $s = -x$ in (3.31), we find

$$\zeta_\eta = \lambda/k \int_0^{|s_\eta|} \left(\frac{x+1}{x}\right)^{1/2} dx \; ; \tag{3.32}$$

thus $\arg \zeta_\eta = \arg \lambda = 0$, and $\zeta_\eta = O(k\eta)$. Therefore, we observe from equation (3.7b) that there exist zeros of $v_1(\eta, \lambda)$ in this range of λ if there exist solutions λ_r of the equation

$$\beta_r + O(1/k\eta) = \lambda/k \int_0^{|s_\eta|} \left(\frac{x+1}{x}\right)^{1/2} dx \quad (r > \text{some } r_0) \; , \tag{3.33}$$

where the β_r are large positive zeros of $J_0(\beta)$ ($\beta_{r_0} = O(k\eta)$). Such λ_r exist if

$$g(\lambda) = \lambda/k \int_0^{k^2\eta/\lambda} \left(\frac{x+1}{x}\right)^{1/2} dx$$

is an increasing function of λ. But

$$g'(\lambda) = \frac{1}{k} \int_0^{|s_\eta|} \left(\frac{x+1}{x}\right)^{1/2} dx + \frac{\lambda}{k} \left(\frac{|s_\eta|+1}{|s_\eta|}\right)^{1/2} \left(-\frac{k^2\eta}{\lambda^2}\right)$$

or

$$g'(\lambda) = \frac{1}{k} \int_0^{|s_\eta|} \left(\frac{x+1}{x}\right)^{1/2} dx - \frac{1}{k} \sqrt{|s_\eta|\left(|s_\eta|+1\right)} \; .$$

Integrating by parts, we find

$$\int_0^{|s_\eta|} \left(\frac{x+1}{x}\right)^{1/2} dx = \sqrt{|s_\eta|\left(|s_\eta|+1\right)} + \frac{1}{2} \int_0^{|s_\eta|} \frac{dx}{\sqrt{x(x+1)}} \; .$$

Thus $g'(\lambda) > 0$, which implies that $g(\lambda)$ is an increasing function of λ and that there exist solutions λ_r of (3.33). These solutions are the zeros of $v_1(\eta, \lambda)$ in the given range of λ. An explicit representation of these zeros is shown by Buchholz (1953, Chapter 17).

An analogous argument applies for the zeros of $v_1'(\eta, \lambda)$. There exist zeros Λ_r of $v_1'(\eta, \lambda)$ which are solutions of the equation

$$\zeta_r + O(1/k\eta) = \frac{\lambda}{k}\int_0^{|s_\eta|}\left(\frac{x+1}{x}\right)^{1/2} dx \qquad (r > \text{some } r_o) , \qquad (3.34)$$

where the ζ_r are large positive zeros of $J_o(\zeta)$ ($\zeta_{r_o} = O(k\eta)$).

3.2 Convergence of Residue Series

We can now investigate whether or not the residue series (3.1) or (3.2) can be used to obtain an asymptotic representation of the field if $k\eta_o \gg 1$. We also assume $k\eta \gg 1$ in (3.1) and $k\xi \gg 1$ in (3.2). These mathematical conditions imply that the field point is far (with respect to wavelength) from the focus. Thus there are two possible choices of source and field points for each series. They correspond to the conditions

$k\xi_1 \gg 1,\ k\xi_2 \gg 1$ or $k\xi_1 \ll 1,\ k\xi_2 \gg 1$ in equation (3.1),

$k\eta_1 \gg 1,\ k\eta_2 \gg 1$ or $k\eta_1 \ll 1,\ k\eta_2 \gg 1$ in equation (3.2).

The first possibility for (3.1) corresponds to the source far from the focus as well as the field point far from the axis. The second possibility demands either the source to be near (with respect to wavelength) the focus or the field point to be near the axis. If we set $\eta = \eta_o$ to obtain the field on the surface, the axis becomes the tip in the above possibilities. Similarly, the first possibility for (3.2) corresponds to the source far from the focus as well as the field point far from the axis. The second possibility demands either the source to be near the focus or the field point to be near the axis. If we set $\eta = \eta_o$ in (3.2), the first possibility then corresponds to the source far from the focus while the second possibility corresponds to the source near the focus.

We shall show that the first possibility for each series implies the existence of a large number of terms which must be summed in order to calculate the field. This slow convergence prohibits these series from being of practical computational value. It also negates the possibility of obtaining any asymptotic representations. The second alternative for each series involves a difficulty of an entirely different nature. Instead of a large number of terms, we encounter a degree of uncertainty as to how many terms are needed. The nature of this uncertainty will be discussed later in this section.

We consider first equation (3.1) if $k\xi_1 \gg 1$, $k\xi_2 \gg 1$. To demonstrate what was indicated above, we study the terms originating from zeros in the range $\arg\lambda = o(\arg s_\eta = \pi)$, $\lambda/k = O(k\eta_0)$, $\lambda < k^2\xi_1$, $k^2\xi_1/\lambda - 1 \gg 1/(k\eta_0)^{2/3}$. In this range we note from equation (3.34) that the separation of two adjacent zeros is defined by

$$\Delta\frac{\Lambda}{k} = \frac{\Lambda_{r+1}}{k} - \frac{\Lambda_r}{k} = O(1) \ .$$

Thus there are $O(k\eta_0)$ zeros in this range of λ. It remains to evaluate the order of each of the residue terms. Hence, we find asymptotic representations for the Whittaker functions that appear in these terms. We observe that

$$\frac{ds_\eta}{d\lambda} = \frac{ik\eta}{1/ik}\left(-\frac{1}{\lambda^2}\right) = \frac{1}{k}\,O(1/k\eta)$$

and by (3.32)

$$\frac{d\zeta_\eta}{d\lambda} = \frac{1}{k}\int_0^{|s_\eta|}\left(\frac{x+1}{x}\right)^{1/2}dx + \frac{\lambda}{k}\left(\frac{|s_\eta|+1}{|s_\eta|}\right)^{1/2}\cdot\frac{1}{k}\,O(1/k\eta) \ .$$

Thus differentiation with respect to λ in equation (3.11) yields

$$\frac{d}{d\lambda} v_1'(\eta,\lambda) = (2ik\eta)^{-1/2}\left[\left(\frac{\lambda}{ik}\right)^{1/2}\left[\frac{\Phi(s_\eta)}{\phi(s_\eta)}\right]^{1/2} J_0''(\zeta_\eta)\cdot\frac{1}{2}\left(\frac{1-s_\eta}{s_\eta}\right)^{1/2}\frac{d\zeta_\eta}{d\lambda} + \right.$$

$$\left. + \frac{1}{k} O\left(\frac{1}{(k\eta)^{1/2}}\right)\right] . \tag{3.35}$$

If we substitute $\eta = \eta_0$, set $\lambda = \Lambda_r$, observe that (3.34) implies

$$(\zeta_{\eta_0})_{\lambda=\Lambda_r} = \zeta_r + O(1/k\eta_0) ,$$

perform a Taylor expansion of $J_0''\left(\zeta_r + O(1/k\eta_0)\right)$ about ζ_r, and use Bessel's differential equation, then (3.35) becomes

$$\left(\frac{d}{d\lambda} v_1'(\eta_0,\lambda)\right)_{\lambda=\Lambda_r} = (2ik\eta_0)^{-1/2}\left[-\left(\frac{\Lambda_r}{ik}\right)^{1/2}\left[\frac{\Phi(s_{\eta_0}^{(r)})}{\phi(s_{\eta_0}^{(r)})}\right]^{1/2} J_0(\zeta_r)\cdot\frac{1}{2}\left(\frac{1-s_{\eta_0}^{(r)}}{s_{\eta_0}^{(r)}}\right)^{1/2} \cdot\right.$$

$$\left. \cdot\left(\frac{d\zeta_{\eta_0}}{d\lambda}\right)_{\lambda=\Lambda_r} + \frac{1}{k} O\left(\frac{1}{(k\eta_0)^{1/2}}\right)\right]. \tag{3.36}$$

The asymptotic representations for $v_1(\eta,\lambda_r)$ and $v_1(\eta_0, \Lambda_r)$ follow from equation (3.7b). However, to find representations for $v_1(\xi_1,-\Lambda_r)$ and $v_2(\xi_2,-\Lambda_r)$ we need some additional analysis. We again identify the parameters of Appendix D as in Section 2.2. Thus $\arg \Lambda_r = 0$ implies

$$\arg s_{\xi_j}^{(r)} = 0, \quad j = 1,2, \qquad \left(s_{\xi_j}^{(r)} = \frac{2ik\xi_j}{4(-\Lambda_r/2ik)}\right)$$

while $\Lambda_r < k^2\xi_1 < k^2\xi_2$ implies $s_{\xi_2}^{(r)} > s_{\xi_1}^{(r)} > 1$. Also

$$\frac{k^2\xi_1}{\Lambda_r} - 1 \gg \frac{1}{(k\eta_o)^{2/3}} \quad \text{implies} \quad \frac{k^2\xi_2}{\Lambda_r} - 1 \gg \frac{1}{(k\eta_o)^{2/3}} ,$$

or

$$s_{\xi_1}^{(r)} - 1 \gg \frac{1}{(k\eta_o)^{2/3}} \quad \text{implies} \quad s_{\xi_2}^{(r)} - 1 \gg \frac{1}{(k\eta_o)^{2/3}} .$$

Finally, equation (D.3) yields

$$\bar{\zeta}_{\xi_j}^{(r)} = \frac{\Lambda_r}{k} \int_1^{s_{\xi_j}^{(r)}} \left(\frac{s-1}{s}\right)^{1/2} ds \; ; \tag{3.37}$$

the above inequalities imply that $\bar{\zeta}_{\xi_j}^{(r)} \gg 1$. Therefore, equations (D.15b) and (D.18b) yield

$$v_2(\xi_2, -\Lambda_r) \sim (2ik\xi_2)^{-1/2} \exp\left\{-\frac{\Lambda_r}{2ik} \log -\frac{\Lambda_r}{2ike}\right\} \frac{e^{-i\bar{\zeta}_{\xi_2}^{(r)}}}{\left(\bar{\phi}(s_{\xi_2}^{(r)})\right)^{1/2}} , \tag{3.38}$$

and

$$v_1(\xi_1, -\Lambda_r) \sim (2ik\xi_1)^{-1/2} \frac{e^{\pi\Lambda_r/2k}}{\sqrt{2\pi}} \frac{\left[e^{i\bar{\zeta}_{\xi_1}^{(r)}} + ie^{-i\bar{\zeta}_{\xi_1}^{(r)}}\right]}{\left(\bar{\phi}(s_{\xi_1}^{(r)})\right)^{1/2}} , \tag{3.39}$$

respectively, where

$$\bar{\phi}^2(s_{\xi_j}^{(r)}) = \frac{1}{s_{\xi_j}^{(r)}}\left((s_{\xi_j}^{(r)} - 1)\right).$$

Now into each residue term

$$\frac{1}{2ik\eta_o} \Gamma\left(\frac{\Lambda_r}{2ik} + \frac{1}{2}\right) \frac{v_1(\xi_1, -\Lambda_r) v_2(\xi_2, -\Lambda_r)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_r}} \cdot \frac{v_1(\eta, \Lambda_r)}{v_1(\eta_o, \Lambda_r)}$$

we substitute for $v_1(\eta, \Lambda_r)$ and $v_1(\eta_o, \Lambda_r)$ the asymptotic representations obtained from equation (3.7b), together with the expressions obtained from equations (3.36), (3.38), and (3.39) for the remaining Whittaker functions. This enables us to observe, upon using Stirling's formula for $\Gamma\left(\frac{\Lambda_r}{2ik} + \frac{1}{2}\right)$, that these residue terms have order

$$\frac{1}{(\xi_1 \xi_2)^{1/2}} \, O\left(\frac{1}{(k\eta)^{1/2}}\right) .$$

Hence, each residue term is small (not exponentially small). But since there are $O(k\eta_o)$ terms, their sum is not necessarily small. Therefore, to obtain the field we must calculate the sum of a large number of terms.

We consider next equation (3.2) if $k\eta_1 \gg 1$, $k\eta_2 \gg 1$. The large number of terms which contribute to the field now originate from the range $\arg\lambda = \pi$ ($\arg s_\eta = 0$), $|\lambda/k| = O(k\eta_o)$, $\lambda > -k^2\eta_1$, $-k^2\eta_o/\lambda - 1 \gg 1/(k\eta_o)^{2/3}$. To show this, we first observe from equation (3.29) that, in the given range of λ, the separation of two adjacent zeros is given by

$$\left|\frac{\Lambda_p}{k}\right| - \left|\frac{\Lambda_{p+1}}{k}\right| = O(1) .$$

This follows since for large p

$$\sum_p = O(p^{2/3})$$

(Abramowitz and Stegun, 1964).

Thus there are $O(k\eta_o)$ zeros in this range of λ. It again remains to evaluate the order of each of the residue terms by finding the asymptotic representations of the Whittaker functions that appear in them. As above, we note that $ds_\eta/d\lambda = \frac{1}{k} O(1/k\eta)$ and by (3.17) that

$$\frac{d\bar{\zeta}_\eta}{d\lambda} = -\frac{1}{k}\int_1^{s_\eta}\left(\frac{s-1}{s}\right)^{1/2} ds - \frac{\lambda}{k}\left(\frac{s_\eta - 1}{s_\eta}\right)^{1/2} \cdot \frac{1}{k} O(1/k\eta) .$$

Hence, differentiation with respect to λ in equation (3.16) yields

$$\frac{d}{d\lambda} v_1'(\eta,\lambda) = \frac{(2ik\eta)^{-1/2}}{\sqrt{2\pi}} C\left(\frac{\lambda}{2ik}\right) e^{\pi i/4} e^{-\pi\lambda/2k} \bar{\psi}(s_\eta) \left\{-\left(\frac{3}{2}\bar{\zeta}_\eta\right)^{-1/3}\right\}^2 \cdot$$

$$\cdot \, Ai''(-\bar{\sigma}_\eta) \cdot \frac{1}{2}\left(\frac{1-s_\eta}{s_\eta}\right)^{1/2} \frac{d\bar{\zeta}_\eta}{d\lambda}\left[1+O(1/k\eta)\right] . \qquad (3.40)$$

If we substitute $\eta = \eta_o$, set $\lambda = \Lambda_p$, observe that (3.29) implies

$$(-\bar{\sigma}_{\eta_o})_{\lambda=\Lambda_p} = -\sum\nolimits_p + O(1/k\eta_o) ,$$

perform a Taylor expansion of $Ai''\left(-\sum_p + O(1/k\eta_o)\right)$ about $-\sum_p$, and use Airy's differential equation, then (3.40) becomes

$$\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\Lambda_p} = \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}} C\left(\frac{\Lambda_p}{2ik}\right) e^{\pi i/4} e^{-\pi\Lambda_p/2k} \bar{\psi}(s_{\eta_o}^{(p)}) \cdot$$

$$\cdot \left\{-Ai(-\sum\nolimits_p)\right\} \cdot \frac{1}{2}\left(\frac{1-s_{\eta_o}^{(p)}}{s_{\eta_o}^{(p)}}\right)^{1/2} \left(\frac{d\bar{\zeta}_{\eta_o}}{d\lambda}\right)_{\lambda=\Lambda_p}\left[1+O(1/k\eta_o)\right] . \qquad (3.41)$$

The asymptotic representations for $v_1(H, \Lambda_p)$ and $v_1(\eta, \Lambda_p)$ follow from equation (3.12). Since we assumed $\lambda > -k^2\eta_1$ and since by definition $\eta_1 = \min(\eta, H)$, both $s_H > 1$ and $s_\eta > 1$. Thus, the Airy functions in the representations for $v_1(H, \Lambda_p)$ and $v_1(\eta, \Lambda_p)$ are oscillatory. Finally, $\arg\lambda = \pi$ implies $\arg s_\xi^{(p)} = \pi$. Here

$$s_\xi^{(p)} = \frac{2ik\xi}{4(-\Lambda_p/2ik)} \quad .$$

Hence (D.25) becomes

$$\zeta_\xi^{(p)} = \frac{\Lambda_p}{k} \int_0^{s_\xi^{(p)}} \left(\frac{s-1}{s}\right)^{1/2} ds \; , \qquad (3.42)$$

which shows $\zeta_\xi^{(p)} \gg 1$. Then from (D.42b) we find

$$v_2(\xi, -\Lambda_p) \sim (2ik\xi_2)^{-1/2}(-i)^{1/2} \exp\left\{-\frac{\Lambda_p}{2ik}\log\frac{\Lambda_p}{2ike}\right\} \frac{e^{-i\zeta_\xi^{(p)}}}{\left(\phi(s_\xi^{(p)})\right)^{1/2}} \; , \qquad (3.43)$$

with

$$\phi^2(s_\xi^{(p)}) = \frac{1}{s_\xi^{(p)}} (1 - s_\xi^{(p)}) \; .$$

Now into each residue term

$$\frac{1}{2ik\eta_o} \Gamma\left(\frac{\Lambda_p}{2ik} + \frac{1}{2}\right) \frac{v_2(\xi, -\Lambda_p) v_1(H, \Lambda_p)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda = \Lambda_p}} \cdot \frac{v_1(\eta, \Lambda_p)}{v_1(\eta_o, \Lambda_p)}$$

we substitute for $v_1(H, \Lambda_p)$ and $v_1(\eta, \Lambda_p)$ the asymptotic representations obtained from equation (3.12), together with the expressions obtained from equations (3.41) and (3.43) for the remaining Whittaker functions. This enables us to observe, upon using Stirling's formula for

$$\Gamma\left(\frac{\Lambda_p}{2ik}+\frac{1}{2}\right) \quad ,$$

that these residue terms have order

$$\frac{1}{\left(\xi\sqrt{\eta_1\eta_2}\right)^{1/2}}\; O\left(\frac{1}{\left(k\sqrt{\eta_1\eta_2}\right)^{1/2}}\right) \quad .$$

But since there are $O(k\eta_o)$ terms, their sum is not necessarily small. Therefore, to obtain the field we must again calculate the sum of a large number of terms.

We now investigate the residue series (3.1) if $k\xi_1 \ll 1$, $k\xi_2 \gg 1$. We begin by noting that, independently of the condition on $k\xi_1$, the terms of this series have exponentially decreasing order in the range $\arg\lambda = \pi$, $|\lambda/k| = O(k\eta_o)$, $\lambda > -k^2\eta_o$. This is shown by first substituting for $v_1(\eta, \Lambda_p)$ and $v_1(\eta_o, \Lambda_p)$ from (3.12), using (3.40) for

$$\left(\frac{d}{d\lambda}\, v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_p} \quad ,$$

(3.43) for $v_2(\xi_2, -\Lambda_p)$, and observing that equations (D.34) imply

$$v_1(\xi_1, -\Lambda_p) \sim (2ik\xi_1)^{-1/2}\left(-\frac{\Lambda_p}{ik}\right)^{1/2}\left[\frac{\Phi(s_{\xi_1}^{(p)})}{\phi(s_{\xi_1}^{(p)})}\right]^{1/2} J_o(\zeta_{\xi_1}^{(p)}) \;, \tag{3.44}$$

with $\zeta_{\xi_1}^{(p)}$ given by (3.42). Then, by Stirling's formula for $\Gamma\left(\frac{\Lambda_p}{2ik}+\frac{1}{2}\right)$, we see that each residue term has order $O(e^{\pi\Lambda_p/2k})$. Thus, since $\arg\Lambda_p = \pi$, they are exponentially small with negative exponents that decrease monotonically.

We consider next the range ($\arg\lambda = 0$, etc) previously discussed, but with the condition $k\xi_1 \ll 1$ replacing $k\xi_1 \gg 1$. All asymptotic representations remain the same with the exception of (3.39) for $v_1(\xi_1, -\Lambda_r)$. Since $s_{\xi_1} = k\xi_1 \cdot \frac{k}{\lambda} \ll 1$ if $k\xi_1 \ll 1$, equations (D.34) yield

$$v_1(\xi_1, -\Lambda_r) \sim (2ik\xi_1)^{-1/2} \left(-\frac{\Lambda_r}{ik}\right)^{1/2} \left[\frac{\Phi(s_{\xi_1}^{(r)})}{\phi(s_{\xi_1}^{(r)})}\right]^{1/2} J_0(\zeta_{\xi_1}^{(r)}) , \tag{3.45}$$

where (by (D.23))

$$\zeta_{\xi_1}^{(r)} = -\frac{\Lambda_r}{ik} \int_0^{s_{\xi_1}^{(r)}} \left(\frac{1-s}{s}\right)^{1/2} ds . \tag{3.45a}$$

But

$$s_{\xi_1}^{(r)} \ll 1 \quad \text{implies} \quad \left|\zeta_{\xi_1}^{(r)}\right| \sim 2 \frac{\Lambda_r}{k} (s_{\xi_1}^{(r)})^{1/2} .$$

This in turn implies $\left|\zeta_{\xi_1}^{(r)}\right| \ll \pi\Lambda_r/2k$. Hence, each residue term has order $O(e^{-\pi\Lambda_r/2k})$, and is exponentially small. Therefore, all residue terms in the range $|\lambda/k| = O(k\eta_0)$ are exponentially small. The same is true for terms in the range $|\lambda/k| \gg O(k\eta_0)$ (Appendix C); thus only terms in the range $|\lambda/k| \ll k\eta_0$ need be considered. That there now exists a degree of uncertainty is reflected in the discussion of this range of λ. We did not derive an asymptotic representation for the zeros of $v_1'(\eta, \lambda)$. In all probability, equation (3.6) can only be solved numerically. Hence, the problem of extracting any information from the residue series (3.1) if $k\xi_1 \ll 1$, $k\xi_2 \gg 1$ remains open, since we cannot say how many terms are needed to obtain the field.

If we recall the possibilities for source and field points, we see that the above discussion applies to a source near (with respect to wavelength) the focus and field point far (with respect to wavelength) from the axis. The source exactly at the focus is governed by the condition $\xi_1 = 0$. This implies $v_1(\xi_1, \lambda) = 1$ for all values of λ. If we use $v_1(\xi_1, \lambda) = 1$ instead of (3.44) or (3.45) in the above discussion, the results do not change. Thus for the source at the focus and field point far from the axis, the conclusions of the previous paragraph apply.

We conclude this section with a similar discussion to the one above for the residue series (3.2) if $k\eta_1 \ll 1$, $k\eta_2 \gg 1$. We first observe that, independently of the condition on $k\eta_1$, the terms of this series have exponentially decreasing order in the range $\arg\lambda = 0$, etc. This follows by substituting for $v_1(H, \Lambda_r)$, $v_1(\eta, \Lambda_r)$ and $v_1(\eta_o, \Lambda_r)$ from equation (3.7b), and using (3.36) for

$$\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda = \Lambda_r}$$

as well as (3.38) for $v_2(\xi, -\Lambda_r)$. Then, on using Stirling's formula for $\Gamma\left(\frac{\Lambda_r}{2ik} + \frac{1}{2}\right)$, we see that each residue term has order $O(e^{-\pi\Lambda_r/2k})$ and is exponentially small. We consider finally the range $(\arg\lambda = \pi,\ \lambda > -k^2\eta_o)$ where the residues were not exponentially small, but with the condition $k\eta_1 \gg 1$ replaced by $k\eta_1 \ll 1$. All asymptotic representations remain the same with the exception of the one for $v_1(\eta_1, \Lambda_p)$ $(\eta_1 = \min(\eta, H))$. Since $s_{\eta_1} = -k\eta_1 \cdot \frac{k}{\lambda} \ll 1$ if $k\eta_1 \ll 1$, equations (D.34) yield

$$v_1(\eta_1, \Lambda_p) \sim (2ik\eta_1)^{-1/2}\left(\frac{\Lambda_p}{ik}\right)^{1/2}\left[\frac{\Phi(s^{(p)}_{\eta_1})}{\phi(s^{(p)}_{\eta_1})}\right]^{1/2} J_o(\zeta^{(p)}_{\eta_1}) , \qquad (3.46)$$

where (by (D.23))

$$\zeta^{(p)}_{\eta_1} = \frac{\Lambda_p}{ik}\int_0^{s^{(p)}_{\eta_1}} \left(\frac{1-s}{s}\right)^{1/2} ds . \qquad (3.46a)$$

But

$$s^{(p)}_{\eta_1} \ll 1 \qquad \text{implies} \qquad \left|\zeta^{(p)}_{\eta_1}\right| \sim 2\,\frac{(-\Lambda_p)}{k}\,(s^{(p)}_{\eta_1})^{1/2} .$$

This in turn implies

$$\left|\zeta^{(p)}_{\eta_1}\right| \ll \frac{\pi(-\Lambda_p)}{2k} \quad .$$

Hence, each residue term has order $O(e^{\pi\Lambda_p/2k})$ and, since $\arg \Lambda_p = \pi$, is exponentially small. Therefore, all residue terms in the range $|\lambda/k| = O(k\eta_o)$ are exponentially small, and the above conclusions apply to the residue series (3.2) if $k\eta_1 \ll 1$, $k\eta_2 \gg 1$.

3.3 Equivalent Integral Representations

The discussion of Section 3.2 shows that the behavior of the residue series (3.1) ((3.2)) if $k\xi_1 \gg 1$, $k\xi_2 \gg 1$ ($k\eta_1 \gg 1$, $k\eta_2 \gg 1$) is analogous to the behavior of the Mie type residue series found in the scattering by smooth, bounded, convex bodies (Ritt and Kazarinoff, 1959; 1960). Since we wish to derive an asymptotic representation of the field, the possibility of a further analogy suggests that we study the residue series obtained by closing the contour of integration around the alternate or Γ-function poles of the integrand of the representation (2.1) ((2.5)). We prove in Appendix C that we can close the contour and thus replace the integral representation (2.1) by such a convergent residue series expansion only provided the inequality $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$, or $\sqrt{\xi_1} + \sqrt{\xi_2} < 2\sqrt{\eta_o} - \sqrt{\eta}$ if we write (2.1) as

$$v_N(\xi,\eta,\Xi,0) = \frac{-e^{-ikR}}{R} - \frac{1}{2\pi i}\int_{-\infty-i\sigma}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) \cdot \qquad (0<\sigma<k)$$

$$\cdot \;\frac{v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)}{v_1'(\eta_o,\lambda)}\, v_1(\eta,\lambda)v_2'(\eta_o,\lambda) \quad ,$$

is satisfied. In the same manner it can be shown that for the integral representation (2.5) the relevant inequality is $\sqrt{\xi} + \sqrt{\eta_1} < \sqrt{\eta_2}$ ($\sqrt{\xi} + \sqrt{\eta_1} < 2\sqrt{\eta_o} - \sqrt{\eta_2}$). This latter inequality can be satisfied only in a limited region of the interior of the paraboloid of revolution. The same is true for the former inequality, which also has the property that for source distances $\Xi > \eta_0$ ($\Xi > 4\eta_0$) it cannot be satisfied in any portion of the interior. Thus, any asymptotic representation of the field derived directly from the above residue series is subject to the same restriction and cannot be used to describe the field throughout the interior. This situation is somewhat analogous to that of the integral representation for $-e^{-ikR}/R$. However, in Appendix E we show that an asymptotic representation of $-e^{-ikR}/R$ can be derived, without using the Γ-function poles of the integrand, directly from the integral representation by evaluating a saddle point contribution. This method can be extended to derive an asymptotic representation of the total field (scattered plus incident). As in the analysis of Appendix E, the extension depends only on the evaluation of saddle point contributions to the integral representation and is independent of the Γ-function poles of the integrand.

We now note that instead of considering the integral representation (2.1) ((2.5)) for the field anywhere in the interior of the paraboloid of revolution, we shall study the integral representation (2.2) ((2.6)) which governs the special case of the field on the surface of the paraboloid. The behavior of the field on the surface is itself important. Moreover, the relation between the surface field and the field anywhere in the interior is well known (Morse and Feshbach, 1953, Chapter 7). However, the main reason for investigating (2.2) ((2.6)) is that the derivation of the asymptotic representation may be demonstrated with much less detail. It is then not difficult to see how the method may be extended to the more general representation (2.1) ((2.5)). . Although we consider only the Neumann problem, the Dirichlet problem is susceptible to similar treatement, the details of which are illuminated by the discussion of the Neumann problem.

3.3.1 Equivalent Integral Representation for the Source at $(\Xi, 0)$

We begin the proof of Theorem 5 by starting with equation (2.2) and proceeding quite formally at first. A discussion of the motivation will appear later. If we let

$$F(\xi_1, \xi_2, \eta_o, k, \lambda) = \frac{1}{2\pi i(2ik\eta_o)} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) v_1(\xi_1, -\lambda) v_2(\xi_2, -\lambda) , \tag{3.47}$$

and denote the path $-\infty - i\sigma$ to $\infty - i\sigma$ $(0 < \sigma < k)$ by C, then (2.2) becomes

$$v_N(\xi, \eta_o, \Xi, 0) = \int_C d\lambda \, \frac{F(\xi_1, \xi_2, \eta_o, k, \lambda)}{v_1'(\eta_o, \lambda)} . \tag{3.48}$$

According to Buchholz (1953, Chapter 2, equation 20a), $v_1(\eta_o, \lambda)$ has the following decomposition:

$$v_1(\eta_o, \lambda) = \frac{e^{-\pi\lambda/2k} v_2(\eta_o e^{-\pi i}, -\lambda)}{e^{\pi i/2} \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} + \frac{e^{-\pi\lambda/2k} e^{\pi i/2} v_2(\eta_o, \lambda)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)} . \tag{3.49}$$

If we recall that

$$v_1'(\eta_o, \lambda) = \left(\frac{d}{d(2ik\eta)} v_1(\eta, \lambda)\right)_{\eta = \eta_o} ,$$

then (3.49) leads to the relation

$$v_1'(\eta_o, \lambda) = \frac{e^{-\pi\lambda/2k} v_2'(\eta_o e^{-\pi i}, -\lambda)}{e^{3\pi i/2} \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} + \frac{e^{-\pi\lambda/2k} e^{\pi i/2} v_2'(\eta_o, \lambda)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)} . \tag{3.50}$$

This can be written as

$$v_1'(\eta_o, \lambda) = \frac{e^{-\pi\lambda/2k} v_2'(\eta_o e^{-\pi i}, -\lambda)}{e^{3\pi i/2} \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} \left[1 + \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) v_2'(\eta_o, \lambda)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) v_2'(\eta_o e^{-\pi i}, -\lambda)}\right] , \tag{3.51}$$

which upon defining

$$g(\eta_o, k, \lambda) = \frac{e^{-\pi\lambda/2k} v_2'(\eta_o e^{-\pi i}, -\lambda)}{e^{3\pi i/2} \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)} ,$$

$$X(\eta_o, k, \lambda) = - \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) v_2'(\eta_o, \lambda)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) v_2'(\eta_o e^{-\pi i}, -\lambda)} ,$$

becomes

$$v_1'(\eta_o, \lambda) = g(\eta_o, k, \lambda) \left[1 - X(\eta_o, k, \lambda)\right] . \tag{3.52}$$

Substituting (3.52) into (3.48) yields

$$v_N(\xi, \eta_o, \Xi, 0) = \int_C d\lambda \, \frac{F(\xi_1, \xi_2, \eta_o, k, \lambda)}{g(\eta_o, k, \lambda)\left[1 - X(\eta_o, k, \lambda)\right]} , \tag{3.53}$$

or more briefly

$$v_N(\xi, \eta_o, \Xi, 0) = \int_C d\lambda \, \frac{F}{g(1 - X)} . \tag{3.53a}$$

This form of equation (2.2) simplifies the proof of Lemma 1 below (the convergence of the integrals appearing in Lemma 1 is shown in Appendix C).

Lemma 1: For all integral values of $M \geqslant 1$,

$$v_N(\xi, \eta_o, \Xi, 0) - \sum_{n=0}^{M-1} \int_C d\lambda \, \frac{F}{g} \cdot X^n = \int_C d\lambda \, \frac{F}{g(1 - X)} \cdot X^M . \tag{3.54}$$

Proof:

From equation (3.53) we obtain

$$v_N(\xi, \eta_o, \Xi, 0) - \int_C d\lambda \frac{F}{g} = \int_C d\lambda \left[\frac{F}{g(1-X)} - \frac{F}{g}\right] ,$$

or

$$v_N(\xi, \eta_o, \Xi, 0) - \int_C d\lambda \frac{F}{g} = \int_C d\lambda \frac{F}{g(1-X)} \cdot X \quad .$$

Thus the lemma is true for $M = 1$. Let us assume it holds for $M-1$. Then

$$v_N(\xi, \eta_o, \Xi, 0) - \sum_{n=0}^{M-2} \int_C d\lambda \frac{F}{g} \cdot X^n = \int_C d\lambda \frac{F}{g(1-X)} \cdot X^{M-1} \quad .$$

Therefore

$$v_N(\xi, \eta_o, \Xi, 0) - \sum_{n=0}^{M-1} \int_C d\lambda \frac{F}{g} \cdot X^n = \int_C d\lambda \left[\frac{F}{g(1-X)} \cdot X^{M-1} - \frac{F}{g} \cdot X^{M-1}\right] ,$$

or

$$v_N(\xi, \eta_o, \Xi, 0) - \sum_{n=0}^{M-1} \int_C d\lambda \frac{F}{g} \cdot X^n = \int_C d\lambda \frac{F}{g(1-X)} \cdot X^M \quad .$$

Consequently the lemma follows by mathematical induction.

We continue the formal procedure by now considering, on the interval $0 < z < \xi_1$, the two equations

$$w(z) \equiv \frac{\sqrt{\xi_1 - z} + \sqrt{\xi_1}}{\sqrt{\xi_2 - z} + \sqrt{\xi_2}} = \frac{\sqrt{\eta_o + z} - \sqrt{\eta_o}}{\sqrt{z}} \cdot \frac{\left(\sqrt{\eta_o + z} - \sqrt{\eta_o}\right)^{2n}}{\left(\sqrt{z}\right)^{2n}} \equiv u_n(z) , \tag{3.55}$$

$$f(z) \equiv \left(\sqrt{\xi_1 - z} + \sqrt{\xi_1}\right)\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right) = z \cdot \frac{\left(\sqrt{z}\right)^{2n+1}}{\left(\sqrt{\eta_o + z} - \sqrt{\eta_o}\right)^{2n+1}} \equiv g_n(z) , \tag{3.56}$$

without regard as to how they arise. In Appendix E, in connection with equation (E.54), we prove that on the given interval $w(z)$ is a decreasing function of z and $u_0(z)$ is an increasing function of z. Thus $u_n(z)$ is also an increasing function of z. Since $u_n(0^+) \to 0$, equation (3.55) has exactly one real solution in the given interval if and only if $w(\xi_1) < u_n(\xi_1)$ or

$$\frac{\sqrt{\xi_1}}{\sqrt{\xi_2-\xi_1}+\sqrt{\xi_2}} < \frac{\sqrt{\eta_0+\xi_1}-\sqrt{\eta_0}}{\sqrt{\xi_1}} \cdot \frac{\left(\sqrt{\eta_0+\xi_1}-\sqrt{\eta_0}\right)^{2n}}{\left(\sqrt{\xi_1}\right)^{2n}} .$$

We write this inequality as

$$\sqrt{\xi_1} < \left(\sqrt{\xi_2-\xi_1}+\sqrt{\xi_2}\right) \cdot \frac{\left(\sqrt{\eta_0+\xi_1}-\sqrt{\eta_0}\right)^{2n+1}}{\left(\sqrt{\xi_1}\right)^{2n+1}} . \tag{3.57}$$

But $\sqrt{\eta_0+\xi_1} < \sqrt{\eta_0} + \sqrt{\xi_1}$. Hence $\sqrt{\eta_0+\xi_1} - \sqrt{\eta_0} < \sqrt{\xi_1}$ or

$$\frac{\sqrt{\eta_0+\xi_1}-\sqrt{\eta_0}}{\sqrt{\xi_1}} < 1 .$$

Thus for fixed ξ_1, ξ_2, the inequality (3.57) will not hold for n large enough. Therefore, there exists an N (depending on ξ_1 and ξ_2) such that equation (3.55) does not have any solution in the given interval for $n \geqslant N$. In addition, we can choose N such that

$$\frac{\left(\sqrt{\eta_0+\frac{5\xi_1}{4}}-\sqrt{\eta_0}\right)^{2n+1}}{\left(\sqrt{\frac{5\xi_1}{4}}\right)^{2n+1}} < \frac{\sqrt{\frac{3\xi_1}{4}}}{2\sqrt{\xi_2}} ,$$

for $n \geqslant N$. Since $u_n(z)$ is increasing for all real z, we observe that for $n \geqslant N$ and ξ_1' satisfying

$$\xi_1 - O\left(\frac{1}{(k\eta_0)^{2/3}}\right) \leqslant \xi_1' \leqslant \xi_1 + O\left(\frac{1}{(k\eta_0)^{2/3}}\right) ,$$

$$\frac{\left(\sqrt{\eta_o+\xi_1'}-\sqrt{\eta_o}\right)^{2n+1}}{\left(\sqrt{\xi_1'}\right)^{2n+1}} < \frac{\left(\sqrt{\eta_o+\frac{5\xi_1}{4}}-\sqrt{\eta_o}\right)^{2n+1}}{\left(\sqrt{\frac{5\xi_1}{4}}\right)^{2n+1}} < \frac{\sqrt{\frac{3\xi_1}{4}}}{2\sqrt{\xi_2}} .$$

Then for these values of n and ξ_1'

$$\left(\sqrt{\xi_2-\xi_1'}+\sqrt{\xi_2}\right)\frac{\left(\sqrt{\eta_o+\xi_1'}-\sqrt{\eta_o}\right)^{2n+1}}{\left(\sqrt{\xi_1'}\right)^{2n+1}} < 2\sqrt{\xi_2}\cdot\frac{\sqrt{\frac{3\xi_1}{4}}}{2\sqrt{\xi_2}}$$

or

$$\sqrt{\xi_1'} > \sqrt{3\xi_1/4} > \left(\sqrt{\xi_2-\xi_1'}+\sqrt{\xi_2}\right)\frac{\left(\sqrt{\eta_o+\xi_1'}-\sqrt{\eta_o}\right)^{2n+1}}{\left(\sqrt{\xi_1'}\right)^{2n+1}} . \qquad (3.57a)$$

In Appendix E we also prove, in connection with equation (E.64), that $g_o(z)$ is an increasing function of z on $0 < z < \xi_1$. However, $g_n(z)$ is not so simple. Differentiating with respect to z, we obtain

$$g_n'(z) = \frac{z^{n+1/2}}{2\sqrt{\eta_o+z}\left(\sqrt{\eta_o+z}-\sqrt{\eta_o}\right)^{2n+z}}\left[(2n+3)\sqrt{\eta_o+z}\left(\sqrt{\eta_o+z}-\sqrt{\eta_o}\right)-(2n+1)z\right] .$$

Thus $g_n(z)$ has an extremum at the solutions of

$$\sqrt{\eta_o+z}\left(\sqrt{\eta_o+z}-\sqrt{\eta_o}\right) = \frac{2n+1}{2n+3}z ,$$

or

$$\eta_o+\frac{2}{2n+3}z = \sqrt{\eta_o(\eta_o+z)} . \qquad (3.58)$$

Squaring both sides, we see that $g_n(z)$ has an extremum at the solutions of

$$\frac{4}{(2n+3)^2}z^2+\frac{(1-2n)}{(2n+3)}\eta_o z = 0 ,$$

or at $z=0$, $z=\frac{(2n-1)(2n+3)}{4}\eta_o$. The extremum at $z=0$ is outside the given interval. For n large enough, $\frac{(2n-1)(2n+3)}{4}\eta_o > \xi_1$; hence the extremum at $z=\frac{(2n-1)(2n+3)}{4}\eta_o$ is outside the given interval. We assume the latter to be true for $n \geqslant n_1$. For z small and positive

$$\sqrt{\eta_o+z} = \sqrt{\eta_o}\left[1+\frac{z}{2\eta_o}+O(z^2)\right]$$

implies

$$(2n+3)\sqrt{\eta_o+z}\left(\sqrt{\eta_o+z}-\sqrt{\eta_o}\right)-(2n+1)z = (\tfrac{1}{2}-n)\,z\left[1+O(z)\right] .$$

Thus for $n \geqslant n_1$, $g_n'(z) < 0$ on $0 < z < \xi_1$, and $g_n(z)$ is a decreasing function of z on this interval. In this case the minimum value of $g_n(z)$ is

$$g_n(\xi_1) = \xi_1 \cdot \frac{\left(\sqrt{\xi_1}\right)^{2n+1}}{\left(\sqrt{\eta_o+\xi_1}-\sqrt{\eta_o}\right)^{2n+1}} .$$

We observed above that

$$\frac{\sqrt{\xi_1}}{\sqrt{\eta_o+\xi_1}-\sqrt{\eta_o}} > 1.$$

Consequently for fixed ξ_1 and $n \geqslant n_1$, $g_n(\xi_1)$ is increasing with n. But $f(z)$ is obviously a decreasing function of z on $0 < z < \xi_1$ with maximum value less than $4\sqrt{\xi_1\xi_2}$. In addition, there exists an n_2 such that for $n \geqslant n_2 \geqslant n_1$ and fixed ξ_1, ξ_2, $g_n(\xi_1) > 4\sqrt{\xi_1\xi_2}$. This negates the possibility of a solution of $f(z) = g_n(z)$ for $n \geqslant n_2$. Therefore, there exists an N_o (depending on ξ_1 and ξ_2, $N_o = \max(n_2, N)$), such that for $n \geqslant N_o$ equations (3.55) and (3.56) do not have any solution in $0 < z < \xi_1$, and the inequality (3.57a) is valid for ξ_1' satisfying

$$\xi_1 - O\left(\frac{1}{(k\eta_o)^{2/3}}\right) \leqslant \xi_1' \leqslant \xi_1 + O\left(\frac{1}{(k\eta_o)^{2/3}}\right) .$$

We conclude the formal procedure by combining the results of the previous two paragraphs together with Lemma 1. We wish to approximate $v_N(\xi, \eta_o, \Xi, 0)$, which is described by equation (3.53). Let $N_o(\xi)$ denote the smallest possible N_o defined in the previous paragraph (with $\xi_1 = \min(\xi, \Xi)$, $\xi_2 = \max(\xi, \Xi)$). Then according to Lemma 1, (3.53) can be replaced by

$$v_N(\xi, \eta_o, \Xi, 0) - \sum_{n=0}^{N_o(\xi)-1} \int_C d\lambda \frac{F}{g} \cdot X^n = \int_C d\lambda \frac{F}{g(1-X)} \cdot X^{N_o(\xi)} . \quad (3.59)$$

We shall show that for $k\eta_o \gg 1$, $k\Xi \gg 1$, $k\xi \gg 1$, the integral defined by

$$R(\xi, \eta_o, \Xi, 0) = \int_C d\lambda \frac{F}{g(1-X)} \cdot X^{N_o(\xi)} \quad (3.60)$$

satisfies

$$\lim_{(k\eta_o) \to \infty} R(\xi, \eta_o, \Xi, 0) = 0.$$

This implies

$$v_N(\xi, \eta_o, \Xi, 0) \sim \sum_{n=0}^{N_o(\xi)-1} \int_C d\lambda \frac{F}{g} \cdot X^n . \quad (3.61)$$

We call equation (3.61) the equivalent integral representation for $v_N(\xi, \eta_o, \Xi, 0)$ if $k\eta_o \gg 1$, $k\Xi \gg 1$, $k\xi \gg 1$; the finite sum is the one in Theorem 5.

Before considering equation (3.60), we shall discuss the motiviation behind the above procedure. We do this by presenting a heuristic argument that begins with the integral representation (2.2), and leads to a series of contour integrals which have the same asymptotic representation as those of equation (3.61). This heuristic argument leads one to develop a rigorous derivation of equation (3.61).

Proceeding heuristically, we observe that one could try to analyze the original representation (2.2) by estimating, as in Appendix E, the integrand along various portions of the path C. Upon doing this, we find that the only contribution to the integral arises from that part of C defined by the condition $|\lambda/k| = O(k\eta_0)$, $\arg\lambda = -\delta(\lambda)$, with $\delta > 0$ and $\delta \ll 1$; in this range $\arg s_\eta = \pi + \delta$, $\arg s_{\xi_1} = \delta$. Then according to (D.34), $v_1'(\eta_0, \lambda)$ is governed by (3.11), and according to (D.21) and (D.20) respectively, $v_1(\xi_1, -\lambda)$ is governed by (E.34) while $v_2(\xi_2, -\lambda)$ is governed by (E.35). Thus for $|\lambda/k| > k\xi_1$ and λ outside the transition region of $v_1(\xi_1, -\lambda)$, the integrand is exponentially small. For $|\lambda/k| < k\xi_1$ and λ outside the transition region of $v_1(\xi_1, -\lambda)$, equations (E.35) and (E.36) apply to $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$ respectively. Since the Bessel function in (3.11) is a sum of exponentials, factoring out the exponential which is larger than one on C, and expanding the other factor into a geometric series leads to integrals of the type that can be evaluated by the method of steepest descent. However, to do this directly involves many computational problems. To avoid these, the above procedure was developed. Equation (3.51) is factorization of $v_1'(\eta_0, \lambda)$ corresponding to the factorization of the Bessel exponentials. Lemma 1 represents the expansion in a geometric series. As one may have recognized, equations (3.55) and (3.56) are the saddle point equations. The choice of $N_0(\xi)$ insures that there are no saddle point contributions in the evaluation of $R(\xi, \eta_0, \Xi, 0)$. Finally, we observe that $g(\eta_0, k, \lambda)$ has no zeros in the region $|\mathrm{Im}\,\lambda| < k$ (Klante, 1959). This avoids the consideration of poles when evaluating the saddle point contributions to the integrals of equation (3.61).

The remainder of this section is devoted to a proof of (3.61), namely a proof that $R(\xi, \eta_0, \Xi, 0)$ is small. In order to estimate $R(\xi, \eta_0, \Xi, 0)$, we first substitute equation (3.52) into (3.60) which gives

$$R(\xi, \eta_0, \Xi, 0) = \int_C d\lambda \, \frac{F(\xi_1, \xi_2, \eta_0, k, \lambda)}{v_1'(\eta_0, \lambda)} \cdot X^{N_0(\xi)}(\eta_0, k, \lambda) \; . \qquad (3.62)$$

Since $0 < \sigma < k$ represents the condition that C lies between the zeros of $v_1'(\eta_0, \lambda)$ and the poles of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, we lose no generality in the estimation of

$R(\xi, \eta_o, \Xi, 0)$ if we assume $\sigma/k = O(1)$. Moreover, we note that for $n = 0$ the extremum of $g_o(z)$ at $z = \frac{(2n-1)(2n+3)}{4} \eta_o$ is negative. Thus, it cannot be greater than ξ_1. Therefore, $N_o(\xi)$ is at least one, and $R(\xi, \eta_o, \Xi, 0)$ differs from zero for all values of η_o, Ξ, ξ.

We shall estimate $R(\xi, \eta_o, \Xi, 0)$ if $\xi_1/\eta_o = O(1)$, $\xi_2/\xi_1 \gg 1$. This configuration corresponds to that considered in Section 3.4.2, where we obtain approximate solutions of the saddle point equations. We write $R(\xi, \eta_o, \Xi, 0)$ as

$$R(\xi, \eta_o, \Xi, 0) = \sum_{j=1}^{7} \int_{C_j} d\lambda \, \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)} , \qquad (3.63)$$

where for some $M_i > 0$ $(i = 1, 2, \ldots 11)$ with $M_5 \gg 1$ and $M_7 \gg 1$

$$C_1 = \left\{ \lambda_r - i\sigma \,\middle|\, -\infty < \lambda_r < -M_1 k^2 \xi_2 < 0 \right\} ,$$

$$C_2 = \left\{ \lambda_r - i\sigma \,\middle|\, -(M_1 + M_2) k^2 \xi_2 < \lambda_r < -M_3 k^2 \eta_o \right\} ,$$

$$C_3 = \left\{ \lambda_r - i\sigma \,\middle|\, -(M_3 + M_4) k^2 \eta_o < \lambda_r < -M_5 k \right\} ,$$

$$C_4 = \left\{ \lambda_r - i\sigma \,\middle|\, -(M_5 + M_6) k < \lambda_r < (M_6 + M_7) k \right\} ,$$

$$C_5 = \left\{ \lambda_r - i\sigma \,\middle|\, M_7 k < \lambda_r < (M_8 + M_9) k^2 \eta_o \right\} ,$$

$$C_6 = \left\{ \lambda_r - i\sigma \,\middle|\, M_9 k^2 \eta_o < \lambda_r < (M_{10} + M_{11}) k^2 \xi_2 \right\} ,$$

$$C_7 = \left\{ \lambda_r - i\sigma \,\middle|\, M_{11} k^2 \xi_2 < \lambda_r < \infty \right\} ,$$

and we examine each integral separately. The estimation of $R(\xi, \eta_o, \Xi, 0)$ if $\xi_1/\eta_o = O(1)$ and $\xi_2/\xi_1 = O(1)$ differs mainly in the definition of the intervals covering C. This difference has been discussed in Appendix E, where we first considered $\xi_1/\eta_o = O(1)$ and $\xi_2/\xi_1 = O(1)$ and then investigated the resulting changes for $\xi_1/\eta_o = O(1)$ and $\xi_2/\xi_1 \gg 1$.

We immediately observe that in Appendix C the integrand of (3.63) over C_1 and C_7 is shown to be exponentially small. Consequently, the integrals over C_1 and C_7 do not contribute materially to the sum.

To estimate the integral over C_4, we use some formulas from Appendix E and two results from Buchholz (1953, Chapter 7):

$$v_1(\eta,\lambda) \sim (2ik\eta)^{-1/2}\left[\frac{(2ik\eta)^{-\lambda/2ik}e^{ik\eta}}{\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)} + \frac{(2ik\eta)^{\lambda/2ik}e^{-\pi\lambda/2k}e^{\pi i/2}e^{-ik\eta}}{\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)}\right], \tag{3.64}$$

and

$$v_2(\eta e^{-\pi i},-\lambda) \sim (2ik\eta e^{-\pi i})^{-1/2}(2ik\eta e^{-\pi i})^{-\lambda/2ik}e^{ik\eta} \quad . \tag{3.65}$$

Substituting (E.8) and (E.9) into the definition of $F(\xi_1,\xi_2,\eta_o,k,\lambda)$, we obtain

$$F(\xi_1,\xi_2,\eta_o,k,\lambda) \sim F^+(\xi_1,\xi_2,\eta_o,k,\lambda)+F^-(\xi_1,\xi_2,\eta_o,k,\lambda)\ , \tag{3.66}$$

where

$$F^+(\xi_1,\xi_2,\eta_o,k,\lambda) = \frac{C(k,\xi_1,\xi_2)}{2ik\eta_o}e^{ik\xi_1}e^{-ik\xi_2}(2ik\xi_1)^{\lambda/2ik}(2ik\xi_2)^{-\lambda/2ik}\ , \tag{3.67}$$

$$F^-(\xi_1,\xi_2,\eta_o,k,\lambda) = \frac{C(k,\xi_1,\xi_2)}{2ik\eta_o}e^{-ik\xi_1}e^{-ik\xi_2}\frac{\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)}{\Gamma\left(\frac{-\lambda}{2ik}+\frac{1}{2}\right)} \ .$$

$$\cdot(2ik\xi_1)^{-\lambda/2ik}(2ik\xi_2)^{-\lambda/2ik}e^{\pi\lambda/2k}e^{\pi i/2}\ , \tag{3.68}$$

with

$$C(k,\xi_1,\xi_2) = \frac{1}{2\pi i(2ik\xi_1)^{1/2}(2ik\xi_2)^{1/2}}\ .$$

Note that over C_4, $v_2(\eta,\lambda)$ is governed by (E.10).

Since $\mathrm{Im}\lambda = -\sigma < 0$ and $\sigma/k = O(1)$, equation (3.64) yields

$$v_1'(\eta_o, \lambda) \sim \frac{(2ik\eta_o)^{-1/2}}{2} \left[\frac{(2ik\eta_o)^{-\lambda/2ik} e^{ik\eta_o}}{\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)} \right] . \tag{3.69}$$

By (3.66) and (3.69)

$$\frac{F}{v_1'(\eta_o, \lambda)} \sim \frac{2e^{-ik\eta_o}}{(2ik\eta_o)^{-1/2}} \Gamma\left(\frac{-\lambda}{2ik} + \frac{1}{2}\right) (2ik\eta_o)^{\lambda/2ik} \left[F^+ + F^-\right] ,$$

or

$$\frac{F(\xi_1, \xi_2, \eta_o, k, \lambda)}{v_1'(\eta_o, \lambda)} \sim 2\left[I_{\pm}^{+}(\lambda, \xi_1, \xi_2, \eta_o) + I_{\pm}^{-}(\lambda, \xi_1, \xi_2, \eta_o)\right] ,$$

(the full expressions for $I_{\pm}^{+}(\lambda, \xi_1, \xi_2, \eta_o)$ and $I_{\pm}^{-}(\lambda, \xi_1, \xi_2, \eta_o)$ are given as equations (E.12) and (E.13)).

In the range C_4 it remains to evaluate $X(\eta_o, k, \lambda)$. From (E.10) and (3.65) we obtain

$$v_2'(\eta_o, \lambda) \sim -\frac{(2ik\eta_o)^{-1/2}}{2} (2ik\eta_o)^{\lambda/2ik} e^{-ik\eta_o} , \tag{3.70}$$

$$v_2'(\eta_o e^{-\pi i}, -\lambda) \sim \frac{(2ik\eta_o e^{-\pi i})^{-1/2}}{2e^{-\pi i}} (2ik\eta_o e^{-\pi i})^{-\lambda/2ik} e^{ik\eta_o} . \tag{3.71}$$

Substituting (3.70) and (3.71) into the definition of $X(\eta_o, k, \lambda)$ we find

$$X(\eta_o, k, \lambda) \sim e^{\pi i/2} \frac{\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)} (2ik\eta_o)^{\lambda/ik} e^{-\pi\lambda/2k} e^{-2ik\eta_o} . \tag{3.72}$$

Since $\mathrm{Im}\lambda = -\sigma < 0$ and $\sigma/k = O(1)$, equation (3.72) yields the estimate that on C_4.

$$X(\eta_o, k, \lambda) \sim O(1) \cdot \frac{1}{(k\eta_o)^{\sigma/k}} \quad . \tag{3.73}$$

But from the definitions (E.12), (E.13) we see that on C_4

$$I_{\pm}^{+}(\lambda, \xi_1, \xi_2, \eta_o) \sim \frac{O(1)}{(k\xi_1)^{1/2}(k\xi_2)^{1/2}(k\eta_o)^{1/2}} \cdot \frac{(k\xi_2)^{\sigma/2k}}{(k\xi_1)^{\sigma/2k}(k\eta_o)^{\sigma/2k}} \ , \tag{3.74}$$

$$I_{\pm}^{-}(\lambda, \xi_1, \xi_2, \eta_o) \sim \frac{O(1)}{(k\xi_1)^{1/2}(k\xi_2)^{1/2}(k\eta_o)^{1/2}} \cdot \frac{(k\xi_1)^{\sigma/2k}(k\xi_2)^{\sigma/2k}}{(k\eta_o)^{\sigma/2k}} \ . \tag{3.75}$$

Therefore, on C_4

$$\frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)} \sim \frac{O(1)}{(k\xi_1)^{1/2}(k\xi_2)^{1/2}(k\eta_o)^{1/2}} \left[\frac{(k\xi_2)^{\sigma/2k}}{(k\xi_1)^{\sigma/2k}(k\eta_o)^{\sigma/2k}} + \right.$$

$$\left. + \frac{(k\xi_1)^{\sigma/2k}(k\xi_2)^{\sigma/2k}}{(k\eta_o)^{\sigma/2k}}\right] \cdot \left[\frac{1}{(k\eta_o)^{\sigma/k}}\right]^{N_o(\xi)} .$$

Since C_4 is of length $O(k)$,

$$\int_{C_4} d\lambda \, \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)} \sim \frac{kO(1)}{(k\xi_1)^{1/2}(k\xi_2)^{1/2}(k\eta_o)^{1/2}} \ \cdot$$

$$\cdot \ \frac{(k\xi_1)^{\sigma/2k}(k\xi_2)^{\sigma/2k}}{(k\eta_o)^{\sigma/2k}} \left[\frac{1}{(k\eta_o)^{\sigma/k}}\right]^{N_o(\xi)} \left[1 + (k\xi_1)^{-\sigma/k}\right],$$

or, since $k\xi_1 \gg 1$ and $\sigma/k > 0$,

$$\int_{C_4} d\lambda \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} \sim \frac{1}{\sqrt{\xi_1\xi_2}} (\xi_2/\xi_1)^{\sigma/2k} \frac{O(1)}{(k\eta_o)^{1/2}} \frac{1}{(k\eta_o)^{\sigma/2k}} \cdot \left[\frac{1}{(k\eta_o)^{\sigma/k}}\right]^{N_o(\xi)-1} . \quad (3.76)$$

The behavior of the Whittaker functions over C_2, C_3, C_5, and C_6 is governed by the asymptotic representations of Appendix D. We consider first the intervals C_2 and C_3. On these intervals $\arg\lambda = \pi + \delta(\lambda)$, where $0 < \delta(\lambda) \ll 1$, so that

$$\arg s_{\xi_j} = \pi - \delta(\lambda) \qquad \left(s_{\xi_j} = \frac{ik\xi_j}{-\lambda/ik}\right) .$$

Thus from (D.34b) we find $v_1(\xi_1,-\lambda)$ is given by (E.5); and using (D.42b) we conclude

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} \frac{(-i)^{1/2}}{(-\lambda/ik)^{1/2}} \exp\left\{-\frac{\lambda}{2ik} \log\frac{\lambda}{2ike}\right\} V^{(2)}(s_{\xi_2}) , \quad (3.77)$$

where for both $v_1(\xi_1,-\lambda)$ and $v_2(\xi_2,-\lambda)$

$$\zeta_{\xi_j} = \frac{\lambda}{k}\int_0^{s_{\xi_j}} \left(\frac{s-1}{s}\right)^{1/2} ds \quad (j=1,2) . \quad (3.78)$$

Moreover, $|s_{\xi_j}|$ has the following possible order of magnitude relations on C_2 and C_3:

(i) $|s_{\xi_j}| \gg 1$, with $|\lambda/k| = O(1)$

(ii) $|s_{\xi_j}| \gg 1$, with $1 \ll |\lambda/k| \ll k\eta_o$,

(iii) $|s_{\xi_1}| = O(1)$, $|s_{\xi_2}| \gg 1$, with $|\lambda/k| = O(k\eta_o)$ (recall that we assume

$\xi_1 = O(\eta_0),\ \ \xi_2 \gg \xi_1)$,

(iv) $\left|s_{\xi_1}\right| \ll 1,\ \left|s_{\xi_2}\right| = O(1)$ with $\left|\lambda/k\right| = O(k\xi_2)$,

(v) $\left|s_{\xi_1}\right| \ll 1,\ \left|s_{\xi_2}\right| \ll 1$, with $\left|\lambda/k\right| \gg O(k\xi_2)$.

Possibility (i) can occur over a portion of C_3 of length $O(k)$. The relations $\left|s_{\xi_j}\right| \gg 1$ ($\left|ik\xi_j\right| \gg \left|-\lambda/ik\right|$) and $\left|\lambda/k\right| = O(1)$ imply that the equations (E.8) and (E.9) are also valid for $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$, respectively. Furthermore, the relation $\left|\lambda/k\right| = O(1)$ indicates that the formulas corresponding to the interval C_4 apply. Therefore, equation (3.76) holds for this case.

For the other possibilities, we begin by using (D.26) in (3.78) to obtain

$$\zeta_{\xi_j} = -\frac{\lambda}{k}\left[\sqrt{-s_{\xi_j}(1-s_{\xi_j})} - \log\left(\sqrt{1-s_{\xi_j}} - \sqrt{-s_{\xi_j}}\right)\right] . \tag{3.78a}$$

Since $\arg s_{\xi_j} = \pi - \delta(\lambda)$, $\arg(-s_{\xi_j}) = -\delta(\lambda)$ and $\arg(1-s_{\xi_j}) \sim 0$. This implies $\arg\left[\sqrt{-s_{\xi_j}(1-s_{\xi_j})} - \log\left(\sqrt{1-s_{\xi_j}} - \sqrt{-s_{\xi_j}}\right)\right] \sim 0$ for all possible values of s_{ξ_j} on C_2 and C_3. Hence on C_2 and C_3, $\arg\zeta_{\xi_j} \sim \arg(-\lambda) \sim 0$; and since $\left|\zeta_{\xi_j}\right| \gtrsim O(k\eta_0)$ (again recall that we assume $\xi_1 = O(\eta_0)$ and $\xi_2 \gg \xi_1$), to approximate $v_1(\xi_1, -\lambda)$ we can use the exponential representation of the Bessel function in (E.5) while equation (3.77) can be replaced by (E.6). If in these equations we set $\zeta_{\xi_j} = \mathrm{Re}\,\zeta_{\xi_j} + i\,\mathrm{Im}\,\zeta_{\xi_j}$, and note that $\mathrm{Re}\,\zeta_{\xi_j}(\lambda) \sim \zeta_{\xi_j}(\lambda_r)$, we find that for λ $(\lambda_r - i\sigma)$ on C_2 and C_3

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}}(-i)^{1/2}\frac{\left[e^{i\zeta_{\xi_1}(\lambda_r)}e^{-\mathrm{Im}\,\zeta_{\xi_1}(\lambda)} + i\,e^{-i\zeta_{\xi_1}(\lambda_r)}e^{\mathrm{Im}\,\zeta_{\xi_1}(\lambda)}\right]}{\left(\phi(s_{\xi_1})\right)^{1/2}} , \tag{3.79}$$

$$v_2(\xi_2,-\lambda) \sim (2ik\xi_2)^{-1/2}(-i)^{1/2} \frac{\exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{-i\zeta_{\xi_2}(\lambda_r)} e^{\mathrm{Im}\,\zeta_{\xi_2}(\lambda)}}{\left(\phi(s_{\xi_2})\right)^{1/2}} . \tag{3.80}$$

If we now consider possibility (ii), we note that the substitution of (D.32b) into (E.5) (exponential form) and (E.6) shows that $v_1(\xi_1,-\lambda)$ and $v_2(\xi_2,-\lambda)$ are again governed by (E.8) and (E.9), respectively (as well as by (3.79) and (3.80)). Moreover, equations (E.8) and (E.9) may be written as

$$v_1(\xi_1,-\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}}(-i)^{1/2}\left[e^{i\zeta_{\xi_1}(\lambda_r)}\frac{|\lambda_r/2ke|^{\sigma/2k}}{(2k\xi_1)^{\sigma/2k}} + \right.$$

$$\left. + i\, e^{-i\zeta_{\xi_1}(\lambda_r)}\frac{(2k\xi_1)^{\sigma/2k}}{|\lambda_r/2ke|^{\sigma/2k}}\right] \cdot \frac{e^{\frac{\sigma}{k}O(1)}}{\left(\phi(s_{\xi_1})\right)^{1/2}} , \tag{3.81}$$

$$v_2(\xi_2,-\lambda) \sim (2ik\xi_2)^{-1/2}(-i)^{1/2} \frac{\exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{-i\zeta_{\xi_2}(\lambda_r)}}{\left(\phi(s_{\xi_2})\right)^{1/2}} .$$

$$\cdot \frac{(2k\xi_2)^{\sigma/2k}}{|\lambda_r/2ke|^{\sigma/2k}} e^{\frac{\sigma}{k}O(1)} , \tag{3.82}$$

which is the result one would obtain by substituting the indicated imaginary parts in (3.79) and (3.80) and retaining the terms which are not of $O(1)$. But in this sub-region

$$k\xi_1 \gg \left|\lambda/k\right| \sim \left|\lambda_r/k\right| \quad ; \quad \text{thus} \quad \frac{(2k\xi_1)^{\sigma/2k}}{\left|\lambda_r/2ke\right|^{\sigma/2k}} \gg \frac{\left|\lambda_r/2ke\right|^{\sigma/2k}}{(2k\xi_1)^{\sigma/2k}} ,$$

and (3.81) reduces to

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} \; \frac{(-i)^{1/2}\, i\, e^{-i\zeta_{\xi_1}(\lambda_r)}}{\left(\phi(s_{\xi_1})\right)^{1/2}} \cdot \frac{((2k\xi_1)^{\sigma/2k}}{\left|\lambda_r/2ke\right|^{\sigma/2k}} \; e^{\frac{\sigma}{k}O(1)} . \tag{3.81a}$$

For possibility (iii), we see at once that $v_2(\xi_2, -\lambda)$ is again governed by equation (3.82). It remains to consider $\left|s_{\xi_1}\right| = O(1)$. In addition, we can consider simultaneously $\left|s_{\xi_2}\right| = O(1)$. Thus for $\left|s_{\xi_j}\right| = O(1)$, we first note that $\left|\operatorname{Im} \zeta_{\xi_j}\right| = O\big(\delta(\lambda)\big)\, \zeta_{\xi_j}(\lambda_r)$. Then by (3.78), $\zeta_{\xi_j}(\lambda_r) = \left|\lambda_r/k\right| O(1)$, and since $O\big(\delta(\lambda)\big) = O(\sigma / \left|\lambda_r\right|)$, equations (3.79) and (3.80) become

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} \; \frac{(-i)^{1/2}\left[e^{i\zeta_{\xi_1}(\lambda_r)} + i\, e^{-i\zeta_{\xi_1}(\lambda_r)}\right]}{\left(\phi(s_{\xi_1})\right)^{1/2}} \; e^{\frac{\sigma}{k}O(1)} , \tag{3.83}$$

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} \; \frac{(-i)^{1/2} \exp\left\{-\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{-i\zeta_{\xi_2}(\lambda_r)}}{\left(\phi(s_{\xi_2})\right)^{1/2}} \cdot e^{\frac{\sigma}{k}O(1)} . \tag{3.84}$$

Finally, we can complete the study of possibilities (iv) and (v) by considering $\left|s_{\xi_j}\right| \ll 1$. But by (D.27), $\left|s_{\xi_j}\right| \ll 1$ implies

$$\frac{1}{\left|\phi(s_{\xi_j})\right|^{1/2}} \sim \left|s_{\xi_j}\right|^{1/4} < 1 .$$

In addition by (D.28a), $\left|\zeta_{\xi_j}(\lambda_r)\right| = O\left(\sqrt{|\lambda_r|\xi}\right)$. Thus

$$O\left(\delta(\lambda)\right)\left|\zeta_{\xi_j}(\lambda_r)\right| = O\left(\frac{\sigma}{|\lambda_r|^{1/2}}\,\xi_j^{1/2}\right) < \frac{\sigma}{k}\,O(1) \quad ,$$

since $\left|s_{\xi_j}\right| \ll 1$ implies $|\lambda_r| \gg k^2\xi_2$. Therefore, equations (3.83) and (3.84) apply.

To find the representations of the Whittaker functions which depend on η_o, we first recall that on C_2 and C_3 $\arg\lambda = \pi + \delta(\lambda)$, where $0 < \delta(\lambda) \ll 1$, so that

$$\arg s_{\eta_o} = -\delta(\lambda) \qquad \left(s_{\eta_o} = \frac{ik\eta_o}{\lambda/ik}\right) \quad .$$

Then equations (D.21) show that equations (3.12) are valid. This in turn shows that $v_1'(\eta_o, \lambda)$ is governed by (3.16) together with (3.10). In addition, we see from (D.15) that $v_2(\eta, \lambda)$ is given by (E.7), while from (D.14) we obtain

$$v_2(\eta e^{-\pi i}, -\lambda) \sim (2ik\eta e^{-\pi i})^{-1/2}(-\lambda/k)^{1/6}\exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} \cdot$$

$$\cdot\, e^{\pi\lambda/2k}\,\bar{V}^{(1)}(s_\eta) \ . \tag{3.85}$$

The behavior of the derivatives $v_2'(\eta_o, \lambda)$, $v_2'(\eta_o e^{-\pi i}, -\lambda)$ follows from that of $v_2(\eta, \lambda)$, $v_2(\eta e^{-\pi i}, -\lambda)$ in the same manner as the behavior of $v_1'(\eta_o, \lambda)$ follows from that of $v_1(\eta, \lambda)$ in Section 3.1. However, the value of $\bar{\zeta}_{\eta_o}$ depends strongly on the position of λ with respect to the transition region. This must be considered since the width of the transition region, defined by

$$\Delta\lambda \equiv -k^2\eta_o - \lambda = \lambda\, O\left(\frac{1}{(k\eta_o)^{2/3}}\right) = k\, O\left((k\eta_o)^{1/3}\right) ,$$

is $\Delta\frac{\lambda}{k} = O\left((k\eta_o)^{1/3}\right)$, while the minimum distance from the path C to the turning point $-k^2\eta_o$ is $\Delta\frac{\lambda}{k} = \frac{\sigma}{k} = O(1)$. Consequently, in this range we need to study three cases, namely $|s_{\eta_o}| > 1$ and away from the transition region

$$\left(\left|s_{\eta_o} - 1\right| \gg \frac{1}{(k\eta_o)^{2/3}}\right) ,$$

$|s_{\eta_o}| < 1$ and away from the transition region, s_{η_o} in the transition region.

For $|s_{\eta_o}| > 1$ and away from the transition region, we consider first $|s_{\eta_o}| \gg 1$ with $|\lambda/k| = O(1)$. This can occur over a portion of C_3 of length $O(k)$ and has already been discussed (possibility (i)) when studying the Whittaker functions which depend on ξ_1 and ξ_2.

We next consider $|s_{\eta_o}| > 1$ and away from the transition region with either $1 \ll |\lambda/k| \ll k\eta_o$ ($|s_{\eta_o}| \gg 1$), or $|\lambda/k| = O(k\eta_o)$ ($|s_{\eta_o}| = O(1)$). In either case, equations (D.3) show that $\bar{\xi}_{\eta_o}$ is given by (3.17). By (D.5),

$$\bar{\xi}_{\eta_o} = -\lambda/k \left[\sqrt{s_{\eta_o}(s_{\eta_o} - 1)} - \log\left(\sqrt{s_{\eta_o} - 1} + \sqrt{s_{\eta_o}}\right)\right] . \qquad (3.86)$$

But $\arg s_{\eta_o} = -\delta(\lambda)$ and $\arg(s_{\eta_o} - 1) \sim 0$. The latter relation is obvious if $|s_{\eta_o}| \gg 1$. If $|s_{\eta_o}| = O(1)$, it follows from the definition of $\delta(\lambda) = \sigma/|\lambda_r|$, since $|s_{\eta_o}| = O(1)$ implies $|\lambda_r/k| = O(k\eta_o)$, or $\delta(\lambda) = \frac{\sigma}{k} O(1/k\eta_o) = O(1/k\eta_o)$. Hence

$$s_{\eta_o} - 1 \sim \left(-\frac{k^2\eta_o}{\lambda_r} - 1\right) - iO(1/k\eta_o) ,$$

and $\arg(s_{\eta_o} - 1)$ not close to zero implies

$$-\frac{k^2\eta_o}{\lambda_r} - 1 = O(1/k\eta_o) .$$

This is contrary to $\left|s_{\eta_o}\right|$ being away from the transition region. These estimates for the arguments of s_{η_o} and $s_{\eta_o} - 1$ show that in the cases under consideration $\arg\left[\sqrt{s_{\eta_o}(s_{\eta_o}-1)} - \log\left(\sqrt{s_{\eta_o}-1} + \sqrt{s_{\eta_o}}\right)\right] \sim 0$. Consequently $\arg \bar{\zeta}_{\eta_o} \sim \arg(-\lambda) \sim 0$, and

$$\arg(-\bar{\sigma}_{\eta_o}) \sim \pi \qquad \left(\bar{\sigma}_{\eta_o} = \left(\frac{3}{2}\bar{\zeta}_{\eta_o}\right)^{2/3}\right) .$$

Since $\left|\bar{\zeta}_{\eta_o}\right| = O(k\eta_o)$, the asymptotic form of the Airy function given by Erdélyi and Swanson (1957, equation 4.6) can be used in (3.12) to obtain

$$v_1'(\eta_o, \lambda) \sim \frac{(2ik\eta_o)^{-1/2} e^{-\pi\lambda/2k}}{\sqrt{2\pi}} \left[e^{-i\bar{\zeta}_{\eta_o}} + i e^{i\bar{\zeta}_{\eta_o}}\right] \cdot \frac{(-i)}{2} \left(\bar{\phi}(s_{\eta_o})\right)^{1/2} , \tag{3.87}$$

while the asymptotic forms of the Hankel functions given in Section D.1 can be used in (E.7) and (3.85) to obtain

$$v_2'(\eta_o, \lambda) \sim (2ik\eta_o)^{-1/2} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} (-i) e^{-i\bar{\zeta}_{\eta_o}} \cdot \frac{(-i)}{2} \left(\bar{\phi}(s_{\eta_o})\right)^{1/2} , \tag{3.88}$$

$$v_2'(\eta_o e^{-\pi i}, -\lambda) \sim (2ik\eta_o e^{-\pi i})^{-1/2} \exp\left\{-\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{\pi\lambda/2k} (i) e^{i\bar{\zeta}_{\eta_o}} \cdot \frac{(i)}{2} \left(\bar{\phi}(s_{\eta_o})\right)^{1/2} , \tag{3.89}$$

where in all three equations we have substituted $\bar{\phi}(s_{\eta_o})$ for $\left((s_{\eta_o}-1)/s_{\eta_o}\right)^{1/2}$.

If $\left|s_{\eta_o}\right| \gg 1$, the substitution of (D.10a) into (3.87), (3.88), and (3.89) shows that $v_1'(\eta_o, \lambda)$, $v_2'(\eta_o, \lambda)$, and $v_2'(\eta_o e^{-\pi i}, -\lambda)$ are governed by (3.5), (3.70) and (3.71) respectively. Moreover, equations (3.5), (3.70) and (3.71) may be written as

$$v_1'(\eta_o,\lambda) \sim \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}}\, e^{-\pi\lambda/2k} \left[e^{-i\bar{\zeta}_{\eta_o}(\lambda_r)} \frac{|\lambda_r/2ke|^{\sigma/2k}}{(2k\eta_o)^{\sigma/2k}} + \right.$$

$$\left. + \, i\, e^{i\bar{\zeta}_{\eta_o}(\lambda_r)} \frac{(2k\eta_o)^{\sigma/2k}}{|\lambda_r/2ke|^{\sigma/2k}} \right] e^{\frac{\sigma}{k} O(1)} \frac{(-i)}{2} \left(\bar{\phi}(s_{\eta_o})\right)^{1/2} , \tag{3.90}$$

$$v_2'(\eta_o,\lambda) \sim (2ik\eta_o)^{-1/2} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} (-i)\, e^{-i\bar{\zeta}_{\eta_o}(\lambda_r)} \frac{|\lambda_r/2ke|^{\sigma/2k}}{(2k\eta_o)^{\sigma/2k}} \,.$$

$$\cdot\, e^{\frac{\sigma}{k} O(1)} \frac{(-i)}{2} \left(\bar{\phi}(s_{\eta_o})\right)^{1/2} , \tag{3.91}$$

$$v_2'(\eta_o e^{-\pi i}, -\lambda) \sim (2ik\eta_o e^{-\pi i})^{-1/2} \exp\left\{-\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{\pi\lambda/2k} (i)\, e^{i\bar{\zeta}_{\eta_o}(\lambda_r)} .$$

$$\cdot\, \frac{(2k\eta_o)^{\sigma/2k}}{|\lambda_r/2ke|^{\sigma/2k}} \cdot e^{\frac{\sigma}{k} O(1)} \cdot \frac{(i)}{2} \left(\bar{\phi}(s_{\eta_o})\right)^{1/2} , \tag{3.92}$$

which is the result one would obtain by setting $\bar{\zeta}_{\eta_o} = \mathrm{Re}\,\bar{\zeta}_{\eta_o} + i\,\mathrm{Im}\,\bar{\zeta}_{\eta_o}$ in (3.87), (3.88), and (3.89), and noting that $\mathrm{Re}\,\bar{\zeta}_{\eta_o}(\lambda) \sim \bar{\zeta}_{\eta_o}(\lambda_r)$, while retaining the terms which are not of $O(1)$ in $\mathrm{Im}\,\bar{\zeta}_{\eta_o}$. But in this sub-region $k\eta_o \gg |\lambda/k| \sim |\lambda_r/k|$; thus

$$\frac{(2k\eta_o)^{\sigma/2k}}{|\lambda_r/2ke|^{\sigma/2k}} \gg \frac{|\lambda_r/2ke|^{\sigma/2k}}{(2k\eta_o)^{\sigma/2k}} \,,$$

and equation (3.90) reduces to

$$v_1'(\eta_o,\lambda) \sim \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}}\, e^{-\pi\lambda/2k}\, i\, e^{i\bar{\zeta}_{\eta_o}(\lambda_r)} \frac{(2k\eta_o)^{\sigma/2k}}{|\lambda_r/2ke|^{\sigma/2k}}\, e^{\frac{\sigma}{k} O(1)} \,.$$

$$\cdot\, \frac{(-i)}{2} \left(\bar{\phi}(s_{\eta_o})\right)^{1/2} . \tag{3.90a}$$

We now estimate the behavior of the integrand in the case corresponding to formulas (3.90), (3.91), and (3.92). In this sub-region $1 \ll |\lambda/k| \ll k\eta_o$, thus equations (3.81a) and (3.82) are valid for $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$, respectively. Substituting these equations into the definition of $F(\xi_1, \xi_2, \eta_o, k, \lambda)$, and using (3.90a) for $v_1'(\eta_o, \lambda)$ yields

$$\frac{F}{v_1'(\eta_o,\lambda)} \sim 2C(k,\xi_1,\xi_2,\eta_o)\,\frac{(2k\xi_1)^{\sigma/2k}(2k\xi_2)^{\sigma/2k}}{(2k\eta_o)^{\sigma/2k}\left|\lambda_r/2ke\right|^{\sigma/2k}}\;\cdot$$

$$\cdot\;\frac{\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\exp\left\{\frac{-\lambda}{2ik}\log\frac{\lambda}{2ike}\right\}}{e^{-\pi\lambda/2k}\left[\phi(s_{\xi_1})\phi(s_{\xi_2})\bar{\phi}(s_{\eta_o})\right]^{1/2}}\cdot\frac{e^{-i\zeta_{\xi_1}(\lambda_r)}}{e^{i\zeta_{\eta_o}(\lambda_r)}}\;e^{-i\zeta_{\xi_2}(\lambda_r)}\;e^{\frac{\sigma}{k}O(1)}\;.$$

But $\arg\lambda = \pi+\delta(\lambda)$ implies $\arg\frac{\lambda}{2ik} = \frac{\pi}{2}+\delta(\lambda)$, as well as $\arg-\frac{\lambda}{2ik} = -\frac{\pi}{2}+\delta(\lambda)$. Hence, by Stirling's formula (Erdélyi et al, 1953)

$$\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) \sim \sqrt{2\pi}\,\exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\}\;,\qquad\text{and}$$

$$\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) \sim \sqrt{2\pi}\,\exp\left\{-\frac{\lambda}{2ik}\log-\frac{\lambda}{2ike}\right\}\;.$$

Inserting the first form into the above estimate of $F/v_1'(\eta_o,\lambda)$ shows that in the case under consideration

$$\left|\frac{F}{v_1'(\eta_o,\lambda)}\right| \leq \frac{O(1)}{(k\xi_1)^{\frac{1}{2}-\frac{\sigma}{2k}}(k\xi_2)^{\frac{1}{2}-\frac{\sigma}{2k}}(k\eta_o)^{\frac{1}{2}+\frac{\sigma}{2k}}}\cdot\frac{e^{\pi\frac{\lambda_r}{2k}}}{\left|\lambda_r/2ke\right|^{\sigma/2k}}\;,$$

since $\lambda_r < 0$ and $1 \ll |\lambda_r/k| \ll k\eta_o$. But $\sigma > 0$ and $0 < \sigma < k$. Consequently, $F/v_1'(\eta_o,\lambda)$ is exponentially small.

To estimate $X(\eta_o, k, \lambda)$, we substitute equations (3.91) and (3.92) into the definition of $X(\eta_o, k, \lambda)$. This gives

$$X \sim -\frac{\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\}}{\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{\pi\lambda/2k}} \cdot \frac{\left|\lambda_r/2ke\right|^{\sigma/k}}{(2k\eta_o)^{\sigma/k}} \cdot$$

$$\cdot \frac{e^{-2i\bar{\zeta}_{\eta_o}(\lambda_r)}}{(i)} e^{\frac{\sigma}{k}O(1)} .$$

Using the above forms for the Γ-functions, we obtain

$$X \sim i e^{-2i\bar{\zeta}_{\eta_o}(\lambda_r)} e^{\frac{\sigma}{k}O(1)} \cdot \frac{\left|\lambda_r/2ke\right|^{\sigma/k}}{(2k\eta_o)^{\sigma/k}} = O\left(\frac{\left|\lambda_r/2ke\right|^{\sigma/k}}{(2k\eta_o)^{\sigma/k}}\right) .$$

Since $\left|\lambda_r/k\right| \ll k\eta_o$, $|X| < 1$, and thus $\left|X^{N_o(\xi)}\right| < 1$. Therefore, $\left(\not{R}/v_1'(\eta_o, \lambda)\right)X^{N_o(\xi)}$ is also exponentially small in the case under consideration, namely, $\left|s_{\eta_o}\right| \gg 1$ with $1 \ll \left|\lambda/k\right| \ll k\eta_o$.

We now consider $\left|s_{\eta_o}\right| > 1$ and away from the transition region, but with $\left|s_{\eta_o}\right| = O(1)$. If in (3.87), (3.88), and (3.89) we set $\bar{\zeta}_{\eta_o} = \mathrm{Re}\,\bar{\zeta}_{\eta_o} + i\,\mathrm{Im}\,\bar{\zeta}_{\eta_o}$, and note that $\left|\mathrm{Im}\,\bar{\zeta}_{\eta_o}\right| = O(\sigma/\left|\lambda_r\right|)\mathrm{Re}\,\bar{\zeta}_{\eta_o}$, $\mathrm{Re}\,\bar{\zeta}_{\eta_o}(\lambda) \sim \bar{\zeta}_{\eta_o}(\lambda_r)$, $\bar{\zeta}_{\eta_o}(\lambda_r) = \left|\lambda_r/k\right| O(1)$, then these equations become

$$v_1'(\eta_o, \lambda) \sim \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}} e^{-\pi\lambda/2k}\left[e^{-i\bar{\zeta}_{\eta_o}(\lambda_r)} + i e^{i\bar{\zeta}_{\eta_o}(\lambda_r)}\right] e^{\frac{\sigma}{k}O(1)} \cdot$$

$$\cdot \frac{(-i)}{2}\left(\not{p}(s_{\eta_o})\right)^{1/2} , \qquad (3.93)$$

$$v_2'(\eta_o,\lambda) \sim (2ik\eta_o)^{-1/2} \exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{-i\bar{\zeta}_{\eta_o}(\lambda_r)} e^{\frac{\sigma}{k}O(1)} \cdot$$

$$\cdot\ (-1/2)\left(\bar{\phi}(s_{\eta_o})\right)^{1/2} , \qquad (3.94)$$

$$v_2'(\eta_o e^{-\pi i},-\lambda) \sim (2ik\eta_o e^{-\pi i})^{-1/2} \exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} \cdot$$

$$\cdot\ e^{\pi\lambda/2k} e^{i\bar{\zeta}_{\eta_o}(\lambda_r)} e^{\frac{\sigma}{k}O(1)} \cdot (-1/2)\left(\bar{\phi}(s_{\eta_o})\right)^{1/2} . \qquad (3.95)$$

To estimate the behavior of the integrand in the case corresponding to the above formulas, we first observe that in accordance with the configuration of η_o, ξ_1 and ξ_2 under consideration the relation $\left|s_{\eta_o}\right| = O(1)$ $\left(\left|\lambda/k\right| = O(k\eta_o)\right)$ implies that the relations $\left|s_{\xi_1}\right| = O(1)$ and $\left|s_{\xi_2}\right| >> 1$ are valid. Thus $v_1(\xi_1,-\lambda)$ is governed by equation (3.83), while $v_2(\xi_2,-\lambda)$ is governed by equation (3.82). Substituting these equations into the definition of $F(\xi_1,\xi_2,\eta_o,k,\lambda)$, and using (3.93) for $v_1'(\eta_o,\lambda)$ yields

$$\frac{F}{v_1'(\eta_o,\lambda)} \sim 2C(k,\xi_1,\xi_2,\eta_o)\,\frac{(2k\xi_2)^{\sigma/2k}}{\left|\lambda_r/2ke\right|^{\sigma/2k}} \cdot \frac{\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\}}{e^{-\pi\lambda/2k}\left[\phi(s_{\xi_1})\phi(s_{\xi_2})\bar{\phi}(s_{\eta_o})\right]^{1/2}} \cdot$$

$$\cdot\ \frac{\left[e^{i\zeta_{\xi_1}(\lambda_r)} + i e^{-i\zeta_{\xi_1}(\lambda_r)}\right]}{\left[e^{-i\bar{\zeta}_{\eta_o}(\lambda_r)} + i e^{i\bar{\zeta}_{\eta_o}(\lambda_r)}\right]} \cdot e^{-i\zeta_{\xi_2}(\lambda_r)} e^{\frac{\sigma}{k}O(1)} .$$

Inserting the above asymptotic form for $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$ into this estimate for $F/v_1'(\eta_o,\lambda)$ shows that in the case under consideration

$$\left|\frac{F}{v_1'(\eta_o,\lambda)}\right| \lesssim \frac{O(1)}{(k\xi_1)^{1/2}(k\xi_2)^{\frac{1}{2}-\frac{\sigma}{2k}}(k\eta_o)^{\frac{1}{2}+\frac{\sigma}{2k}}} \cdot e^{\pi\frac{\lambda_r}{2k}} ,$$

since $\lambda_r < 0$ and $|\lambda_r/k| = O(k\eta_o)$. Consequently, $F/v_1'(\eta_o,\lambda)$ is exponentially small.

To estimate $X(\eta_o, k, \lambda)$, we substitute equations(3.94) and (3.95) into the definition of $X(\eta_o, k, \lambda)$, and using the above forms for the Γ-functions, we obtain

$$X \sim i e^{-2i\bar{\zeta}_{\eta_o}(\lambda_r)} e^{\frac{\sigma}{k}O(1)} = O(1) .$$

Therefore $X^{N_o(\xi)} = O(1)$, and $\left(F/v_1'(\eta_o,\lambda)\right)X^{N_o(\xi)}$ is also exponentially small in the case under consideration, namely, $|s_{\eta_o}| > 1$ and away from the transition region but $|s_{\eta_o}| = O(1)$.

For $|s_{\eta_o}| < 1$ and away from the transition region, we consider first the case where $|s_{\eta_o}|$ is not much smaller than 1, $|s_{\eta_o}| = O(1)$. In this case equations (D.3) and (D.6) show that $\bar{\zeta}_{\eta_o}$ is given by equation (3.14). If we perform the integration indicated in (3.14), we find

$$\bar{\zeta}_{\eta_o} = i\frac{\lambda}{k}\left[\frac{\pi}{2} - \sin^{-1}\sqrt{s_{\eta_o}} - \sqrt{s_{\eta_o}(1-s_{\eta_o})}\right] . \qquad (3.96)$$

But $\arg s_{\eta_o} = -\delta(\lambda)$, while the previous argument indicates that for $|s_{\eta_o}| = O(1)$, $\arg(1-s_{\eta_o}) \sim 0$. Since

$$\mathrm{Re}\left[\frac{\pi}{2} - \sin^{-1}\sqrt{s_{\eta_o}} - \sqrt{s_{\eta_o}(1-s_{\eta_o})}\right] \sim \int_{s_{\eta_o}^{(r)}}^{1} \left(\frac{1-t}{t}\right)^{1/2} dt$$

$$\left(s_{\eta_o}^{(r)} = -\frac{k^2\eta_o}{\lambda_r}, \quad \lambda = \lambda_r - i\sigma\right) ,$$

$$0 < O(1) = \int_{s_{\eta_o}^{(r)}}^{1} \left(\frac{1-t}{t}\right)^{1/2} dt = \left[\frac{\pi}{2} - \sin^{-1}\sqrt{s_{\eta_o}^{(r)}} - \sqrt{s_{\eta_o}^{(r)}(1-s_{\eta_o}^{(r)})}\right] ,$$

these estimates for the arguments of s_{η_o} and $1-s_{\eta_o}$ show that

$$\arg\left[\frac{\pi}{2} - \sin^{-1}\sqrt{s_{\eta_o}} - \sqrt{s_{\eta_o}(1-s_{\eta_o})}\right] \sim 0.$$

Then $\arg \bar{\zeta}_{\eta_o} \sim \arg(i\lambda) \sim 3\pi/2$, and $\arg(-\bar{\sigma}_{\eta_o}) \sim 0 \;\left(\bar{\sigma}_{\eta_o} = (\frac{3}{2}\bar{\zeta}_{\eta_o})^{2/3}\right)$.
Since $\left|\bar{\zeta}_{\eta_o}\right| = O(k\eta_o)$, the asymptotic form of the Airy function given by Erdélyi and Swanson (1957, equation 4.5) can be used in (3.12) to obtain

$$v_1'(\eta_o, \lambda) \sim \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}} \; \frac{e^{-\pi\lambda/2k}\, e^{-i\bar{\zeta}_{\eta_o}}}{\left(\bar{\phi}(s_{\eta_o})\right)^{1/2}} \cdot \frac{1}{2}\left(\frac{1-s_{\eta_o}}{s_{\eta_o}}\right)^{1/2} , \tag{3.97}$$

while the asymptotic forms of the Hankel functions given in Section D.1 together with the continuation formula (Erdélyi et al, 1953)

$$H_{1/3}^{(2)}(z\, e^{\pi i}) = e^{\pi i/3} H_{1/3}^{(1)}(z) + H_{1/3}^{(2)}(z) ,$$

can be used in (E.7) and (3.85) to obtain

$$v_2'(\eta_o, \lambda) \sim (2ik\eta_o)^{-1/2} \exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} \frac{(i)\, e^{i\zeta_{\eta_o}}}{e^{-\pi i/2}\left(\bar{\phi}(s_{\eta_o})\right)^{1/2}} \cdot$$

$$\cdot \frac{1}{2}\left(\frac{1-s_{\eta_o}}{s_{\eta_o}}\right)^{1/2} , \tag{3.98}$$

$$v_2'(\eta_o e^{-\pi i}, -\lambda) \sim (2ik\eta_o e^{-\pi i})^{-1/2} \exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} \cdot$$

$$\cdot \frac{e^{\pi\lambda/2k}(i)\, e^{i\bar{\zeta}_{\eta_o}}}{(\bar{\phi}(s_{\eta_o}))^{1/2}} \cdot \frac{1}{2e^{-\pi i}}\left(\frac{1-s_{\eta_o}}{s_{\eta_o}}\right)^{1/2}, \qquad (3.99)$$

where we have neglected the term containing $e^{-i\bar{\zeta}_{\eta_o}}$ in (3.99), since $\arg\bar{\zeta}_{\eta_o} \sim 3\pi/2$ and $|\bar{\zeta}_{\eta_o}| = O(k\eta_o)$ implies that this term is exponentially small.

We now estimate the behavior of the integrand for $|s_{\eta_o}| < 1$, $|s_{\eta_o}|$ not much smaller than 1 and s_{η_o} away from the transition region. Using the asymptotic forms of the Γ-functions together with equation (3.98) and (3.99), we immediately observe $X(\eta_o, k, \lambda) \sim 1$. If in equation (3.97) we set $\bar{\zeta}_{\eta_o} = \mathrm{Re}\,\bar{\zeta}_{\eta_o} + i\,\mathrm{Im}\,\bar{\zeta}_{\eta_o}$, and note that $|\mathrm{Re}\,\bar{\zeta}_{\eta_o}| = O(\sigma/|\lambda_r|)\,|\mathrm{Im}\,\bar{\zeta}_{\eta_o}|$, $|\mathrm{Im}\,\bar{\zeta}_{\eta_o}| \sim |\bar{\zeta}_{\eta_o}(\lambda_r)|$, $|\bar{\zeta}_{\eta_o}(\lambda_r)| = |\lambda_r/k|\,O(1)$, this equation becomes

$$v_1'(\eta_o, \lambda) \sim \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}}\, e^{-\pi\lambda/2k}\, e^{-i\bar{\zeta}_{\eta_o}(\lambda_r)}\, \frac{e^{i\frac{\sigma}{k}O(1)}}{(\bar{\phi}(s_{\eta_o}))^{1/2}} \cdot \frac{1}{2}\left(\frac{1-s_{\eta_o}}{s_{\eta_o}}\right)^{1/2}. \qquad (3.100)$$

As above, $v_1(\xi_1, -\lambda)$ is governed by equation (3.83) and $v_2(\xi_2, -\lambda)$ is governed by equation (3.82). Substituting these equations into the definition of $F(\xi_1, \xi_2, \eta_o, k, \lambda)$, using (3.100) for $v_1'(\eta_o, \lambda)$ and the above asymptotic form for $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, we find that

$$\left|\frac{F}{v_1'(\eta_o, \lambda)}\right| \leq \frac{O(1)}{(k\xi_1)^{1/2}(k\xi_2)^{\frac{1}{2}-\frac{\sigma}{2k}}(k\eta_o)^{\frac{1}{2}+\frac{\sigma}{2k}}} \cdot \frac{\exp\left\{-\frac{\lambda_r}{k}\int_{s^{(r)}_{\eta_o}}^{1} dt\left(\frac{1-t}{t}\right)^{1/2}\right\}}{e^{-\pi\frac{\lambda_r}{2k}}},$$

since $\lambda_r < 0$ and $|\lambda_r/k| = O(k\eta_o)$. But in Appendix E we show (investigation of C_2) that the ratio

$$\frac{\exp\left\{-\frac{\lambda_r}{k}\int_{s_{\eta_o}^{(r)}}^{1} dt \left(\frac{1-t}{t}\right)^{1/2}\right\}}{e^{-\pi\frac{\lambda_r}{2k}}}$$

is exponentially small in this case. Therefore, $F/v_1'(\eta_o, \lambda)$ and $\left(F/v_1'(\eta_o, \lambda)\right) X^{N_o(\xi)}$ are also exponentially small for $|s_{\eta_o}| < 1$, $|s_{\eta_o}|$ not much smaller than 1 and s_{η_o} away from the transition region.

We complete the study of $|s_{\eta_o}| < 1$ and away from the turning point $\lambda = -k^2\eta_o$ by considering $|s_{\eta_o}| \ll 1$ $\left(|s_{\eta_o}| \gg 1/(k\eta_o)^{2/3}\right)$. But $\arg s_{\eta_o} = -\delta(\lambda)$, and this together with $|s_{\eta_o}| \ll 1$ implies that $\arg(1-s_{\eta_o}) \sim 0$. Hence, equations (3.97), (3.98), and (3.99) also apply in this range. We immediately observe $X(\eta_o, k, \lambda) \sim 1$. For $|s_{\eta_o}| \ll 1$, $|\bar{\phi}(s_{\eta_o})|^{1/2} \sim 1/|s_{\eta_o}|^{1/4}$, and since the relation $|\bar{\zeta}_{\eta_o}(\lambda_r)| = |\lambda_r/k| O(1)$ remains valid in this range, equation (3.100) is replaced by

$$v_1'(\eta_o, \lambda) \sim \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}} e^{-\pi\lambda/2k} e^{-i\bar{\zeta}_{\eta_o}(\lambda_r)} e^{i\frac{\sigma}{k}O(1)} \cdot \frac{1}{2} \frac{1}{|s_{\eta_o}|^{1/4}} . \quad (3.101)$$

Furthermore, in accordance with the configuration under consideration $\left[\xi_1/\eta_o = O(1),\ \xi_2/\xi_1 \gg 1\right]$, the relation $|s_{\eta_o}| \ll 1$ implies that the relation $|s_{\xi_1}| \ll 1$ is valid, while $|s_{\xi_2}|$ satisfies either $|s_{\xi_2}| = O(1)$ or $|s_{\xi_2}| \ll 1$. Thus in this case $v_1(\xi_1, -\lambda)$ is governed by (3.83), and $v_2(\xi_2, -\lambda)$ (either possibility) is governed by (3.84). Substituting these equations into the definition of $F(\xi_1, \xi_2, \eta_o, k, \lambda)$, using (3.101) for $v_1'(\eta_o, \lambda)$ and the above asymptotic form of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, we find that

$$\left|\frac{F}{v_1'(\eta_o,\lambda)}\right| \leqslant \frac{O(1)}{(k\xi_1)^{1/2}(k\xi_2)^{1/2}(k\eta_o)^{1/2}} \cdot \frac{\exp\left\{-\left[-\frac{\lambda_r}{k}\int_{s_{\eta_o}(r)}^{1} dt\left(\frac{1-t}{t}\right)^{1/2}\right]\right\}}{e^{-\pi\frac{\lambda_r}{2k}}},$$

since $\lambda_r < 0$ and $|\lambda_r/k| \gg O(k\eta_o)$. Then the argument of Appendix E shows that $F/v_1'(\eta_o,\lambda)$ and consequently $\left(F/v_1'(\eta_o,\lambda)\right) X^{N_o(\xi)}$ is exponentially small for $|s_{\eta_o}| \ll 1$ $\left(|s_{\eta_o}| \gg 1/(k\eta_o)^{2/3}\right)$.

The case where s_{η_o} is in the transition region ($|s_{\eta_o}| \sim 1$) can be handled with much less detail than the above ones. If s_{η_o} is in the transition region, then equations (3.18) and (3.19) show that $\bar{\zeta}_{\eta_o} = O(1)$, $\bar{\psi}(s_{\eta_o}) \sim (2/3)^{1/6}$. Thus the Airy and Hankel functions in (3.12), (E.7), and (3.85) are bounded independently of $k\eta_o$, and equation (3.16) implies $v_1' = O\left(1/(k\eta_o)^{2/3}\right)$. Using the asymptotic forms of the Γ-functions, we obtain $|X(\eta_o,k,\lambda)| \leqslant \bar{M}_1$, for some $\bar{M}_1$ independent of $k\eta_o$. But $|s_{\eta_o}| \sim 1$ implies $|s_{\xi_1}| = O(1)$, $|s_{\xi_2}| \gg 1$. Hence substituting (3.83) for $v_1(\xi_1,-\lambda)$ and (3.82) for $v_2(\xi_2,-\lambda)$, the above order of $v_1'(\eta_o,\lambda)$, and the asymptotic form of $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$ into $F/v_1'(\eta_o,\lambda)$, we observe

$$\left|\frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)}\right| \leqslant \frac{O(1)}{(k\xi_1)^{1/2}(k\xi_2)^{\frac{1}{2}-\frac{\sigma}{2k}}(k\eta_o)^{\frac{1}{2}+\frac{\sigma}{2k}}} \cdot e^{\pi\frac{\lambda_r}{2k}},$$

since $\lambda_r < 0$ and $|\lambda_r/k| = O(k\eta_o)$. Therefore, $\left(F/v_1'(\eta_o,\lambda)\right) X^{N_o(\xi)}$ is exponentially small for $|s_{\eta_o}| \sim 1$.

The above range (the range for which $|s_{\eta_o}| \sim 1$) exhausts the possible sub-regions of C_2 and C_3. In these sub-regions we have shown that the integrand is exponentially small, except possibly in a sub-region of C_3 where it behaves as in the interval C_4. Therefore the integral over C_2 is not contributing materially to the sum (3.63), while the integral over C_3 is either not contributing materially to the sum or equation (3.76) also exists for C_3.

We continue the study of equation (3.63) by considering the integrals over C_5 and C_6. In this range $\arg\lambda = -\delta(\lambda)$, where $0 < \delta(\lambda) \ll 1$, so that

$$\arg s_{\eta_o} = \pi + \delta(\lambda) \qquad \left(s_{\eta_o} = \frac{ik\eta_o}{\lambda/ik} \right) .$$

Thus equation (D.34) shows that $v_1'(\eta_o, \lambda)$ is governed by (3.11) together with (3.10), where ζ_{η_o} is given by (3.31). Moreover, equations (D.41) and (D.42) yield

$$v_2(\eta e^{-\pi i}, -\lambda) \sim (2ik\eta e^{-\pi i})^{-1/2} \frac{(i)^{1/2}}{(\lambda/ik)^{1/2}} \exp\left\{-\frac{\lambda}{2ik}\log -\frac{\lambda}{2ike}\right\} e^{\pi\lambda/2k} V^{(1)}(s_\eta) \tag{3.102}$$

$$v_2(\eta, \lambda) \sim (2ik\eta)^{-1/2} \frac{(-i)^{1/2}}{(\lambda/ik)^{1/2}} \exp\left\{\frac{\lambda}{2ik}\log -\frac{\lambda}{2ike}\right\} V^{(2)}(s_\eta) \quad , \tag{3.103}$$

with ζ_{η_o} again given by (3.31). The behavior of the derivatives $v_2'(\eta_o e^{-\pi i}, -\lambda)$, $v_2'(\eta_o, \lambda)$ follows from that of $v_2(\eta e^{-\pi i}, -\lambda)$, $v_2(\eta, \lambda)$ in the same manner as the behavior of $v_1'(\eta_o, \lambda)$ follows from that of $v_1(\eta, \lambda)$ in Section 3.1. In addition, by (D.26)

$$\zeta_{\eta_o} = \frac{\lambda}{k}\left[\sqrt{-s_{\eta_o}(1-s_{\eta_o})} - \log\left(\sqrt{1-s_{\eta_o}} - \sqrt{-s_{\eta_o}}\right)\right] . \tag{3.104}$$

Since $\arg s_{\eta_o} = \pi + \delta(\lambda)$, then $\arg -s_{\eta_o} = \delta(\lambda)$ and $\arg(1-s_{\eta_o}) \sim 0$. This implies

$$\arg\left[\sqrt{-s_{\eta_o}(1-s_{\eta_o})} - \log\left(\sqrt{1-s_{\eta_o}} - \sqrt{-s_{\eta_o}}\right)\right] \sim 0$$

for all possible values of s_{η_o} in C_5 and C_6. Hence on C_5 and C_6 we find

$\arg \zeta_{\eta_0} \sim \arg \lambda \sim 0$, and since $\left|\zeta_{\eta_0}\right| \geq O(k\eta_0)$ we can use the exponential representation of the Bessel function in (3.11), the asymptotic forms of the Hankel functions (Section D. 2) in (3.102) and (3.103) to obtain

$$v_1'(\eta_0, \lambda) \sim -\frac{(2ik\eta_0)^{-1/2}}{\sqrt{2\pi}} \cdot \frac{e^{i\zeta_{\eta_0}} e^{-i\pi/4}}{2i} \left[1 - e^{-2i\zeta_{\eta_0}} e^{\pi i/2}\right] \cdot \left(\phi(s_{\eta_0})\right)^{1/2}, \tag{3.105}$$

$$v_2'(\eta_0 e^{-\pi i}, -\lambda) \sim (2ik\eta_0 e^{-\pi i})^{-1/2} (-i)^{1/2} \exp\left\{-\frac{\lambda}{2ik} \log -\frac{\lambda}{2ike}\right\} \cdot$$
$$\cdot e^{\pi\lambda/2i} (i) e^{-i\zeta_{\eta_0}} \cdot (-1/2)\left(\phi(s_{\eta_0})\right)^{1/2}, \tag{3.106}$$

$$v_2'(\eta_0, -\lambda) \sim (2ik\eta_0)^{-1/2} (-i)^{1/2} \exp\left\{\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike}\right\} (-i) e^{-i\zeta_{\eta_0}} \cdot (1/2)\left(\phi(s_{\eta_0})\right)^{1/2}, \tag{3.107}$$

where in all three relations we have substituted $\phi(s_{\eta_0})$ for $\left(\frac{1 - s_{\eta_0}}{s_{\eta_0}}\right)^{1/2}$.

We now investigate how the right hand members of (3.105), (3.106), and (3.107) (or $X(\eta_0, k, \lambda)$) behave in the various sub-regions in which we estimate the behavior of the integrand. We first note that $\arg \lambda = -\delta(\lambda)$ implies $\arg \lambda/2ik = -\frac{\pi}{2} - \delta(\lambda)$ and $\arg -\frac{\lambda}{2ik} = \frac{\pi}{2} - \delta(\lambda)$, which in turn implies the validity of the asymptotic forms

$$\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \sim \sqrt{2\pi} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\},$$

and

$$\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \sim \sqrt{2\pi} \exp\left\{-\frac{\lambda}{2ik} \log -\frac{\lambda}{2ike}\right\}.$$

Substituting these forms together with equations (3.106) and (3.107) into the definitions of $X(\eta_0, k, \lambda)$ and $g(\eta_0, k, \lambda)$, we find

$$X(\eta_o, k, \lambda) \sim e^{-2i\zeta_{\eta_o}} e^{\pi i/2} \;, \tag{3.108}$$

$$g(\eta_o, k, \lambda) \sim -\frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}} \cdot \frac{e^{i\zeta_{\eta_o}} e^{-i\pi/4}}{2i} \cdot \left(\phi(s_{\eta_o})\right)^{1/2} . \tag{3.109}$$

Inserting (3.108) and (3.109) into (3.52) and comparing with (3.105), we see that equations (3.105) through (3.109) are consistent.

In the sub-regions of C_5 and C_6 there are four possibilities for $|s_{\eta_o}|$, namely, $|s_{\eta_o}| \gg 1$ with $|\lambda/k| = O(1)$, $|s_{\eta_o}| \gg 1$ with $1 \ll |\lambda/k| \ll k\eta_o$, $|s_{\eta_o}| = O(1) \left(|\lambda/k| = O(k\eta_o)\right)$, and $|s_{\eta_o}| \ll 1 \left(|\lambda/k| \gg k\eta_o\right)$. The first can occur over a portion of C_5 of length $O(k)$. The relations $|s_{\eta_o}| \gg 1$ ($|ik\eta_o| \gg |\lambda/ik|$) and $|\lambda/k| = O(1)$ imply that in this case $v_1'(\eta_o, \lambda)$, $v_2'(\eta_o, \lambda)$, and $v_2'(\eta_o e^{-\pi i}, -\lambda)$ are also governed by (3.5), (3.70), and (3.71), respectively. Furthermore, the relation $|\lambda/k| = O(1)$ indicates that the formulas corresponding to the interval C_4 apply. Therefore, equation (3.76) holds for that portion of C_5 where $|\lambda/k| = O(1)$.

For $|s_{\eta_o}| \gg 1$ with $1 \ll |\lambda/k| \ll k\eta_o$, we observe that the substitution of (D.32b) into (3.105), (3.106), and (3.107) shows that $v_1'(\eta_o, \lambda)$, $v_2'(\eta_o, \lambda)$, and $v_2'(\eta_o e^{-\pi i}, -\lambda)$ are again governed by (3.5), (3.70), and (3.71), respectively. Moreover, these equations may be written as

$$v_1'(\eta_o, \lambda) \sim -\frac{(2ik\eta_o)^{+1/2}}{\sqrt{2\pi}} \; \frac{e^{i\zeta_{\eta_o}(\lambda_r)} e^{-i\pi/4}}{2i} \cdot \frac{(2k\eta_o)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}} \; e^{\frac{\sigma}{k}O(1)} \cdot$$

$$\cdot \left[1 - e^{-2i\zeta_{\eta_o}(\lambda_r)} \frac{(\lambda_r/2ke)^{\sigma/k}}{(2k\eta_o)^{\sigma/k}} \; e^{\frac{\sigma}{k}O(1)}\right] \left(\phi(s_{\eta_o})\right)^{1/2} , \tag{3.110}$$

$$v_2'(\eta_o e^{-\pi i}, -\lambda) \sim (2ik\eta_o e^{-\pi i})^{-1/2} (-i)^{1/2} \exp\left\{-\frac{\lambda}{2ik}\log -\frac{\lambda}{2ike}\right\} e^{\pi\lambda/2k} \cdot$$

$$\cdot (i) e^{i\zeta_{\eta_o}(\lambda_r)} \frac{(2k\eta_o)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}} e^{\frac{\sigma}{k} O(1)} \cdot (-1/2) \left(\phi(s_{\eta_o})\right)^{1/2} ,$$

$$v_2'(\eta_o, \lambda) \sim (2ik\eta_o)^{-1/2} (-i)^{1/2} \exp\left\{\frac{\lambda}{2ik} \log -\frac{\lambda}{2ike}\right\} (-i) e^{-i\zeta_{\eta_o}(\lambda_r)} \cdot$$

$$\cdot \frac{(\lambda_r/2ke)^{\sigma/2k}}{(2k\eta_o)^{\sigma/2k}} e^{\frac{\sigma}{k} O(1)} \cdot (1/2) \left(\phi(s_{\eta_o})\right)^{1/2} ,$$

which is the result one would obtain by setting $\zeta_{\eta_o} = \mathrm{Re}\,\zeta_{\eta_o} + i\,\mathrm{Im}\,\zeta_{\eta_o}$ in (3.105), (3.106), and (3.107), and noting that $\mathrm{Re}\,\zeta_{\eta_o}(\lambda) \sim \zeta_{\eta_o}(\lambda_r)$ $(\lambda = \lambda_r - i\sigma)$, while retaining the terms which are not of $O(1)$ in $\mathrm{Im}\,\zeta_{\eta_o}$. Thus

$$X(\eta_o, k, \lambda) \sim e^{-2i\zeta_{\eta_o}(\lambda_r)} e^{\pi i/2} \frac{(\lambda_r/2ke)^{\sigma/k}}{(2k\eta_o)^{\sigma/k}} e^{\frac{\sigma}{k} O(1)} ; \qquad (3.111)$$

and since $k\eta_o \gg |\lambda/k| \sim \lambda_r/k$, equation (3.110) reduces to

$$v_1'(\eta_o, \lambda) \sim -\frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}} \frac{e^{i\zeta_{\eta_o}(\lambda_r)}}{2i} e^{-i\pi/4} \frac{(2k\eta_o)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}} e^{\frac{\sigma}{k} O(1)} . \qquad (3.110a)$$

If $s_{\eta_o} = O(1)$, then in (3.105) and (3.108) we set $\zeta_{\eta_o} = \mathrm{Re}\,\zeta_{\eta_o} + i\,\mathrm{Im}\,\zeta_{\eta_o}$, and noting that $|\mathrm{Im}\,\zeta_{\eta_o}| = O(\sigma/\lambda_r)|\mathrm{Re}\,\zeta_{\eta_o}|$ $(\lambda = \lambda_r - i\sigma,\ \lambda_r > 0)$, $\mathrm{Re}\,\bar{\zeta}_{\eta_o} \sim \zeta_{\eta_o}(\lambda_r)$,

$\zeta_{\eta_o}(\lambda_r) = \frac{\lambda_r}{k} O(1)$, we obtain

$$X(\eta_o, k, \lambda) \sim e^{-2i\zeta_{\eta_o}(\lambda_r)} e^{\pi i/2} e^{\frac{\sigma}{k} O(1)} , \qquad (3.112)$$

$$v_1'(\eta_o, \lambda) \sim - \frac{(2ik\eta_o)^{-1/2}}{\sqrt{2\pi}} \cdot \frac{e^{i\zeta_{\eta_o}(\lambda_r)}}{2i} e^{-\pi i/4} e^{\frac{\sigma}{k} O(1)} .$$

$$\cdot \left[1 - e^{-2i\zeta_{\eta_o}(\lambda_r)} e^{\pi i/2} e^{\frac{\sigma}{k} O(1)} \right] \left(\phi(s_{\eta_o}) \right)^{1/2} . \qquad (3.113)$$

This latter equation contains the function

$$Y(\eta_o, k, \lambda_r, \sigma) \equiv 1 - e^{-2i\zeta_{\eta_o}(\lambda_r)} e^{\pi i/2} e^{\frac{\sigma}{k} O(1)} .$$

We know (Section 3.1) that there exists values of λ_r in this range for which $1 - e^{-2i\zeta_{\eta_o}(\lambda_r)} e^{\pi i/2} \sim 0$. But since $\sigma/k = O(1)$, no such λ_r can exist for $Y(\eta_o, k, \lambda_r, \sigma)$.

Finally, if $\left| s_{\eta_o} \right| \ll 1$ $\left(\left| \lambda/k \right| \gg k\eta_o \right)$, the substitution of (D.28a) into (3.105) and (3.108) shows that we can use equations (C.4) and (C.21) for $v_1'(\eta_o, \lambda)$ and $X(\eta_o, k, \lambda)$, respectively.

In order to find the representations of the Whittaker functions which depend on ξ_j, $j = 1, 2$, we first observe that $\arg\lambda = -\delta(\lambda)$ implies

$$\arg s_{\xi_j} = \delta(\lambda) \qquad \left(s_{\xi_j} = \frac{ik\xi_j}{-\lambda/ik} \right) .$$

Then equations (D.21) and (D.20) show that $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$ are given by (E.34) and (E.35), respectively, with $\bar{\zeta}_{\xi_j}$ given by

$$\bar{\zeta}_{\xi_j} = \frac{\lambda}{k}\int_1^{s_{\xi_j}} \left(\frac{s-1}{s}\right)^{1/2} ds \ .$$

However, in this range $\bar{\zeta}_{\xi_j}$ depends on the position of λ with respect to the turning point $\lambda = k^2\xi_j$. This must be considered since the widths of the transition regions (defined by (E.78)) are again $\Delta\frac{\lambda}{k} = O\left((k\eta_o)^{1/3}\right)$, while the minimum distance from the path to the turning points is $\Delta\frac{\lambda}{k} = \sigma/k = O(1)$. As indicated in Appendix E, we do not allow $k^2\xi_2$ to be in the transition region of $v_1(\xi_1,-\lambda)$. Consequently, in this range we need to study the following cases: $|\lambda| < k^2\xi_1$ and away from the transition region of $v_1(\xi_1,-\lambda)$, λ in the transition region of $v_1(\xi_1,-\lambda)$, $|\lambda| > k^2\xi_1$, $|\lambda| < k^2\xi_2$ and away from the transition region of both $v_1(\xi_1,-\lambda)$ and $v_2(\xi_2,-\lambda)$, λ in the transition region of $v_2(\xi_2,-\lambda)$, $|\lambda| > k^2\xi_2$ and away from the transition region of $v_2(\xi_2,-\lambda)$.

For $|\lambda| < k^2\xi_1$ and away from the transition region of $v_1(\xi_1,-\lambda)$, we consider first the case $|\lambda/k| = O(1)$. This can occur over a portion of C_5 of length $O(k)$ and has already been discussed when studying the Whittaker functions which depend on η_o.

We next consider $|\lambda| < k^2\xi_1$ and away from the transition region of $v_1(\xi_1,-\lambda)$ with either $1 \ll |\lambda/k| \ll k\xi_1$ $\left(\left|s_{\xi_1}\right| \gg 1,\ \left|s_{\xi_2}\right| \gg 1\right)$, or $|\lambda/k| = O(k\xi_1) = O(k\eta_o)$ $\left(\left|s_{\xi_1}\right| = O(1),\ \left|s_{\xi_2}\right| \ll 1\right.$, recall that the configuration of ξ_1, ξ_2, and η_o under consideration assumes $\left.\xi_1 = O(\eta_o),\ \xi_2 \gg \xi_1\right)$. In either case, equation (D.5) shows that

$$\bar{\zeta}_{\xi_j} = \frac{\lambda}{k}\left[\sqrt{s_{\xi_j}(s_{\xi_j}-1)} - \log\left(\sqrt{s_{\xi_j}-1} + \sqrt{s_{\xi_j}}\right)\right] \ .$$

But $\arg s_{\xi_j} = \delta(\lambda)$ and $\arg(s_{\xi_j} - 1) \sim 0$. The latter relation is obvious if $\left|s_{\xi_j}\right| \gg 1$. If $\left|s_{\xi_j}\right| = O(1)$ (although the relation $\left|s_{\xi_2}\right| = O(1)$ need not be considered for this case, we develop, for future use, the estimate of $v_2(\xi_2,-\lambda)$ for $\left|s_{\xi_2}\right| = O(1)$

along with the estimate of $v_1(\xi_1, -\lambda)$ for $\left|s_{\xi_1}\right| = O(1)$), it follows since $\arg(s_{\xi_j} - 1)$ not close to zero is contrary to s_{ξ_j} being away from the transition region of $v_j(\xi_j, -\lambda)$ (see the previous argument for s_{η_o} and λ in the interval C_3). These estimates for $\arg s_{\xi_j}$ and $\arg(s_{\xi_j} - 1)$ imply

$$\arg\left[\sqrt{s_{\xi_j}(s_{\xi_j} - 1)} - \log\left(\sqrt{s_{\xi_j} - 1} + \sqrt{s_{\xi_j}}\right)\right] \sim 0 .$$

As a result, $\arg \bar{\zeta}_{\xi_j} \sim \arg\lambda \sim 0$. Then since $\left|\bar{\zeta}_{\xi_j}\right| \geq O(k\eta_o)$, $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$ are governed by (E.38) and (E.39), respectively.

If $1 \ll |\lambda/k| \ll k\xi_1$ ($\left|s_{\xi_j}\right| \gg 1$), the substitution of (D.10a) into (E.38) and (E.39) shows that (E.8) is valid for $v_1(\xi_1, -\lambda)$, while (E.9) is valid for $v_2(\xi_2, -\lambda)$. Moreover, these equations can be written as

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} e^{\pi\lambda/2k} \left[e^{i\bar{\zeta}_{\xi_1}(\lambda_r)} \frac{(\lambda_r/2ke)^{\sigma/2k}}{(2k\xi_1)^{\sigma/2k}} + \right.$$

$$\left. + i e^{-i\bar{\zeta}_{\xi_1}(\lambda_r)} \frac{(2k\xi_1)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}} \right] \frac{e^{\frac{\sigma}{k} O(1)}}{\left(\bar{\phi}(s_{\xi_1})\right)^{1/2}} , \qquad (3.114)$$

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} \exp\left\{-\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike}\right\} \cdot \frac{e^{-i\bar{\zeta}_{\xi_2}(\lambda_r)}}{\left(\bar{\phi}(s_{\xi_2})\right)^{1/2}} \cdot$$

$$\cdot \frac{(2k\xi_2)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}} e^{\frac{\sigma}{k} O(1)} , \qquad (3.115)$$

which is the result one would obtain by setting $\bar{\zeta}_{\xi_j} = \mathrm{Re}\,\bar{\zeta}_{\xi_j} + i\,\mathrm{Im}\,\bar{\zeta}_{\xi_1}$ in (E.38) and (E.39), and noting that $\mathrm{Re}\,\bar{\zeta}_{\xi_j}(\lambda) \sim \zeta_{\xi_j}(\lambda_r)$, while retaining the terms which are not of $O(1)$ in $\mathrm{Im}\,\bar{\zeta}_{\xi_j}$. But since $k\xi_1 \gg |\lambda/k| \sim \lambda_r/k$, equation (3.114) reduces to

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} e^{\pi\lambda/2k} \cdot \frac{i\,e^{-i\bar{\zeta}_{\xi_1}(\lambda_r)}}{\left(\bar{\phi}(s_{\xi_1})\right)^{1/2}} \cdot \frac{(2k\xi_1)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}}\, e^{\frac{\sigma}{k} O(1)} . \tag{3.114a}$$

We now estimate the behavior of the integrand in the case corresponding to equations (3.114) and (3.115). In accordance with the configuration of ξ_1, ξ_2 and η_o under consideration, the relation $1 \ll |\lambda/k| \ll k\eta_o$ is also valid. Thus inserting equations (3.114a) and (3.115) into the definition of $F(\xi_1, \xi_2, \eta_o, k, \lambda)$, using (3.110a) for $v_1'(\eta_o, \lambda)$, and the above asymptotic form for $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, we observe

$$\frac{F}{v_1'(\eta_o, \lambda)} \sim \frac{2\sqrt{2\pi}\; C(k, \xi_1, \xi_2, \eta_o)}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}} \; \frac{(2k\xi_1)^{\sigma/2k}}{(2k\eta_o)^{\sigma/2k}} \; \frac{(2k\xi_2)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}} \cdot (-i)^{1/2} e^{\frac{\sigma}{k} O(1)} \cdot e^{i\zeta^-(\lambda_r)} ,$$

with

$$\zeta^-(\lambda_r) = -\bar{\zeta}_{\xi_1}(\lambda_r) - \bar{\zeta}_{\xi_2}(\lambda_r) - \zeta_{\eta_o}(\lambda_r) .$$

But in this case

$$s_{\xi_1}^{(r)} = \frac{k^2\xi_1}{\lambda_r} > 1 \qquad \text{and} \qquad s_{\xi_2}^{(r)} = \frac{k^2\xi_2}{\lambda_r} > 1 .$$

Hence, equation (E.48) shows that $\bar{\zeta}_{\xi_j}(\lambda_r) = \lambda_r/k\; \bar{\Phi}_o(s_{\xi_j}^{(r)})$. In addition, equations

(3.104) and (E.49) give $\zeta_{\eta_o}(\lambda_r) = \lambda_r/k\, \Phi_o(s^{(r)}_{\eta_o})$. Therefore, (3.111) and the above estimate of $F/v_1'(\eta_o,\lambda)$ yield

$$\frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} \sim \frac{2\sqrt{2\pi}\; C(k,\xi_1,\xi_2,\eta_o)}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}} \; \frac{(2k\xi_1)^{\sigma/2k}}{(2k\eta_o)^{\sigma/2k}} \; \frac{(2k\xi_2)^{\sigma/2k}}{(\lambda_r/2ke)^{\sigma/2k}} \cdot$$

$$\cdot \frac{(\lambda_r/2ke)^{N_o(\xi)\frac{\sigma}{k}}}{(2k\eta_o)^{N_o(\xi)\frac{\sigma}{k}}} \, (-i)^{1/2} \, e^{\frac{\sigma}{k}O(1)} \, e^{\frac{\pi i N_o(\xi)}{2}} \, e^{i\Phi_o^{N_o^-}(\lambda_r)} , \tag{3.116}$$

where

$$\Phi_o^{N_o^-}(\lambda_r) = -\frac{\lambda_r}{k}\bar{\Phi}_o(s^{(r)}_{\xi_1}) - \frac{\lambda_r}{k}\bar{\Phi}_o(s^{(r)}_{\xi_2}) - \left(2N_o(\xi)+1\right)\frac{\lambda_r}{k}\Phi_o(s^{(r)}_{\eta_o}) .$$

The estimate given by equation (3.116) enables us to estimate the integral

$$\int_{C_5'} d\lambda \, \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} ,$$

where C_5' is the sub-region of C_5 discussed above $(1 \ll |\lambda/k| \ll k\xi_1)$. For $\lambda = \lambda_r - i\sigma$, $d\lambda = d\lambda_r$, and this integral becomes

$$\int_{(C_5)_r'} d\lambda_r \, \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} ,$$

with

$$(C_5)_r' = \left\{\lambda_r \,\middle|\, \lambda_r - i\sigma \in C_5,\; 1 \ll \frac{\lambda_r}{k} \ll k\xi_1\right\} .$$

Substituting equation (3.116) into this latter integral, we see that it may be estimated by the method of stationary phase. It follows from the computations in Appendix E ((E.50) through (E.53)), that $\Phi_o^{N_o^-}(\lambda_r)$ has a stationary point in $(C_5)'_r$ provided the equation

$$\left(\sqrt{\xi_1 - z} + \sqrt{\xi_1}\right)\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right) = \frac{z \cdot \left(\sqrt{z}\right)^{2N_o(\xi)+1}}{\left(\sqrt{\eta_o + z} - \sqrt{\eta_o}\right)^{2N_o(\xi)+1}} ,$$

has a solution in $0 < z < \xi_1$. But $N_o(\xi)$ was chosen so that no such solution exists. Therefore, the integral can be estimated by integration by parts and we obtain

$$\int_{C_5'} d\lambda \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)} = \int_{(C_5)'_r} d\lambda_r \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)}$$

$$= \frac{1}{(\xi_1)^{\frac{1}{2}(1+\frac{\sigma}{k})} (\xi_2)^{\frac{1}{2}(1-\frac{\sigma}{k})}} O\left(\frac{1}{(k\eta_o)^{1/2}}\right) . \tag{3.117}$$

We now consider $|\lambda| < k^2\xi_1$ and away from the transition region of $v_1(\xi_1, -\lambda)$, but with $|\lambda/k| = O(k\xi_1)$ ($|s_{\xi_1}| = O(1)$, $|s_{\xi_2}| \gg 1$). We also derive, but do not use, the estimate of $v_2(\xi_2, -\lambda)$ if $|s_{\xi_2}| > 1$ and away from the transition region of $v_2(\xi_2, -\lambda)$, but $|s_{\xi_2}| = O(1)$. Thus in (E.38) and (E.39) we set $\bar{\zeta}_{\xi_j} = \mathrm{Re}\,\bar{\zeta}_{\xi_j} + i\,\mathrm{Im}\,\bar{\zeta}_{\xi_j}$, and if we note that $|\mathrm{Im}\,\bar{\zeta}_{\xi_j}| = O(\sigma/\lambda_r)\,\mathrm{Re}\,\bar{\zeta}_{\xi_j}$, $\mathrm{Re}\,\bar{\zeta}_{\xi_j}(\lambda) \sim \bar{\zeta}_{\xi_j}(\lambda_r)$, $\bar{\zeta}_{\xi_j}(\lambda_r) = \frac{\lambda_r}{k} O(1)$, we find

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} e^{\pi\lambda/2k} \frac{\left[e^{i\bar{\zeta}_{\xi_1}(\lambda_r)} + i e^{-i\bar{\zeta}_{\xi_1}(\lambda_r)}\right]}{\left(\bar{\phi}(s_{\xi_1})\right)^{1/2}} e^{\frac{\sigma}{k} O(1)} , \tag{3.118}$$

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} \exp\left\{-\frac{\lambda}{2ik}\log - \frac{\lambda}{2ike}\right\} \frac{e^{-i\bar{\zeta}_{\xi_2}(\lambda_r)}}{\left(\bar{\phi}(s_{\xi_2})\right)^{1/2}} e^{\frac{\sigma}{k} O(1)} \quad . \tag{3.119}$$

To estimate the behavior of the integrand for $|\lambda| < k^2\xi_1$ and away from the transition region of $v_1(\xi_1, -\lambda)$, but $|\lambda/k| = O(k\xi_1)$ ($|s_{\xi_1}| = O(1)$), we note that in accordance with the configuration of ξ_1, ξ_2, and η_o under consideration, the relations $|\lambda/k| = O(k\eta_o)$ ($|s_{\eta_o}| = O(1)$) and $|\lambda/k| \ll k\xi_2$ ($|s_{\xi_2}| \gg 1$) are also valid. Then $v_2(\xi_2, -\lambda)$ is still governed by (3.115) $\left[\text{with } (\lambda_r/2ke) = O(k\xi_1)\right]$, while $v_1(\xi_1, -\lambda)$ is governed by (3.118), and $v_1'(\eta_o, \lambda)$ is governed by (3.113). Using these equations and the above asymptotic form for $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, we find

$$\frac{F}{v_1'(\eta_o, \lambda)} \sim \frac{2\sqrt{2\pi}\, C(k, \xi_1, \xi_2, \eta_o)(2k\xi_2)^{\sigma/2k}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}(2k\xi_1)^{\sigma/2k}} \cdot \frac{(-i)^{1/2} e^{\frac{\sigma}{k} O(1)}}{Y(\eta_o, k, \lambda_r, \sigma)} \cdot \left[e^{i\zeta^+(\lambda_r)} + i e^{i\zeta^-(\lambda_r)}\right] ,$$

with

$$\zeta^+(\lambda_r) = \bar{\zeta}_{\xi_1}(\lambda_r) - \bar{\zeta}_{\xi_2}(\lambda_r) - \zeta_{\eta_o}(\lambda_r) ,$$

$$\zeta^-(\lambda_r) = -\bar{\zeta}_{\xi_1}(\lambda_r) - \bar{\zeta}_{\xi_2}(\lambda_r) - \zeta_{\eta_o}(\lambda_r) .$$

But in this case the relations

$$s_{\xi_1}^{(r)} = \frac{k^2\xi_1}{\lambda_r} > 1 \quad \text{and} \quad s_{\xi_2}^{(r)} = \frac{k^2\xi_2}{\lambda_r} > 1$$

are again valid. Hence, the equalities of the previous case ($1 \ll |\lambda/k| \ll k\xi_1$) together with equation (3.112) gives

$$\frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} \sim \frac{2\sqrt{2\pi}\, C(k,\xi_1,\xi_2,\eta_o)(2k\xi_2)^{\sigma/2k}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}(2k\xi_1)^{\sigma/2k}} \cdot \frac{(-i)^{1/2} e^{\frac{\sigma}{k}O(1)}}{Y(\eta_o,k,\lambda_r,\sigma)} \cdot$$

$$\cdot\, e^{\frac{\pi i N_o(\xi)}{2}} \left[e^{i\Phi_o^{N_o^+}(\lambda_r)} + i\, e^{i\Phi_o^{N_o^-}(\lambda_r)} \right] , \qquad (3.120)$$

where

$$\Phi_o^{N_o^+}(\lambda_r) = \frac{\lambda_r}{k}\bar{\Phi}_o(s_{\xi_1}^{(r)}) - \frac{\lambda_r}{k}\bar{\Phi}_o(s_{\xi_2}^{(r)}) - \left(2N_o(\xi)+1\right)\frac{\lambda_r}{k}\Phi_o(s_{\eta_o}^{(r)}) ,$$

$$\Phi_o^{N_o^-}(\lambda_r) = -\frac{\lambda_r}{k}\bar{\Phi}_o(s_{\xi_1}^{(r)}) - \frac{\lambda_r}{k}\bar{\Phi}_o(s_{\xi_2}^{(r)}) - \left(2N_o(\xi)+1\right)\frac{\lambda_r}{k}\Phi_o(s_{\eta_o}^{(r)}) .$$

The estimate given by equation (3.120) enables us to estimate the integral

$$\int_{(C_5+C_6)'} d\lambda \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} ,$$

where $(C_5+C_6)'$ is the sub-region of C_5 and C_6 discussed above $\left[\left|s_{\xi_1}\right| = O(1),\ \left|s_{\eta_o}\right| = O(1),\ \left|s_{\xi_2}\right| \gg 1 \right]$. For $\lambda = \lambda_r - i\sigma$, $d\lambda = d\lambda_r$, and this integral becomes

$$\int_{(C_5+C_6)_r'} d\lambda_r \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} ,$$

with

$$(C_5+C_6)_r' = \left\{ \lambda_r \,\middle|\, \lambda_r - i\sigma \in C_5+C_6,\ \lambda_r < k^2\xi_1 \text{ and away from the transition region of } v_1(\xi_1,-\lambda), \text{ but } \lambda_r/k = O(k\xi_1) \right\} .$$

Substituting (3.120) in this latter integral, we see that it becomes two integrals which may be estimated by stationary phase since $Y(\eta_o, k, \lambda_r, \sigma)$ has no zeros in $(C_5+C_6)'_r$. It follows from the computations in Appendix E ((E.50) through (E.53)), that one integrand has a stationary point in $(C_5+C_6)'_r$ provided either of these equations

$$\frac{\sqrt{\xi_1 - z} + \sqrt{\xi_1}}{\sqrt{\xi_2 - z} + \sqrt{\xi_2}} = \frac{\sqrt{\eta_o + z} - \sqrt{\eta_o}}{\sqrt{z}} \left(\frac{\sqrt{\eta_o + z} - \sqrt{\eta_o}}{\sqrt{z}} \right)^{2N_o(\xi)} ,$$

$$\left(\sqrt{\xi_1 - z} + \sqrt{\xi_1}\right)\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right) = \frac{z \cdot \left(\sqrt{z}\right)^{2N_o(\xi)+1}}{\left(\sqrt{\eta_o + z} - \sqrt{\eta_o}\right)^{2N_o(\xi)+1}} ,$$

have a solution in $0 < z < \xi_1$. But $N_o(\xi)$ was chosen such that these equations do not have any solution in $0 < z < \xi_1$. Therefore, the integrals can be estimated by integration by parts and we obtain

$$\int_{(C_5+C_6)'} d\lambda \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)} = \int_{(C_5+C_6)'_r} d\lambda_r \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)}$$

$$= \frac{1}{\xi_1^{\frac{1}{2}(1+\frac{\sigma}{k})} \xi_2^{\frac{1}{2}(1-\frac{\sigma}{k})}} O\left(\frac{1}{(k\eta_o)^{1/2}}\right) . \tag{3.121}$$

For λ in the transition region of $v_1(\xi_1, -\lambda)$, $\left|s_{\xi_1}\right| \sim 1$. Thus in accordance with the configuration of ξ_1, ξ_2, and η_o under consideration, $\left|s_{\xi_2}\right| \gg 1$ and $\left|s_{\eta_o}\right| = O(1)$. In this case $\left|\bar{\zeta}_{\xi_1}\right| = O(1)$. Hence, the Airy function in (E.34) is bounded independent of $k\eta_o$, while (D.7) and (D.8) imply $\bar{\mathcal{J}}(s_{\xi_1}) \sim (2/3)^{1/6}$. Moreover, equation (3.115) $\left[\text{with } (\lambda_r/2ke) = O(k\xi_1)\right]$ is again valid for $v_2(\xi_2, -\lambda)$, while $v_1'(\eta_o,\lambda)$ is again governed by (3.113). Using these equations and the above asymptotic form for $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$, we find

$$\frac{F}{v_1'(\eta_o,\lambda)} \sim \frac{C(k,\xi_1,\xi_2,\eta_o)}{\left[\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}} \frac{(2k\xi_2)^{\sigma/2k}}{(2k\xi_1)^{\sigma/2k}} \frac{(\lambda/k)^{1/6}Ai(-\bar{\sigma}_{\xi_1})e^{\frac{\sigma}{k}O(1)}}{Y(\eta_o,k,\lambda_r,\sigma)} \cdot e^{i\zeta(\lambda_r)} ,$$

with

$$\zeta(\lambda_r) = -\bar{\zeta}_{\xi_2}(\lambda_r) - \zeta_{\eta_o}(\lambda_r) .$$

But in this case the relation

$$s_{\xi_2}^{(r)} = \frac{k^2\xi_2}{\lambda_r} > 1$$

is again valid. Thus, the equalities of the case $1 \ll |\lambda/k| \ll k\xi_1$ together with equation (3.112) give

$$\frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} \sim \frac{C(k,\xi_1,\xi_2,\eta_o)(2k\xi_2)^{\sigma/2k}}{\left[\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}(2k\xi_1)^{\sigma/2k}} \cdot \frac{(\lambda/k)^{1/6}Ai(-\bar{\sigma}_{\xi_1})}{Y(\eta_o,k,\lambda_r,\sigma)} \cdot$$
$$\cdot e^{\frac{\sigma}{k}O(1)} \cdot e^{i\Phi_o^{N_o}(\lambda_r)} , \qquad (3.122)$$

where

$$\Phi_o^{N_o}(\lambda_r) = -\frac{\lambda_r}{k}\bar{\Phi}_o(s_{\xi_2}^{(r)}) - \left(2N_o(\xi)+1\right)\frac{\lambda_r}{k}\Phi_o(s_{\eta_o}^{(r)}) .$$

It remains to estimate the integral

$$\int_{(C_5+C_6)''} d\lambda \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} ,$$

where $(C_5+C_6)''_r$ is the sub-region of C_5 and C_6 discussed above $\left[\text{on it } |s_{\xi_1}| \sim 1,\ |s_{\xi_2}| >> 1,\ |s_{\eta_0}| = O(1)\right]$. For $\lambda = \lambda_r - i\sigma$, $d\lambda = d\lambda_r$, and this integral becomes

$$\int_{(C_5+C_6)''_r} d\lambda_r \frac{F}{v_1'(\eta_0,\lambda)} \cdot X^{N_0(\xi)} ,$$

with

$$(C_5+C_6)''_r = \left\{\lambda_r \middle| \lambda_r \text{ is in the transition region of } v_1(\xi_1,-\lambda)\right\} .$$

Examining the estimate of the integrand (equation (3.122)), we observe that the latter integral can be estimated by stationary phase since $Ai(-\bar{\sigma}_{\xi_1})$ is independent of $k\eta_0$ and not oscillating rapidly in $(C_5+C_6)''_r$. It follows from the computations in Appendix E ((E.50) through (E.53)) that $\Phi_0^{N_0}(\lambda_r)$ has a stationary point in $(C_5+C_6)''_r$ provided the equation

$$\left(\sqrt{\xi_2-\xi_1} + \sqrt{\xi_2}\right)\frac{\left(\sqrt{\eta_0+z} - \sqrt{\eta_0}\right)^{2N_0(\xi)+1}}{\left(\sqrt{z}\right)^{2N_0(\xi)+1}} = \sqrt{z} .$$

has a solution in

$$\xi_1 - O\left(\frac{1}{(k\eta_0)^{2/3}}\right) \leqslant z \leqslant \xi_1 + O\left(\frac{1}{(k\eta_0)^{2/3}}\right) .$$

But $N_0(\xi)$ was chosen so that no such solution exists. Therefore the integrand does not have any stationary points in $(C_5+C_6)''_r$, and the integral can be estimated by integration by parts. This gives

$$\int_{(C_5+C_6)''} d\lambda \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)} = \int_{(C_5+C_6)''_r} d\lambda_r \frac{F}{v_1'(\eta_o,\lambda)} \cdot X^{N_o(\xi)}$$

$$= \frac{1}{\xi_1^{\frac{1}{2}(1+\frac{\sigma}{k})}\xi_2^{\frac{1}{2}(1-\frac{\sigma}{k})}} O\left(\frac{1}{(k\eta_o)^{1/3}}\right). \qquad (3.123)$$

For $|\lambda| > k^2\xi_1$, $|\lambda| < k^2\xi_2$ and away from the transition region of both $v_1(\xi_1,-\lambda)$ and $v_2(\xi_2,-\lambda)$, we first consider $|\lambda/k| = O(k\xi_1)\left(\left|s_{\xi_1}\right| = O(1)\right)$ and thus in accordance with the configuration of ξ_1, ξ_2 and η_o under consideration $\left|s_{\xi_2}\right| \gg 1$, $\left|s_{\eta_o}\right| = O(1)$. In this case we can again use equation (3.115) for $v_2(\xi_2,-\lambda)$. However by (D.6)

$$\bar{\zeta}_{\xi_1} = i(-\lambda/k)\int_{s_{\xi_1}}^{1} \left(\frac{1-s}{s}\right)^{1/2} ds\ ,$$

and performing the indicated integration, we obtain

$$\bar{\zeta}_{\xi_1} = i(-\lambda/k)\left[\frac{\pi}{2} - \sin^{-1}\sqrt{s_{\xi_1}} - \sqrt{s_{\xi_1}(1-s_{\xi_1})}\right]\ .$$

Now $\arg s_{\xi_1} = \delta(\lambda)$, and since $\arg(1-s_{\xi_1})$ not close to zero is contrary to s_{ξ_1} being away from the transition region of $v_1(\xi_1,-\lambda)$ (see the previous argument for s_{η_o} and λ in the interval C_3), we find $\arg(1-s_{\xi_1}) \sim 0$. Then these estimates for $\arg s_{\xi_1}$ and $\arg(1-s_{\xi_1})$ show that

$$\arg\left[\frac{\pi}{2} - \sin^{-1}\sqrt{s_{\xi_1}} - \sqrt{s_{\xi_1}(1-s_{\xi_1})}\right] \sim 0$$

(see the previous argument for s_{η_o} and λ in the interval C_3). Consequently in in this case $\arg\bar{\zeta}_{\xi_1} \sim \pi/2 + \arg(-\lambda) \sim 3\pi/2$, and $\arg(-\bar{\sigma}_{\xi_1}) \sim 0$ $\left[\bar{\sigma}_{\xi_1} = (\frac{3}{2}\bar{\zeta}_{\xi_1})^{2/3}\right]$. Since $|\bar{\zeta}_{\xi_1}| = O(k\eta_o)$, the asymptotic form of the Airy function given by Erdélyi and Swanson (1957, equation 4.5) can be used in (E.34) to derive

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} e^{\pi\lambda/2k} \cdot \frac{i e^{-i\bar{\zeta}_{\xi_1}}}{\left(\phi(s_{\xi_1})\right)^{1/2}} .$$

If in this equation we set $\bar{\zeta}_{\xi_1} = \mathrm{Re}\,\bar{\zeta}_{\xi_1} + i\,\mathrm{Im}\,\bar{\zeta}_{\xi_1}$, and note that $\left|\mathrm{Re}\,\bar{\zeta}_{\xi_1}\right| = O(\sigma/\lambda_r)\left|\mathrm{Im}\,\bar{\zeta}_{\xi_1}\right|$, $\left|\mathrm{Im}\,\bar{\zeta}_{\xi_1}\right| \sim \left|\bar{\zeta}_{\xi_1}(\lambda_r)\right|$, $\left|\bar{\zeta}_{\xi_1}(\lambda_r)\right| = \frac{\lambda_r}{k} O(1)$, the equation becomes

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} e^{\pi\lambda/2k} \cdot \frac{i\exp\left\{-\frac{\lambda_r}{k}\int_{s^{(r)}_{\xi_1}}^{1} ds\left(\frac{1-s}{s}\right)^{1/2}\right\} e^{i\frac{\sigma}{k}O(1)}}{\left(\phi(s_{\xi_1})\right)^{1/2}} . \tag{3.124}$$

Inserting (3.124) together with (3.115) into the definition of $F(\xi_1, \xi_2, \eta_o, k, \lambda)$, using (3.113) for $v_1'(\eta_o, \lambda)$, (3.112) for $X(\eta_o, k, \lambda)$, and the above asymptotic form for $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, we observe

$$\left|\frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)}\right| \leqslant \frac{1}{(k\xi_1)^{\frac{1}{2} + \frac{\sigma}{2k}} (k\xi_2)^{\frac{1}{2} - \frac{\sigma}{2k}} (k\eta_o)^{1/2}} \, O\left(\exp\left[-\frac{\lambda_r}{k}\int_{s^{(r)}_{\xi_1}}^{1} ds\left(\frac{1-s}{s}\right)^{1/2}\right]\right)$$

since $\lambda_r > 0$, $\lambda_r/k = O(k\eta_o)$, $s^{(r)}_{\xi_1} < 1$ and not in a neighborhood of 1. Hence the integrand is exponentially small if $|\lambda| > k^2\xi_1$, $|\lambda| < k^2\xi_2$, $|\lambda/k| = O(k\xi_1)$ and away from the transition regions of both $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$. Since equation (3.124) remains valid for $\left|s_{\xi_1}\right| \ll 1$ $\left(\left|s_{\xi_1}\right| \gg 1/(k\eta_o)^{2/3}\right)$, this conclusion is also true if $\left|\lambda/k\right| = O(k\xi_1)$ is replaced by $k\xi_1 \ll \left|\lambda/k\right| \ll k\xi_2$.

We complete the case $|\lambda| > k^2\xi_1$, $|\lambda| < k^2\xi_2$ and away from both transition regions by considering $\left|\lambda/k\right| = O(k\xi_2)$ $\left(\left|s_{\xi_2}\right| = O(1)\right)$. Then in accordance with the configuration of ξ_1, ξ_2, and η_o under consideration, the relations $\left|s_{\xi_1}\right| \ll 1$ $\left(\left|s_{\xi_1}\right| \gg 1/(k\eta_o)^{2/3}\right)$ and $\left|s_{\eta_o}\right| \ll 1$ are also valid. Thus equations (C.4) and (C.21) govern the behavior of $v_1'(\eta_o, \lambda)$ and $X(\eta_o, k, \lambda)$, respectively. Furthermore, equation (3.124) is still valid for $v_1(\xi_1, -\lambda)$ while $v_2(\xi_2, -\lambda)$ is now governed by (3.119). Using these equations and the above asymptotic form for $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, we find

$$\left|\frac{F}{v_1'(\eta_o, \lambda)} \cdot X^{N_o(\xi)}\right| \lesssim \frac{1}{(k\xi_1)^{1/2}(k\xi_2)^{1/2}(k\eta_o)^{1/2}} \; \frac{e^{4N_o(\xi)\,\mathrm{Im}\sqrt{\lambda\eta_o}}}{e^{-2\mathrm{Im}\sqrt{\lambda\eta_o}}}$$

$$\cdot\, O\left(\exp\left\{-\frac{\lambda_r}{k}\int_{s_{\xi_1}^{(r)}}^{1} ds \left(\frac{1-s}{s}\right)^{1/2}\right\}\right) \qquad , \qquad (3.125)$$

since $\mathrm{Im}\sqrt{\lambda} < 0$, $\lambda_r > 0$, $\lambda_r/k = O(k\eta_o)$, $s_{\xi_1}^{(r)} < 1$ and not in a neighborhood of 1. Hence the integrand is exponentially small if $|\lambda| > k^2\xi_1$, $|\lambda| < k^2\xi_2$, $|\lambda/k| = O(k\xi_2)$ and away from the transition regions of both $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$.

For λ in the transition region of $v_2(\xi_2, -\lambda)$, we can again use (3.124) for $v_1(\xi_1, -\lambda)$, (C.4) for $v_1'(\eta_o, \lambda)$, and (C.21) for $X(\eta_o, k, \lambda)$. We also know that in this case $\left|\zeta_{\xi_1}\right| = O(1)$. Thus the Airy function in (E.35) is bounded independent of $k\eta_o$, while (D.7) and (D.8) imply $\bar{\psi}(s_{\xi_2}) \sim (2/3)^{1/6}$. Therefore, equation

(3.125) with $(k\xi_2)^{1/2}$ replaced by $(k\xi_2)^{1/3}$ describes the behavior of the integrand in this case. Consequently, for λ in the transition region of $v_2(\xi_2, -\lambda)$ the integrand is exponentially small.

For $|\lambda| > k^2 \xi_2$ and away from the transition region of $v_2(\xi_2, -\lambda)$, equations (3.124), (C.4), and (C.21) remain valid. Moreover, $\bar{\zeta}_{\xi_2}$ is now given by

$$\bar{\zeta}_{\xi_2} = i(-\lambda/k) \left[\frac{\pi}{2} - \sin^{-1} \sqrt{s_{\xi_2}} - \sqrt{s_{\xi_2}(1 - s_{\xi_2})} \right] .$$

Hence, the previous arguments $\left[|\lambda| > k^2 \xi_1, \ |\lambda/k| = O(k\xi_1) \text{ or } |\lambda/k| >> k\xi_1 \right]$ show that in this case $\arg \bar{\zeta}_{\xi_2} \sim 3\pi/2$. Since $\left| \bar{\zeta}_{\xi_2} \right| > O(k\eta_o)$, the asymptotic forms of the Hankel functions given in Section D.1 together with the continuation formula (Erdélyi et al, 1953)

$$H^{(2)}_{1/3}(z e^{\pi i}) = e^{\pi i/3} H^{(1)}_{1/3}(z) + H^{(2)}_{1/3}(z) ,$$

show that (E.35) becomes

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} \exp\left\{ -\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike} \right\} e^{5\pi i/6} \cdot \frac{\exp\left\{ \frac{\lambda_r}{k} \int_{s^{(r)}_{\xi_2}}^{1} ds \left(\frac{1-s}{s} \right)^{1/2} \right\}}{\left(\bar{\phi}(s_{\xi_2}) \right)^{1/2}} e^{i \frac{\sigma}{k} O(1)} ,$$

where we have set $\bar{\zeta}_{\xi_2} = \text{Re}\,\bar{\zeta}_{\xi_2} + i\,\text{Im}\,\bar{\zeta}_{\xi_2}$, noted that $\left| \text{Re}\,\bar{\zeta}_{\xi_2} \right| = O(\sigma/\lambda_r) \left| \text{Im}\,\bar{\zeta}_{\xi_2} \right|$, $\left| \text{Im}\,\bar{\zeta}_{\xi_2} \right| \sim \left| \bar{\zeta}_{\xi_2}(\lambda_r) \right|$, $\left| \bar{\zeta}_{\xi_2}(\lambda_r) \right| = \lambda_r/k\, O(1)$, and neglected the term containing

$$\exp\left\{ -\frac{\lambda_r}{k} \int_{s^{(r)}_{\xi_2}}^{1} ds \left(\frac{1-s}{s} \right)^{1/2} \right\}.$$

Thus

$$\left|\frac{F}{v_1'(\eta_o,\lambda)}\cdot X^{N_o(\xi)}\right| \lesssim \frac{1}{(k\xi_1)^{1/2}(k\xi_2)^{1/2}(k\eta_o)^{1/2}} \frac{e^{4N_o(\xi)\,\mathrm{Im}\sqrt{\lambda\eta_o}}}{e^{-2\,\mathrm{Im}\sqrt{\lambda\eta_o}}} .$$

$$\cdot\, O\left(\exp\left\{-\frac{\lambda_r}{k}\int_{s^{(r)}_{\xi_1}}^{s^{(r)}_{\xi_2}} ds\left(\frac{1-s}{s}\right)^{1/2}\right\}\right)$$

since $\mathrm{Im}\sqrt{\lambda} < 0$, $\lambda_r > 0$, $\lambda_r/k > O(k\eta_o)$, $s^{(r)}_{\xi_1} < s^{(r)}_{\xi_2}$ and not in a neighborhood of $s^{(r)}_{\xi_2}$. Therefore the integrand is exponentially small if $|\lambda| > k^2\xi_2$ and away from the transition region of $v_2(\xi_2, -\lambda)$.

The above range exhausts the possible sub-regions of C_5 and C_6, and thus completes the study of the intervals C_2 through C_6. In these intervals the integrand is either exponentially small and the integral is not contributing materially to the sum in equation (3.63), or the contributions are described by equations (3.76) (for C_4, and possibly C_3 and C_5), (3.117), (3.121) and (3.123). Examining these equations, we finally observe that

$$\lim_{(k\eta_o)\to\infty} R(\xi,\eta_o,\Xi,0) = 0 ,$$

as previously asserted.

3.3.2 Equivalent Integral Representation for the Source at (0, H)

If in the integral representation (2.6) we let

$$G(\xi,\eta_o,H,k,\lambda) = \frac{1}{2\pi i(2ik\eta_o)}\,\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) v_2(\xi,-\lambda)v_1(H,\lambda) , \qquad (3.126)$$

and again denote the path $-\infty - i\sigma$ to $\infty - i\sigma$ $(0<\sigma<k)$ by C, then (2.6) becomes

$$v_N(\xi,\eta_o,0,H) = \int_C d\lambda\,\frac{G(\xi,\eta_o,H,k,\lambda)}{v_1'(\eta_o,\lambda)} . \qquad (3.127)$$

Substituting (3.52) into (3.127) yields

$$v_N(\xi, \eta_0, 0, H) = \int_C d\lambda \, \frac{G(\xi, \eta_0, H, k, \lambda)}{g(\eta_0, k, \lambda)\left[1 - X(\eta_0, k, \lambda)\right]} \quad , \tag{3.128}$$

or more briefly

$$v_N(\xi, \eta_0, 0, H) = \int_C d\lambda \, \frac{G}{g(1-X)} \quad . \tag{3.128a}$$

Then the proof of Lemma 1 can be repeated to obtain Lemma 2. (The convergence of the integrals appearing in Lemma 2 is again shown in Appendix C).

Lemma 2: For all integral values of $M \geqslant 1$,

$$v_N(\xi, \eta_0, 0, H) - \sum_{n=0}^{M-1} \int_C d\lambda \, \frac{G}{g} \cdot X^n = \int_C d\lambda \, \frac{G}{g(1-X)} \cdot X^M \quad . \tag{3.128b}$$

We now consider, on the interval $0 < z < H$, the two equations

$$\bar{w}(z) \equiv \frac{\sqrt{H-z} + \sqrt{H}}{\sqrt{\eta_0 - z} + \sqrt{\eta_0}} = \frac{\left(\sqrt{\eta_0 - z} + \sqrt{\eta_0}\right)^{2n}}{\left(\sqrt{z}\right)^{2n}} \cdot \frac{\sqrt{\xi + z} - \sqrt{\xi}}{\sqrt{z}} \equiv v_n(z) \cdot \bar{u}(z) \; , \tag{3.129}$$

$$\bar{f}(z) \cdot v_n(z) \equiv \left(\sqrt{H-z} + \sqrt{H}\right)\left(\sqrt{\eta_0 - z} + \sqrt{\eta_0}\right) \cdot \frac{\left(\sqrt{\eta_0 - z} + \sqrt{\eta_0}\right)^{2n}}{\left(\sqrt{z}\right)^{2n}}$$

$$= \frac{z\sqrt{z}}{\sqrt{\xi + z} - \sqrt{\xi}} \equiv \bar{g}(z) \; , \tag{3.130}$$

together with the assumption that H is bounded away from η_0 and satisfies the relation

$$\frac{\eta_0}{H} - 1 \gg \frac{1}{(k\eta_0)^{2/3}} \quad .$$

This case can be studied by using the method employed in Section 3.3.1, and will be discussed immediately below. The same is not true when H is in a neighborhood of η_0 governed by

$$\frac{\eta_0}{H} - 1 = O\left(\frac{1}{(k\eta_0)^{2/3}}\right) .$$

This configuration is studied in Section 3.5.

Since $H < \eta_0$, we know from Appendix E (analysis of w(z), (E.54)) that on the given interval $\bar{w}(z)$ is a decreasing function of z. Similarly, we know from Appendix E (analysis of u(z), (E.54)) that on the given interval $\bar{u}(z)$ is an increasing function of z. It is obvious that $v_n(z)$ is a decreasing function of z on this same interval for all $n \geqslant 1$. Thus for n large enough, the product $v_n(z) \cdot \bar{u}(z)$ will be a decreasing function of z with minimum value

$$v_n(H) \cdot \bar{u}(H) = \frac{\left(\sqrt{\eta_0 - H} + \sqrt{\eta_0}\right)^{2n}}{\left(\sqrt{H}\right)^{2n}} \cdot \frac{\sqrt{\xi + H} - \sqrt{\xi}}{\sqrt{H}} .$$

But

$$\frac{\sqrt{\eta_0 - H} + \sqrt{\eta_0}}{\sqrt{H}} > 1 ,$$

thus $v_n(H)$, and with it $v_n(H) \cdot \bar{u}(H)$, increases with n. In particular, we can choose n large enough such that $v_n(z) \cdot \bar{u}(z)$ is a decreasing function of z on $0 < z < H$ with minimum value $v_n(H)\bar{u}(H) > 1$. On this same interval, the maximum value of $\bar{w}(z)$ approaches $\sqrt{H/\eta_0} < 1$. Therefore, there exists an n_3 such that for $n \geqslant n_3$ equation (3.129) has no solution in $0 < z < H$.

We also know from Appendix E (analysis of g(z), (E.64)) that $\bar{g}(z)$ is an increasing function of z. Again it is obvious that $\bar{f}(z) \cdot v_n(z)$ is a decreasing function of z on $0 < z < H$. Since $\bar{g}(0^+) \to 0$, equation (3.130) has one real solution in $0 < z < H$ if and only if $\bar{f}(H) \cdot v_n(H) < \bar{g}(H)$ or

$$\frac{\left(\sqrt{\eta_o - H} + \sqrt{\eta_o}\right)\left(\sqrt{\eta_o - H} + \sqrt{\eta_o}\right)^{2n}}{\left(\sqrt{H}\right)^{2n}} < \frac{H}{\sqrt{\xi + H} - \sqrt{\xi}} \quad . \tag{3.131}$$

But we can again chose n large enough so that this inequality cannot hold. In addition, we can choose n large enough so that

$$\frac{\left(\sqrt{\eta_o - H'} + \sqrt{\eta_o}\right)\left(\sqrt{\eta_o - H'} + \sqrt{\eta_o}\right)^{2n}}{\left(\sqrt{H'}\right)^{2n}} > \frac{H'}{\sqrt{\xi + H'} - \sqrt{\xi}} \tag{3.131a}$$

is valid for H' satisfying

$$H - O\left(\frac{1}{(k\eta_o)^{2/3}}\right) \leqslant H' \leqslant H + O\left(\frac{1}{(k\eta_o)^{2/3}}\right) \quad .$$

Hence, there exists an n_4 such that for $n \geqslant n_4$ equation (3.130) has no solution in $0 < z < H$ and the inequality (3.131a) is valid. Then for $n \geqslant N_1 = \max(n_3, n_4)$, the equations (3.129) and (3.130) have no solution in $0 < z < H$, and the inequality (3.131a) is valid.

Let $N_1(\xi)$ be the smallest possible N_1 defined above. Then according to Lemma 2, equation (3.128) can be replaced by

$$v_N(\xi, \eta_o, 0, H) - \sum_{n=0}^{N_1(\xi)-1} \int_C d\lambda\, \frac{G}{g} \cdot X^n = \int_C d\lambda\, \frac{G}{g(1-X)} \cdot X^{N_1(\xi)} \quad . \tag{3.132}$$

We can show (see discussion below) that for $k\eta_o \gg 1$, $kH \gg 1$, $k\xi \gg 1$, $\frac{\eta_o}{H} - 1 \gg 1/(k\eta_o)^{2/3}$, the integral defined by

$$R(\xi, \eta_o, 0, H) = \int_C d\lambda\, \frac{G}{g(1-X)} \cdot X^{N_1(\xi)} \tag{3.133}$$

satisfies $\lim_{(k\eta_o)\to\infty} R(\xi, \eta_o, 0, H) = 0$. This implies

$$v_N(\xi, \eta_o, 0, H) \sim \sum_{n=0}^{N_1(\xi)-1} \int_C d\lambda \, \frac{G}{g} \cdot X^n \, . \tag{3.134}$$

We call equation (3.134) the equivalent integral representation for $v_N(\xi, \eta_o, 0, H)$ if $k\eta_o \gg 1$, $kH \gg 1$, $k\xi \gg 1$, and

$$\frac{\eta_o}{H} - 1 \gg \frac{1}{(k\eta_o)^{2/3}} \, .$$

The details of the proof that

$$\lim_{(k\eta_o) \to \infty} R(\xi, \eta_o, 0, H) = 0$$

follow quite closely those of the proof that

$$\lim_{(k\eta_o) \to \infty} R(\xi, \eta_o, \Xi, 0) = 0 \, .$$

Because of this and the extreme length of the proof we do not present it. The necessary asymptotic forms and arguments have all been developed in Section 3.3.1. The only difference (as one might expect from the difference between the cases of Appendix E) is that now the integrand is exponentially small for the entire range $C_5 + C_6$ while the various arguments involving the stationary point equations and transition regions arise in the range $C_2 + C_3$.

3.4 Saddle Point Analysis

In this section we demonstrate how an asymptotic representation of the total field can be obtained by estimating the integrals occurring in the equivalent integral representations. The details of these estimations do not differ very much from those of Section 3.3 and Appendix E, where all the necessary asymptotic forms have been developed. We will refer to Section 3.3 or Appendix E whenever possible, and discuss only the differences in the estimations. We find that each of the integrals can be estimated by a saddle point integration. We show that the first integral ($n = 0$) is approximately equal to twice the incident wave. For each remaining integral ($n = 1, 2, 3, \ldots$), we consider the field point to be "far" from the source if the saddle point equations have an approximate solution. This criterion, different for each n, is precisely defined in Section 3.4.2 (for $n = 0$, see Appendix E, equation (E.55b)). These approximate solutions yield terms in the asymptotic representation of the total field which, by a comparison of their phase with the geometrical path length, can be identified as the reflected rays of geometric optics. We consider explicitly the source at $(\Xi, 0)$. The same results can be established for the source at $(0, H)$, provided we redefine the concept of field point "far" from the source (see the end of Appendix E for $n = 0$).

3.4.1 Saddle Point Integration

We complete the proof of Theorem 5 by first writing equation (3.61) as

$$v_N(\xi, \eta_o, \Xi, 0) \sim \int_C d\lambda \frac{F}{g} + \sum_{n=1}^{N_o(\xi)-1} \int_C d\lambda \frac{F}{g} \cdot X^n , \tag{3.135}$$

and considering the first term separately. Since $g(\eta_o, k, \lambda)$ has no zeros in the region $|\text{Im}\,\lambda| < k$ (Klante, 1959), and F/g is exponentially decreasing for $|\lambda| \to \infty$ in the same region (Appendix C), the first term is equal to

$$\int_{-\infty}^{\infty} d\lambda \frac{F}{g} .$$

Using the definitions of $F(\xi_1, \xi_2, \eta_o, k, \lambda)$ and $g(\eta_o, k, \lambda)$, we see that

$$J^{(0)}(\xi_1, \xi_2, \eta_o, k) \equiv \int_{-\infty}^{\infty} d\lambda \, \frac{F}{g} = \frac{1}{2\pi i(2ik\eta_o)} \int_{-\infty}^{\infty} d\lambda \, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \cdot$$

$$\cdot \frac{e^{3\pi i/2} v_1(\xi_1, -\lambda) v_2(\xi_2, -\lambda)}{e^{-\pi\lambda/2k} v_2'(\eta_o e^{-\pi i}, -\lambda)} \quad . \tag{3.136}$$

We estimate $J^{(0)}(\xi_1, \xi_2, \eta_o, k)$ by setting

$$J^{(0)}(\xi_1, \xi_2, \eta_o, k) = \sum_{j=1}^{5} \int_{C_j} d\lambda \, \frac{F}{g} \; , \tag{3.136a}$$

where the C_j have been defined in connection with equation (E.4a) (with $\eta = \eta_o$), and examining each integral separately. We have already mentioned that the integrand is exponentially small over C_1 and C_5. Thus, the integrals over these arcs do not contribute materially to the approximation of $J^{(0)}(\xi_1, \xi_2, \eta_o, k)$.

On C_2, we can use equation (E.5) and (E.6) for $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$, respectively. The various asymptotic forms of $v_2'(\eta_o e^{-\pi i}, -\lambda)$ (depending on the position of λ with respect to the turning point $\lambda = -k^2\eta_o^2$) are given by equations (3.89) and (3.99). The arguments of Appendix E and Section 3.3.1 then show that the integrand is exponentially small over C_2. This implies that the integral over C_2 also does not contribute materially to the estimation of $J^{(0)}(\xi_1, \xi_2, \eta_o, k)$.

To estimate the integrand over C_3, we substitute (E.8) for $v_1(\xi_1, -\lambda)$, (E.9) for $v_2(\xi_2, -\lambda)$, and (3.71) for $v_2'(\eta_o e^{-\pi i}, -\lambda)$. We find

$$\frac{F(\xi_1, \xi_2, \eta_o, k, \lambda)}{g(\eta_o, k, \lambda)} \sim 2\left[I_{\Xi}^{+}(\lambda, \xi_1, \xi_2, \eta_o) + I_{\Xi}^{-}(\lambda, \xi_1, \xi_2, \eta_o)\right] \quad , \tag{3.137}$$

where $I_{\Xi}^{+}(\lambda, \xi_1, \xi_2, \eta)$ and $I_{\Xi}^{-}(\lambda, \xi_1, \xi_2, \eta)$ are given by (E.12) and (E.13), respectively. Hence, the argument of Appendix E implies

$$\int_{C_3} d\lambda \, \frac{F}{g} = \frac{1}{\sqrt{\xi_1 \xi_2}} \, O\left(\frac{1}{(k\eta_o)^{1/2}}\right) \quad . \tag{3.138}$$

Therefore, equation (3.136a) can be written as

$$J^{(0)}(\xi_1, \xi_2, \eta_o, k) \sim \int_{C_4} d\lambda \, \frac{F}{g} \; . \tag{3.139}$$

On C_4, $v_1(\xi_1, -\lambda)$ and $v_2(\xi_2, -\lambda)$ are governed by (E.34) and (E.35) respectively, while $g(\eta_o, k, \lambda)$ is given by (3.109). Using these equations, we observe that on C_4

$$\frac{F}{g} \sim 2C(k, \xi_1, \xi_2, \eta_o) \left[C\left(-\frac{\lambda}{2ik}\right)\right]^2 e^{\pi i/4} \bar{\psi}(s_{\xi_1}) Ai(-\bar{\sigma}_{\xi_1}) e^{\pi i/12} \bar{\psi}(s_{\xi_2}) Ai(-\bar{\sigma}_{\xi_2}) \cdot$$

$$\cdot (-i)^{1/2} \cdot \frac{e^{-i\zeta_{\eta_o}}}{\left(\phi(s_{\eta_o})\right)^{1/2}} , \tag{3.140}$$

or

$$\frac{F}{g} \sim 2\left[\text{Asymptotic Representation of } I_{\Xi}(\lambda, \xi_1, \xi_2, \eta_o)\right] , \tag{3.140a}$$

where $I_{\Xi}(\lambda, \xi_1, \xi_2, \eta)$ is defined by equation (E.3). Consequently, the arguments of Appendix E apply and equation (3.139) becomes

$$J^{(0)}(\xi_1, \xi_2, \eta_o, k) \sim \frac{-2\, e^{-ik\sqrt{(\xi_1+\xi_2+\eta_o)^2 - 4\xi_1\xi_2}}}{\sqrt{(\xi_1+\xi_2+\eta_o)^2 - 4\xi_1\xi_2}} , \tag{3.141}$$

provided

$$\frac{1}{4\xi_2\eta_o} \left| \xi_2 - (\xi_1 + \eta_o) \right|^2 \gg \frac{1}{(k\eta_o)^{2/3}} ,$$

and

$$J^{(0)}(\xi_1, \xi_2, \eta_0, k) \sim 2\int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta_0) , \tag{3.141a}$$

if

$$\frac{1}{4\xi_2\eta_0}\left|\xi_2 - (\xi_1 + \eta_0)\right|^2 = O\left(\frac{1}{(k\eta_0)^{2/3}}\right) .$$

The latter integral calls to mind formula (E.42) from Appendix E. But for those ξ_1, ξ_2, and η_0 for which (3.141a) holds, I_{Ξ} (in (E.42)) has no stationary point on C_4'', and therefore (E.42) becomes

$$\frac{-e^{-ikR_{\Xi}}}{R_{\Xi}} \sim \int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta_0) . \tag{3.141b}$$

Further, according to the definition of R_{Ξ} (Buchholz, 1953, Chapter 4, equation 3), $R_{\Xi} = \sqrt{(\xi + \Xi + \eta_0)^2 - 4\xi\Xi}$. Therefore equations (3.141a) and (3.141) imply

$$J^{(0)}(\xi_1, \xi_2, \eta_0, k) \sim \frac{-2e^{-ikR_{\Xi}}}{R_{\Xi}} , \tag{3.142}$$

for all values of ξ and Ξ.

We suggest a geometric interpretation of the occurence of the distinct representations (3.141) and (3.141a). We do this by showing that if $\xi < \Xi$, the relation $\xi_2 = \xi_1 + \eta_0$, which now becomes $\Xi = \xi + \eta_0$, implies that the line joining the two points $(\Xi, 0)$ and (ξ, η_0) is perpendicular to the surface of the paraboloid of revolution. Such a ray path gives rise to a caustic, and in analogy with the analysis of other diffraction problems it is reasonable to conjecture that a caustic should be associated with the condition under which (3.141a) holds. Thus let (ξ, η_0) be a point on the surface of the parabola $x^2 = 4\eta_0(\eta_0 + z)$, with x coordinate given by $+2\sqrt{\xi\eta_0}$, and z coordinate given by $\xi - \eta_0$ (see Section 1.3). Then the slope of the line perpendicular to (ξ, η_0) is

$$-1\Big/\left(\frac{dx}{dz}\right)_{(\xi,\eta_0)} = -\sqrt{\xi/\eta_0} \quad .$$

If this line intersects the z-axis at $(\Xi, 0)$, then necessarily $\Xi > \xi$, and

$$2\sqrt{\xi\eta_0} - 0 = -\sqrt{\xi/\eta_0}\,(\xi - \eta_0 - \Xi) \; .$$

Hence $2\eta_0 = \Xi + \eta_0 - \xi$, or $\Xi = \xi + \eta_0$. The significance of this geometrical relation is further discussed in Section 3.4.3. If $\xi > \Xi$, the equality $\xi_2 = \xi_1 + \eta_0$ becomes $\xi = \Xi + \eta_0$. This has no special geometric interpretation; it only reflects the inability of the Green's function to distinguish between the two configurations.

We now consider the terms in equation (3.135) which have the form

$$J^{(n)}(\xi_1, \xi_2, \eta_0, k) \equiv \int_C d\lambda \, \frac{F}{g} \cdot X^n \, , \qquad n = 1, 2, 3, \ldots \; . \tag{3.143}$$

We can estimate $J^{(n)}(\xi_1, \xi_2, \eta_0, k)$ by setting

$$J^{(n)}(\xi_1, \xi_2, \eta_0, k) = \sum_{j=1}^{7} \int_{C_j} d\lambda \, \frac{F}{g} \cdot X^n \, , \tag{3.143a}$$

where the C_j have been defined in connection with equation (3.63), and examining each integral separately. We know from Appendix C that the integrand is exponentially small over C_1 and C_7. Hence, these integrals do not contribute materially to the approximation. On C_4, equation (3.137) together with the argument of Section 3.3.1 yields

$$\int_{C_4} d\lambda \, \frac{F}{g} \cdot X^n \sim \frac{1}{\sqrt{\xi_1 \xi_2}} (\xi_2/\xi_1)^{\sigma/2k} \frac{O(1)}{(k\eta_0)^{1/2}} \, \frac{1}{(k\eta_0)^{\sigma/2k}} \left[\frac{1}{(k\eta_0)^{\sigma/k}} \right]^{n-1} . \tag{3.144}$$

Finally, the arguments of Section 3.3.1 show that the integrand is exponentially small over C_2 and C_3. Therefore, the integrals over C_2 and C_3 do not contribute materially to the estimation, and using (3.144) we see that (3.143a) can be written as

$$J^{(n)}(\xi_1, \xi_2, \eta_0, k) \sim \int_{C_5+C_6} d\lambda \frac{F}{g} \cdot X^n \ . \qquad (3.145)$$

To estimate the integral over C_5+C_6, we first observe that the arguments of Section 3.3.1 can be applied directly to (3.145) to obtain

$$J^{(n)}(\xi_1, \xi_2, \eta_0, k) \sim \int_{(C_5+C_6)^1} d\lambda \frac{F}{g} \cdot X^n + \int_{(C_5+C_6)''} d\lambda \frac{F}{g} \cdot X^n \ , \qquad (3.146)$$

where $(C_5+C_6)^1 = \left\{\lambda \epsilon C \middle| 1 \ll |\lambda/k| < k\xi_1 \text{ and away from the transition region of } v_1(\xi_1, -\lambda)\right\}$, and in $(C_5+C_6)^1$

$$\frac{F}{g} \cdot X^n \sim \frac{2\sqrt{2\pi}\, C(k, \xi_1, \xi_2, \eta_0)(-i)^{1/2} e^{\pi i \frac{n}{2}}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_0})\right]^{1/2}} \cdot \left[e^{i\zeta_n^+(\lambda)} + ie^{i\zeta_n^-(\lambda)}\right] \ , \qquad (3.147)$$

with

$$\zeta_n^+(\lambda) = \bar{\zeta}_{\xi_1}(\lambda) - \bar{\zeta}_{\xi_2}(\lambda) - \zeta_{\eta_0}(\lambda) - 2n\zeta_{\eta_0}(\lambda) \ , \qquad (3.148a)$$

$$\zeta_n^-(\lambda) = -\bar{\zeta}_{\xi_1}(\lambda) - \bar{\zeta}_{\xi_2}(\lambda) - \zeta_{\eta_0}(\lambda) - 2n\zeta_{\eta_0}(\lambda) \ , \qquad (3.148b)$$

while in the range $(C_5+C_6)''$ (defined in Section 3.3.1)

$$\frac{F}{g} \cdot X^n \sim \frac{2\sqrt{2\pi}\, C\left(-\frac{\lambda}{2ik}\right) e^{\pi i/4} \bar{\psi}(s_{\xi_1}) Ai(-\bar{\sigma}_{\xi_1})(-i)^{1/2} e^{\pi i \frac{n}{2}} e^{i\zeta_n(\lambda)}}{\left[\bar{\phi}(s_{\xi_2})\phi(s_{\eta_0})\right]^{1/2}} \ , \qquad (3.149)$$

with

$$\zeta_n(\lambda) = -\bar{\zeta}_{\xi_2}(\lambda) - \zeta_{\eta_o}(\lambda) - 2n\zeta_{\eta_o}(\lambda) \ . \tag{3.150}$$

For these values of n, the definition of $N_o(\xi)$ implies that at least one of the following three equations has a solution in the indicated interval:

(i) (3.55) in $0 < z < \xi_1$,

(ii) (3.56) in $0 < z < \xi_1$,

$$\text{(iii)} \quad \frac{\sqrt{z}}{\sqrt{\xi_2 - z} + \sqrt{\xi_2}} = \frac{\sqrt{\eta_o + z} - \sqrt{\eta_o}}{\sqrt{z}} \cdot \frac{\left(\sqrt{\eta_o + z} - \sqrt{\eta_o}\right)^{2n}}{\left(\sqrt{z}\right)^{2n}}$$

$$\text{in } \xi_1 - O\left(\frac{1}{(k\eta_o)^{2/3}}\right) \leqslant z \leqslant \xi_1 + O\left(\frac{1}{(k\eta_o)^{2/3}}\right) \ . \tag{3.151}$$

Equation (3.151) results from seeking real solutions of $d\zeta_n(\lambda)/d\lambda = 0$ in the transition region of $v_1(\xi_1, -\lambda)$. We shall not explicitly consider these solutions, but will discuss their significance in Section 3.4.3. Thus, let us assume that the triplet (ξ, Ξ, η_o) is such that possibility (iii) cannot be true. Then the argument of Section 3.3.1 yields

$$\int_{(C_5+C_6)''} d\lambda \frac{F}{g} \cdot X^n = \frac{1}{\xi_1^{\frac{1}{2}(1+\frac{\sigma}{k})} \xi_2^{\frac{1}{2}(1-\frac{\sigma}{k})}} O\left(\frac{1}{(k\eta_o)^{1/3}}\right) \ .$$

which, when inserted into (3.146) gives

$$J^{(n)}(\xi_1, \xi_2, \eta_o, k) \sim \int_{(C_5+C_6)^1} d\lambda \frac{F}{g} \cdot X^n \ . \tag{3.152}$$

In order to analyze equation (3.152), we write it as

$$J^{(n)}(\xi_1,\xi_2,\eta_o,k) \sim \int_{(C_5+C_6)^1} d\lambda \frac{F^+}{g}\cdot X^n + \int_{(C_5+C_6)^1} d\lambda \frac{F^-}{g}\cdot X^n , \tag{3.152a}$$

where

$$\frac{F^+}{g}\cdot X^n \equiv \frac{2\sqrt{2\pi}\; C(k,\xi_1,\xi_2,\eta_o)(-i)^{1/2}\, e^{\pi i \frac{n}{2}}\, e^{i\zeta_n^+(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\phi(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}} , \tag{3.147a}$$

$$\frac{F^-}{g}\cdot X^n \equiv \frac{2\sqrt{2\pi}\; C(k,\xi_1,\xi_2,\eta_o)(-i)^{1/2}\, e^{\pi i \frac{n}{2}}\, e^{i\zeta_n^-(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}} . \tag{3.147b}$$

If equation (3.55) ((3.56)) does not have a solution in $0 < z < \xi_1$, then the arguments of Section 3.3.1 imply

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^+}{g}\cdot X^n \left(\int_{(C_5+C_6)^1} d\lambda \frac{F^-}{g}\cdot X^n\right) =$$

$$= \frac{1}{\xi_1^{\frac{1}{2}(1+\frac{\sigma}{k})}\,\xi_2^{\frac{1}{2}(1-\frac{\sigma}{k})}}\; O\left(\frac{1}{(k\eta_o)^{1/2}}\right). \tag{3.153}$$

However if equation (3.55) ((3.56)) does have a solution $z_n^+(z_n^-)$ in $0 < z < \xi_1$, then $\zeta_n^+(\lambda)$ $\left(\zeta_n^-(\lambda)\right)$ has a saddle point at $\lambda_n^+ = k^2 z_n^+$ $\left(\lambda_n^- = k^2 z_n^-\right)$. Thus, the integral

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^+}{g}\cdot X^n \quad \left(\int_{(C_5+C_6)^1} d\lambda \frac{F^-}{g}\cdot X^n\right)$$

can be estimated by deforming the contour $(C_5+C_6)^1$ to a new contour $\tilde{C}$ shown in Fig. 3-1. This is possible because the fact that $g(\eta_0, k, \lambda)$ has no zeros in $|\mathrm{Im}\,\lambda| < k$ also means that $X(\eta_0, k, \lambda)$ has no poles in the same region. The arguments of Section 3.3 apply directly to the nonreal horizontal pieces of $\tilde{C}$. To estimate the integral over the nonreal vertical pieces, we first note that the estimates of $|F^+/g|$ $(|F^-/g|)$ and $|X|$ on these vertical pieces are obtainable directly from Section 3.3 by replacing σ with σ', where $0 \leqslant \sigma' \leqslant \sigma$. These estimates combined with the fact that the vertical pieces have length equal to $O(k)$, yields

$$\frac{1}{\sqrt{\xi_1 \xi_2}}\, O\left(\frac{1}{(k\eta_0)^{1/2}}\right)$$

as the estimate of the integral over the nonreal vertical pieces. But since the real piece of $\tilde{C}$ contains a stationary point $\lambda_n^+ = k^2 z_n^+$ of $\frac{F^+}{g} \cdot X^n$, we find that

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^+}{g} \cdot X^n = \int_{\tilde{C}} d\lambda \frac{F^+}{g} \cdot X^n$$

$$\sim \frac{2 \cdot 2\pi\, C(k, \xi_1, \xi_2, \eta_0)(-i)^{1/2} e^{\pi i \frac{n}{2}} e^{i\zeta_n^+(\lambda_n^+)} e^{i\frac{\pi}{4}\,\mathrm{sign}\,\zeta_n^{+''}(\lambda_n^+)}}{\left(\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_0})\right]^{1/2}\right)_{\lambda=\lambda_n^+} \cdot \sqrt{\left|\zeta_n^{+''}(\lambda_n^+)\right|}}, \quad (3.154)$$

provided $\zeta_n^{+''}(\lambda_n^+) \neq 0$. There may be more than one solution of (3.56). If we call the solutions $z_n^-(i)$, the corresponding result for

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^-}{g} \cdot X^n \quad \text{is}$$

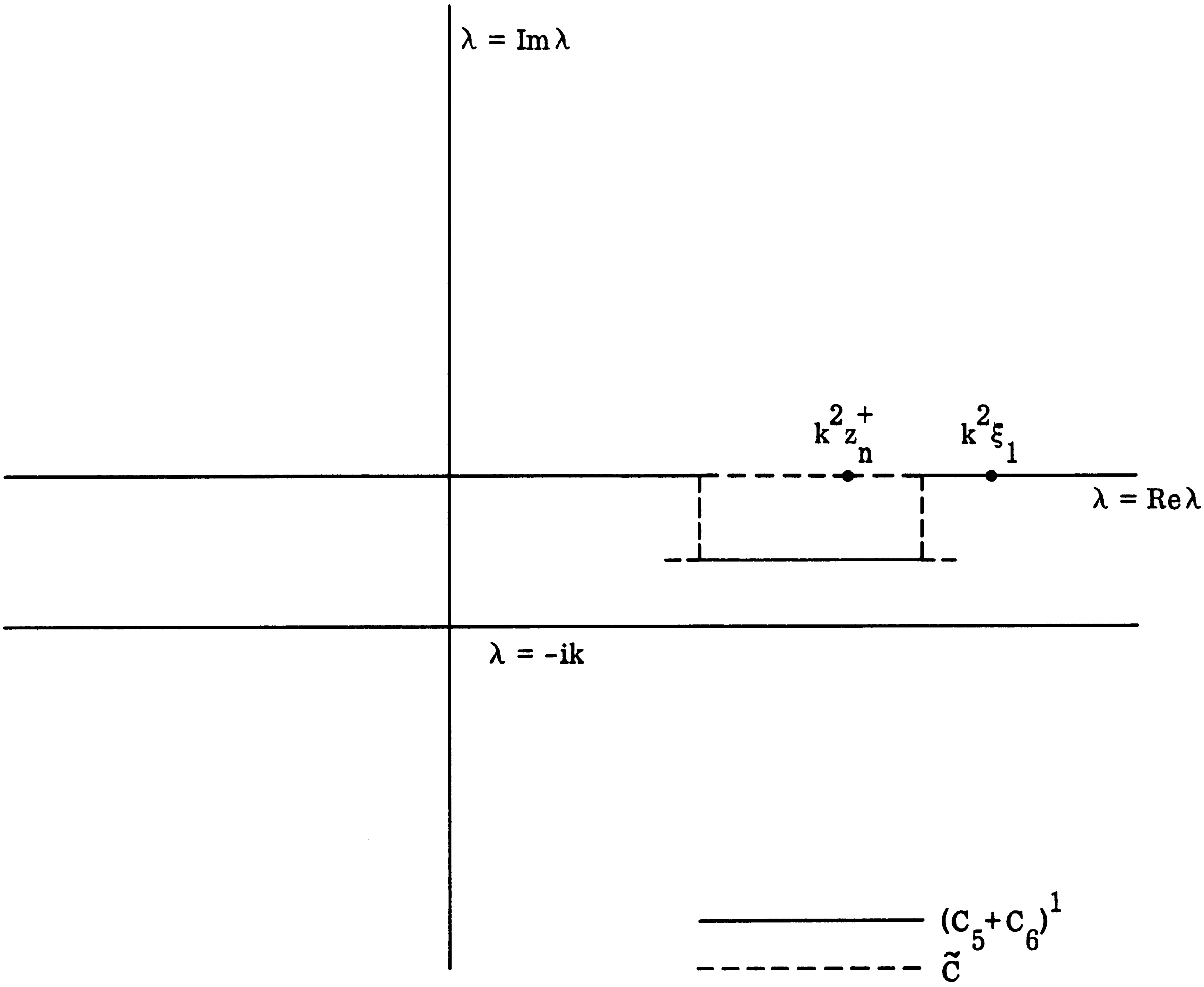

FIG. 3-1: CONTOURS OF INTEGRATION IN THE λ-PLANE.

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^-}{g} \cdot X^n = \int_{\tilde{C}} d\lambda \frac{F^-}{g} \cdot X^n$$

$$\sim \sum_i \frac{2 \cdot 2\pi\, C(k, \xi_1, \xi_2, \eta_o)(-i)^{1/2} e^{\pi i \frac{n}{2}} e^{i\zeta_n^-\left(\lambda_n^-(i)\right)} e^{i\frac{\pi}{4}\,\mathrm{sign}\,\zeta_n^{-''}\left(\lambda_n^-(i)\right)}}{\left(\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]^{1/2}\right)_{\lambda=\lambda_n^-(i)} \cdot \sqrt{\left|\zeta_n^{-''}\left(\lambda_n^-(i)\right)\right|}}, \tag{3.155}$$

provided $\zeta_n^{-''}\left(\lambda_n^-(i)\right) \neq 0$. The existence of solutions $\lambda_n^+ \left(\lambda_n^-(i)\right)$ such that $\zeta_n^{+''}(\lambda_n^+) = 0$ $\left(\zeta_n^{-''}\left(\lambda_n^-(i)\right) = 0\right)$ will be discussed in Section 3.4.3. However, such solutions will not be explicitly considered. Thus we assume that the triplet (ξ, Ξ, η_o) is such that $\zeta_n^{+''}(\lambda_n^+) \neq 0$, $\zeta_n^{-''}\left(\lambda_n^-(i)\right) \neq 0$, for each i. Then defining the terms of equations (3.154) and (3.155) as $E^+(\lambda_n^+)$, and $E^-\left(\lambda_n^-(i)\right)$, respectively, and substituting equations (3.153), (3.154), and (3.155) into (3.152a), we observe that

$$J^{(n)}(\xi_1, \xi_2, \eta_o, k) \sim E^+(\lambda_n^+), \text{ if possibility (ii) is not true}, \tag{3.156a}$$

$$J^{(n)}(\xi_1, \xi_2, \eta_o, k) \sim \sum_i E^-\left(\lambda_n^-(i)\right), \text{ if possibility (i) is not true}, \tag{3.156b}$$

$$J^{(n)}(\xi_1, \xi_2, \eta_o, k) \sim E^+(\lambda_n^+) + \sum_i E^-\left(\lambda_n^-(i)\right), \text{ if both possibilities are true.} \tag{3.156c}$$

This completes the proof of Theorem 5.

3.4.2 Approximate Solutions of the Saddle Point Equations

The exact form of the representations (3.154) and (3.155) depend, ,of course, on the exact solutions of (3.55) and (3.56). However, even for $n = 1$, these equations appear too difficult to solve exactly, and we investigate approximate solutions of the type discussed in Appendix E. Thus we assume that

$\Xi/\eta_0 = O(1)$ and $\Xi/\xi \ll 1$ (here the condition implied by the symbols $\ll$ is understood to be independent of $k\eta_0$); this represents a configuration of considerable physical interest, the configuration studied in Section 3.3.1 and subsequently. Let us consider (3.55) and (3.56) for $n=1$. They are

$$w(z) \equiv \frac{\sqrt{\xi_1 - z} + \sqrt{\xi_1}}{\sqrt{\xi_2 - z} + \sqrt{\xi_2}} = \frac{\sqrt{\eta_0 + z} - \sqrt{\eta_0}}{\sqrt{z}} \cdot \frac{\left(\sqrt{\eta_0 + z} - \sqrt{\eta_0}\right)^2}{\left(\sqrt{z}\right)^2} \equiv u_1(z) , \tag{3.157}$$

$$f(z) \equiv \left(\sqrt{\xi_1 - z} + \sqrt{\xi_1}\right)\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right) = z \cdot \frac{\left(\sqrt{z}\right)^3}{\left(\sqrt{\eta_0 + z} - \sqrt{\eta_0}\right)^3} \equiv g_1(z) , \tag{3.158}$$

with $\xi_1 = \Xi$, $\xi_2 = \xi$ for the given configuration. The condition that (3.157) has a solution ((3.57)) becomes

$$\sqrt{\xi_1} < \left(\sqrt{\xi_2 - \xi_1} + \sqrt{\xi_2}\right) \cdot \frac{\left(\sqrt{\eta_0 + \xi_1} - \sqrt{\eta_0}\right)^3}{\left(\sqrt{\xi_1}\right)^3} . \tag{3.159}$$

Examining (3.157) and (3.158), we see that since $g_1(0^+) \rightarrow \infty$, equation (3.158) always has a solution $\hat{z}_1^-$ for small values of z, while $u_1(0^+) \rightarrow 0$ implies that equation (3.157) has a solution $\hat{z}_1^+$ for small values of z provided (3.159) holds. But for fixed Ξ, (3.159) will eventually hold for all ξ larger than some fixed ξ_0. Since we wish our results to be valid for these values of ξ, we assume a priori that the ratio ξ_2/ξ_1 is such that (3.159) is true. To define this mathematically, we first write equation (3.159) as

$$1 < \left(\sqrt{\frac{\xi_2}{\xi_1} - 1} + \sqrt{\frac{\xi_2}{\xi_1}}\right)\left(\sqrt{\frac{\eta_0}{\xi_1} + 1} - \sqrt{\frac{\eta_0}{\xi_1}}\right)^3 , \tag{3.159a}$$

or since $\xi_2/\xi_1 \gg 1$, as

$$1 < 2\sqrt{\frac{\xi_2}{\xi_1}}\left(\sqrt{\frac{\eta_o}{\xi_1}+1} - \sqrt{\frac{\eta_o}{\xi_1}}\right)^3 \quad . \tag{3.159b}$$

But $\eta_o/\xi_1 = O(1)$ implies

$$\sqrt{\frac{\eta_o}{\xi_1}+1} - \sqrt{\frac{\eta_o}{\xi_1}} = O(1) \ .$$

Thus if we consider $(\xi/\Xi)^{1/6} \gg 1$ added to the defining conditions, we observe

$$1 < 2^{1/3}(\xi_2/\xi_1)^{1/6}\left(\sqrt{\frac{\eta_o}{\xi_1}+1} - \sqrt{\frac{\eta_o}{\xi_1}}\right), \tag{3.159c}$$

which implies that (3.159b) holds and consequently that the inequality (3.159) is valid. This inequality not only guarantees the existence of a solution to equation (3.157) but also that equation (3.158) has only one solution. This is so because: (1) $f(z)$ is decreasing from $4\sqrt{\xi_1\xi_2}$ on $0 < z < \xi_1$; (2) $g_1(z)$ is either decreasing from ∞, or decreasing from ∞ to a minimum and then increasing, on $0 < z < \xi_1$; and (3) (3.159) can be written as $f(\xi_1) > g(\xi_1)$. In addition, (3.159c) implies that equation (3.151) cannot have a solution for $n = 1$, since the condition for a solution to this equation can now ($\xi_2 \gg \xi_1$) be written as

$$\frac{\sqrt{\xi_1}}{2\sqrt{\xi_2}} = \frac{\left(\sqrt{\eta_o+\xi_1} - \sqrt{\eta_o}\right)^3}{\left(\sqrt{\xi_1}\right)^3}\left[1+O\left(\frac{1}{(k\eta_o)^{1/3}}\right)\right].$$

Finally, the solutions $\hat{z}_1^+$ and $\hat{z}_1^-$ can be approximated by writing (3.157) and (3.158) as

$$\sqrt{\frac{\xi_1}{\xi_2}}\left[1+O(z)\right] = \left(\frac{\sqrt{z}}{2\sqrt{\eta_o}}\right)^3\left[1+O(z)\right] \quad , \tag{3.157a}$$

$$4\sqrt{\xi_1\xi_2}\left[1+O(z)\right] = z\cdot\left(\frac{2\sqrt{\eta_o}}{\sqrt{z}}\right)^3\left[1+O(z)\right] \ . \tag{3.158a}$$

Therefore

$$\hat{z}_1^+ = 4\eta_o(\xi_1/\xi_2)^{1/3}\left[1+O\left((\xi_1/\xi_2)^{1/3}\right)\right] \quad , \tag{3.160}$$

$$\hat{z}_1^- = \frac{4\eta_o^3}{\xi_1^2}(\xi_1/\xi_2)\left[1+O(\xi_1/\xi_2)\right] \quad . \tag{3.161}$$

We now evaluate the result of the stationary phase integration if we use the stationary points given by (3.160) and (3.161). For positive real values of λ, equations (3.148a) and (3.148b) become

$$\zeta_n^+(\lambda) = \Phi_o^+(\lambda) - 2n\frac{\lambda}{k}\Phi_o(s_{\eta_o}) \; , \tag{3.162a}$$

$$\zeta_n^-(\lambda) = \Phi_o^-(\lambda) - 2n\frac{\lambda}{k}\Phi_o(s_{\eta_o}) \; , \tag{3.162b}$$

where $\Phi_o^+(\lambda)$, $\Phi_o^-(\lambda)$ and $\Phi_o(s_{\eta_o})$ are given by equations (E.46), (E.47), and (E.49), respectively. Then the computations in Appendix E (equations (E.50) through (E.53)) show that

$$\frac{d^2\zeta_n^+(\lambda)}{d\lambda^2} = \frac{\left[\bar{\phi}(s_{\xi_2})i\phi(s_{\eta_o}) - \bar{\phi}(s_{\xi_1})i\bar{\phi}(s_{\eta_o}) + (1+2n)\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\right]}{2ik\lambda\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]} \; , \tag{3.163a}$$

$$\frac{d^2\zeta_n^-(\lambda)}{d\lambda^2} = \frac{\left[-\bar{\phi}(s_{\xi_2})i\phi(s_{\eta_o}) - \bar{\phi}(s_{\xi_1})i\phi(s_{\eta_o}) + (1+2n)\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\right]}{2ik\lambda\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_{\eta_o})\right]} \; . \tag{3.163b}$$

In addition, these computations give

$$\zeta_n^+(\lambda_n^+) = k\left[\sqrt{\xi_1(\xi_1 - z_n^+)} - \sqrt{\xi_2(\xi_2 - z_n^+)} - \sqrt{\eta_o(\eta_o + z_n^+)} - 2n\sqrt{\eta_o(\eta_o + z_n^+)}\right] \tag{3.164a}$$

$$\zeta_n^-(\lambda_n^-) = k\left[-\sqrt{\xi_1(\xi_1 - z_n^-)} - \sqrt{\xi_2(\xi_2 - z_n^-)} - \sqrt{\eta_o(\eta_o + z_n^-)} - 2n\sqrt{\eta_o(\eta_o + z_n^-)}\right] \; . \tag{3.164b}$$

But

$$\left(\bar{\phi}(s_{\xi_1})\right)_{z=\hat{z}_1^+} = \sqrt{\frac{\xi_1-\hat{z}_1^+}{\xi_1}} = 1+O\left((\xi_1/\xi_2)^{1/3}\right) ,$$

$$\left(\bar{\phi}(s_{\xi_2})\right)_{z=\hat{z}_1^+} = \sqrt{\frac{\xi_2-\hat{z}_1^+}{\xi_2}} = 1+O\left((\xi_1/\xi_2)^{4/3}\right) ,$$

$$\left(i\phi(s_{\eta_o})\right)_{z=\hat{z}_1^+} = \sqrt{\frac{\eta_o+\hat{z}_1^+}{\eta_o}} = 1+O\left((\xi_1/\xi_2)^{1/3}\right) ,$$

$$\left(\bar{\phi}(s_{\xi_1})\right)_{z=\hat{z}_1^-} = \sqrt{\frac{\xi_1-\hat{z}_1^-}{\xi_1}} = 1+O(\xi_1/\xi_2) ,$$

$$\left(\bar{\phi}(s_{\xi_2})\right)_{z=\hat{z}_1^-} = \sqrt{\frac{\xi_2-\hat{z}_1^-}{\xi_2}} = 1+O\left((\xi_1/\xi_2)^{2}\right) ,$$

$$\left(i\phi(s_{\eta_o})\right)_{z=\hat{z}_1^-} = \sqrt{\frac{\eta_o+\hat{z}_1^-}{\eta_o}} = 1+O(\xi_1/\xi_2) .$$

Substituting these equations into equations (3.163) and (3.164), we see that

$$\left(\frac{d^2\zeta_1^+(\lambda)}{d\lambda^2}\right)_{\lambda=\hat{\lambda}_1^+} = \frac{1}{8k^3\eta_o}\cdot(\xi_2/\xi_1)^{1/3}\left[1+O\left((\xi_1/\xi_2)^{1/3}\right)\right] , \qquad (3.165a)$$

$$\left(\frac{d^2\zeta_1^-(\lambda)}{d\lambda^2}\right)_{\lambda=\hat{\lambda}_1^-} = \frac{\xi_1^2}{8k^3\eta_o^3}\cdot(\xi_2/\xi_1)\left[(-1+2\cdot 1)+O(\xi_1/\xi_2)\right] , \qquad (3.165b)$$

$$\zeta_1^+(\hat{\lambda}_1^+) = -k\xi_2+k\xi_1-k\eta_o-k(2\eta_o)+k\eta_o O\left((\xi_1/\xi_2)^{1/3}\right) , \qquad (3.166a)$$

$$\zeta_1^-(\hat{\lambda}_1^-) = -k\xi_2-k\xi_1-k\eta_o-k(2\eta_o)+k\eta_o O\ (\xi_1/\xi_2) . \qquad (3.166b)$$

Therefore, equations (3.154) and (3.155) yield

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^+}{g} \cdot X^1 \sim \frac{-2}{\sqrt{\xi_1 \xi_2}} (\xi_1/\xi_2)^{1/6} e^{-ik(\xi_2 - \xi_1 + \eta_o + 2\eta_o)} e^{\pi i/2} ,$$

which for $\xi_1 = \Xi$, $\xi_2 = \xi$ becomes

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^+}{g} \cdot X^1 \sim \frac{-2}{\sqrt{\Xi} \cdot \sqrt{\xi}} (\Xi/\xi)^{1/6} e^{-ik(\xi - \Xi + \eta_o + 2\eta_o)} e^{\pi i/2} , \tag{3.167a}$$

and

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^-}{g} \cdot X^1 \sim \frac{2\eta_o}{(\xi_1)^{3/2}(\xi_2)^{1/2}} (\xi_1/\xi_2)^{1/2} e^{-ik(\xi_2 + \xi_1 + \eta_o + 2\eta_o)} ,$$

which for $\xi_1 = \Xi$, $\xi_2 = \xi$ becomes

$$\int_{(C_5+C_6)^1} d\lambda \frac{F^-}{g} \cdot X^1 \sim \frac{2\eta_o}{(\Xi)^{3/2} \cdot \sqrt{\xi}} (\Xi/\xi)^{1/2} e^{-ik(\xi + \Xi + \eta_o + 2\eta_o)} . \tag{3.167b}$$

Inserting equations (3.167) into equation (3.152a), we obtain

$$J^{(1)}(\Xi, \xi, \eta_o, k) \sim \frac{-2}{\sqrt{\Xi} \cdot \sqrt{\xi}} (\Xi/\xi)^{1/6} e^{-ik(\xi - \Xi + \eta_o + 2\eta_o)} e^{\pi i/2} \left[1 + O\left((\Xi/\xi)^{1/3}\right)\right] ,$$

or since $(\Xi/\xi)^{1/6} \ll 1$

$$J^{(1)}(\Xi, \xi, \eta_o, k) \sim \frac{-2}{\sqrt{\Xi} \cdot \sqrt{\xi}} (\Xi/\xi)^{1/6} e^{-ik(\xi - \Xi + \eta_o + 2\eta_o)} e^{\pi i/2} . \tag{3.168}$$

In order to establish Theorem 6, we now investigate the asymptotic representation of $J^{(n)}(\Xi, \xi, \eta_o, k)$ for $n > 1$. We need only follow the argument for

$n = 1$. The part of that argument essential to the case $n = 1$ was the appropriate condition for (3.57) to hold and for equation (3.55) to have a solution, namely that $(\xi/\Xi)^{1/6} \gg 1$. As above, we observe that if $n > 1$, the corresponding sufficient condition is $(\xi_2/\xi_1)^{1/2(2n+1)} \gg 1$. Under this condition, as for $n = 1$, equation (3.56) has only one solution in $0 < z < \xi_1$, equation (3.151) does not have any solution, and the two solutions $\hat{z}_n^+$ and $\hat{z}_n^-$ are approximated by

$$\hat{z}_n^+ = 4\eta_o(\xi_1/\xi_2)^{1/2n+1}\left[1 + O\left((\xi_1/\xi_2)^{1/2n+1}\right)\right], \tag{3.169}$$

$$\hat{z}_n^- = 4\eta_o(\eta_o^2/\xi_1^2)^{1/2n-1}(\xi_1/\xi_2)^{1/2n-1}\left[1 + O\left((\xi_1/\xi_2)^{1/2n-1}\right)\right]. \tag{3.170}$$

Consequently, the stationary phase approximation gives

$$J^{(n)}(\Xi, \xi, \eta_o, k) \sim \frac{-2}{\sqrt{\Xi}\cdot\sqrt{\xi}}(\Xi/\xi)^{\frac{1}{2(2n+1)}} e^{-ik(\xi - \Xi + \eta_o + 2n\eta_o)} e^{\pi i \frac{n}{2}} . \tag{3.171}$$

This result enables us to obtain an asymptotic representation of the field $v_N(\xi, \eta_o, \Xi, 0)$. We first substitute definitions (3.136) and (3.143) into equation (3.135) and find

$$v_N(\xi, \eta_o, \Xi, 0) \sim \sum_{n=0}^{N_o(\xi)-1} J^{(n)}(\Xi, \xi, \eta_o, k) .$$

Using equation (3.142), we observe that

$$v_N(\xi, \eta_o, \Xi, 0) \sim \frac{-2\,e^{-ikR_\Xi}}{R_\Xi} + \sum_{n=1}^{N_o(\xi)-1} J^{(n)}(\Xi, \xi, \eta_o, k) . \tag{3.172}$$

Then if $n_1(\xi)$ is such that $(\xi/\Xi)^{1/2(2n+1)} \gg 1$ for $n < n_1(\xi)$, we can substitute equations (3.168) and (3.171) into (3.172) to find

$$v_N(\xi,\eta_0,\Xi,0) \sim \frac{-2e^{-ikR_\Xi}}{R_\Xi} - \frac{2}{\sqrt{\Xi}\cdot\sqrt{\xi}} \sum_{n=1}^{n_1(\xi)} (\Xi/\xi)^{\frac{1}{2(2n+1)}} e^{-ik\psi(\Xi,\xi,\eta_0,n)} e^{\pi i \frac{n}{2}} +$$

$$+ \sum_{n_1(\xi)+1}^{N_0(\xi)-1} J^{(n)}(\Xi,\xi,\eta_0,k) \,, \qquad (3.173)$$

with $\psi(\Xi,\xi,\eta_0,n) = \xi - \Xi + \eta_0 + 2n\eta_0$. This completes the proof of Theorem 6.

3.4.3 Interpretation of Results

We conclude Section 3.4 with a partial interpretation of equation (3.173). Included in this interpretation is a discussion of the asymptotic representation which arises from a real solution of $d\zeta_n(\lambda)/d\lambda = 0$ (defined by equation (3.151)) in the transition region of $v_1(\xi_1,-\lambda)$, as well as a discussion of the asymptotic representation which arises from a real solution λ_n^- of $d\zeta_n^-(\lambda)/d\lambda = 0$, with the additional property that

$$\left(\frac{d^2\zeta_n^-(\lambda)}{d\lambda^2}\right)_{\lambda=\lambda_n^-} = 0.$$

We need not consider the latter case for $\zeta_n^+(\lambda)$ —— we shall show that solutions of this type are not possible. The model for the interpretation is suggested by the first term of equation (3.173). Thus we try to identify the successive terms with the reflected rays of geometric optics. To do this in general is very difficult. If $n > n_1(\xi)$, we cannot even hope for such an identification without additional analysis of the stationary point equations. Such additional analysis is quite complicated even for $J^{(1)}(\Xi,\xi,\eta_0,k)$ if $\xi/\Xi = O(1)$, $\Xi/\eta_0 = O(1)$, and we cannot use the approximate solutions given by equations (3.160) and (3.161). For $n \leqslant n_1(\xi)$, the geometric analysis involved in the identification grows rapidly with n. Consequently, we investigate only the $n=1$ term.

We first prove Theorem 7, i.e. for a triplet (Ξ, ξ, η_0) such that $\Xi/\eta_0 = O(1)$, $(\Xi/\xi)^{1/6} \ll 1$, the approximate phase of $J^{(1)}(\Xi, \xi, \eta_0, k)$ which depends explicitly on distance, $\psi(\Xi, \xi, \eta_0, 1)$, is equal to the approximate path length of any ray that travels from $(\Xi, 0)$ to (ξ, η_0) via a single reflection. A ray of this type is shown in Fig. 3-2. The point (ξ', η_0) must be determined in terms of (ξ_1, η_0) and (ξ_2, η_0). If $\tan\psi_1$ denotes the slope of the incident ray from $(\Xi, 0)$ to (ξ', η_0), $\tan\psi_2$ denotes the slope of the normal to $\eta = \eta_0$ at (ξ', η_0), and $\tan\psi_3$ denotes the slope of the reflected ray from (ξ', η_0) to (ξ, η_0), then the condition that the angle of incidence equal the angle of reflection implies

$$\psi_3 = 2\psi_2 - \psi_1 ,$$

or

$$\tan\psi_3 = \frac{2\tan\psi_2 - \tan\psi_1 + \tan\psi_1 \tan^2\psi_2}{1 - \tan^2\psi_2 + 2\tan\psi_2 \tan\psi_1} .$$

Since $\tan\psi_1 = x/z - \xi_1$, $\tan\psi_2 = -x/2\eta_0$, $\tan\psi_3 = x_2 - x/z_2 - z$, this equation can be written as

$$\frac{x_2 - x}{z_2 - z} = \frac{-\dfrac{x}{\eta_0} - \dfrac{x}{z - \xi_1} + \dfrac{x^3}{4\eta_0^2(z - \xi_1)}}{1 - \dfrac{x^2}{4\eta_0^2} - \dfrac{x^2}{\eta_0(z - \xi_1)}} . \tag{3.174}$$

But

$$z = \frac{x^2}{4\eta_0} - \eta_0 , \qquad z_2 = \frac{x_2^2}{4\eta_0} - \eta_0 ,$$

thus equation (3.174) becomes

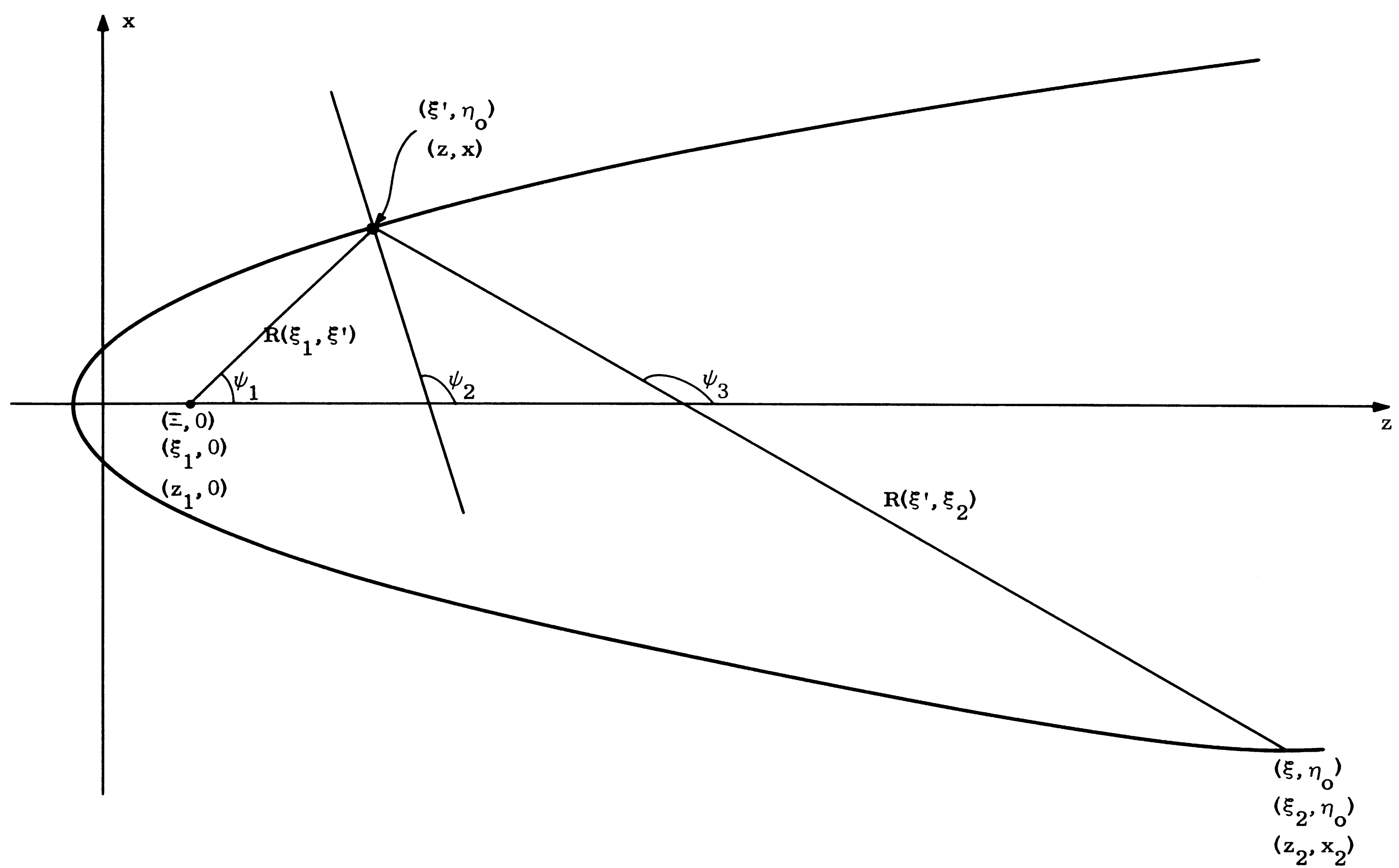

FIG. 3-2: A RAY TRAVELING FROM $(\Xi, 0)$ TO (ξ, η_o) VIA ONE REFLECTION.

$$\left[4\eta_o^2\left(\frac{x^2}{4\eta_o}-\eta_o-\xi_1\right)-\left(\frac{x^2}{4\eta_o}-\eta_o-\xi_1\right)x^2-4\eta_o x^2\right]=$$

$$=\frac{1}{4}(x_2+x)\left[-4\left(\frac{x^2}{4\eta_o}-\eta_o-\xi_1\right)x-4\eta_o x+\frac{x^3}{\eta_o}\right],$$

which reduces to

$$\frac{x^4}{4\eta_o}+2\eta_o x^2+\xi_1 x_2 x+4\eta_o^2(\xi_1+\eta_o)=0\ . \tag{3.175}$$

The solution of equation (3.175) gives the coordinate ξ' ($\xi'=x^2/4\eta_o$) in terms of ξ_1 and ξ_2.

We do not solve equation (3.175) exactly, but instead estimate the order of the solution. Since we are seeking a $\xi'>\Xi$, the two possibilities for a solution are $x=O(\xi_1)$ or $x\gg\xi_1$. If $x=O(\xi_1)$, the terms $A=x^4/4\eta_o$, $B=2\eta_o x^2$, $C=\xi_1 x_2 x$, $D=4\eta_o^2(\xi_1+\eta_o)$, have orders

$$A=O(\xi_1^3),\quad B=O(\xi_1^3),\quad D=O(\xi_1^3),\quad C=O(\xi_1^{5/2}\xi_2^{1/2})\ .$$

This shows that C dominates A, B, and D, which implies that $x=O(\xi_1)$ cannot be a solution. Hence $x\gg\xi_1$, which shows that $A\gg O(\xi_1)x^2$, and so dominates $B=O(\xi_1)x^2$ and $D=O(\xi_1^3)$. Therefore the solution x obeys $x^4/4\eta_o\approx\xi_1|x_2|x$ (x_2 negative), or $x^3/4\eta_o\approx\xi_1|x_2|$. Consequently,

$$x^3=O(\xi_1^2 x_2)\Rightarrow(\xi')^{3/2}=O(\xi_1\cdot\xi_2^{1/2})\Rightarrow\xi'=O(\xi_1^{2/3}\cdot\xi_2^{1/3})\ ,$$

and

$$\xi'/\xi_1=O\left((\xi_2/\xi_1)^{1/3}\right)\gg1,\qquad \xi_2/\xi'=O\left((\xi_2/\xi_1)^{2/3}\right)\gg1\ .$$

These conditions imply that the distance $R(\xi_1, \xi')$ from $(\xi_1, 0)$ to (ξ', η_o), as well as the distance $R(\xi', \xi_2)$ from (ξ', η_o) to (ξ_2, η_o), can be written as

$$R(\xi_1, \xi') \equiv \sqrt{(\xi_1+\xi'+\eta_o)^2 - 4\xi_1\xi'} = \xi'+\eta_o-\xi_1+\eta_o O(\xi_1/\xi') ,$$

$$R(\xi', \xi_2) \equiv \sqrt{(\xi_2-\xi')^2+4\eta_o(\sqrt{\xi_2}+\sqrt{\xi'})^2} = \xi_2-\xi'+2\eta_o+\eta_o O\left((\xi'/\xi_2)^{1/2}\right) .$$

Thus, the approximate path length is given by

$$d_1 = \xi'+\eta_o-\xi_1+(\xi_2-\xi'+2\eta_o) = \xi_2-\xi_1+\eta_o+2\eta_o ,$$

or

$$d_1 = \xi-\Xi+\eta_o+2\eta_o ,$$

which is equal to $\psi(\Xi, \xi, \eta_o, 1)$. This ends the proof of Theorem 7.

The latter part of the above derivation may be extended to $\psi(\Xi, \xi, \eta_o, n)$. Suppose we assume that a ray leaves the source at $(\Xi, 0)$ and travels to a point on the surface (ξ, η_o) via n reflections at the points $(\xi^{(n)}, \eta_o)$, where the angle of incidence of the initial ray and each subsequent reflected ray is small. This latter condition is equivalent to assuming that the ratios $\xi^{(0)}/\xi^{(1)}$, $\sqrt{\xi^{(j)}/\xi^{(j+1)}}$ are small for $1 \leqslant j \leqslant n$ ($\xi^{(0)} = \Xi$, $\xi^{(n+1)} = \xi$). But then the distance $R(\xi^{(0)}, \xi^{(1)})$ from $(\xi^{(0)}, 0)$ to $(\xi^{(1)}, \eta_o)$ as well as the distance $R(\xi^{(j)}, \xi^{(j+1)})$ from $(\xi^{(j)}, \eta_o)$ to $(\xi^{(j+1)}, \eta_o)$, can be written as

$$R(\xi^{(0)}, \xi^{(1)}) \equiv \sqrt{(\xi^{(1)}+\xi^{(0)}+\eta_o)^2 - 4\xi^{(1)}\xi^{(0)}} = \xi^{(1)}+\eta_o-\xi^{(0)}+\eta_o O(\xi^{(0)}/\xi^{(1)}) ,$$

$$R(\xi^{(j)}, \xi^{(j+1)}) \equiv \sqrt{(\xi^{(j+1)}-\xi^{(j)})^2+4\eta_o\left(\sqrt{\xi^{(j+1)}}+\sqrt{\xi^{(j)}}\right)}$$

$$= \xi^{(j+1)}-\xi^{(j)}+2\eta_o+\eta_o O\left(\sqrt{\xi^{(j)}/\xi^{(j+1)}}\right) .$$

Thus, the approximate path length traveled by such a ray is given by

$$d_n = (\xi^{(1)} - \xi^{(0)} + \eta_0) + \sum_{j=1}^{n} (\xi^{(j+1)} - \xi^{(j)} + 2\eta_0) ,$$

and this "telescopes" to yield

$$d_n = \xi^{(n+1)} - \xi^{(0)} + \eta_0 + 2n\eta_0 = \xi - \Xi + \eta_0 + 2n\eta_0 ,$$

which is equal to $\psi(\Xi, \xi, \eta_0, n)$.

To complete the proof that for $n \leqslant n_1(\xi)$, $\psi(\Xi, \xi, \eta_0, n)$ is equal to the approximate path length of any ray that travels from $(\Xi, 0)$ to (ξ, η_0) via n reflections, we would need to show, as we did for $n = 1$, that the condition $(\xi/\Xi)^{1/2(2n+1)} \gg 1$ implies that the ratios $\xi^{(0)}/\xi^{(1)}$, $\sqrt{\xi^{(j)}/\xi^{(j+1)}}$ are small for $1 \leqslant j \leqslant n$. This will not be attempted here.

The above equality between path length and phase which depends explicitly on distance is an important step in the identification of the $n = 1$ term of equation (3.173) with the corresponding reflected rays of geometric optics. To further complete this identification, one ought to show that the amplitude of the $n = 1$ term can be obtained by applying the optical form of the principle of conservation of energy to a bundle of rays which eminate from the source at $(\Xi, 0)$, strike the surface at (ξ', η_0), and are reflected to the point (ξ, η_0) (Levy and Keller, 1959). The geometry of the paraboloid of revolution suggests that this is a difficult problem, and its resolution will not be attempted here. Finally, the additional phase factor which appears in the $n = 1$ term (as well as the other terms for $n \leqslant n_1(\xi)$) must be accounted for. This again is a difficult problem, we simply discuss the nature of the difficulty involved.

In most cases when one uses a geometrical interpretation of the asymptotic representation of a total or scattered field, one finds that there exist field points where the ray picture is no longer valid. These occur either as isolated points, curves, or surfaces, called caustics. They are discussed by Levy and Keller (1959) and by Kline and Kay (1965). An example of caustic surfaces in the case of diffraction by a paraboloid of revolution is the envelope of all the rays which are reflected n times. For each n there is one such caustic. A caustic line in the paraboloid of revolution occurs when a ray strikes the surface and is reflected back along its own incident path through the source. Then the caustic is the path this reflected ray travels after passing through the source. Therefore, the portion of the axis from the source to ∞ is a caustic, while other caustic lines are generated by rays, discussed in Section 3.3.1, perpendicular to the surface. The intersections of the caustic surfaces and lines with the surface of the paraboloid of revolution form caustic circles on this surface.

For the source at (0, H), we immediately note, from the failure of the analytic geometry problem of Section 3.3.1 to have a solution intersecting the z-axis at $z = -H$: no ray leaving the source can be incident perpendicular to the surface. Therefore, the only caustic line in this case is that part of the axis from the source to ∞. However, the caustic surfaces formed by the rays reflected n times remain.

The behavior of the field at a caustic is characterized by an enhancement of its amplitude by a factor dependent upon ka, where a is the characteristic dimension of the body under consideration, compared to the amplitude of the field given by the geometric rays. Thus for the paraboloid of revolution, we expect the amplitude of the field at a caustic to depend on $(k\eta_0)^b$, with $b > 0$. This behavior cannot be determined by stationary contributions of the type discussed if $\zeta_n^{+''}(\lambda_n^+) \neq 0$ and $\zeta_n^{-''}(\lambda_n^-) \neq 0$, since equations (3.163) together with equations (3.154) and (3.155) show that these contributions have no dependence on $k\eta_0$. Therefore we investigate equations (3.163a) and (3.163b) with regard to zeros.

The forms below equation (3.163a) indicate that the zeros of this equation correspond to zeros of

$$\left(\sqrt{\frac{\xi_2 - z}{\xi_2}} - \sqrt{\frac{\xi_1 - z}{\xi_1}}\right)\sqrt{\frac{\eta_0 + z}{\eta_0}} + (1+2n)\sqrt{\frac{\xi_2 - z}{\xi_2}} \cdot \sqrt{\frac{\xi_1 - z}{\xi_1}} \quad . \tag{3.176}$$

But the function $(\xi_2 - z)/(\xi_1 - z)$ has a derivative equal to

$$\frac{\xi_2 - \xi_1}{(\xi_1 - z)^2} > 0 .$$

Thus it is increasing on $0 < z < \xi_1$, and

$$\frac{\xi_1}{\xi_2}\left[\frac{\xi_2 - z}{\xi_1 - z}\right] > 1$$

on this interval. Hence

$$\frac{\xi_2 - z}{\xi_2} > \frac{\xi_1 - z}{\xi_1}$$

on $0 < z < \xi_1$, and (3.176) has no zeros in $0 < z < \xi_1$. Consequently, the integral

$$\int_{(C_5 + C_6)^1} d\lambda \, \frac{F^+}{g} \cdot X^n$$

cannot determine behavior at a caustic.

Similarly, the zeros of equation (3.163b) correspond to zeros of

$$-\sqrt{\frac{\eta_o + z}{\eta_o}}\left(\sqrt{\frac{\xi_2 - z}{\xi_2}} + \sqrt{\frac{\xi_1 - z}{\xi_1}}\right) + (1+2n)\sqrt{\frac{\xi_2 - z}{\xi_2}} \cdot \sqrt{\frac{\xi_1 - z}{\xi_1}} \quad ,$$

or the solutions of

$$\frac{1}{\sqrt{\frac{\xi_1 - z}{\xi_1}}} + \frac{1}{\sqrt{\frac{\xi_2 - z}{\xi_2}}} = (1+2n)\sqrt{\frac{\eta_o}{\eta_o + z}} \quad , \tag{3.177}$$

in $0 < z < \xi_1$. Since the right side is a decreasing function of z on $0 < z < \xi_1$, with maximum value $(1+2n)$, and the left side is an increasing function with maximum value approaching ∞, and minimum value approaching 1, on $0 < z < \xi_1$, equation (3.177) has exactly one real solution in this interval. Thus the condition that (3.177) and (3.56) have a simultaneous solution, would correspond to the relation between the parameters ξ, Ξ, η_0 at the intersection of a caustic with the surface of the paraboloid of revolution.

There is another possibility for the description of the behavior at a caustic. For positive real values of λ, equation (3.150) becomes

$$\zeta_n(\lambda) = -\frac{\lambda}{k}\bar{\Phi}_0(s_{\xi_2}) - \frac{\lambda}{k}\Phi_0(s_{\eta_0}) - 2n\frac{\lambda}{k}\Phi_0(s_{\eta_0}) , \tag{3.178}$$

with $\bar{\Phi}(s_{\xi_2})$ and $\Phi_0(s_{\eta_0})$ given by equations (E.48) and (E.49), respectively. The computations in Appendix E (equations (E.50) through (E.53)) show that

$$\frac{d^2\zeta_n(\lambda)}{d\lambda^2} = \frac{\left[-i\phi(s_{\eta_0}) + (1+2n)\bar{\phi}(s_{\xi_2})\right]}{2ik\lambda\left[\bar{\phi}(s_{\xi_2})\phi(s_{\eta_0})\right]} . \tag{3.179}$$

Then from the form of equation (3.149) and the analysis of Section 3.3.1 leading to equation (3.123), we observe that a solution of (3.151), z_n, with $\zeta_n''(\lambda_n) \neq 0$, gives rise to a term with an amplitude which depends on $(k\eta_0)^{1/6}$. Hence the condition

$$\frac{\sqrt{\xi_1}}{\sqrt{\xi_2-\xi_1} + \sqrt{\xi_2}} = \frac{\sqrt{\eta_0+\xi_1} - \sqrt{\eta_0}}{\sqrt{\xi_1}} \cdot \frac{\left(\sqrt{\eta_0+\xi_1} - \sqrt{\eta_0}\right)^{2n}}{\left(\sqrt{\xi_1}\right)^{2n}} ,$$

would describe the intersection of a caustic with the surface of the paraboloid of revolution.

An additional property of caustics is the appearance of a phase jump of $\pi/2$ in the geometric optics field as the field point passes through a caustic. This phase jump has not been explained geometrically, but appears in the asymptotic representation associated with the given problem. This is the origin of the factor $e^{\pi in/2}$ which appears in the $n \leq n_1(\xi)$ terms of equation (3.173).

The approximate analysis of Section 3.4.2 is fruitful precisely because, for a given n, it corresponds to field points which are well away from not only the nth caustic line but also from the region of influence of the nth caustic surface. This is not true for an n such that $n > n_1(\xi)$ in equation (3.173). Thus a physical description of the statement (which we have defined mathematically) that a field point (ξ, η_0) is "far" from the source is that it be so far that it is far from the caustic lines and the region of influence of the caustic surfaces corresponding to the integers $n < n_1(\xi)$. For other field points, we most likely need a careful numerical study to obtain any information from the saddle point analysis.

3.5 Whispering Gallery Waves

In this section we prove Theorem 8, that is we investigate the source at (0, H) where H is in a neighborhood of η_0 governed by

$$\frac{\eta_0}{H} - 1 = O\left(\frac{1}{(k\eta_0)^{2/3}}\right) \quad \text{or} \quad \frac{\eta_0}{H} - 1 = \frac{h}{(k\eta_0)^{2/3}} \quad (h = O(1)) .$$

This implies that the $N_1(\xi)$ of equation (3.134) is increasing with $k\eta_0$. Therefore, the use of the representation (3.134) is subject to the question of the convergence of the series therein. We do not consider this question. Instead, we investigate the residue series directly. We show that there exist residues, exponentially small if

$$\frac{\eta_0}{H} - 1 \gg \frac{1}{(k\eta_0)^{2/3}} ,$$

which have an amplitude dependence of $(k\eta_0)^{1/6}$. The phase of these residues which depends on distance is approximately equal to the arc length from the tip of the paraboloid to the field point (ξ, η_0) under consideration. Moreover, for field points (ξ, η_0) such that $\xi/\eta_0 \gg 1$, the distance-dependent amplitude of these residues is approximately proportional to $1/\rho$, where ρ is the radius of the cylindrical cross section of the paraboloid of revolution. Therefore, these residue terms can be interpreted as cylindrical waves which are traveling along the surface of the paraboloid of revolution. We shall call these waves "whispering gallery waves" after similarly behaving waves of other concave surface diffraction problems (Section 1.1).

3.5.1 Asymptotic Form of Residue Terms

If $\eta = \eta_o$, then the residue series (3.2) becomes

$$v_N(\xi, \eta_o, 0, H) = \frac{1}{2ik\eta_o} \sum_{p=1}^{\infty} \Gamma\left(\frac{\Lambda_p}{2ik} + \frac{1}{2}\right) \frac{v_2(\xi, -\Lambda_p) v_1(H, \Lambda_p)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_p}} \tag{3.180}$$

We have shown (Section3.2) that the zeros given by equation (3.34) give rise to residues which are exponentially small. Thus (3.180) can be written as

$$v_N(\xi, \eta_o, 0, H) \sim \frac{1}{2ik\eta_o} \sum_{\Lambda_p < Mk} \Gamma\left(\frac{\Lambda_p}{2ik} + \frac{1}{2}\right) \frac{v_2(\xi, -\Lambda_p) v_1(H, \Lambda_p)}{\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_p}} , \tag{3.181}$$

for some $M > 0$. We define the parameter H_1 by the following conditions: H_1 has the dimensions of η_o,

$$\frac{\eta_o}{H_1} - 1 > O\left(\frac{1}{(k\eta_o)^{2/3}}\right) ,$$

$-k^2 H_1$ is not a zero of $v_1'(\eta_o, \lambda)$. Then we further decompose equation (3.181) into

$$v_N(\xi,\eta_o,0,H) \sim \frac{1}{2ik\eta_o} \sum_{\Lambda_p<-k^2H_1} \Gamma\left(\frac{\Lambda_p}{2ik}+\frac{1}{2}\right) \frac{v_2(\xi,-\Lambda_p)v_1(H,\Lambda_p)}{\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=\Lambda_p}} +$$

$$+ \frac{1}{2ik\eta_o} \sum_{-k^2H_1<\Lambda_p<Mk} \Gamma\left(\frac{\Lambda_p}{2ik}+\frac{1}{2}\right) \frac{v_2(\xi,-\Lambda_p)v_1(H,\Lambda_p)}{\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=\Lambda_p}} . \quad (3.182)$$

In this section we consider only the sum

$$v_N^*(\xi,\eta_o,0,H_1) = \frac{1}{2ik\eta_o} \sum_{\Lambda_p<-k^2H_1} \Gamma\left(\frac{\Lambda_p}{2ik}+\frac{1}{2}\right) \frac{v_2(\xi,-\Lambda_p)v_1(H,\Lambda_p)}{\left(\frac{d}{d\lambda}v_1'(\eta_o,\lambda)\right)_{\lambda=\Lambda_p}} . \quad (3.183)$$

The zeros Λ_p are given by equation (3.27) (in order to avoid confusion of notation, we replace the $\sum_p$ of (3.27) by ω_p). The asymptotic form of $v_2(\xi,-\Lambda_p)$ is governed by equation (3.43) with $\zeta_\xi^{(p)}$ given by (3.42). The asymptotic forms of the Whittaker functions in equation (3.183) which depend on H and η_o have also been derived in Section 3.1 and 3.2. The function $v_1(H,\lambda)$ is governed by equation (3.12) (for $\eta = H$), with $\bar{\zeta}_H$ given by (3.17) (for $\eta = H$). Since $\bar{\zeta}_{\eta_o}$ is also given by (3.17) (for $\eta = \eta_o$), we observe that

$$\frac{d\bar{\zeta}_\eta}{d(2ik\eta)} = -\frac{\lambda}{k}\left(\frac{s_\eta - 1}{s_\eta}\right)^{1/2} \cdot \frac{1}{2\lambda/ik} = -i\left(\frac{s_\eta - 1}{s_\eta}\right)^{1/2} \cdot \frac{1}{2} ,$$

and as a result equation (3.41) for $\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_p}$ can be replaced by

$$\left(\frac{d}{d\lambda} v_1'(\eta_o, \lambda)\right)_{\lambda=\Lambda_p} \sim (2ik\eta_o)^{-1/2} \frac{C\left(\frac{\Lambda_p}{2ik}\right)}{\sqrt{2\pi}} e^{\pi i/4} e^{-\pi \frac{\Lambda_p}{2k}} \bar{\psi}(s_{\eta_o}^{(p)}) \left\{-Ai(-\omega_p)\right\} \cdot$$

$$\cdot -i\left(\frac{s_{\eta_o}^{(p)}-1}{s_{\eta_o}^{(p)}}\right)^{1/2} \cdot \frac{1}{2}\left(\frac{d\bar{\zeta}_{\eta_o}}{d\lambda}\right)_{\lambda=\Lambda_p} . \quad (3.184)$$

Now by (3.17) and the definition of s_{η_o},

$$\frac{d\bar{\zeta}_{\eta_o}}{d\lambda} = -\frac{1}{k}\int_1^{s_{\eta_o}} \left(\frac{s-1}{s}\right)^{1/2} ds + \frac{1}{k}\left[s_{\eta_o}(s_{\eta_o}-1)^{1/2}\right] . \quad (3.185)$$

However in this range of λ, $s_{\eta_o} - 1 = O\left(1/(k\eta_o)^{2/3}\right)$, thus equations (3.18) and (3.19) are valid (for $\eta = \eta_o$), and equation (3.185) can be written as

$$\frac{d\bar{\zeta}_{\eta_o}}{d\lambda} \sim \frac{1}{k}\left[s_{\eta_o}(s_{\eta_o}-1)^{1/2}\right] . \quad (3.185a)$$

Moreover, equation (3.27) implies

$$s_{\eta_o}^{(p)} - 1 \sim \frac{\omega_p}{(k\eta_o)^{2/3}} ;$$

Consequently,

$$\left(\frac{s_{\eta_o}^{(p)}-1}{s_{\eta_o}^{(p)}}\right)^{1/2} \cdot \frac{1}{2}\left(\frac{d\bar{\zeta}_{\eta_o}}{d\lambda}\right)_{\lambda=\Lambda_p} \sim \frac{1}{2k} \cdot \frac{\omega_p}{(k\eta_o)^{2/3}} . \quad (3.186)$$

Then using equation (3.186) together with the fact that equations (3.18) and (3.19) imply $\bar{\psi}(s_{\eta_o}^{(p)}) \sim (2/3)^{1/6}$, we find that (3.184) becomes

$$\left(\frac{d}{d\lambda} v_1'(\eta_o,\lambda)\right)_{\lambda=\Lambda_p} \sim (2ik\eta_o)^{-1/2} \frac{C\left(\frac{\Lambda_p}{2ik}\right)}{\sqrt{2\pi}} e^{\pi i/4} e^{-\pi\frac{\Lambda_p}{2k}} (2/3)^{1/6} \left\{-\omega_p Ai(\omega_p)\right\} \cdot$$

$$\cdot \frac{-i}{2k(k\eta_o)^{2/3}} \qquad (3.187)$$

In order to estimate $s_H^{(p)}$

$$\left(s_H^{(p)} \equiv \frac{2ikH}{4\Lambda_p/2ik}\right) ,$$

we write it as

$$s_H^{(p)} = -\frac{k^2 H}{\Lambda_p} = -\frac{k^2\eta_o}{\Lambda_p} \cdot \frac{H}{\eta_o} ,$$

or

$$s_H^{(p)} = \left[1 + \frac{\omega_p}{(k\eta_o)^{2/3}} + O\left(\frac{1}{(k\eta_o)^{4/3}}\right)\right] \cdot \left[1 - \frac{h}{(k\eta_o)^{2/3}} + O\left(\frac{1}{(k\eta_o)^{4/3}}\right)\right],$$

which implies that

$$s_H^{(p)} - 1 \sim \frac{\omega_p - h}{(k\eta_o)^{2/3}} .$$

Therefore, equation (3.18) and (3.19) are valid (for $\eta = H$). This shows that $\bar{\psi}(s_H^{(p)}) \sim (2/3)^{1/6}$, and

$$\bar{\sigma}_H^{(p)} = \left(\frac{3}{2}\bar{\zeta}_H^{(p)}\right)^{2/3} \sim \left(\frac{-\Lambda_p}{k}(s_H^{(p)} - 1)^{3/2}\right)^{2/3} \sim \omega_p - h .$$

Using these results in (3.12), and performing a Taylor expansion of the Airy function about $\omega_p - h$, we obtain

$$v_1(H, \Lambda_p) \sim (2ikH)^{-1/2} \frac{C\left(\frac{\Lambda_p}{2ik}\right)}{\sqrt{2\pi}} e^{\pi i/4} e^{-\pi \frac{\Lambda_p}{2k}} (2/3)^{1/6} Ai(\omega_p - h) . \qquad (3.188)$$

Then inserting (3.187), (3.188) and equation (3.43) for $v_2(\xi, -\Lambda_p)$ into equation (3.183), and using the definition of $\phi(s_\xi^{(p)})$, we find

$$v_N^*(\xi, \eta_o, 0, H_1) \sim (k\eta_o)^{1/6} e^{-\pi i/4} \left(\frac{\pi}{\xi\eta_o}\right)^{1/2} \left(\frac{\xi}{\xi+\eta_o}\right)^{1/4} \sum_{\Lambda_p < -k^2 H_1} \cdot \frac{Ai(-\omega_p + h)}{(-\omega_p)Ai(-\omega_p)} e^{-i\zeta_\xi^{(p)}} , \qquad (3.189)$$

or, upon defining the dimensionless parameter ω_{H_1} by $k(\eta_o - H_1) = (k\eta_o)^{1/3}\omega_{H_1}$,

$$v_N^*(\xi, \eta_o, 0, H_1) \sim (k\eta_o)^{1/6} e^{-\pi i/4} \left(\frac{\pi}{\xi\eta_o}\right)^{1/2} \left(\frac{\xi}{\xi+\eta_o}\right)^{1/4} \sum_{\omega_p < \omega_{H_1}} \cdot \frac{Ai(-\omega_p + h)}{(-\omega_p)Ai(-\omega_p)} e^{-i\zeta_\xi^{(p)}} . \qquad (3.189a)$$

3.5.2 Interpretation of Results

We first note that the residues of equation (3.183) are exponentially small if

$$\frac{\eta_o}{H} - 1 \gg \frac{1}{(k\eta_o)^{2/3}} .$$

This follows because now $H < H_1$, $s_H = -k^2 H/\lambda < 1$ for $\lambda < -k^2 H_1$, and λ away from the transition region of $v_1(H,\lambda)$ implies that $v_1(H,\lambda)$ is governed by the decreasing exponential form of the Airy function in equation (3.12). The same conclusion can be drawn if we consider the residue series (3.2) for the field point at (ξ,η). Unless both

$$\frac{\eta_0}{H} - 1 = O\left(\frac{1}{(k\eta_0)^{2/3}}\right) \quad \text{and} \quad \frac{\eta_0}{\eta} - 1 = O\left(\frac{1}{(k\eta_0)^{2/3}}\right) ,$$

the residues arising from zeros of equation (3.27) are exponentially small. Thus any physical consequence arising from residues of this type is confined to an η-layer near the surface of the paraboloid of revolution defined by

$$\frac{\eta_0}{\eta} - 1 = O\left(\frac{1}{(k\eta_0)^{2/3}}\right) .$$

Finally, from the form of the residue series (3.1) for the source at $(\Xi, 0)$, we see that these residues are exponentially small for all values of Ξ. This is because the asymptotic form of $v_1(\xi_1, -\Lambda_p)$ is governed by equation (3.44).

We now investigate the distance-dependent phase of the terms in equation (3.189a). Using (D.26), we see that (3.42) becomes

$$\zeta_\xi^{(p)} = \frac{\Lambda_p}{k}\left[-\sqrt{-s_\xi^{(p)}(1-s_\xi^{(p)})} + \log\left(\sqrt{1-s_\xi^{(p)}} - \sqrt{-s_\xi^{(p)}}\right)\right]. \qquad (3.190)$$

But since $\left(\sqrt{1-s} - \sqrt{-s}\right)\left(\sqrt{1-s} + \sqrt{-s}\right) = 1$, $\arg s = \pi$ implies

$$\log\left(\sqrt{1-s} - \sqrt{-s}\right) = -\log\left(\sqrt{1-s} + \sqrt{-s}\right) .$$

Hence we can write equation (3.190) as

$$\zeta_\xi^{(p)} = \frac{-\Lambda_p}{k}\left[\sqrt{-s_\xi^{(p)}(1-s_\xi^{(p)})} + \log\left(\sqrt{1-s_\xi^{(p)}} - \sqrt{-s_\xi^{(p)}}\right)\right]. \qquad (3.190a)$$

Then using equation (3.27) together with the definition of $s_\xi^{(p)}$,

$$s_\xi^{(p)} = \frac{2ik\xi}{4\Lambda_p/2ik} \quad ,$$

we observe that

$$\zeta_\xi^{(p)} \sim k\eta_o\left[\sqrt{\frac{\xi}{\eta_o}\left(1+\frac{\xi}{\eta_o}\right)} + \log\left(\sqrt{1+\frac{\xi}{\eta_o}} + \sqrt{\frac{\xi}{\eta_o}}\right)\right],$$

for each p, and consequently

$$-i\zeta_\xi^{(p)} \sim -ik\left[\sqrt{\xi(\xi+\eta_o)} + \eta_o \log\left(\sqrt{1+\frac{\xi}{\eta_o}} + \sqrt{\frac{\xi}{\eta_o}}\right)\right], \tag{3.191}$$

for each p. The arc length from the vertex of a parabola $x^2 = 4\eta_o z'$ to a point $(z', \pm\sqrt{4\eta_o z'}\ )$ on the parabola is

$$L = \sqrt{z'(z'+\eta_o)} + \eta_o \log\left(\sqrt{1+\frac{z'}{\eta_o}} + \sqrt{\frac{z'}{\eta_o}}\right) .$$

(See the Smithsonian Institution Tables, 1922, p. 46, eq. 2.404). In our coordinate system (Section 1.3), $z' = \eta_o + z$, $z = \xi - \eta_o$; thus $z' = \xi$ and

$$L = \sqrt{\xi(\xi+\eta_o)} + \eta_o \log\left(\sqrt{1+\frac{\xi}{\eta_o}} + \sqrt{\frac{\xi}{\eta_o}}\right), \tag{3.192}$$

which shows that for a field point (ξ, η_o),

$$-i\zeta_\xi^{(p)} \sim -ikL, \qquad \text{for each } p\ . \tag{3.193}$$

There is another factor in the right-hand member of equation (3.189a) which can be interpreted geometrically. This is the distance-dependent amplitude of each term in (3.189a) which we write as

$$A(\xi, \eta_o) = \frac{1}{(\xi\eta_o)^{1/2}} \cdot \left(\frac{\xi}{\xi+\eta_o}\right)^{1/4} . \qquad (3.194)$$

If we assume that the field point (ξ, η_o) under consideration is such that $\xi/\eta_o \gg 1$, then (3.194) becomes

$$A(\xi, \eta_o) = \frac{1}{(\xi\eta_o)^{1/2}} \left[1 + O(\eta_o/\xi)\right] ,$$

or

$$A(\xi, \eta_o) \sim \frac{1}{(\xi\eta_o)^{1/2}} . \qquad (3.195)$$

But according to the definitions in Section 1.3, the radius ρ of the cylindrical cross section of the paraboloid of revolution is given by $\rho = 2\sqrt{\xi\eta_o}$. Thus if $\xi/\eta_o \gg 1$, $A(\xi, \eta_o)$ is approximately proportional to $1/\rho$.

Equations (3.193) and (3.195) have an important consequence. They show that: (i) each residue terms has a distance-dependent phase approximately equal to the arc length from the tip of the paraboloid to the field point (ξ, η_o) under consideration; and (ii) if $\xi/\eta_o \gg 1$ the distance-dependent amplitude of each residue term is approximately proportional to the inverse of the radius of the cylindrical cross section of the paraboloid of revolution. But this is exactly the distance-dependent amplitude and phase that one would expect of a cylindrical wave traveling along the surface of the paraboloid of revolution. Therefore, we can interpret the residue terms of equation (3.189a) as cylindrical waves which are traveling along the surface of the paraboloid of revolution. By equation (3.193), we see that each wave is arriving at the field point (ξ, η_o) approximately in phase. Waves of this type were first noticed by Rayleigh (1896, 1910) in connection with the propagation of sound in a room with a concave ceiling. He called them "whispering gallery waves", a name which we keep. Kimber (1961a, b) shows the existence of such waves for the interior of a circular cylinder under line source excitation, and for the interior of a sphere under point source excitation. Keller and Rubinow (1960) discuss not only these domains, but also show the existence of waves of this type in the case of bounded cylindrical convex regions.

APPENDIX A
NORMALIZATION (POINT SOURCE NORMALIZATION)

In the text, we first assumed the point source in question to be represented by $\rho(\underline{r})$. Let $J(\xi, \eta, \phi)$ denote the volume Jacobian in the coordinates of the paraboloid of revolution. Then for the point source at $(\Xi, 0)$

$$\rho(\xi, \eta) = \frac{1}{J(\xi, \eta, \phi)} C' \delta(\xi - \Xi)\delta(\eta) = \frac{C}{(\xi + \eta)} \delta(\xi - \Xi)\delta(\eta) \ ,$$

and so

$$\iiint \rho(\xi, \eta) dV = \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} 2C\delta(\xi - \Xi)\delta(\eta) d\xi \, d\eta \, d\phi = 4\pi C$$

$$= 4\pi, \quad \text{for} \quad C = 1 \ .$$

Therefore, the choice of a point source such that $C = 1$ implies

$$\iiint \rho(\underline{r}) dV = \iiint C'' \delta(\underline{r} - \underline{r}_0) dV = 4\pi \ ,$$

which in turn implies $C'' = 4\pi$ or $\rho(\underline{r}) = 4\pi\delta(\underline{r} - \underline{r}_0)$. Since the free space Green's function for this $\rho(\underline{r})$ is $-e^{-ikR}/R$ $(R = |\underline{r} - \underline{r}_0|)$, we demonstrate consistency (indicated by the agreement of the integrals of **equations (2.5) and (2.16)) by** showing that the solution to

$$-L_\eta v - L_\xi v = \delta(\xi - \Xi)\delta(\eta) \ ,$$

$$\text{(*)}$$

$$\int_{\text{all space}} |v(\xi, \eta, s)|^2 dV < \infty \ ,$$

has $-e^{-ikR}/R$ $(R = |\underline{r} - \underline{r}_0|)$, for its limit as $s \to 0^+$.

According to the theory of Section **2.4**, the solution to (*) can be represented as

$$v(\xi,\eta,s) = \frac{1}{2\pi i}\int_{\Gamma} \tilde{G}(\xi,\Xi,-\lambda)G(\eta,0,\lambda)d\lambda \ ,$$

where Γ is a straight line contour between the poles of $\tilde{G}(\xi,\Xi,-\lambda)$ and $G(\eta,0,\lambda)$. But

$$G(\eta,0,\lambda) = \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)}{2i\gamma}\ y_1(0,\lambda)y_2(\eta,\lambda) \qquad (0<\eta)$$

is analytic in $\mathrm{Im}\,\lambda < k$,* and

$$\tilde{G}(\xi,\Xi,-\lambda) = \frac{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)}{2i\gamma}\begin{cases} y_1(\xi,-\lambda)y_2(\Xi,-\lambda) & (\xi<\Xi) \\ y_1(\Xi,-\lambda)y_2(\xi,-\lambda) & (\xi>\Xi)\end{cases}$$

$$\left(\gamma = \frac{\omega - is}{c}\right)$$

is analytic in $\mathrm{Im}\,\lambda > -k$.* Therefore if Γ is a path defined by

$$-\infty - i\sigma < \lambda < \infty - i\sigma, \quad |\sigma| < k,$$

$$v(\xi,\eta,s) = \frac{1}{2\pi i}(2\pi i)^{-3/2}\int_{\Gamma} \Gamma\left(\frac{1}{2}-\frac{\lambda}{2i\gamma}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2i\gamma}\right)y_1(\xi_1,-\lambda)y_2(\xi_2,-\lambda)y_2(\eta,\lambda)d\lambda \ .$$

Arguing as in Section 1.4, we see that the limit as $s \rightarrow 0^+$ may be taken inside the integral by replacing the parameter γ with the parameter k. Then using the integral representation for e^{-ikR}/R in Buchholz (1953, Chapter 16, equation 9), we observe that

$$\lim_{s\rightarrow 0^+} v(\xi,\eta,s) = -\frac{e^{-ikR}}{R} \ ,$$

which demonstrates consistency.

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* See Appendix B.

APPENDIX B
ANALYTICITY OF RESOLVENT GREEN'S FUNCTION $\tilde{R}_\lambda$

The analyticity of $\tilde{R}_\lambda$ follows that of $\tilde{G}(\xi, \xi'.\lambda)$ which was represented as

$$\tilde{G}(\xi, \xi', \lambda) = \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)}{2i\gamma} \begin{cases} y_1(\xi, \lambda) y_2(\xi', \lambda) \, , & \xi < \xi' \, , \\ y_1(\xi', \lambda) y_2(\xi, \lambda) \, , & \xi > \xi' \, . \end{cases}$$

According to Buchholz (1953), the functions y_1 and y_2 are entire functions of λ. Thus the singularities of $\tilde{G}(\xi, \xi', \lambda)$ are those of $\Gamma\left(\frac{1}{2} - \frac{\lambda}{2i\gamma}\right)$, which are simple poles at the points $\frac{\lambda}{2i\gamma} = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$. $\tilde{G}(\xi, \xi', \lambda)$ will be analytic in any domain which excludes these points. Consider then the expression $\lambda/2i\gamma$ with $\lambda = x + iy$ and $\gamma = \frac{1}{c}(\omega - is)$. We observe

$$\frac{\lambda}{2i\gamma} = \frac{-i\lambda}{2\gamma} = \frac{-i\lambda\gamma^*}{2|\gamma|^2} = \frac{-i(x+iy)\left(\frac{1}{c}(\omega + is)\right)}{2|\gamma|^2} \, .$$

Thus

$$\frac{\lambda}{2i\gamma} = \frac{\frac{sx}{c} + \frac{\omega y}{c} - \frac{i\omega x}{c} + \frac{isy}{c}}{2|\gamma|^2} \, ,$$

and if this is to be real $\frac{\omega x}{c} = \frac{sy}{c}$, implying $x = \frac{s}{\omega} y$. Then

$$\frac{\lambda}{2i\gamma} = \frac{\frac{s^2 y}{\omega c} + \frac{\omega^2 y}{\omega c}}{2|\gamma|^2} = \frac{y}{2\omega c |\gamma|^2} (s^2 + \omega^2) \, .$$

But $|\gamma|^2 = \gamma\gamma^* = \frac{1}{c^2}(\omega^2 + s^2)$, and so for real $\lambda/2i\gamma$ we find $\frac{\lambda}{2i\gamma} = \frac{yc}{2\omega}$.

To exclude the poles of the Γ-function we need, for real $\lambda/2i\gamma$,

$$\frac{\lambda}{2i\gamma} < \frac{1}{2} . \quad \text{This implies} \quad \frac{yc}{2\omega} < \frac{1}{2} \quad \text{or} \quad y < \frac{\omega}{c} = k \, .$$

But since $\lambda = x + iy$, $y = \text{Im}\,\lambda$. Therefore, the condition for analyticity in a half plane becomes $\text{Im}\,\lambda < k$.

APPENDIX C

CLOSING THE CONTOUR (CONVERGENCE OF RESIDUE SERIES)

In this appendix, we consider the behavior as $|\lambda| \to \infty$ of the integrands appearing in

(i) the representations (1.18) through (1.23),

(ii) the equations (3.54) (Lemma 1) and (3.128b) (Lemma 2).

Due to the similarity of the various integrands involved, we shall study explicitly only the integrands of the representation (1.18) (of which the integrand of (1.17) is a factor) and equation (3.54). The other integrands are susceptible to similar treatment, the details of which are illuminated by the analysis below. For convenience, we repeat the representation (1.18)

$$v_N(\xi, \eta, \Xi, 0) = \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \frac{y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda)}{y_1'(\eta_o, \lambda)} \cdot \quad (0 < \sigma < k)$$

$$\cdot \left[y_2(\eta, \lambda) y_1'(\eta_o, \lambda) - y_1(\eta, \lambda) y_2'(\eta_o, \lambda) \right] ,$$

and equation (3.54)

$$v_N(\xi, \eta, \Xi, 0) - \sum_{n=0}^{M-1} \int_C d\lambda \frac{F}{g} \cdot X^n = \int_C d\lambda \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^M ,$$

where C is the path defined by $-\infty - i\sigma$ to $\infty - i\sigma$ $(0 < \sigma < k)$, and $F(\xi_1, \xi_2, \eta_o, k, \lambda)$ is given by equation (3.47), while $g(\eta_o, k, \lambda)$ and $X(\eta_o, k, \lambda)$ are defined following equation (3.51)

We shall show that as $|\lambda| \to \infty$, the integrand of equation (1.18) is exponentially decreasing in the upper half plane ($\text{Im}\,\lambda \geq -\sigma$) for all values of ξ_1, ξ_2, η and η_o ($\xi_2 > \xi_1$, $\eta > 0$). We shall also show that in the upper half plane this integrand has poles only at the zeros of the function $y_1'(\eta_o, \lambda)$. These zeros lie on the real line (Buchholz, 1942/3; 1953). For $|\lambda| \to \infty$, we investigate their behavior and find that they lie only on the positive half of the real line. The form of these zeros together with the fact that the above exponential decrease is true for positive real values of λ away from the zeros of $y_1'(\eta_0, \lambda)$, enables us to replace the integral representation (1.16) by a convergent residue series expansion.

The behavior as $|\lambda| \to \infty$ of the integrand of equation (1.16) in the lower half plane ($\mathrm{Im}\,\lambda \leqslant -\sigma$) is different from that above. In this half plane, the integrand has poles at the poles of the Γ-function $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ which lie on the negative imaginary axis. We shall show that for a portion of the lower half plane which includes the negative imaginary axis away from the poles of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, the integrand of equation (1.16) is exponentially decreasing as $|\lambda| \to \infty$ if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$. Thus, the integral representation (1.16) can be replaced by a second convergent residue series expansion if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$. If $\sqrt{\xi_1} + \sqrt{\xi_2} > \sqrt{\eta}$, we observe that the integrand is exponentially increasing over the above portion of the lower half plane. If $\sqrt{\xi_1} + \sqrt{\xi_2} = \sqrt{\eta}$, we find that the integrand is decreasing like some power of $|\lambda|$ as $|\lambda| \to \infty$, but that the second residue series is not absolutely convergent.

Finally, we shall show that as $|\lambda| \to \infty$ on C, the integrands appearing in equation (3.54) are exponentially decreasing for all values of ξ_1, ξ_2 and η_0. Therefore, the integrals appearing in equation (3.54) are indeed convergent.

C.1 Behavior in the Upper Half Plane

We begin by noting that the function $\bar{f}(\eta, \eta_0, k, \lambda)$ defined by

$$\bar{f}(\eta, \eta_0, k, \lambda) = \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)\left[y_2(\eta, \lambda) y_1'(\eta_0, \lambda) - y_1(\eta, \lambda) y_2'(\eta_0, \lambda)\right], \quad (C.1)$$

is analytic in the complex λ-plane. Since $y_1(\eta, \lambda)$ and $y_2(\eta, \lambda)$ are analytic functions of λ (Buchholz, 1953, Chapter 2), the only possible singularities of $\bar{f}(\eta, \eta_0, k, \lambda)$ are poles which can occur at the poles of $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$. These poles lie on the positive imaginary axis at $\lambda = ik(2n+1)$, $n = 0, 1, 2, \ldots$. But at these points

$$y_1(\eta, \lambda) = y_1\left[\eta, ik(2n+1)\right] = \eta^{-1/2} M_{n+1/2, 0}(2ik\eta) = (2ik)^{1/2} e^{-ik\eta} L_n^{(0)}(2ik\eta), \tag{C.2a}$$

$$y_2(\eta, \lambda) = y_2\left[\eta, ik(2n+1)\right] = \eta^{-1/2} W_{n+1/2, 0}(2ik\eta) = (-1)^n n!\,(2ik)^{1/2} e^{-ik\eta} L_n^{(0)}(2ik\eta) , \tag{C.2b}$$

where $L_n^{(0)}(2ik\eta)$ is the corresponding Laguerre polynomial. Thus

$$y_2\left[\eta, ik(2n+1)\right] y_1'\left[\eta_o, ik(2n+1)\right] - y_1\left[\eta, ik(2n+1)\right] y_2'\left[\eta_o, ik(2n+1)\right]$$

is equal to

$$(-1)^n n!\,(2ik)\, e^{-ik\eta} L_n^{(0)}(2ik\eta)\left[\frac{d}{d\eta}\left(e^{-ik\eta} L_n^{(0)}(2ik\eta)\right)\right]_{\eta=\eta_o} -$$

$$-(-1)^n n!\,(2ik)\, e^{-ik\eta} L_n^{(0)}(2ik\eta)\left[\frac{d}{d\eta}\left(e^{-ik\eta} L_n^{(0)}(2ik\eta)\right)\right]_{\eta=\eta_o} = 0 ,$$

and therefore cancels the simple pole of $\Gamma\left(\frac{-\lambda}{2ik} + \frac{1}{2}\right)$ at $\lambda = ik(2n+1)$, $n = 0, 1, 2, \ldots$. This implies the analyticity of $\bar{f}(\eta, \eta_o, k, \lambda)$.

Before proceeding with any more calculations, we consider some conventions to be used in this appendix. We recall that the λ-plane is to be cut at $\lambda = -3\pi/4$. Thus, $\arg\lambda$ satisfies $-3\pi/4 < \arg\lambda \leqslant 5\pi/4$; this convention does not change in any appendix. Hence for $|\lambda| \to \infty$, the upper half plane ($\operatorname{Im}\lambda \geqslant -\sigma$) can be characterized by $-\delta(\lambda) \leqslant \arg\lambda \leqslant \pi + \delta(\lambda)$, where $\delta(\lambda)$ is a small positive angle which decreases as $|\lambda|$ increases. Similarly, for $|\lambda| \to \infty$ the lower half plane ($\operatorname{Im}\lambda \leqslant -\sigma$) is characterized by $-3\pi/4 < \arg\lambda \leqslant -\delta(\lambda)$, $\pi + \delta(\lambda) \leqslant \arg\lambda \leqslant 5\pi/4$, where $\delta(\lambda)$ is as above. We shall use this characterization throughout this appendix.

We now consider the behavior of $y_1'(\eta_0, \lambda)$ as $|\lambda| \to \infty$. For $-3\pi/4 < \arg\lambda \leq 5\pi/4$, $y_1(\eta_0, \lambda)$ has the representation (Buchholz, 1953, p. 98, eq. 17a)

$$y_1(\eta, \lambda) \sim \eta^{-1/2} \left(-\frac{4k^2\eta}{\pi^2\lambda}\right)^{1/4} \cos\left[2\sqrt{\eta\lambda} - \frac{\pi}{4}\right], \tag{C.3}$$

which implies

$$y_1'(\eta_0, \lambda) \sim -\frac{1}{\eta_0}\left(-\frac{4k^2\eta_0\lambda}{\pi^2}\right)^{1/4} \sin\left[2\sqrt{\eta_0\lambda} - \frac{\pi}{4}\right]. \tag{C.4}$$

Therefore, as $|\lambda| \to \infty$ the zeros of $y_1'(\eta_0, \lambda)$ are the zeros of $\sin\left[2\sqrt{\eta_0\lambda} - \frac{\pi}{4}\right]$ which obey the relation

$$\lambda_N = \frac{1}{\eta_0}\left(\frac{\pi}{8} \pm \frac{N\pi}{2}\right)^2, \qquad N \to \infty, \tag{C.4a}$$

and consequently lie on the positive real axis. For $-3\pi/4 < \arg\lambda < 0$, the representations (C.3) and (C.4) may be replaced by

$$y_1(\eta, \lambda) \sim e^{2i\sqrt{\lambda\eta}}, \qquad y_1'(\eta_0, \lambda) \sim e^{2i\sqrt{\lambda\eta_0}},$$

while for $0 < \arg\lambda \leq 5\pi/4$, the representations (C.3) and (C.4) may be replaced by

$$y_1(\eta, \lambda) \sim e^{-2i\sqrt{\lambda\eta}}, \qquad y_1'(\eta_0, \lambda) \sim e^{-2i\sqrt{\lambda\eta_0}},$$

where we have considered only the exponential factor in λ. We do this whenever the nonexponential factors or exponential factors not dependent on λ are not essential to our arguments.

To demonstrate the exponential decrease of the integrand of equation (1.15), we first examine the factor

$$\frac{f(\xi_1, \xi_2, k, \lambda)}{y_1'(\eta_0, \lambda)},$$

where the function $f(\xi_1, \xi_2, k, \lambda)$ is defined by

$$f(\xi_1, \xi_2, k, \lambda) = \frac{(2ik)^{-3/2}}{2\pi i} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda) \ . \tag{C.5}$$

With this definition of $f(\xi_1, \xi_2, k, \lambda)$, we note that equation (1.18) can be written as

$$v_N(\xi, \eta, \Xi, 0) = \int_C d\lambda \, \frac{f(\xi_1, \xi_2, k, \lambda)}{y_1'(\eta_o, \lambda)} \cdot \bar{f}(\eta, \eta_o, k, \lambda) \ .$$

Now we need to investigate $y_1(\xi_1, -\lambda)$ and $y_2(\xi_2, -\lambda)$. For $\pi/4 < \arg\lambda \leqslant 5\pi/4$, we can use Buchholz (1953, p. 98, eq. 17b) to find that as $|\lambda| \to \infty$

$$y_1(\xi_1, -\lambda) \sim \cos\left[2\sqrt{\xi_1\lambda}\, e^{-i\pi/2} - \frac{\pi}{4}\right] \sim e^{2\sqrt{\xi_1\lambda}} e^{-i\pi/4} + e^{-2\sqrt{\xi_1\lambda}} e^{i\pi/4} \ , \tag{C.6}$$

while for $-3\pi/4 < \arg\lambda \leqslant \pi/4$, the same equation shows that as $|\lambda| \to \infty$

$$y_1(\xi_1, -\lambda) \sim \cos\left[2\sqrt{\xi_1\lambda}\, e^{i\pi/2} - \frac{\pi}{4}\right] \sim e^{-2\sqrt{\xi_1\lambda}} e^{-i\pi/4} + e^{2\sqrt{\xi_1\lambda}} e^{i\pi/4} \ . \tag{C.7}$$

The function $y_2(\xi_2, -\lambda)$ is not so simple. In order to examine its behavior and then that of $f(\xi_1, \xi_2, k, \lambda)$ as $|\lambda| \to \infty$, we divide the upper half plane into the sub-regions $-\delta(\lambda) \leqslant \arg\lambda < 0$, $0 \leqslant \arg\lambda < \pi/2$, $\arg\lambda = \pi/2$, $\pi/2 < \arg\lambda \leqslant \pi$, $\pi < \arg\lambda \leqslant \pi + \delta(\lambda)$. On each of these sectors we first obtain the behavior of $y_2(\xi_2, -\lambda)$, and then study the function $f(\xi_1, \xi_2, k, \lambda)$.

We begin by noting that we can obtain the behavior of $y_2(\xi_2, -\lambda)$ on (i) $-\delta(\lambda) \leqslant \arg\lambda < 0$ and (ii) $0 \leqslant \arg\lambda < \pi/2$ with one calculation. However, on these sectors, we cannot obtain the behavior of $y_2(\xi_2, -\lambda)$ directly. Instead, we make use of equation (21b), page 19 of Buchholz (1953) which asserts

$$y_2(\xi_2, -\lambda) = \frac{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)}{2\pi i} \left\{ e^{-\pi\lambda/2k} y_2(\xi_2 e^{\pi i}, \lambda) - \right.$$
$$\left. - e^{\pi\lambda/2k} y_2(\xi_2 e^{-\pi i}, \lambda) \right\} \ .$$

Now $-\delta(\lambda) \leqslant \arg\lambda < \pi/2 \Longrightarrow -\pi/2 - \delta(\lambda) \leqslant \arg \frac{\lambda}{2ik} < 0 \Longrightarrow \mathrm{Im}\, \frac{\lambda}{2ik} < 0$. In addition, $\arg \frac{\lambda}{2ik}(2ik\xi_2 e^{\pi i}) = \arg(\lambda e^{\pi i}) \in [\pi - \delta(\lambda),\ 3\pi/2)$ and $\arg \frac{\lambda}{2ik}(2ik\xi_2 e^{-\pi i}) = \arg(\lambda e^{-\pi i}) \in [-\pi - \delta(\lambda), -\pi/2)$. Therefore, equation (19a), page 99 of Buchholz (1953) applies to both $y_2(\xi_2 e^{\pi i}, \lambda)$ and $y_2(\xi_2 e^{-\pi i}, \lambda)$. But $\arg\lambda \in [-\delta(\lambda), \pi/2)$ implies $\mathrm{Re}\,\lambda > 0$, thus

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{\pi\lambda/2k} y_2(\xi_2 e^{-\pi i}, \lambda) .$$

Since the above-mentioned equation for $y_2(\xi_2 e^{-\pi i}, \lambda)$ can be written as

$$y_2(\xi_2 e^{-\pi i}, \lambda) \sim \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{\pi\lambda/2k} e^{-2\sqrt{\lambda\xi_2}} ,$$

we find

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{\pi\lambda/2k} e^{\pi\lambda/2k} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} e^{-2\sqrt{\lambda\xi_2}} .$$

The range of $\arg \frac{\lambda}{2ik}$ implies that Stirling's approximation is valid for the Γ-function $\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)$, thus from Erdélyi et al (1953) we observe

$$\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) \sim \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} . \tag{C.8}$$

Consequently, the above estimate for $y_2(\xi_2, -\lambda)$ can be written as

$$y_2(\xi_2, -\lambda) \sim \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) \Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) e^{\pi\lambda/2k}\ \Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right) e^{\pi\lambda/2k} e^{-2\sqrt{\lambda\xi_2}} .$$

However, from Erdélyi et al (1953) we also observe

$$\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right) = \frac{\pi}{\cos \pi \frac{\lambda}{2ik}} , \tag{C.9}$$

and using the exponential representation of the cosine, we see that as $|\lambda| \to \infty$ on $-\delta(\lambda) \leqslant \arg\lambda < \pi/2$

$$y_2(\xi_2,-\lambda) \sim \Gamma\left(\frac{1}{2}-\frac{\lambda}{2ik}\right) e^{\pi\lambda/2k} e^{-2\sqrt{\lambda\xi_2}} . \qquad (C.10)$$

Therefore, using (C.4), (C.7), (C.9) and (C.10), we find that on $-\delta(\lambda) \leqslant \arg\lambda < 0$

$$\frac{f}{y_1'(\eta_o,\lambda)} \sim \frac{e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}}{e^{-2\mathrm{Im}\sqrt{\lambda\eta_o}}} ,$$

since $-\delta(\lambda)/2 \leqslant \arg\sqrt{\lambda} < 0 \Longrightarrow \mathrm{Re}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} < 0$.

Similarly, using (C.4), (C.6) or (C.7), (C.9) and (C.10), we find that on $0 \leqslant \arg\lambda < \pi/2$ and provided λ is not a zero of $y_1'(\eta_o,\lambda)$

$$\frac{f}{y_1'(\eta_o,\lambda)} \sim \frac{e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}}{e^{2\mathrm{Im}\sqrt{\lambda\eta_o}}} ,$$

since $0 \leqslant \arg\sqrt{\lambda} < \pi/4 \Longrightarrow \mathrm{Re}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} \geqslant 0$. In either case, since $\xi_2 > \xi_1$, $f/y_1'(\eta_o,\lambda)$ is exponentially decreasing as $|\lambda| \to \infty$ away from a zero of $y_1'(\eta_o,\lambda)$.

(iii) $\arg\lambda = \pi/2$.

For $\arg\lambda = \pi/2$, we set $\lambda = i|\lambda|$. Thus $-\lambda/2ik = -|\lambda|/2k$, $\lambda/2ik = |\lambda|/2k$, and equation (20), page 100 of Buchholz (1953) can be used to give

$$y_2(\xi_2,-\lambda) \sim \exp\left\{-\frac{|\lambda|}{2k}\log\frac{|\lambda|}{2ke}\right\} e^{-2\sqrt{|\lambda|\xi_2}\cos\pi/4} . \qquad (C.11)$$

But (C.8) applies for $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$, and using this together with (C.4), (C.6) and (C.11), we note

$$\frac{f}{y_1'(\eta_o,\lambda)} \sim \frac{e^{-2\sqrt{|\lambda|\xi_2}\cos\pi/4}\, e^{2\sqrt{|\lambda|\xi_1}\cos\pi/4}}{e^{2\sqrt{|\lambda|\eta_o}\cos\pi/4}} ,$$

which is exponentially small as $|\lambda| \to \infty$ since $\xi_2 > \xi_1$.

We conclude by observing that we can obtain the behavior of $y_2(\xi_2, -\lambda)$ on the sectors (iv) $\pi/2 < \arg\lambda \leqslant \pi$ and (v) $\pi < \arg\lambda \leqslant \pi + \delta(\lambda)$ with one calculation. In these cases, we note that $0 < \arg \frac{\lambda}{2ik} \leqslant \frac{\pi}{2} + \delta(\lambda)$, which in turn implies $\mathrm{Im}\,\lambda/2ik > 0$. But in addition

$$-\frac{\lambda}{2ik} = e^{-i\pi}\frac{\lambda}{2ik} ,$$

and $\arg \frac{\lambda}{2ik}(2ik\xi_2) = \arg\lambda \in (\pi/2, \pi + \delta(\lambda)] \subset (-\pi, 3\pi)$. Hence, equation (20), page 100 Of Buchholz (1953) applies to $y_2(\xi_2, -\lambda)$, giving

$$y_2(\xi_2, -\lambda) \sim \exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} e^{-2\sqrt{\lambda\xi_2}} . \qquad (C.12)$$

But equation (C.8) also applies; thus, using it together with (C.4), (C.6) and (C.12), we find that on $\pi/2 < \arg\lambda \leqslant \pi$

$$\frac{f}{y'(\eta_o, \lambda)} \sim \frac{e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\; e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}}{e^{2\mathrm{Im}\sqrt{\lambda\eta_o}}} ,$$

since $\pi/4 < \arg\sqrt{\lambda} \leqslant \pi/2 \Longrightarrow \mathrm{Re}\sqrt{\lambda} \geqslant 0,\ \mathrm{Im}\sqrt{\lambda} > 0$. This is exponentially small as $|\lambda| \to \infty$ since $\xi_2 > \xi_1$ and $\mathrm{Im}\sqrt{\lambda} > 0$.

For $\pi < \arg\lambda \leqslant \pi + \delta(\lambda)$, we must argue slightly differently. In this case (C.8), (C.4), (C.6) and (C.12) give

$$\frac{f}{y_1'(\eta_o, \lambda)} \sim \frac{e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\; e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}}}{e^{2\mathrm{Im}\sqrt{\lambda\eta_o}}} ,$$

since $\pi/2 < \arg\sqrt{\lambda} \leqslant \frac{\pi}{2} + \frac{\delta(\lambda)}{2} \Longrightarrow \mathrm{Re}\sqrt{\lambda} < 0,\ \mathrm{Im}\sqrt{\lambda} > 0$. But as $|\lambda| \to \infty$ on $\pi < \arg\lambda \leqslant \pi + \delta(\lambda)$, $|\mathrm{Re}\sqrt{\lambda}| \leqslant k$ while $\mathrm{Im}\sqrt{\lambda}$ increases without bound. Therefore,

$$\frac{e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}}}{e^{2\mathrm{Im}\sqrt{\lambda\eta_0}}} \leqslant \frac{e^{2k\sqrt{\xi_2}}\, e^{2k\sqrt{\xi_1}}}{e^{2\mathrm{Im}\sqrt{\lambda\eta_0}}} ,$$

which is exponentially small since $\mathrm{Im}\sqrt{\lambda}$ is increasing without bound. Consequently, $f/y_1'(\eta_0, \lambda)$ is exponentially decreasing as $|\lambda| \to \infty$ in the upper half plane.

We now examine the behavior as $|\lambda| \to \infty$ of the function $\bar{f}(\eta, \eta_0, k, \lambda)$. We shall find a representation for $\bar{f}(\eta, \eta_0, k, \lambda)$ which is valid in the entire half-plane $-\delta(\lambda) \leqslant \arg\lambda \leqslant \pi + \delta(\lambda)$. On this half-plane, $y_1(\eta, \lambda)$ and $y_1'(\eta_0, \lambda)$ are governed by (C.3) and (C.4) respectively. It remains only to find representations for $y_2(\eta, \lambda)$ and $y_2'(\eta_0, \lambda)$. For $-\delta(\lambda) \leqslant \arg\lambda \leqslant \pi + \delta(\lambda)$, $-\frac{\pi}{2} + \delta(\lambda) \leqslant \arg \frac{\lambda}{2ik} \leqslant \frac{\pi}{2} + \delta(\lambda)$, and $\arg \frac{\lambda}{2ik}(2ik\eta) = \arg\lambda \in \left[-\delta(\lambda), \pi + \delta(\lambda)\right] \subset (-2\pi, 2\pi)$. Hence, equation 19, page 99 of Buchholz (1953) yields

$$y_2(\eta, \lambda) \sim \frac{(2)^{1/2}}{\eta^{1/2}} \left(-\frac{4k^2\eta}{\lambda}\right)^{1/4} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} \cos\left[2\sqrt{\lambda\eta} - \pi \frac{\lambda}{2ik} + \frac{\pi}{4}\right] ,$$

and by differentiation

$$y_2'(\eta, \lambda) \sim -\frac{(2)^{1/2}}{\eta} (-4k^2\eta\lambda)^{1/4} \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} \sin\left[2\sqrt{\lambda\eta} - \pi \frac{\lambda}{2ik} + \frac{\pi}{4}\right] .$$

But equation (C.8) for the Γ-function $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ is valid, thus these equations can be written as

$$y_2(\eta, \lambda) \sim \frac{(2)^{1/2}}{\eta^{1/2}} \left(-\frac{4k^2\eta}{\lambda}\right)^{1/4} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \cos\left[2\sqrt{\lambda\eta} - \pi \frac{\lambda}{2ik} + \frac{\pi}{4}\right], \tag{C.13}$$

$$y_2'(\eta, \lambda) \sim -\frac{(2)^{1/2}}{\eta} (-4k^2\eta\lambda)^{1/4} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \sin\left[2\sqrt{\lambda\eta} - \pi \frac{\lambda}{2ik} + \frac{\pi}{4}\right] . \tag{C.14}$$

Therefore, using (C.3), (C.4), (C.13) and (C.14), we find

$$\bar{f} \sim \Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\left[\cos\left[2\sqrt{\lambda\eta}-\pi\frac{\lambda}{2ik}+\frac{\pi}{4}\right]\sin\left[2\sqrt{\lambda\eta_0}-\frac{\pi}{4}\right]-\right.$$

$$\left.-\cos\left[2\sqrt{\lambda\eta}-\frac{\pi}{4}\right]\sin\left[2\sqrt{\lambda\eta_0}-\pi\frac{\lambda}{2ik}+\frac{\pi}{4}\right]\right],$$

which upon using the exponential representations of the sine and cosine becomes

$$\bar{f} \sim \Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)e^{\pi\lambda/2k}\left[e^{-2i\sqrt{\lambda\eta}}e^{2i\sqrt{\lambda\eta_0}}+e^{-2i\sqrt{\lambda\eta_0}}e^{2i\sqrt{\lambda\eta}}\right]+$$

$$+\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)e^{-\pi\lambda/2k}\left[e^{2i\sqrt{\lambda\eta}}e^{-2i\sqrt{\lambda\eta_0}}+e^{2i\sqrt{\lambda\eta_0}}e^{-2i\sqrt{\lambda\eta}}\right]. \tag{C.15}$$

To examine the behavior as $|\lambda|\to\infty$ of the integrand of equation (2.18) $\left(\frac{f}{y_1'(\eta_0,\lambda)}\cdot\bar{f}\right)$, we again divide $-\delta(\lambda)\leq\arg\lambda\leq\pi+\delta(\lambda)$ into the sub-regions $-\delta(\lambda)\leq\arg\lambda<0$, $0\leq\arg\lambda<\pi/2$, $\arg\lambda=\pi/2$, $\pi/2<\arg\lambda\leq\pi$, $\pi<\arg\lambda\leq\pi+\delta(\lambda)$, and we examine each sub-region separately.

(i) $-\delta(\lambda)\leq\arg\lambda<0$.

In this sub-region, $\operatorname{Re}\lambda>0$, and $\frac{-\delta(\lambda)}{2}\leq\arg\sqrt{\lambda}<0$ implies $\operatorname{Im}\sqrt{\lambda}<0$, $\operatorname{Re}\sqrt{\lambda}>0$. Then using (C.9) and (C.15) together with the exponential representation of the cosine in (C.9), we find

$$\bar{f}\sim e^{-2i\sqrt{\lambda\eta}}e^{2i\sqrt{\lambda\eta_0}},$$

since $\eta\leq\eta_0$. This together with the previous representation of $\frac{f}{y_1'(\eta_0,\lambda)}$ shows that on $-\delta(\lambda)\leq\arg\lambda<0$

$$\frac{f}{y_1'(\eta_0,\lambda)}\cdot\bar{f}\sim e^{-2\operatorname{Re}\sqrt{\lambda\xi_2}}e^{2\operatorname{Re}\sqrt{\lambda\xi_1}}e^{2\operatorname{Im}\sqrt{\lambda\eta}},$$

which is exponentially decreasing as $|\lambda|\to\infty$ since $\operatorname{Im}\sqrt{\lambda}<0$, $\operatorname{Re}\sqrt{\lambda}>0$, and $\xi_2>\xi_1$.

(ii) $0 \leqslant \arg\lambda < \pi/2$

In this sub-region, $\mathrm{Re}\,\lambda > 0$, and $0 \leqslant \arg\sqrt{\lambda} < \pi/4$ implies $\mathrm{Im}\sqrt{\lambda} \geqslant 0$, $\mathrm{Re}\sqrt{\lambda} > 0$. Then using (C.9) and (C.15) together with the exponential representation of the cosine in (C.9), we find

$$\bar{f} \sim e^{2i\sqrt{\lambda\eta}}\, e^{-2i\sqrt{\lambda\eta_0}} ,$$

since $\eta \leqslant \eta_0$. This together with the previous representation of $f/y_1'(\eta_0, \lambda)$ shows that on $0 \leqslant \arg\lambda < \pi/2$

$$\frac{f}{y_1'(\eta_0,\lambda)} \cdot \bar{f} \sim e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\eta}} ,$$

which is exponentially decreasing as $|\lambda| \to \infty$ (provided λ is not a zero of $y_1'(\eta_0,\lambda)$) since $\mathrm{Im}\sqrt{\lambda} \geqslant 0$, $\mathrm{Re}\sqrt{\lambda} > 0$ and $\xi_2 > \xi_1$.

(iii) $\arg\lambda = \pi/2$

In this sub-region, $\mathrm{Re}\,\lambda > 0$, and $\arg\sqrt{\lambda} = \pi/4$ implies $\mathrm{Im}\sqrt{\lambda} = \mathrm{Re}\sqrt{\lambda} = \sqrt{|\lambda|}\,\cos\pi/4$. Using (C.9) and (C.15) as above, we find

$$\bar{f} \sim e^{-2\sqrt{|\lambda|\eta}\,\cos\pi/4}\, e^{2\sqrt{|\lambda|\eta_0}\,\cos\pi/4} ,$$

since $\eta \leqslant \eta_0$. This together with the previous estimate of $f/y_1'(\eta_0,\lambda)$ shows that for $\arg\lambda = \pi/2$

$$\frac{f}{y_1'(\eta_0,\lambda)} \cdot \bar{f} \sim e^{-2\sqrt{|\lambda|\xi_2}\,\cos\pi/4}\, e^{2\sqrt{|\lambda|\xi_1}\,\cos\pi/4}\, e^{-2\sqrt{|\lambda|\eta}\,\cos\pi/4} ,$$

which is exponentially small as $|\lambda| \to \infty$ since $\xi_2 > \xi_1$.

(iv) $\pi/2 < \arg\lambda \leqslant \pi$

In this sub-region, $\mathrm{Re}\,\lambda < 0$ and $\pi/4 < \arg\sqrt{\lambda} \leqslant \pi/2$ implies $\mathrm{Re}\sqrt{\lambda} \geqslant 0$, $\mathrm{Im}\sqrt{\lambda} > 0$. Using (C.9) and (C.15) as above, we find

$$\bar{f} \sim e^{2i\sqrt{\lambda\eta}}\, e^{-2i\sqrt{\lambda\eta_o}},$$

since $\eta \leq \eta_o$. Thus, the argument follows closely that of (ii). If $\mathrm{Re}\sqrt{\lambda} = 0$. the exponential decrease is implied by $\mathrm{Im}\sqrt{\lambda} > 0$, $\eta > 0$.

(v) $\pi < \arg\lambda \leq \pi + \delta(\lambda)$

In this sub-region, $\mathrm{Re}\lambda < 0$ and $\pi/2 < \arg\sqrt{\lambda} \leq \frac{\pi}{2} + \frac{\delta(\lambda)}{2}$ implies $\mathrm{Re}\sqrt{\lambda} < 0$, $\mathrm{Im}\sqrt{\lambda} > 0$. Using (C.9) and (C.15) as above, we find

$$\bar{f} \sim e^{2i\sqrt{\lambda\eta}}\, e^{-2i\sqrt{\lambda\eta_o}},$$

since $\eta \leq \eta_o$. This together with the previous estimate of $f/y_1'(\eta_o, \lambda)$ shows that on $\pi < \arg\lambda \leq \pi + \delta(\lambda)$

$$\frac{f}{y_1'(\eta_o, \lambda)} \cdot \bar{f} \sim e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\eta}} .$$

But as $|\lambda| \to \infty$ on $\pi < \arg\lambda \leq \pi + \delta(\lambda)$, $\pi/2 < \arg\lambda \leq \frac{\pi}{2} + \frac{\delta(\lambda)}{2}$, or $|\mathrm{Re}\sqrt{\lambda}| \leq k$, while $\mathrm{Im}\sqrt{\lambda}$ increases without bound. Therefore,

$$e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\eta}} \leq e^{2k\sqrt{\xi_2}}\, e^{2k\sqrt{\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\eta}},$$

which is exponentially decreasing as $|\lambda| \to \infty$ since $\eta > 0$. Consequently, the integrand of equation (1.16) is exponentially decreasing as $|\lambda| \to \infty$ in the upper half-plane for all values of ξ_1, ξ_2, η and η_o ($\xi_2 > \xi_1$, $\eta > 0$).

Before proceeding further with equation (1.16), we discuss the convergence of the integrals appearing in equation (3.54). This convergence will be demonstrated by showing that the integrands appearing in equation (3.54) are exponentially decreasing as $|\lambda| \to \infty$ on C. If we compare the definition of $F(\xi_1, \xi_2, \eta_o, k, \lambda)$ (equation (3.47)) with that of $f(\xi_1, \xi_2, k, \lambda)$ (equation (C.5)), we observe that the exponential behavior in λ as $|\lambda| \to \infty$ is the same for both $F/v_1'(\eta_o, \lambda)$ and

$f/y_1'(\eta_o, \lambda)$. Thus we have already proved that $F/v_1'(\eta_o, \lambda)$ is exponentially decreasing as $|\lambda| \to \infty$ on C. It remains to consider $g(\eta_o, k, \lambda)$ and $X(\eta_o, k, \lambda)$. Examining the definitions of these functions given following equation (3.51), we see that in order to derive representations for them as $|\lambda| \to \infty$ on C, we need only to derive a representation for $y_2'(\eta_o e^{-\pi i}, -\lambda)$ as $|\lambda| \to \infty$ on C.

We consider first $\arg\lambda = -\delta(\lambda)$. Then

$$\arg - \frac{\lambda}{2ik} = \arg e^{i\pi} \frac{\lambda}{2ik} = \frac{\pi}{2} - \delta(\lambda) , \qquad \text{and}$$

$$\arg \left(-\frac{\lambda}{2ik}\right)(2ik\eta_o e^{-\pi i}) = \arg \left(e^{i\pi} \frac{\lambda}{2ik} \cdot 2ik\eta_o e^{-\pi i}\right) = \arg\lambda = -\delta(\lambda) .$$

Thus, we can use equation (19), page 99 of Buchholz (1953) to assert that as $|\lambda| \to \infty$

$$y_2(\eta e^{-\pi i}, -\lambda) \sim \exp\left\{-\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike}\right\} \cos\left[2\sqrt{\lambda\eta} + \frac{\pi\lambda}{2ik} + \frac{\pi}{4}\right] ,$$

and by differentiation,

$$y_2'(\eta_o e^{-\pi i}, -\lambda) \sim \exp\left\{-\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike}\right\} \sin\left[2\sqrt{\lambda\eta_o} + \frac{\pi\lambda}{2ik} + \frac{\pi}{4}\right] . \tag{C.16}$$

Moreover, for $\arg - \frac{\lambda}{2ik} = \frac{\pi}{2} - \delta(\lambda)$, Stirling's formula is applicable to the Γ-function $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$ (Erdélyi et al, 1953). This gives

$$\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \sim \exp\left\{-\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike}\right\} . \tag{C.17}$$

Using (C.17) together with the fact that $\arg\lambda = -\delta(\lambda)$ implies $\mathrm{Re}\,\lambda > 0$, we observe that (C.16) can be written as

$$y_2'(\eta_o e^{-\pi i}, -\lambda) \sim \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) e^{\pi\lambda/2k} e^{2i\sqrt{\lambda\eta_o}} . \tag{C.18}$$

Similarly, $\mathrm{Re}\,\lambda > 0$ implies that (C.14) can be written as (with $\eta = \eta_o$)

$$y_2'(\eta_0,\lambda) \sim \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) e^{\pi\lambda/2k} e^{-2i\sqrt{\lambda\eta_0}} . \tag{C.19}$$

Substituting (C.18) into the definition of $g(\eta_0, k, \lambda)$, and (C.18) together with (C.19) into the definition of $X(\eta_0, k, \lambda)$, we find that as $|\lambda| \to \infty$ on $\arg\lambda = -\delta(\lambda)$

$$g(\eta_0, k, \lambda) \sim e^{2i\sqrt{\lambda\eta_0}} , \tag{C.20}$$

and

$$X(\eta_0, k, \lambda) \sim e^{-4i\sqrt{\lambda\eta_0}} . \tag{C.21}$$

Comparing (C.20) with (C.4) (for $\arg\lambda = -\delta(\lambda)$), we see that F/g is exponentially decreasing on $\arg\lambda = -\delta(\lambda)$ as $|\lambda| \to \infty$. But $\arg\lambda = -\delta(\lambda)$ implies $\arg\sqrt{\lambda} = \frac{-\delta(\lambda)}{2}$, which in turn implies $\operatorname{Im}\sqrt{\lambda} < 0$. Hence, $X(\eta_0, k, \lambda)$ is also exponentially decreasing as $|\lambda| \to \infty$ on $\arg\lambda = -\delta(\lambda)$, and with it the integrands appearing in equation (3.54)

$$\left(\frac{F}{g} \cdot X^n \quad \text{and} \quad \frac{F}{v_1'(\eta_0,\lambda)} \cdot X^n \quad \text{for any } n \geq 0 \quad .\right)$$

We now consider $\arg\lambda = \pi + \delta(\lambda)$. Then

$$\arg -\frac{\lambda}{2ik} = \arg e^{-i\pi} \frac{\lambda}{2ik} = -\frac{\pi}{2} + \delta(\lambda), \quad \text{and}$$

$$\arg\left(-\frac{\lambda}{2ik}\right)(2ik\eta_0 e^{-\pi i}) = \arg(e^{-2\pi i}\lambda) = -2\pi + \arg\lambda = -\pi + \delta(\lambda) .$$

Thus, we can again use equation (19), page 99, of Buchholz (1953) to assert that as $|\lambda| \to \infty$

$$y_2(\eta e^{-\pi i}, -\lambda) \sim \exp\left\{-\frac{\lambda}{2ik}\log -\frac{\lambda}{2ike}\right\} \cos\left[2\sqrt{\lambda\eta}\, e^{-\pi i} + \frac{\pi\lambda}{2ik} + \frac{\pi}{4}\right] ,$$

and by differentiation

$$y_2'(\eta_0 e^{-\pi i}, -\lambda) \sim \exp\left\{-\frac{\lambda}{2ik}\log -\frac{\lambda}{2ike}\right\} \sin\left[2\sqrt{\lambda\eta_0}\, e^{-\pi i} + \frac{\pi\lambda}{2ik} + \frac{\pi}{4}\right] . \tag{C.22}$$

In addition, $\arg - \frac{\lambda}{2ik} = -\frac{\pi}{2} + \delta(\lambda)$ implies that (C.17) is again valid for $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$. This together with the fact that $\arg\lambda = \pi + \delta(\lambda)$ implies $\mathrm{Re}\,\lambda < 0$, shows that (C.22) can be written as

$$y_2'(\eta_o e^{-\pi i}, -\lambda) \sim \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) e^{-\pi\lambda/2k} e^{2i\sqrt{\lambda\eta_o}} . \tag{C.23}$$

Similarly, $\mathrm{Re}\,\lambda < 0$ implies that (C.14) can be written as (with $\eta = \eta_o$)

$$y_2'(\eta_o, \lambda) \sim \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) e^{-\pi\lambda/2k} e^{-2i\sqrt{\lambda\eta_o}} . \tag{C.24}$$

Substituting (C.23) into the definition of $g(\eta_o, k, \lambda)$, and (C.23) together with (C.24) into the definition of $X(\eta_o, k, \lambda)$, we find that as $|\lambda| \to \infty$ on $\arg\lambda = \pi + \delta(\lambda)$

$$g(\eta_o, k, \lambda) \sim e^{-\pi\lambda/k} e^{2i\sqrt{\lambda\eta_o}} , \tag{C.25}$$

and

$$X(\eta_o, k, \lambda) \sim 1 . \tag{C.26}$$

Examining (C.25), we see that the dominant term is $e^{-\pi\lambda/k}$, and so $|g(\eta_o, k, \lambda)|$ is exponentially increasing with $|\lambda|$ on $\arg\lambda = \pi + \delta(\lambda)$. Thus, F/g is exponentially decreasing as $|\lambda| \to \infty$ on $\arg\lambda = \pi + \delta(\lambda)$, since in this case we showed that

$$F \sim e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}} e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}} \lesssim e^{2k\sqrt{\xi_2}} e^{2k\sqrt{\xi_1}} .$$

Since we have already proved that $F/v_1'(\eta_o, \lambda)$ is exponentially decreasing as $|\lambda| \to \infty$ on C, then equation (C.26) implies that both

$$\frac{F}{g} \cdot X^n \quad (n \geq 0) \qquad \text{and} \qquad \frac{F}{v_1'(\eta_o, \lambda)} \cdot X^n \quad (n \geq 0)$$

are also exponentially decreasing as $|\lambda| \to \infty$ on $\arg\lambda = \pi + \delta(\lambda)$. Consequently, the integrals appearing in equation (3.54) converge.

We continue the study of equation (1.16) by considering the implication of having shown that the integrand is exponentially decreasing as $|\lambda| \to \infty$ in the upper half plane, and, in particular, for $\arg\lambda = 0$ away from a zero of $y_1'(\eta_0, \lambda)$. Let $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ denote the zeros of $y_1'(\eta_0, \lambda)$ along the real axis. For large $|\lambda|$, these zeros have the form given by equation (C.4a). Therefore, by use of the residue theorem

$$v_N(\xi, \eta, \Xi, 0) = (2ik)^{-3/2} \sum_{n=1}^{\infty} \Gamma\left(\frac{\lambda_n}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda_n}{2ik} + \frac{1}{2}\right) \frac{y_1(\xi_1, -\lambda_n) y_2(\xi_2, -\lambda_n)}{\left[\frac{d}{d\lambda} y_1'(\eta_0, \lambda)\right]_{\lambda=\lambda_n}} \cdot \left[-y_1(\eta, \lambda_n) y_2'(\eta_0, \lambda_n)\right] .$$

Furthermore, the Wronskian relation between $y_1(\eta_0, \lambda)$ and $y_2(\eta_0, \lambda)$ implies

$$-y_1(\eta_0, \lambda_n) y_2'(\eta_0, \lambda_n) = \frac{2ik}{\eta_0 \, \Gamma\left(-\frac{\lambda_n}{2ik} + \frac{1}{2}\right)} .$$

Hence, $v_N(\xi, \eta, \Xi, 0)$ can be written as

$$v_N(\xi, \eta, \Xi, 0) = \frac{(2ik)^{-1/2}}{\eta_0} \sum_{n=1}^{\infty} \Gamma\left(\frac{\lambda_n}{2ik} + \frac{1}{2}\right) \frac{y_1(\xi_1, -\lambda_n) y_2(\xi_2, -\lambda_n)}{\left[\frac{d}{d\lambda} y_1'(\eta_0, \lambda)\right]_{\lambda=\lambda_n}} \cdot \frac{y_1(\eta, \lambda_n)}{y_1(\eta_0, \lambda_n)} . \tag{C.27}$$

But in (C.27), $\arg\lambda_n = 0$ for n sufficiently large. Thus, the previously developed asymptotic forms can be used to investigate the convergence of the series in (C.27). By arguing as above, it is seen without difficulty that the series is absolutely convergent provided $\xi_2 > \xi_1$, $\eta > 0$. Moreover, these are the conditions assumed when deriving (1.16).

For the most obvious attempt to represent the series in (C.27) as the sum of the incident plus scattered fields, we use the integral representation of

$-e^{-ikR}/R$ (Appendix A) to write equation (1.18) as

$$v_N(\xi, \eta, \Xi, 0) = -\frac{e^{-ikR}}{R} + \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \cdot$$
$$(0 < \sigma < k)$$
$$\cdot \frac{y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda)}{y_1'(\eta_0, \lambda)} \left[-y_1(\eta, \lambda) y_2'(\eta_0, \lambda)\right] .$$

Then the previous calculations can be used to show that the integrand of the remaining integral is exponentially decreasing in the upper half plane if and only if $\sqrt{\xi_1} - \sqrt{\xi_2} + \sqrt{\eta} < 0$. But in the upper half plane this integrand possesses poles not only at the zeros of $y_1'(\eta_0, \lambda)$, but also at the poles of the Γ-function $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$ which lie along the positive imaginary axis at the points $\lambda = ik(2n+1)$, $n = 0, 1, 2, \ldots$. However, the contribution of these latter poles to the residue series (absolutely convergent if $\sqrt{\xi_1} - \sqrt{\xi_2} + \sqrt{\eta} < 0$) simply cancels $-e^{-ikR}/R$. This follows immediately from the existence of a residue series (in terms of Laguerre polynomials) for $-e^{-ikR}/R$ if $\sqrt{\xi_1} - \sqrt{\xi_2} + \sqrt{\eta} < 0$, and the calculation which shows $\bar{f}(\eta, \eta_0, k, \lambda)$ analytic in the upper half plane. The final result is the previously obtained residue series.

C.2 Behavior in the Lower Half Plane

In the lower half plane, the integrand of equation (1.18) has poles at the poles of the Γ-function $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ which lie along the negative imaginary axis at the points $\lambda = -ik(2n+1)$, $n = 0, 1, 2, \ldots$. To investigate the behavior of the integrand as $|\lambda| \to \infty$, we consider first the region $\pi + \delta(\lambda) \lesssim \arg\lambda \lesssim 5\pi/4$. As always, $y_1(\eta, \lambda)$ and $y_1'(\eta_0, \lambda)$ are governed by (C.3) and (C.4), respectively. In addition, $y_1(\xi_1, -\lambda)$ is governed by (C.6). It remains to find $y_2(\xi_2, -\lambda)$, $y_2(\eta, \lambda)$ and $y_2'(\eta_0, \lambda)$. We immediately see that the arguments leading to equations (C.12), (C.13) and (C.14) can be repeated for $\pi + \delta(\lambda) \lesssim \arg\lambda \lesssim 5\pi/4$. Thus as $|\lambda| \to \infty$ in this region

$$\frac{f}{y_1'(\eta_o,\lambda)} \sim \frac{e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}}}{e^{2\mathrm{Im}\sqrt{\lambda\eta_o}}} ,$$

and

$$\frac{f}{y_1'(\eta_o,\lambda)} \cdot \bar{f} \sim e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\eta}} .$$

But $\pi+\delta(\lambda) \leqslant \arg\lambda \leqslant 5\pi/4$ implies $\frac{\pi}{2}+\frac{\delta(\lambda)}{2} \leqslant \arg\sqrt{\lambda} \leqslant 5\pi/8$, which in turn implies $\mathrm{Re}\sqrt{\lambda} < 0$, $\mathrm{Im}\sqrt{\lambda} > 0$ and $-\mathrm{Re}\sqrt{\lambda} < \mathrm{Im}\sqrt{\lambda}$. Hence,

$$e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{-2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\eta}} < e^{2\mathrm{Im}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Im}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\eta}} .$$

Therefore on $\pi+\delta(\lambda) \leqslant \arg\lambda \leqslant 5\pi/4$, the integrand of equation (1.18) is exponentially decreasing as $|\lambda| \to \infty$ if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$.

We now consider the remainder of the lower half plane which can be characterized by $-3\pi/4 < \arg\lambda \leqslant -\delta(\lambda)$. In this region, we first examine the factor

$$\frac{f(\xi_1, \xi_2, k, \lambda)}{y_1'(\eta_o,\lambda)} .$$

Since $y_1'(\eta_o,\lambda)$ is governed by (C.4) and $y_1(\xi_1,-\lambda)$ is governed by (C.7), it remains only to find a representation for $y_2(\xi_2,-\lambda)$. We can derive a representation for $y_2(\xi_2,-\lambda)$ which is valid for the entire region $-3\pi/4 < \arg\lambda \leqslant -\delta(\lambda)$. In this case $-\lambda = e^{i\pi}\lambda$, and so $-\frac{\lambda}{2ik} = e^{i\pi}\frac{\lambda}{2ik}$. Then $-3\pi/4 < \arg\lambda \leqslant -\delta(\lambda)$ implies $+\pi/4 < \arg -\frac{\lambda}{2ik} \leqslant \frac{\pi}{2} - \delta(\lambda)$, while

$$\arg\left(-\frac{\lambda}{2ik}\right)(2ik\xi_2) = \arg(e^{i\pi}\lambda) = \pi+\arg\lambda \in (\pi/4, \pi-\delta(\lambda)] \subset [0, 2\pi) .$$

Hence, equation (19), page 99 of Buchholz (1953) may be used for $y_2(\xi_2,-\lambda)$. This gives

$$y_2(\xi_2,-\lambda) \sim \exp\left\{-\frac{\lambda}{2ik}\log -\frac{\lambda}{2ike}\right\} \cos\left[2\sqrt{e^{\pi i}\lambda\xi_2} + \frac{\pi\lambda}{2ik} + \frac{\pi}{4}\right] .$$

Since in this case (C.17) is valid for $\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)$, this estimate for $y_2(\xi_2,-\lambda)$ can be written as

$$y_2(\xi_2,-\lambda) \sim \Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right)\cos\left[2i\sqrt{\lambda\xi_2}+\frac{\pi\lambda}{2ik}+\frac{\pi}{4}\right] \quad . \qquad (C.28)$$

Then to examine the factor

$$\frac{f(\xi_1,\xi_2,k,\lambda)}{y_1'(\eta_o,\lambda)} ,$$

we divide the region $-3\pi/4 < \arg\lambda \leqslant -\delta(\lambda)$ into the sub-regions $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda)$, $\arg\lambda = -\pi/2$, $-3\pi/4 < \arg\lambda < -\pi/2$, and consider each sub-region separately,.

(i) $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda)$

Since $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda)$ implies $\mathrm{Re}\,\lambda > 0$, then (C.28) can be written as

$$y_2(\xi_2,-\lambda) \sim \Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) e^{\pi\lambda/2k}\, e^{-2\sqrt{\lambda\xi_2}} . \qquad (C.29)$$

Using (C.4), (C.7), (C.29), (C.9) and the exponential representation of the cosine in (C.9), we find

$$\frac{f}{y_1'(\eta_o,\lambda)} \sim \frac{e^{-2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}}{e^{-2\mathrm{Im}\sqrt{\lambda\eta_o}}} ,$$

since $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda) \Longrightarrow -\pi/4 < \arg\sqrt{\lambda} \leqslant \frac{-\delta(\lambda)}{2} \Longrightarrow \mathrm{Re}\sqrt{\lambda} > 0$, $\mathrm{Im}\sqrt{\lambda} < 0$. But $\xi_2 > \xi_1$; thus, $f/y_1'(\eta_o,\lambda)$ is exponentially decreasing as $|\lambda| \to \infty$ on $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda)$.

(ii) $\arg\lambda = -\pi/2$

In this case $\lambda = -i|\lambda|$, hence (C.28) can be written as

$$y_2(\xi_2,-\lambda) \sim \Gamma\left(\frac{|\lambda|}{2k}+\frac{1}{2}\right) e^{2\sqrt{|\lambda|\xi_2}\cos\pi/4} . \qquad (C.30)$$

Then using (C.4), (C.7), (C.9) and (C.30), we find that away from a pole of $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$

$$\frac{f}{y_1'(\eta_0,\lambda)} \sim \frac{e^{2\sqrt{|\lambda|\xi_2}\cos\pi/4}\, e^{2\sqrt{|\lambda|\xi_1}\cos\pi/4}}{e^{2\sqrt{|\lambda|\eta_0}\cos\pi/4}} \quad ,$$

which is exponentially decreasing as $|\lambda| \to \infty$ if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta_0}$.

(iii) $-3\pi/4 < \arg\lambda < -\pi/2$

Since $-3\pi/4 < \arg\lambda < -\pi/2$ implies $\mathrm{Re}\lambda < 0$, then (C.28) can be written as

$$y_2(\xi_2,-\lambda) \sim \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right) e^{-\pi\lambda/2k}\, e^{2\sqrt{\lambda\xi_2}} \quad . \tag{C.31}$$

Using (C.4), (C.7), (C.31), (C.9) and the exponential representation of the cosine in (C.9), we find

$$\frac{f}{y_1'(\eta_0,\lambda)} \sim \frac{e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_2}}}{e^{-2\mathrm{Im}\sqrt{\lambda\eta_0}}} \quad ,$$

since $-3\pi/4 < \arg\lambda < -\pi/2 \Longrightarrow -3\pi/8 < \arg\sqrt{\lambda} < -\pi/4 \Longrightarrow \mathrm{Re}\sqrt{\lambda} > 0,\ \mathrm{Im}\sqrt{\lambda} < 0$. But in this case $\mathrm{Re}\sqrt{\lambda} < -\mathrm{Im}\sqrt{\lambda}$, thus

$$\frac{e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_2}}}{e^{-2\mathrm{Im}\sqrt{\lambda\eta_0}}} < \frac{e^{-2\mathrm{Im}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\xi_2}}}{e^{-2\mathrm{Im}\sqrt{\lambda\eta_0}}} \quad ,$$

which decreases exponentially as $|\lambda| \to \infty$ if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta_0}$. But as $\arg\sqrt{\lambda}$ approaches $-\pi/4$, $-\mathrm{Im}\sqrt{\lambda}$ approaches $\mathrm{Re}\sqrt{\lambda}$. Hence, $f/y_1'(\eta_0,\lambda)$ decreases exponentially as $|\lambda| \to \infty$ only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta_0}$.

To examine the behavior as $|\lambda| \to \infty$ of $\bar{f}(\eta,\eta_0,k,\lambda)$, as well as that of the integrand of equation (1.16) $\left(\frac{f}{y_1'(\eta_0,\lambda)} \cdot \bar{f}\right)$, we again divide the region $-3\pi/4 < \arg\lambda \lesssim -\delta(\lambda)$ into the sectors $-\pi/2 < \arg\lambda \lesssim -\delta(\lambda)$, $\arg\lambda = -\pi/2$, $-3\pi/4 < \arg\lambda < -\pi/2$, and we examine each sector separately. As always (C.3) and (C.4) are valid for $y_1(\eta,\lambda)$ and $y_1'(\eta_0,\lambda)$, respectively. It remains only to find a representation for $y_2(\eta,\lambda)$ and $y_2'(\eta_0,\lambda)$.

(i) $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda)$

In this case $-\pi < \arg \frac{\lambda}{2ik} \leqslant -\frac{\pi}{2} - \delta(\lambda)$, and $\arg \frac{\lambda}{2ik}(2ik\eta) = \arg\lambda \in (-\pi/2, -\delta(\lambda)] \subset (-2\pi, 2\pi)$. Thus the arguments leading to equations (C.13), (C.14) and (C.15) can be repeated as $|\lambda| \to \infty$ on this sector. But $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda)$ implies $\operatorname{Re}\lambda > 0$ and $-\pi/4 < \arg\lambda \leqslant -\delta(\lambda)/2$, which in turn implies $\operatorname{Re}\sqrt{\lambda} > 0$, $\operatorname{Im}\sqrt{\lambda} < 0$, Then using (C.9) and (C.15) together with the exponential representation of the cosine in (C.9), we find

$$\bar{f} \sim e^{-2i\sqrt{\lambda\eta}}\, e^{2i\sqrt{\lambda\eta_0}}$$

since $\eta \leqslant \eta_0$. This together with the previous estimate of $f/y_1'(\eta_0, \lambda)$ shows that on $-\pi/2 < \arg\lambda \leqslant -\delta(\lambda)$

$$\frac{f}{y_1'(\eta_0, \lambda)} \cdot \bar{f} \sim e^{-2\operatorname{Re}\sqrt{\lambda\xi_2}}\, e^{2\operatorname{Re}\sqrt{\lambda\xi_1}}\, e^{2\operatorname{Im}\sqrt{\lambda\eta}},$$

which is exponentially decreasing as $|\lambda| \to \infty$ since $\operatorname{Im}\sqrt{\lambda} < 0$, $\operatorname{Re}\sqrt{\lambda} > 0$, and $\xi_2 > \xi_1$.

(ii) $\arg\lambda = -\pi/2$

In this case $\lambda = -i|\lambda|$ and $\frac{\lambda}{2ik} = -\frac{|\lambda|}{2k}$. Thus, we can use equation (20), page 100 of Buchholz (1953) to assert that as $|\lambda| \to \infty$ on $\arg\lambda = -\pi/2$

$$y_2(\eta, \lambda) \sim \exp\left\{-\frac{|\lambda|}{2k} \log \frac{|\lambda|}{2ke}\right\} e^{-2\sqrt{|\lambda|\eta}\cos\pi/4}, \tag{C.32}$$

and by differentiation

$$y_2'(\eta_0, \lambda) \sim \exp\left\{-\frac{|\lambda|}{2k} \log \frac{|\lambda|}{2ke}\right\} e^{-2\sqrt{|\lambda|\eta_0}\cos\pi/4}. \tag{C.33}$$

But equation (C.17) holds for the Γ-function $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$. Hence, substituting (C.17) together with (C.3), (C.4), (C.32) and (C.33) into (C.1), we find

$$\bar{f} \sim e^{-2\sqrt{|\lambda|\eta}\cos\pi/4}\, e^{2\sqrt{|\lambda|\eta_0}\cos\pi/4}$$

since $\eta \leqslant \eta_0$. Therefore, away from a pole of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$

$$\frac{f}{y_1'(\eta_o,\lambda)}\cdot\bar{f} \sim e^{2\sqrt{|\lambda|\xi_2}\cos\pi/4}\, e^{2\sqrt{|\lambda|\xi_1}\cos\pi/4}\, e^{-2\sqrt{|\lambda|\eta}\cos\pi/4},$$

which vanishes exponentially if and only if $\sqrt{\xi_1}+\sqrt{\xi_2}<\sqrt{\eta}$. This equation also shows that as $|\lambda|\to\infty$ on $\arg\lambda=-\pi/2$, the integrand of equation (1.16) is exponentially increasing if $\sqrt{\xi_1}+\sqrt{\xi_2}>\sqrt{\eta}$, and λ is away from a pole of $\Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)$.

(iii) $-3\pi/4<\arg\lambda<-\pi/2$

On this sector, we cannot directly obtain a representation for $y_2(\eta,\lambda)$ as $|\lambda|\to\infty$. However, we recall that for $0<\arg\lambda<\pi/2$ we derived equation (C.10) for $y_2(\xi_2,-\lambda)$ as $|\lambda|\to\infty$. Thus if we consider the substitution $v=\lambda e^{-\pi i}$, and recall that in $0<\arg\lambda<\pi/2$, $-\lambda=e^{-\pi i}\lambda$, then as $|v|\to\infty$ on $-\pi<\arg v<-\pi/2$, (C.10) implies

$$y_2(\xi_2,v)\sim\Gamma\left(\frac{1}{2}+\frac{v}{2ik}\right)e^{-\pi v/2k}\,e^{-2\sqrt{e^{\pi i}v\xi_2}}=\Gamma\left(\frac{1}{2}+\frac{v}{2ik}\right)e^{-\pi v/2k}\,e^{-2i\sqrt{v\xi_2}}.$$

But since v,ξ_2 are dummy variables, we can write for $-\pi<\arg\lambda<-\pi/2$, and in particular for $-3\pi/4<\arg\lambda<-\pi/2$,

$$y_2(\eta,\lambda)\sim\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)e^{-\pi\lambda/2k}\,e^{-2i\sqrt{\lambda\eta}},\tag{C.34}$$

and by differentiation

$$y_2'(\eta_o,\lambda)\sim\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)e^{-\pi\lambda/2k}\,e^{-2i\sqrt{\lambda\eta_o}}.\tag{C.35}$$

Substituting these equations together with (C.3) and (C.4) into (C.1), we obtain

$$\bar{f}\sim\Gamma\left(\frac{1}{2}-\frac{\lambda}{2ik}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2ik}\right)e^{-\pi\lambda/2k}\left[e^{-2i\sqrt{\lambda\eta}}\,e^{2i\sqrt{\lambda\eta_o}}+e^{2i\sqrt{\lambda\eta}}\,e^{-2i\sqrt{\lambda\eta_o}}\right].$$

But $-3\pi/4<\arg\lambda<-\pi/2$ implies $-3\pi/8<\arg\sqrt{\lambda}<-\pi/4$, which in turn implies $\operatorname{Re}\sqrt{\lambda}>0$, $\operatorname{Im}\sqrt{\lambda}<0$, $\operatorname{Re}\sqrt{\lambda}<-\operatorname{Im}\sqrt{\lambda}$. Then using (C.9) together with the exponential representation of the cosine in (C.9), we find

$$\bar{f} \sim e^{-2i\sqrt{\lambda\eta}}\, e^{2i\sqrt{\lambda\eta_o}} ,$$

since $\eta \leqslant \eta_o$. Therefore,

$$\frac{f}{y_1'(\eta_o, \lambda)} \cdot \bar{f} \sim e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Im}\sqrt{\lambda\eta}} .$$

In addition,

$$e^{2\mathrm{Re}\sqrt{\lambda\xi_1}}\, e^{2\mathrm{Re}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Im}\sqrt{\lambda\eta}} < e^{-2\mathrm{Im}\sqrt{\lambda\xi_1}}\, e^{-2\mathrm{Im}\sqrt{\lambda\xi_2}}\, e^{2\mathrm{Im}\sqrt{\lambda\eta}} .$$

This implies that as $|\lambda| \to \infty$ on $-3\pi/4 < \arg\lambda < -\pi/2$, the integrand of equation (1.18) is exponentially decreasing if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$. But as $\arg\sqrt{\lambda}$ approaches $-\pi/4$, $-\mathrm{Im}\sqrt{\lambda}$ approaches $\mathrm{Re}\sqrt{\lambda}$. Hence, this integral is exponentially decreasing only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$. Finally, $-\mathrm{Im}\sqrt{\lambda}$ approaching $\mathrm{Re}\sqrt{\lambda}$ also implies that if $\sqrt{\xi_1} + \sqrt{\xi_2} - \sqrt{\eta} = \beta > 0$, the integrand of equation (1.18) is exponentially increasing as $|\lambda| \to \infty$ on $-3\pi/4 < \arg\lambda < -\pi/2$.

We continue to study equation (1.18) by considering the implication of having shown that if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$, the integrand is exponentially decreasing as $|\lambda| \to \infty$ in the lower half-plane, and in particular for $\arg\lambda = -\pi/2$ and λ away from a pole of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$. Thus suppose $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$. Since the poles of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ lie at the points $\lambda = -ik(2n+1)$, $n = 0, 1, 2, \ldots,$ then the residue theorem implies

$$v_N(\xi, \eta, \Xi, 0) = -(2ik)^{-1/2} \sum_{n=0}^{\infty} (-1)^n \frac{y_1\left[\xi_1, ik(2n+1)\right] y_2\left[\xi_2, ik(2n+1)\right]}{y_1'\left[\eta_o, -ik(2n+1)\right]} .$$

$$\cdot \left\{ y_2\left[\eta, -ik(2n+1)\right] y_1'\left[\eta_o, -ik(2n+1)\right] - y_1\left[\eta, -ik(2n+1)\right] y_2'\left[\eta_o, -ik(2n+1)\right] \right\} , \tag{C.36}$$

where we have used the fact that the residues of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ at the poles $\lambda = -ik(2n+1)$ are

$\frac{2k}{(-i)} \frac{(-1)^n}{n!}$. Moreover, the previously developed asymptotic forms can be used to investigate the convergence of the series in (C.36). By arguing as above, it is seen without difficulty that this series is absolutely convergent if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\eta}$.

A representation of the type given in (C.36) may be found for a less restrictive inequality. Using the integral representation of $-e^{-ikR}/R$ (Appendix A), we observed that (1.16) could be written as

$$v_N(\xi, \eta, \Xi, 0) = -\frac{e^{-ikR}}{R} + \frac{(2ik)^{-3/2}}{2\pi i} \int_{-\infty - i\sigma}^{\infty - i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \cdot \qquad (0 < \sigma < k)$$

$$\cdot \frac{y_1(\xi_1, -\lambda) y_2(\xi_2, -\lambda)}{y_1'(\eta_0, \lambda)} \left[-y_1(\eta, \lambda) y_2'(\eta_0, \lambda)\right] \qquad (C.37)$$

The arguments above show that the integrand of this latter integral is exponentially decreasing as $|\lambda| \to \infty$ in the lower half plane if and only if $\sqrt{\xi_1} + \sqrt{\xi_2} < 2\sqrt{\eta_0} - \sqrt{\eta}$. Thus, suppose $\sqrt{\xi_1} + \sqrt{\xi_2} < 2\sqrt{\eta_0} - \sqrt{\eta}$. Then the residue theorem gives

$$v_N(\xi, \eta, \Xi, 0) = -\frac{e^{-ikR}}{R} + (2ik)^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{y_1\left[\xi_1, ik(2n+1)\right] y_2\left[\xi_2, ik(2n+1)\right]}{y_1'\left[\eta_0, -ik(2n+1)\right]} \cdot$$

$$\cdot\, y_1\left[\eta, -ik(2n+1)\right] y_2'\left[\eta_0, -ik(2n+1)\right] , \qquad (C.38)$$

and $\sqrt{\xi_1} + \sqrt{\xi_2} < 2\sqrt{\eta_0} - \sqrt{\eta}$ implies the series in (C.38) converges absolutely.

It is instructive to consider the behavior of the integrand of (1.16) ((C.37)) if $\sqrt{\xi_1} + \sqrt{\xi_2} = \sqrt{\eta}$ ($\sqrt{\xi_1} + \sqrt{\xi_2} = 2\sqrt{\eta_0} - \sqrt{\eta}$). Then the exponential amplitude factors are equal to 1, and the behavior of the integrand as $|\lambda| \to \infty$ is governed by the powers of λ which appear. It is not difficult to show that as $|\lambda| \to \infty$ in the lower half plane, the integrand behaves as $1/|\lambda|^{3/4}$. Consequently, the absolute value of the integrand of (1.18) ((C.37)) is approaching zero as $|\lambda| \to \infty$. Therefore, the residue theorem may not necessarily be used to obtain (C.36) ((C.38)). At the poles, $|\lambda| = k(2n+1)$. Hence, the residue series, whose terms behave as $|\lambda|^{1/4}$ for large λ, does not converge absolutely.

APPENDIX D

UNIFORM ASYMPTOTIC REPRESENTATIONS

In this appendix we will consider separately the two asymptotic representations of the Whittaker functions which are needed in Chapters II and III. The cases studied all correspond to problems investigated in those chapters. The results obtained are based on the work of Langer (1932, 1935). More detailed results are given in the memoir of Erdélyi and Swanson (1957), who discuss the necessity for the two representations, and the paper by Taylor (1939).

D.1 Airy Function Representations

In this section we study the Whittaker functions $M_{\ell,0}(4\ell s)$ and $W_{\ell,0}(4\ell s)$ for $s \in R_1$, R_1 being the region illustrated in Fig. D-1, and $|\ell| \to \infty$ subject to the restriction that $\arg(\ell s) = \pi/2$. These functions satisfy the differential equation

$$\frac{d^2u}{ds^2} + \left[4\ell^2\left(\frac{1-s}{s}\right) + \frac{1}{4s^2}\right] u = 0 . \tag{D.1}$$

If we define $\bar{\rho} = -2i\ell$, (D.1) becomes

$$\frac{d^2u}{ds^2} + \left[\bar{\rho}^2\left(\frac{s-1}{s}\right) + \frac{1}{4s^2}\right] u = 0 . \tag{D.2}$$

For $\bar{\rho}$ a complex parameter such that $|\bar{\rho}| \gg 1$ and s belonging to the closed domain R_1, equation (D.2) is of the type studied by Langer (1932) and his results may be applied. In our applications (Chapters II and III) we have adopted the convention $\arg s \in \left[-\pi/4, 7\pi/4\right)$. For a closed domain containing $s = 1$ in the region defined by $\arg s \in \left[-\pi/4, \pi/4\right]$ Langer's results are applicable provided $|s| \gg 1/|\bar{\rho}|^{2/3}$ throughout the domain in question (Taylor, 1939). Such a region of validity is the region R_1 shown in Fig. D-1.

Let $\bar{\phi}^2(s) = \frac{s-1}{s}$, where $\bar{\phi}$ is the root of $\bar{\phi}^2(s)$ such that

$$\lim_{s\to 1}\left\{\frac{1}{(s-1)^{1/2}}\,\bar{\phi}(s)\right\} = 1 .$$

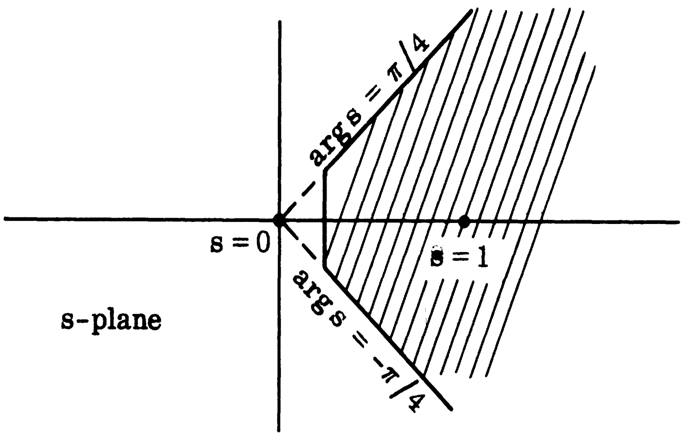

FIG. D-1: REGION OF VALIDITY OF AIRY FUNCTION REPRESENTATION.

We also define

$$\bar{\Phi}(s) = \int_1^s \bar{\phi}(t)\,dt , \qquad \bar{\zeta} = \bar{\rho}\,\bar{\Phi}(s) , \tag{D.3}$$

and

$$\bar{\psi}(s) = \left[\bar{\Phi}(s)\right]^{1/6}\left[\bar{\phi}(s)\right]^{-1/2} \text{ with } \bar{\psi}(1) = \lim_{s\to 1} \bar{\psi}(s); \tag{D.4}$$

these are the functions used in Langer's theory. For $|s| > 1$ we can write

$$\bar{\Phi}(s) = \sqrt{s(s-1)} - \log\left(\sqrt{s-1} + \sqrt{s}\right), \tag{D.5}$$

while for $|s| < 1$ we observe that

$$\bar{\Phi}(s) = i\int_1^s \left(\frac{1-t}{t}\right)^{1/2} dt . \tag{D.6}$$

We can derive the expansions

$$\bar{\phi}(s) = (s-1)^{1/2}\left[1 - \frac{1}{2}(s-1) + O\left((s-1)^2\right)\right] \quad \text{as } s \to 1 , \tag{D.7}$$

and

$$\bar{\Phi}(s) = \frac{2}{3}(s-1)^{3/2} - \frac{1}{5}(s-1)^{5/2} + O\left((s-1)^{7/2}\right) \text{ as } s \to 1 \; . \tag{D.8}$$

We note from (D.8) that as $s \to 1$,

$$|\bar{\zeta}| = O(1) \text{ if and only if } |s-1| = O\left(\frac{1}{|\bar{\rho}|^{2/3}}\right) \; .$$

We also need the behavior of the above functions as $s \to \infty$. Expanding $\left(1 - \frac{1}{s}\right)^{1/2}$ about $s = \infty$ gives

$$\bar{\phi}(s) = 1 - \frac{1}{2s} + O(1/s^2) \quad \text{as} \quad s \to \infty \; , \tag{D.9}$$

and using (D.5) and (D.3), respectively, yields

$$\bar{\Phi}(s) = s - \frac{1}{2} - \log 2\sqrt{s} + O(1/s) \quad \text{as} \quad s \to \infty \; , \tag{D.10}$$

$$\bar{\zeta} = \bar{\rho}\left[s - \frac{1}{2} - \log 2\sqrt{s} + O(1/s)\right] \quad \text{as} \quad s \to \infty \; . \tag{D.10a}$$

We wish to find asymptotic representations of the functions $M_{\ell,0}(4\ell s)$ and $W_{\ell,0}(4\ell s)$ for $|\ell| \to \infty$ $(\arg \ell s = \pi/2)$ valid for all $s \in R_1$. In deriving them we shall use the known asymptotic representations of these functions for $|\ell s| >> |\ell|$. Thus for $|\ell s| >> |\ell|$ we use the result (Buchholz, 1953, Chapter 7)

$$M_{\ell,0}(4\ell s) \sim \frac{(4\ell s)^{-\ell} e^{2\ell s}}{\Gamma\left(\frac{1}{2} - \ell\right)} + \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\Gamma\left(\frac{1}{2} + \ell\right)} e^{-\pi i(\ell - \frac{1}{2})}$$

$$(-\pi/2 < \arg \ell s < 3\pi/2) \; .$$

Similarly for $|\ell s| >> |\ell|$ and $\arg \ell s = \pi/2$,

$$W_{\ell,0}(4\ell s) \sim (4\ell s)^{\ell} e^{-2\ell s} \; .$$

Finally $\arg \ell s = \pi/2$ implies $\arg \ell s\, e^{-\pi i} = -\pi/2 \in (-3\pi/2, 3\pi/2)$; hence for $|\ell s| >> |\ell|$

$$W_{-\ell, o}(4\ell s\, e^{-\pi i}) \sim (4\ell s\, e^{-\pi i})^{-\ell} e^{2\ell s} \quad .$$

These latter two representations enable us to directly find the desired asymptotic representations for $W_{\ell, o}(4\ell s)$ and $W_{-\ell, o}(4\ell s\, e^{-\pi i})$. Once we obtain them we can use the relation (Buchholz, 1953, Chapter 2, equation 20a)

$$M_{\ell, o}(4\ell s) = \frac{e^{-\ell \pi i} W_{-\ell, o}(4\ell s\, e^{-\pi i})}{\Gamma\left(\frac{1}{2} - \ell\right)} + \frac{e^{-\ell \pi i} e^{\pi i/2} W_{\ell, o}(4\ell s)}{\Gamma\left(\frac{1}{2} + \ell\right)} \qquad \text{(D.11)}$$

to obtain an asymptotic representation for $M_{\ell, o}(4\ell s)$.

We derive the desired asymptotic representations for $W_{\ell, o}(4\ell s)$ and $W_{-\ell, o}(4\ell s\, e^{-\pi i})$ by comparing their behavior for $|\ell s| >> |\ell|$ with the asymptotic behavior of the functions

$$\bar{v}^{(j)}(s) = (\pi/2)^{1/2} e^{\pm 5\pi i/12} \bar{\psi}(s) \bar{\zeta}^{1/3} H^{(j)}_{1/3}(\bar{\zeta}) \qquad (j = 1, 2)$$

which are solutions of the related differential equation

$$\frac{d^2Y}{ds^2} + \left[\bar{\rho}^2 \left(\frac{s-1}{s}\right) - \frac{\bar{\psi}''(s)}{\bar{\psi}(s)}\right] Y = 0 \quad .$$

The asymptotic behavior of these functions for large $\bar{\zeta}$ is given by

$$\bar{v}^{(j)}(s) = \bar{\psi}(s)\, \bar{\zeta}^{-1/6} e^{\pm i\bar{\zeta}} \left[1 + O(1/\bar{\zeta})\right] \qquad \begin{array}{l} j = 1, \ -\pi < \arg \bar{\zeta} < 2\pi \quad , \\ j = 2, \ -2\pi < \arg \bar{\zeta} < \pi \quad . \end{array}$$

But since $\arg \ell s = \pi/2$, we see from equation (D.10a) that as $s \to \infty$, $\arg \bar{\zeta} \to 0$. Therefore either of the above expressions is valid as $s \to \infty$; (D.10a) implies that

$\bar{v}^{(2)}(s)$ has the correct exponential behavior in s for $W_{\ell,0}(4\ell s)$ and that $\bar{v}^{(1)}(s)$ has the correct exponential behavior for $W_{-\ell,0}(4\ell s\, e^{-\pi i})$. Hence by Langer (1932), we see that for any $\bar{\zeta}$ such that $\arg \ell s = \pi/2$ and $s \in R_1$,

$$W_{-\ell,0}(4\ell s\, e^{-\pi i}) = \bar{C}_1\left[\bar{v}^{(1)}(s) + O(1/\bar{\rho})\right] \qquad \text{if } |\bar{\zeta}| \leqslant N, \tag{D.12a}$$

$$W_{-\ell,0}(4\ell s\, e^{-\pi i}) = \bar{C}_1\left[\bar{v}^{(1)}(s) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\bar{E}^{(1)}(\bar{\zeta})}{\bar{\rho}}\right] \qquad \text{if } |\bar{\zeta}| > N, \tag{D.12b}$$

$$W_{\ell,0}(4\ell s) = \bar{C}_2\left[\bar{v}^{(2)}(s) + O(1/\bar{\rho})\right] \qquad \text{if } |\bar{\zeta}| \leqslant N, \tag{D.13a}$$

$$W_{\ell,0}(4\ell s) = \bar{C}_2\left[\bar{v}^{(2)}(s) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\bar{E}^{(2)}(\zeta)}{\bar{\rho}}\right] \qquad \text{if } |\bar{\zeta}| > N, \tag{D.13b}$$

where N is a large positive number, the $\bar{E}^{(j)}(\bar{\zeta})$ $(j = 1, 2)$ are of the form

$$\bar{E}^{(j)}(\bar{\zeta}) = \bar{A}^{(j)} e^{i\bar{\zeta}} + \bar{B}^{(j)} e^{-i\bar{\zeta}}$$

with $\bar{A}^{(j)} = O(1)$ or $\bar{A}^{(j)} = 0$, $\bar{B}^{(j)} = O(1)$ or $\bar{B}^{(j)} = 0$, depending on the value of $\arg \bar{\zeta}$, and the $\bar{C}_j$ $(j = 1, 2)$ are determined by the relations

$$\bar{C}_1 = \lim_{s\to\infty} \frac{W_{-\ell,0}(4\ell s\, e^{-\pi i})}{\bar{v}^{(1)}(s)} = \lim_{s\to\infty} \frac{(4\ell s\, e^{-\pi i})^{-\ell} e^{2\ell s}}{\bar{\psi}(s)\,\bar{\zeta}^{-1/6} e^{i\bar{\zeta}}},$$

$$\bar{C}_2 = \lim_{s\to\infty} \frac{W_{\ell,0}(4\ell s)}{\bar{v}^{(2)}(s)} = \lim_{s\to\infty} \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\bar{\psi}(s)\bar{\zeta}^{-1/6} e^{-i\bar{\zeta}}}.$$

Using the definitions (D.3) and D.4) together with the expansions (D.9) and (D.10), we find

$$\bar{C}_1 = (-2i\ell)^{1/6} e^{-\ell \log \ell/e} e^{\ell\pi i}, \qquad \bar{C}_2 = (-2i\ell)^{1/6} e^{\ell \log \ell/e} .$$

Thus (D.12) and (D.13) become, for $\arg \ell s = \pi/2$ and $s \in R_1$

$$W_{-\ell,o}(4\ell s e^{-\pi i}) = (-2i\ell)^{1/6} e^{-\ell\log\ell/e} e^{\ell\pi i}\left[\bar{v}^{(1)}(s) + O(1/\bar{\rho})\right] \left(|\bar{\zeta}| \leqslant N\right), \quad \text{(D.14a)}$$

$$W_{-\ell,o}(4\ell s e^{-\pi i}) = (-2i\ell)^{1/6} e^{-\ell\log\ell/e} e^{\ell\pi i}\left[\bar{v}^{(1)}(s) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\bar{E}^{(1)}(\bar{\zeta})}{\bar{\rho}}\right] \left(|\bar{\zeta}| > N\right), \quad \text{(D.14b)}$$

$$W_{\ell,o}(4\ell s) = (-2i\ell)^{1/6} e^{\ell\log\ell/e}\left[\bar{v}^{(2)}(s) + O(1/\bar{\rho})\right] \qquad \left(|\zeta| \leqslant N\right), \quad \text{(D.15a)}$$

$$W_{\ell,o}(4\ell s) = (-2i\ell)^{1/6} e^{\ell\log\ell/e}\left[\bar{v}^{(2)}(s) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\bar{E}^{(2)}(\bar{\zeta})}{\bar{\rho}}\right]\left(|\zeta| > N\right). \quad \text{(D.15b)}$$

We now use (D.11) to estimate $M_{\ell,o}(4\ell s)$ for $\arg \ell s = \pi/2$ and $s \in R_1$. Since $\left|\arg \ell - \pi/2\right| \leqslant \pi/4$

$$\Gamma\left(\frac{1}{2}+\ell\right) = \sqrt{2\pi}\, e^{\ell\log\ell/e}\left[1 + O(1/\ell)\right] \qquad \left(|\ell| \gg 1\right) \quad \text{(D.16)}$$

(Erdélyi et al, 1953). Using (D.11), (D.14), (D.15), (D.16) and the relation (Erdélyi et al, 1953)

$$\Gamma\left(\frac{1}{2}-\ell\right)\Gamma\left(\frac{1}{2}+\ell\right) = \frac{\pi}{\cos \ell\pi}, \quad \text{(D.17)}$$

we find that for $\arg \ell s = \pi/2$ and $s \in R_1$

$$M_{\ell,o}(4\ell s) = \frac{(-2i\ell)^{1/6} e^{-\ell\pi i}}{\sqrt{2\pi}} \left[\bar{v}^{(1)}(s) + e^{\pi i/2}\, \bar{v}^{(2)}(s) + O(1/\bar{\rho})\right] \text{ if } \quad |\bar{\zeta}| \leqslant N, \quad \text{(D.18a)}$$

$$M_{\ell,o}(4\ell s) = \frac{(-2i\ell)^{1/6} e^{-\ell\pi i}}{\sqrt{2\pi}} \left[\bar{v}^{(1)}(s) + e^{\pi i/2}\bar{v}^{(2)}(s) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\left[\bar{E}^{(1)}(\bar{\zeta}) + \bar{E}^{(2)}(\bar{\zeta})\right]}{\bar{\rho}}\right]$$

$$\text{if} \quad |\bar{\zeta}| > N. \quad \text{(D.18b)}$$

For purposes of calculation, it is often more convenient to represent the Hankel functions $H^{(j)}_{1/3}(\bar{\zeta})$ in terms of the Airy function $Ai(z)$ defined by the integral representation

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}s^3 + zs\right)ds \ .$$

We can use the relations (Abramowitz and Stegun, 1964)

$$H^{(1)}_{1/3}(\bar{\zeta}) = e^{-\pi i/6}\sqrt{3/\bar{\sigma}}\left[Ai(-\bar{\sigma}) - iBi(-\bar{\sigma})\right] \quad \left(\bar{\sigma} = (\tfrac{3}{2}\bar{\zeta})^{2/3}\right),$$

$$\left[Ai(-\bar{\sigma}) - iBi(-\bar{\sigma})\right] = 2e^{-\pi i/3}Ai(-\bar{\sigma}e^{2\pi i/3}) \ ,$$

$$H^{(2)}_{1/3}(\bar{\zeta}) = e^{\pi i/6}\sqrt{3/\bar{\sigma}}\left[Ai(-\bar{\sigma}) + iBi(-\bar{\sigma})\right] \ ,$$

$$\left[Ai(-\bar{\sigma}) + iBi(-\bar{\sigma})\right] = 2e^{\pi i/3}Ai(-\bar{\sigma}e^{-2\pi i/3})$$

($Bi(z)$ is an Airy function linearly independent of $Ai(z)$ which does not enter in the final result and so is not defined here) to find

$$H^{(1)}_{1/3}(\bar{\zeta}) = e^{-\pi i/2}\frac{(2)^{4/3}(3)^{1/6}}{\bar{\zeta}^{1/3}}Ai(-\bar{\sigma}e^{2\pi i/3}) \ ,$$

$$H^{(2)}_{1/3}(\bar{\zeta}) = e^{\pi i/2}\frac{(2)^{4/3}(3)^{1/6}}{\bar{\zeta}^{1/3}}Ai(-\bar{\sigma}e^{-2\pi i/3}) \ .$$

Therefore equations (D.14) and (D.15) become, for $\arg \ell s = \pi/2$ and $s \in R_1$,

$$W_{-\ell,0}(4\ell s e^{-\pi i}) = C_\ell e^{-\pi i/12}e^{-\ell\log \ell/e}e^{\ell\pi i}\left[\bar{\psi}(s)Ai(-\bar{\sigma}e^{2\pi i/3}) + O(1/\bar{\rho})\right]$$
$$\left(|\bar{\zeta}| \leq N\right) \ , \qquad \text{(D.19a)}$$

$$W_{-\ell,0}(4\ell s\, e^{-\pi i}) = C_\ell e^{-\pi i/12} e^{-\ell \log \ell/e} e^{\ell \pi i}\left[\bar{\psi}(s)\mathrm{Ai}(-\bar{\sigma} e^{2\pi i/3}) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\bar{E}^{(1)}(\bar{\zeta})}{\bar{\rho}}\right]$$

$$\left(|\bar{\zeta}| > N\right), \tag{D.19b}$$

$$W_{\ell,0}(4\ell s) = C_\ell e^{\pi i/12} e^{\ell \log \ell/e}\left[\bar{\psi}(s)\mathrm{Ai}(-\bar{\sigma} e^{-2\pi i/3}) + O(1/\bar{\rho})\right] \quad \left(|\bar{\zeta}| < N\right), \tag{D.20a}$$

$$W_{\ell,0}(4\ell s) = C_\ell e^{\pi i/12} e^{\ell \log \ell/e}\left[\bar{\psi}(s)\mathrm{Ai}(-\bar{\sigma} e^{-2\pi i/3}) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\bar{E}^{(2)}(\bar{\zeta})}{\bar{\rho}}\right]$$

$$\left(|\bar{\zeta}| > N\right), \tag{D.20b}$$

with $C_\ell = 2(3)^{1/6}\pi^{1/2}(-i\ell)^{1/6}$ and $\bar{\sigma} = (\frac{3}{2}\bar{\zeta})^{2/3}$. Finally, since (Abramowitz and Stegun, 1964)

$$\mathrm{Ai}(-\bar{\sigma}) = e^{-\pi i/3}\mathrm{Ai}(-\bar{\sigma} e^{2\pi i/3}) + e^{\pi i/3}\mathrm{Ai}(-\bar{\sigma} e^{-2\pi i/3}) ,$$

equations (D.18) become

$$M_{\ell,0}(4\ell s) = D_\ell e^{\pi i/4} e^{-\ell \pi i}\left[\bar{\psi}(s)\mathrm{Ai}(-\bar{\sigma}) + O(1/\bar{\rho})\right] \quad \text{if} \quad |\bar{\zeta}| \leqslant N , \tag{D.21a}$$

$$M_{\ell,0}(4\ell s) = D_\ell e^{\pi i/4} e^{-\ell \pi i}\left[\bar{\psi}(s)\mathrm{Ai}(-\bar{\sigma}) + \frac{\bar{\psi}(s)\bar{\zeta}^{-1/6}\left[\bar{E}^{(1)}(\bar{\zeta}) + \bar{E}^{(2)}(\bar{\zeta})\right]}{\bar{\rho}}\right]$$

$$\text{if} \quad |\bar{\zeta}| > N , \tag{D.21b}$$

where $D_\ell = C_\ell/\sqrt{2\pi}$, $\bar{\sigma}$ is as above, $\arg \ell s = \pi/2$, and $s \in R_1$.

D.2 Bessel Function Representations

In this section we study $M_{\ell,0}(4\ell s)$ and $W_{\ell,0}(4\ell s)$ for $s \in R_0$, the closed domain illustrated in Fig. D-2. We now define $\rho = 2\ell$ in equation (D.1) obtaining

$$\frac{d^2u}{ds^2} + \left[\rho^2\left(\frac{1-s}{s}\right) + \frac{1}{4s^2}\right] u = 0 . \tag{D.22}$$

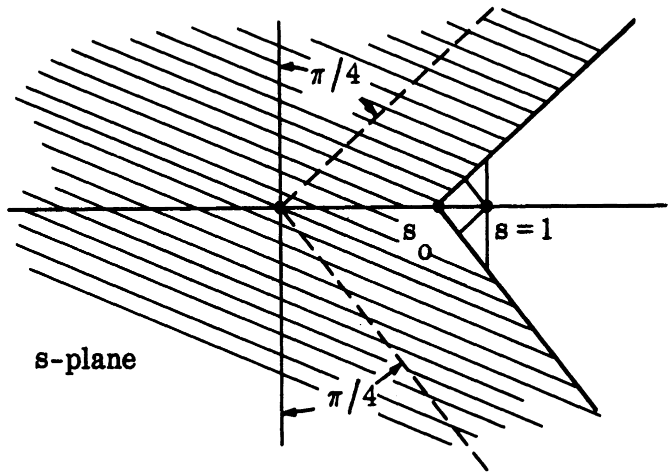

FIG. D-2: REGION OF VALIDITY OF BESSEL FUNCTION REPRESENTATION.

Suppose we consider ρ to be a complex parameter such that $|\rho| \gg 1$, and s to belong to a simply connected, closed domain R_o of the complex plane which includes the point $s = 0$, but excludes the point $s = 1$. Then equation (D.22) is of the type studied in Langer (1935) provided R_o has the required properties. For the s-plane described by $\arg s \in \left[-\pi/4, 7\pi/4\right)$, Langer's results may be applied in a closed domain containing $s = 0$ of the region described as follows (Taylor, 1939): Let s_o be a real, positive point in the s-plane such that

$$\frac{\sqrt{2}}{2}(1 - s_o) \gg \frac{1}{|\rho|^{2/3}} .$$

Then the region R_o is the union of the sub-region

$$\Omega_1 = \left\{ s \,\middle|\, |s| \leqslant \frac{s_o \sin\frac{\pi}{4}}{\sin(\frac{\pi}{4} - \arg s)} \text{ and } 0 \leqslant \arg s \leqslant \pi/4 \text{ , or} \right.$$

$$\left. |s| \leqslant \frac{s_o \sin\frac{\pi}{4}}{\sin(\frac{\pi}{4} + \arg s)} \text{ and } -\pi/4 \leqslant \arg s \leqslant 0 \text{ , or } \pi/4 < \arg s < 3\pi/4 \right\}$$

and the sub-region $\Omega_2 = \left\{ s \,\middle|\, 3\pi/4 \leqslant \arg s \leqslant 7\pi/4 \right\}$. (To emphasize the arguments of s in the regions Ω_1 and Ω_2 we shall simply refer to these sub-regions as $\arg s \in \left[-\pi/4, 3\pi/4\right)$ and $\arg s \in \left[3\pi/4, 7\pi/4\right)$, respectively.) It should be noted that R_0 and the region R_1 for the Airy function representation are not unique regions of validity for the respective representations. We have chosen them simply to ensure a valid representation in any portion of the s-plane arising from the applications.

Let $\phi^2(s) = \frac{1}{s}(1-s)$, where $\phi(s)$ is to be the root of $\phi^2(s)$ determined by the relation

$$\lim_{s \to 0} \left\{ s^{1/2} \phi(s) \right\} = 1.$$

We also define

$$\Phi(s) = \int_0^s \phi(t)dt \,, \qquad \zeta = \rho\Phi(s) \tag{D.23}$$

$$\psi(s) = \left[\phi(s)\Phi(s)\right]^{-1/2} \quad \text{with } \psi(0) = \lim_{s \to 0} \psi(s) \,. \tag{D.24}$$

We show below that for large s we can write

$$\Phi(s) = \pm i \int_0^s \left(\frac{t-1}{t}\right)^{1/2} dt \,, \quad \begin{array}{l} + \text{ if } \arg s \in \left[-\pi/4, 3\pi/4\right) , \\ - \text{ if } \arg s \in \left[3\pi/4, 7\pi/4\right) . \end{array} \tag{D.25}$$

(The relation

$$\Phi(s) = -i \int_0^s \left(\frac{t-1}{t}\right)^{1/2} dt$$

also holds for any s such that arg s is in a neighborhood of π.) Integrating by parts, and then following with an elementary integration, we obtain for any s

$$\int_0^s \left(\frac{t-1}{t}\right)^{1/2} dt = s\left(1-\frac{1}{s}\right)^{1/2} + \log\left(\sqrt{1-s} - \sqrt{-s}\right), \tag{D.26}$$

the square root being defined above. In order to derive the necessary asymptotic representations the behavior of these functions as $s \to 0$ and $s \to \infty$ must be investigated. Clearly,

$$\phi(s) = \frac{1}{s^{1/2}} - \frac{1}{2}s^{1/2} + O(s^{3/2}) \quad \text{as } s \to 0. \tag{D.27}$$

Therefore equations (D.23) and (D.24) yield

$$\Phi(s) = 2s^{1/2} - \frac{1}{3}s^{3/2} + O(s^{5/2}) \qquad \text{as } s \to 0\,, \tag{D.28}$$

$$\zeta = \rho\Phi(s) = 2\rho s^{1/2}\left[1 - \frac{1}{6}s + O(s^2)\right] \quad \text{as } s \to 0\,, \tag{D.28a}$$

$$\Psi(s) = \frac{1}{2^{1/2}}\left[1 + \frac{1}{3}s + O(s^2)\right] \qquad \text{as } s \longrightarrow 0\,, \tag{D.29}$$

and

$$\frac{\Phi(s)}{\phi(s)} = 2s\left[1 + \frac{1}{3}s + O(s^2)\right] \qquad \text{as } s \longrightarrow 0\,. \tag{D.30}$$

In examining the behavior at infinity we note that for $\arg s \in \left[-\pi/4, 3\pi/4\right)$, $-s = e^{\pi i}s$; hence as $s \longrightarrow \infty$, $\phi(s) \longrightarrow e^{\pi i/2}$. For $\arg s \in \left[3\pi/4, 7\pi/4\right)$, $-s = e^{-\pi i}s$; hence as $s \longrightarrow \infty$, $\phi(s) \longrightarrow e^{-\pi i/2}$. Therefore expanding $(1-\frac{1}{s})^{1/2}$ about $s = \infty$ yields

$$\phi(s) = i\left[1 - \frac{1}{2s} + O(1/s^2)\right] \quad \text{as } |s| \longrightarrow \infty,\ \arg s \in \left[-\pi/4, 3\pi/4\right)\,, \tag{D.31a}$$

$$\phi(s) = -i\left[1 - \frac{1}{2s} + O(1/s^2)\right] \quad \text{as } |s| \longrightarrow \infty,\ \arg s \in \left[3\pi/4, 7\pi/4\right)\,. \tag{D.31b}$$

The definitions (D.23) and(D.26) then imply

$$\zeta = \rho\Phi(s) = i\rho\left[s - \frac{1}{2} - \log(2i\sqrt{s}) + O(1/s)\right] \quad \text{as } |s| \to \infty, \ \arg s \in \left[-\pi/4, 3\pi/4\right), \tag{D.32a}$$

$$\zeta = \rho\Phi(s) = -i\rho\left[s - \frac{1}{2} - \log(-2i\sqrt{s}) + O(1/s)\right] \text{ as } |s| \to \infty, \ \arg s \in \left[3\pi/4, 7\pi/4\right). \tag{D.32b}$$

We can now directly find an asymptotic representation of $M_{\ell,0}(4\ell s)$ which is valid in R_0. Because of the regularity of $M_{\ell,0}(4\ell s)$ at the origin $s = 0$, the theory in Langer (1935) asserts that for any ζ such that $\arg \ell s = \pi/2$ and $s \in R_0$

$$M_{\ell,0}(4\ell s) = c\left[v^0(s) + \frac{\psi(s)\zeta^5 O(1)}{\rho^4}\right] \quad \text{if } |\zeta| \leqslant N, \tag{D.33a}$$

$$M_{\ell,0}(4\ell s) = c\left[v^0(s) + \frac{\psi(s)\zeta^{1/2}\left[e^{i\zeta}O(1) + e^{-i\zeta}O(1)\right]}{\rho}\right] \text{ if } |\zeta| > N, \tag{D.33b}$$

where $v^0(s) = \psi(s)\zeta J_0(\zeta)$, N is a large positive number and c is a function of ℓ to be determined by a comparison of the behavior of $v^0(s)$ and $M_{\ell,0}(4\ell s)$ as $s \to 0$. $v^0(s)$ is a solution of the differential equation

$$\frac{d^2Y}{ds^2} + \left[\rho^2\left(\frac{1-s}{s}\right) + \frac{1}{4s^2} - \frac{\psi''(s)}{\psi(s)}\right] Y = 0,$$

which is called the related equation of (D.25). We find

$$c = \lim_{s \to 0} \frac{M_{\ell,0}(4\ell s)}{v^0(s)},$$

and thus

$$c = \frac{(4\ell)^{1/2}}{2^{1/2}\rho} = \frac{1}{\rho^{1/2}}.$$

Therefore, equations (D.33a) and (D.33b) become

$$M_{\ell,o}(4\ell s) = \rho^{1/2}\left[\frac{\Phi(s)}{\phi(s)}\right]^{1/2} J_o(\zeta) + \frac{\Psi(s)\zeta^5 O(1)}{\rho^{9/2}} \quad \left(|\zeta| \leq N\right), \tag{D.34a}$$

$$M_{\ell,o}(4\ell s) = \rho^{1/2}\left[\frac{\Phi(s)}{\phi(s)}\right]^{1/2} J_o(\zeta) + \frac{\Psi(s)\zeta^{1/2}\left[e^{i\zeta}O(1) + e^{-i\zeta}O(1)\right]}{\rho^{3/2}} \quad \left(|\zeta| > N\right), \tag{D.34b}$$

for $\arg \ell s = \pi/2$ and $s \in R_o$.

In Chapter II we are primarily concerned with the case where $|\rho s| \ll 1$ while $|\rho| \to \infty$ with $\arg\rho = \arg s - \pi/2$. In this case $s \to 0$, and we can use expansions (D.28) through (D.30) to find for $|\zeta| \leq N$

$$M_{\ell,o}(4\ell s) = (2\rho s)^{1/2} J_o(\zeta)\left[1 + O(s)\right] + O\left((\rho s)^{1/2} s^2\right) .$$

Upon retaining only the order term of the lowest order in s this becomes

$$M_{\ell,o}(4\ell s) = (2\rho s)^{1/2} J_o(\zeta)\left[1 + O(s)\right] \qquad (\arg \ell s = \pi/2,\ s \in R_o), \tag{D.35a}$$

except at a zero of $J_o(\zeta)$, where then the additional term is the required estimate. For $|\zeta| > N$ and ζ not a zero of $J_o(\zeta)$ we find

$$M_{\ell,o}(4\ell s) = (2\rho s)^{1/2} J_o(\zeta)\left[1 + O(s) + O(1/\rho)\right] ,$$

while at a zero of $J_o(\zeta)$ the estimate

$$M_{\ell,o}(4\ell s) = O\left(\frac{(\rho s)^{1/4}}{\rho^{5/4}}\right) (\arg \ell s = \pi/2,\ s \in R_o)$$

is valid. Upon comparing order terms we find the next to last result reduces to

$$M_{\ell,o}(4\ell s) = (2\rho s)^{1/2} J_o(\zeta)\left[1 + O(1/\rho)\right] \quad (\arg \ell s = \pi/2,\ s \in R_o). \tag{D.35b}$$

In order to derive an asymptotic representation for $W_{\ell,0}(4\ell s)$ for $s \in R_0$ we consider the two sub-regions Ω_1 and Ω_2 separately. In these two sub-regions we can directly find an asymptotic representation of $W_{\ell,0}(4\ell s)$ by a procedure analogous to the one used in Section D.1. We again make use of the fact that $\arg \ell s = \pi/2 \in (-3\pi/2,\ 3\pi/2)$ implies

$$W_{\ell,0}(4\ell s) \sim (4\ell s)^{\ell} e^{-2\ell s}$$

for $|\ell s| \gg |\ell|$. We compare this asymptotic representation with the following solutions of the related equation

$$v^{(j)}(s) = (\pi/2)^{1/2}\, e^{\pm \pi i/4}\, \Psi(s)\zeta\, H_0^{(j)}(\zeta) \qquad (j = 1, 2)$$

whose asymptotic behavior for large $|\zeta|$ is given by

$$v^{(j)}(s) = \Psi(s)\zeta^{1/2} e^{\pm i\zeta}\left[1 + O(1/\rho)\right] \quad \begin{array}{l} j = 1,\ -\pi < \arg\zeta < 2\pi \ , \\ j = 2,\ -2\pi < \arg\zeta < \pi . \end{array}$$

Consider then $\arg s \in \left[-\pi/4, 3\pi/4\right)$. Since $\arg \ell s = \pi/2$, equation (D.32a) implies that as $s \to \infty$, $\arg\zeta \to \pi$; consequently only $v^{(1)}(s)$ has the asymptotic representation given above as $s \to \infty$. But we see from equation (D.32a) that $v^{(1)}(s)$ has the correct exponential dependence in s for $W_{\ell,0}(4\ell s)$ and hence according to Langer (1935) we see that for any ζ such that $\arg \ell s = \pi/2$ and $s \in \Omega_1$ (see the beginning of Section D.2 for the definition of Ω_1)

$$W_{\ell,0}(4\ell s) = D\left[v^{(1)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right] \quad \text{if } |\zeta| \leq N\ , \qquad \text{(D.36a)}$$

$$W_{\ell,0}(4\ell s) = D\left[v^{(1)}(s) + \frac{\Psi(s)\zeta^{1/2} E(\zeta)}{\rho}\right] \quad \text{if } |\zeta| > N\ , \qquad \text{(D.36b)}$$

where N is a large positive number, $E(\zeta)$ is of the form

$$E(\zeta) = A e^{i\zeta} + B e^{-i\zeta}$$

with $A = O(1)$ or $A = 0$, $B = O(1)$ or $B = 0$, depending on the value of $\arg\zeta$, and D is determined by the relation

$$D = \lim_{s\to\infty} \frac{W_{\ell,o}(4\ell s)}{v^{(1)}(s)} = \lim_{s\to\infty} \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\Psi(s)\zeta^{1/2} e^{i\zeta}} .$$

Using the definitions (D.23), (D.24) and the expansions (D.31a), (D.32a), we obtain

$$D = \frac{(i)^{1/2}}{(2\ell)^{1/2}} e^{\ell\log - \ell/e} \quad ;$$

therefore equations (D.36) become

$$W_{\ell,o}(4\ell s) = \frac{(i)^{1/2}}{(2\ell)^{1/2}} e^{\ell\log - \ell/e} \left[v^{(1)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right]\left(|\zeta| \leqslant N\right), \quad \text{(D.37a)}$$

$$W_{\ell,o}(4\ell s) = \frac{(i)^{1/2}}{(2\ell)^{1/2}} e^{\ell\log - \ell/e} \left[v^{(1)}(s) + \frac{\Psi(s)\zeta^{1/2} E(\zeta)}{\rho}\right]\left(|\zeta| > N\right), \quad \text{(D.37b)}$$

for $\arg\ell s = \pi/2$ and $s \in \Omega_1$.

As in the study of $M_{\ell,o}(4\ell s)$, we further consider, for use in Chapter II, equations (D.37) if $|2\ell s| \ll 1$. We can again use expansions (D.28) through (D.30) to find for $|\zeta| \leqslant N$

$$W_{\ell,o}(4\ell s) = i(\pi)^{1/2} e^{\ell\log - \ell/e} (2\ell s)^{1/2} H_o^{(1)}(\zeta) \left[1 + O(s) + O(1/\ell)\right],$$

except at a zero of $H_o^{(1)}(\zeta)$ (there are no zeros of $H_o^{(1)}(\zeta)$ on the principal branch, Erdélyi et al, 1953) where the additional term in (D.37a) is the required estimate. Upon comparing order terms we find the last result becomes

$$W_{\ell,o}(4\ell s) = i(\pi)^{1/2} e^{\ell\log - \ell/e} (2\ell s)^{1/2} H_o^{(1)}(\zeta) \left[1 + O(1/\ell)\right] \quad (\arg\ell s = \pi/2,\ s \in \Omega_1). \quad \text{(D.38a)}$$

Similarly for $|\zeta| > N$ (with the above exception for zeros of $H_0^{(1)}(\zeta)$)

$$W_{\ell,0}(4\ell s) = i(\pi)^{1/2} e^{\ell \log - \ell/e} (2\ell s)^{1/2} H_0^{(1)}(\zeta)\left[1 + O(1/\ell)\right] \quad (\arg \ell s = \pi/2, \; s \in \Omega_1). \tag{D.38b}$$

To complete the investigation of $W_{\ell,0}(4\ell s)$ we now consider $\arg s \in [3\pi/4, 7\pi/4)$. In this region we also need an asymptotic representation of $W_{-\ell,0}(4\ell s e^{-\pi i})$. As previously, we make use of the fact that $\arg \ell s e^{-\pi i} = -\pi/2 \in (-3\pi/2, 3\pi/2)$ implies

$$W_{\ell,0}(4\ell s e^{-\pi i}) \sim (4\ell s e^{-\pi i})^{-\ell} e^{2\ell s}$$

for $|\ell s| \gg |\ell|$. From (D.32b) we see that as $s \to \infty$, $\arg \zeta \to 0$; thus both asymptotic representations for $v^{(j)}(s)$, $j = 1, 2$ are valid as $s \to \infty$. Hence, (D.32b) implies that $v^{(2)}(s)$ ($v^{(1)}(s)$) has the correct exponential dependence in s for $W_{\ell,0}(4\ell s)$ ($W_{-\ell,0}(4\ell s e^{-\pi i})$). Therefore, according to Langer (1935), for any ζ such that $\arg \ell s = \pi/2$ and $s \in \Omega_2$

$$W_{-\ell,0}(4\ell s e^{-\pi i}) = D_1\left[v^{(1)}(s) + \frac{\psi(s)\zeta \log \zeta\, O(1)}{\rho}\right] \qquad \text{if } |\zeta| \leqslant N, \tag{D.39a}$$

$$W_{-\ell,0}(4\ell s e^{-\pi i}) = D_1\left[v^{(1)}(s) + \frac{\psi(s)\zeta^{1/2} E^{(1)}(\zeta)}{\rho}\right] \qquad \text{if } |\zeta| > N, \tag{D.39b}$$

$$W_{\ell,0}(4\ell s) = D_2\left[v^{(2)}(s) + \frac{\psi(s)\zeta \log \zeta\, O(1)}{\rho}\right] \qquad \text{if } |\zeta| \leqslant N, \tag{D.40a}$$

$$W_{\ell,0}(4\ell s) = D_2\left[v^{(2)}(s) + \frac{\psi(s)\zeta^{1/2} E^{(2)}(\zeta)}{\rho}\right] \qquad \text{if } |\zeta| > N, \tag{D.40b}$$

where N is a large positive number, the $E^{(j)}(\zeta)$, $j = 1, 2$ are of the form

$$E^{(j)}(\zeta) = A^{(j)} e^{i\zeta} + B^{(j)} e^{-i\zeta}$$

with $A^{(j)} = O(1)$ or $A^{(j)} = 0$, $B^{(j)} = O(1)$ or $B^{(j)} = 0$ depending on the value of $\arg\zeta$, and the D_j, $j = 1, 2$ are determined by the relations

$$D_1 = \lim_{s\to\infty} \frac{W_{-\ell,0}(4\ell s e^{-\pi i})}{v^{(1)}(s)} = \lim_{s\to\infty} \frac{(4\ell s e^{-\pi i})^{-\ell} e^{2\ell s}}{\Psi(s)\zeta^{1/2} e^{i\zeta}} ,$$

$$D_2 = \lim_{s\to\infty} \frac{W_{\ell,0}(4\ell s)}{v^{(2)}(s)} = \lim_{s\to\infty} \frac{(4\ell s)^{\ell} e^{-2\ell s}}{\Psi(s)\zeta^{1/2} e^{-i\zeta}} .$$

Using the definitions (D.23), (D.24) and the expansions (D.31b), (D.32b) we obtain

$$D_1 = \frac{(-i)^{1/2}}{(2\ell)^{1/2}} e^{-\ell\log -\ell/e} e^{\ell\pi i} , \quad D_2 = \frac{(-i)^{1/2}}{(2\ell)^{1/2}} e^{\ell\log -\ell/e} .$$

Thus equations (D.39) and (D.40) become, for $\arg \ell s = \pi/2$ and $s \in \Omega_2$,

$$W_{-\ell,0}(4\ell s e^{-\pi i}) = \frac{(-i)^{1/2}}{(2\ell)^{1/2}} e^{-\ell\log -\ell/e} e^{\ell\pi i}\left[v^{(1)}(s) + \frac{\Psi(s)\zeta \log\zeta\, O(1)}{\rho}\right]\left(|\zeta| \leq N\right), \tag{D.41a}$$

$$W_{-\ell,0}(4\ell s e^{-\pi i}) = \frac{(-i)^{1/2}}{(2\ell)^{1/2}} e^{-\ell\log -\ell/e} e^{\ell\pi i}\left[v^{(1)}(s) + \frac{\Psi(s)\zeta^{1/2} E^{(1)}(\zeta)}{\rho}\right]\left(|\zeta| > N\right), \tag{D.41b}$$

$$W_{\ell,0}(4\ell s) = \frac{(-i)^{1/2}}{(2\ell)^{1/2}} e^{\ell\log -\ell/e}\left[v^{(2)}(s) + \frac{\Psi(s)\zeta\log\zeta\, O(1)}{\rho}\right] \left(|\zeta| \leq N\right), \tag{D.42a}$$

$$W_{\ell,0}(4\ell s) = \frac{(-i)^{1/2}}{(2\ell)^{1/2}} e^{\ell\log -\ell/e} \left[v^{(2)}(s) + \frac{\Psi(s)\zeta^{1/2} E^{(2)}(\zeta)}{\rho}\right]\left(|\zeta| > N\right) . \tag{D.42b}$$

APPENDIX E

ASYMPTOTIC REPRESENTATION OF $-e^{-ikR}/R$

In Appendix A, as well as in Buchholz (1953), an integral representation for $-e^{-ikR_{\Xi}}/R_{\Xi}$ (R_{Ξ} being the distance from $(\Xi, 0)$ to (ξ, η)) is derived. The result is

$$\frac{-e^{-ikR_{\Xi}}}{R_{\Xi}} = \frac{1}{2\pi i}\int_{\substack{-\infty-i\sigma \\ (|\sigma|<k)}}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) v_1(\xi_1,-\lambda)v_2(\xi_2,-\lambda)v_2(\eta,\lambda)\ . \tag{E.1}$$

In a similar fashion we can derive

$$\frac{-e^{-ikR_H}}{R_H} = \frac{1}{2\pi i}\int_{\substack{-\infty-i\sigma \\ (|\sigma|<k)}}^{\infty-i\sigma} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(-\frac{\lambda}{2ik}+\frac{1}{2}\right) v_1(\xi,-\lambda)v_1(\eta_1,\lambda)v_2(\eta_2,\lambda)\ . \tag{E.2}$$

where R_H is the distance from $(0, H)$ to (ξ, η). If we refer to the asymptotic representations of Appendix C, we see that the contour of integration in (E.1) can be closed around the top (bottom), and thus $-e^{ikR_{\Xi}}/R_{\Xi}$ can be replaced by a convergent residue series expansion, provided the inequality $\sqrt{\xi_1}-\sqrt{\xi_2}+\sqrt{\eta}<0$ $(\sqrt{\xi_1}+\sqrt{\xi_2}-\sqrt{\eta}<0)$ is satisfied. The same is true for (E.2) provided the inequality $-\sqrt{\xi}+\sqrt{\eta_1}+\sqrt{\eta_2}<0$ $(\sqrt{\xi}+\sqrt{\eta_1}-\sqrt{\eta_2}<0)$ holds. We note that there are regions of the interior of the paraboloid of revolution in which none of the inequalities listed above (top or bottom) are satisfied. Thus there are no convergent residue series representations in these regions. Hence if $k\xi_1 \gg 1$, $k\xi_2 \gg 1$, $k\eta \gg 1$ ($k\eta_1 \gg 1$, $k\eta_2 \gg 1$, $k\xi \gg 1$), the derivation of an asymptotic representation of

$$\frac{-e^{ikR_{\Xi}}}{R_{\Xi}}\left(\frac{-e^{ikR_H}}{R_H}\right)$$

directly from the integral representation (E.1) ((E.2)) is considered in order to provide some insight into the derivation of an asymptotic representation of the field.

E. 1 Asymptotic Representation of $-e^{-ikR}/R$ for the Source at $(\Xi, 0)$

Since we only want to use this derivation as a guide, we assume $\sigma = 0$ for brevity. Let us define

$$I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) = \frac{1}{2\pi i} \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) v_1(\xi_1, -\lambda) v_2(\xi_2, -\lambda) v_2(\eta, \lambda) \tag{E.3}$$

so that (E. 1) becomes

$$-\frac{e^{\mp ikR_{\Xi}}}{R_{\Xi}} = \int_{-\infty}^{\infty} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) \ . \tag{E.4}$$

We write (E.4) as

$$-\frac{e^{-ikR_{\Xi}}}{R_{\Xi}} = \sum_{j=1}^{5} \int_{C_j} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) \ , \tag{E.4a}$$

where for some $M_i > 0$ $(i = 1, \ldots, 7)$ with $M_3 \gg 1$ and $M_5 \gg 1$

$$C_1 = \left\{\lambda \middle| -\infty < \lambda < -M_1 k^2 \eta < 0\right\} \ ,$$

$$C_2 = \left\{\lambda \middle| -(M_1 + M_2) k^2 \eta < \lambda < -M_3 k\right\} \ ,$$

$$C_3 = \left\{\lambda \middle| -(M_3 + M_4) k < \lambda < (M_4 + M_5) k\right\} ,$$

$$C_4 = \left\{\lambda \middle| M_5 k < \lambda < (M_6 + M_7) k^2 \eta\right\} ,$$

$$C_5 = \left\{\lambda \middle| M_7 k^2 \eta < \lambda < \infty\right\} ,$$

and we examine each integral separately.

The behavior of the Whittaker functions over C_1 and C_5 is governed by the asymptotic representations of Appendix C (equations (C.6), (C, 12), (C.13)) respectively for C_1 and (C.6), (C.10), (C.13) respectively for C_5). Using these representations, we see that the integrand over C_1 and C_5 is exponentially small. The behavior of the Whittaker functions over C_2 and C_4 is governed by the asymptotic representations of Appendix D. We consider first C_2. Since $\arg\lambda = \pi$, $\arg s_{\xi_j} = \pi$ ($s_{\xi_j} = k^2\lambda^{-1}\xi_j$). Then from (D.34b) and (D.42b), respectively, we find

$$v_1(\xi_1, -\lambda) \sim (2ik\xi_1)^{-1/2} \left(-\frac{\lambda}{ik}\right)^{1/2} \left[\frac{\Phi(s_{\xi_1})}{\phi(s_{\xi_1})}\right]^{1/2} J_o(\zeta_{\xi_1}) , \qquad \text{(E.5)}$$

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} (-i)^{1/2} \exp\left\{-\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} \frac{e^{-i\zeta_{\xi_2}}}{\left(\phi(s_{\xi_2})\right)^{1/2}} , \qquad \text{(E.6)}$$

with

$$\phi^2(s_{\xi_j}) = \frac{1}{s_{\xi_j}}(1 - s_{\xi_j}) ,$$

$$\Phi(s_{\xi_j}) = \int_0^{s_{\xi_j}} \left(\frac{t-1}{t}\right)^{1/2} dt ,$$

$$\zeta_{\xi_j} = \frac{\lambda}{k}\Phi(s_{\xi_j}) .$$

In addition, $\arg\lambda = \pi$ implies $\arg s_\eta = 0$. Thus, from (D.15) we obtain

$$v_2(\eta, \lambda) \sim (2ik\eta)^{-1/2} (-\lambda/k)^{1/6} \exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\} \cdot (\pi/2)^{1/2} \cdot$$

$$\cdot e^{-5\pi i/12}\, \bar{\psi}(s_\eta)\bar{\zeta}_\eta^{1/3} H^{(2)}_{1/3}(\bar{\zeta}_\eta) , \qquad \text{(E.7)}$$

with

$$\bar{\phi}^2(s_\eta) = \frac{s_\eta - 1}{s_\eta} ,$$

$$\bar{\Phi}(s_\eta) = \int_1^{s_\eta} \left(\frac{s-1}{s}\right)^{1/2} ds \quad \text{if} \quad s_\eta > 1 ,$$

$$\bar{\Phi}(s_\eta) = i \int_1^{s_\eta} \left(\frac{1-s}{s}\right)^{1/2} ds \quad \text{if} \quad s_\eta < 1 ,$$

$$\bar{\zeta}_\eta = -\frac{\lambda}{k} \bar{\Phi}(s_\eta) ,$$

$$\bar{\psi}(s_\eta) = \left[\bar{\Phi}(s_\eta)\right]^{1/6} \left[\bar{\phi}(s_\eta)\right]^{-1/2} .$$

If $s_\eta > 1$, then $\arg \bar{\zeta}_\eta = 0$, and a fixed distance away from $s_\eta = 1$, $H^{(2)}_{1/3}(\bar{\zeta}_\eta)$ is proportional to $e^{if(\eta,\lambda)}$, where $f(\eta, \lambda)$ is real. Hence, substituting (E.5), (E.6) and (E.7) into the integrand, we see that it is exponentially small for $s_\eta > 1$ and bounded away from $s_\eta = 1$. On the other hand, $s_\eta < 1$ implies $\arg \bar{\zeta}_\eta = 3\pi/2$, which in turn implies that away from $s_\eta = 1$, $H^{(2)}_{1/3}(\bar{\zeta}_\eta)$ is exponentially increasing. This exponential increase is proportional to

$$\exp\left\{-\frac{\lambda}{k} \int_{s_\eta}^{1} dt \left(\frac{1-t}{t}\right)^{1/2}\right\} .$$

Consequently, the integrand has order

$$O\left(\frac{\exp\left\{-\frac{\lambda}{k} \int_{s_\eta}^{1} dt \left(\frac{1-t}{t}\right)^{1/2}\right\}}{e^{-\pi\lambda/2k}}\right) .$$

But in the range of C_2, $s_\eta = O(1)$. Since the trigonometric substitution $t = \sin^2\theta$ shows that $\int_0^1 \left(\frac{1-t}{t}\right)^{1/2} dt = \pi/2$, the integrand is also exponentially small for $s_\eta < 1$ and bounded away from $s_\eta = 1$. Finally, $H^{(2)}_{1/3}(\bar{\zeta}_\eta)$ is bounded around $s_\eta = 1$, and so the integrand is exponentially small there. Therefore, the integral over C_2, as well as the integrals over C_1 and C_5, does not contribute materially to the asymptotic representation of $-e^{-ikR_{\underline{=}}}/R_{\underline{=}}$.

To estimate the contribution of C_3, we first recall (Buchholz, 1953, Chapter 7) the asymptotic representations valid in the range $|\lambda/k| \leqslant O(1)$. They are

$$v_1(\xi_1, -\lambda) \sim (2ik\xi_1)^{-1/2} \left[\frac{(2ik\xi_1)^{\lambda/2ik}\, e^{ik\xi_1}}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2ik}\right)} + \frac{(2ik\xi_1)^{-\lambda/2ik}\, e^{\pi\lambda/2k}\, e^{\pi i/2}}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2ik}\right)}\, e^{-ik\xi_1} \right], \tag{E.8}$$

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} (2ik\xi_2)^{-\lambda/2ik}\, e^{-ik\xi_2}, \tag{E.9}$$

$$v_2(\eta, \lambda) \sim (2ik\eta)^{-1/2} (2ik\eta)^{\lambda/2ik}\, e^{-ik\eta}. \tag{E.10}$$

Substituting (E.8), (E.9) and (E.10) into (E.3), we find

$$I_{\underline{=}}(\lambda, \xi_1, \xi_2, \eta) \sim I^+_{\underline{=}}(\lambda, \xi_1, \xi_2, \eta) + I^-_{\underline{=}}(\lambda, \xi_1, \xi_2, \eta), \tag{E.11}$$

where

$$I^+_{\underline{=}}(\lambda, \xi_1, \xi_2, \eta) = C(k, \xi_1, \xi_2, \eta)\, e^{ik\xi_1}\, e^{-ik(\xi_2+\eta)}\, \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \cdot$$
$$\cdot (2ik\xi_1)^{\lambda/2ik} (2ik\xi_2)^{-\lambda/2ik} (2ik\eta)^{\lambda/2ik}, \tag{E.12}$$

$$I_{\mp}^{-}(\lambda, \xi_1, \xi_2, \eta) = C(k, \xi_1, \xi_2, \eta)\, e^{-ik\xi_1} e^{-ik(\xi_2+\eta)}\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \cdot$$

$$\cdot\, (2ik\xi_1)^{-\lambda/2ik} (2ik\xi_2)^{-\lambda/2ik} (2ik\eta)^{\lambda/2ik} e^{\pi\lambda/2k} e^{\pi i/2}, \tag{E.13}$$

with

$$C(k, \xi_1, \xi_2, \eta) = \frac{(2ik\xi_1)^{-1/2} (2ik\xi_2)^{-1/2} (2ik\eta)^{-1/2}}{2\pi i} .$$

We consider first the integral $\int_{C_3} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta)$.

Inserting (E. 12) we obtain

$$C(k, \xi_1, \xi_2, \eta)\, e^{ik\xi_1} e^{-ik(\xi_2+\eta)} \int_{C_3} d\lambda\ \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \cdot$$

$$\cdot\, (2ik\xi_1)^{\lambda/2ik} (2ik\xi_2)^{-\lambda/2ik} (2ik\eta)^{\lambda/2ik} .$$

We estimate this integral by using the relation

$$\int_{C_3} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) = \sum_{j=1}^{5} \int_{C_j'} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) - \sum_{j=1,2,4,5} \int_{C_j'} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta), \tag{E. 14}$$

where

C_1' is a straight line path from $|\lambda| = \infty$, $3\pi/4 < \arg\lambda < \pi$, to a point where $|\lambda/k| \gg O(k\eta)$, $\arg\lambda = \pi$,

$C_2' = C_2$, $C_3' = C_3$, $C_4' = C_4$,

C_5' is a straight line path from a point where $|\lambda/k| \gg O(k\eta)$, $\arg\lambda = 0$, to $|\lambda| = \infty$, $0 < \arg\lambda < \pi/4$.

We evaluate the sum

$$\sum_{j=1}^{5} \int_{C_j'} d\lambda\, I_{\pm}^{+}(\lambda, \xi_1, \xi_2, \eta)$$

as a single contour integral

$$\int_{C'} d\lambda\, I_{\pm}^{+}(\lambda, \xi_1, \xi_2, \eta) \qquad \left(C' = \sum_{j=1}^{5} C_j'\right) .$$

In order to estimate the integral over C_1', we observe that $\pi/2 < \arg\lambda < \pi$ implies $-\lambda = e^{-i\pi}\lambda$; thus $\arg(-\lambda/2ik) = -\alpha \in (-\pi, -\pi/2)$. Hence (C.17) applies to $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$, and the dependence of the integrand on λ can be written as

$$I_{\pm}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{-\frac{\lambda}{2ik} \log -\frac{\lambda}{2ike}\right\} \exp\left\{\frac{\lambda}{2ik} \log \frac{2k\xi_1\eta}{\xi_2}\right\} e^{\lambda\pi/4k} ,$$

which is equal to

$$I_{\pm}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{-\frac{\lambda}{2ik} \log\left|\frac{\lambda}{2ike}\right|\right\} \exp\left\{\frac{\lambda}{2ik} \log \frac{2k\xi_1\eta}{\xi_2}\right\} e^{\lambda\pi/4k}\, e^{\lambda\alpha/2k} . \tag{E.15}$$

By choosing the end point of C_1' so that $|\lambda/k|$ is large enough, namely so that $\frac{|\lambda/k|}{k\eta} \cdot \frac{1}{4e} > 1$ (such a choice is implied in the original definition of $C_1, C_2, \ldots, C_5$) we then observe that the integrand is exponentially small over C_1'. Therefore, the integral over C_1' does not contribute materially to the second sum of (E.14).

To estimate the integral over C_5', we observe that $0 < \arg\lambda \leqslant \pi/2$ implies $\arg\frac{\lambda}{2ik} = -\beta \in (-\pi/2, 0]$. Thus we use (C.9) to find $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$, and the dependence of the integrand on λ can be written as

$$I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \frac{1}{e^{\lambda\pi/2k}\exp\left\{\frac{\lambda}{2ik}\log\frac{\lambda}{2ike}\right\}} \cdot \exp\left\{\frac{\lambda}{2ik}\log\frac{2k\xi_1\eta}{\xi_2}\right\} e^{\lambda\pi/4k} ,$$

$$\sim \exp\left\{-\frac{\lambda}{2ik}\log\left|\frac{\lambda}{2ike}\right|\right\} \exp\left\{\frac{\lambda}{2ik}\log\frac{2k\xi_1\eta}{\xi_2}\right\} \frac{e^{\lambda\beta/2k}}{e^{\pi\lambda/4k}} . \quad \textbf{(E.16)}$$

By choosing the end point of C_5' so that $|\lambda/k|$ is large enough, namely so that $\frac{|\lambda/k|}{k\eta} \cdot \frac{1}{4e} > 1$, (such a choice is implied in the original definition of $C_1, C_2, \ldots, C_5$), we then observe that the integrand is exponentially small over C_5'. Consequently, the integral over C_5' does not contribute materially to the second sum of (E.14).

The last two paragraphs also show that $I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta)$ is exponentially small for all values of ξ_1, ξ_2, η on a large arc from C_1' to C_5' in the upper half plane. Thus we can evaluate $\int_{C'} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta)$ as a sum of residues of $I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta)$ at the poles of $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$. The residues of $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)$ at the poles $\lambda = ik(2n+1)$, $n = 0, 1, 2, \ldots$, are $\frac{2k}{i}\frac{(-1)^n}{n!}$. Hence, the residue theorem yields

$$\int_{C'} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \frac{e^{ik\xi_1} e^{-ik(\xi_2+\eta)}}{2ik\xi_2} \sum_{n=0}^{\infty} \frac{2k}{i}\frac{(-1)^n}{n!} \cdot (2ik\xi_1)^n (2ik\xi_2)^{-n} (2ik\eta)^n ,$$

which reduces to

$$\int_{C'} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \frac{-e^{ik\xi_1} e^{-ik(\xi_2+\eta)}}{\xi_2} \sum_{n=0}^{\infty} \frac{\left(\frac{-2ik\xi_1\eta}{\xi_2}\right)^n}{n!}$$

$$= \frac{-e^{ik\xi_1} e^{-ik(\xi_2+\eta)} e^{-2ik\xi_1\eta/\xi_2}}{\xi_2} . \quad \textbf{(E.17)}$$

Inserting (E.17) into (E.14) we obtain

$$\int_{C_3} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \frac{-e^{ik\xi_1} e^{-ik(\xi_2+\eta)} e^{-2ik\xi_1\eta/\xi_2}}{\xi_2} - \sum_{j=2,4} \int_{C_j'} d\lambda\, I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta)\,, \tag{E.18}$$

since the integrals over C_1' and C_5' do not contribute.

In order to estimate the contributions from C_2' and C_4', we first note that in these ranges of λ the dependence of the integrand on λ can be written as

$$I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{\frac{\lambda}{2ik} \log \frac{2ik\xi_1}{\frac{-\lambda}{2ike}}\right\} \exp\left\{-\frac{\lambda}{2ik} \log \xi_2/\eta\right\},$$

which, upon using the definition of s_{ξ_1}, becomes

$$I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{\frac{\lambda}{2ik} \log 4s_{\xi_1} e\right\} \exp\left\{-\frac{\lambda}{2ik} \log \xi_2/\eta\right\}. \tag{E.19}$$

If we define $\psi_0^{+}(\lambda) = \frac{\lambda}{k}\cdot\frac{1}{2}\log \xi_2/\eta - \frac{\lambda}{k}\cdot\frac{1}{2}\log 4s_{\xi_1} e$, then substituting in (E.19) yields

$$I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim e^{i\psi_0^{+}(\lambda)}\,. \tag{E.20}$$

For $\arg\lambda = \pi$, $\arg s_{\xi_1} = \pi$; thus $\log 4s_{\xi_1} e = \log|4s_{\xi_1} e| + i\pi$, implying

$$I_{\mp}^{+}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{\frac{i\lambda}{k}\left[\frac{1}{2}\log\frac{\xi_2}{\eta} - \frac{1}{2}\log|4s_{\xi_1} e|\right]\right\} e^{\pi\lambda/2k}\,,$$

which is exponentially small. Therefore the integral over C_2' does not contribute materially.

However, for $\arg\lambda = 0$, $\psi_0^{+}(\lambda)$ is real valued, and since $\lambda/k = O(k\eta)$, we study the integral over C_4' by using the method of stationary phase (Jeffreys and Jeffreys, 1956). Differentiating $\psi_0^{+}(\lambda)$ we obtain

$$\frac{d\psi_o^+(\lambda)}{d\lambda} = \frac{1}{k}\cdot\frac{1}{2}\log\frac{\xi_2}{\eta} - \frac{1}{k}\cdot\frac{1}{2}\log 4s_{\xi_1} \quad . \tag{E.21}$$

Therefore, the stationary point equation $d\psi_o^+(\lambda)/d\lambda = 0$ has a solution given by

$$\frac{\lambda_o^+}{k} = 4k\eta\frac{\xi_1}{\xi_2} \quad .$$

This value is contained in C_4' because of the choice of end points of C_4'. Since

$$\frac{d^2\psi_o^+(\lambda)}{d\lambda^2} = \frac{1}{2k\lambda} \, , \qquad \psi_o^+(\lambda_o^+) = -\frac{2k\eta\xi_1}{\xi_2} \quad ,$$

the method of stationary phase gives

$$\int_{C_4'} d\lambda\, I_{\pm}^+(\lambda, \xi_1, \xi_2, \eta) = \frac{-1}{\xi_2}\, e^{ik\xi_1}\, e^{-ik(\xi_2+\eta)}\, e^{-2ik\eta\xi_1/\xi_2} + \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right) . \tag{E.22}$$

Substituting (E.22) into (E.18), we find

$$\int_{C_3} d\lambda\, I_{\pm}^+(\lambda, \xi_1, \xi_2, \eta) = \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right) \quad . \tag{E.22a}$$

We now wish to consider

$$\int_{C_3} d\lambda\, I_{\pm}^-(\lambda, \xi_1, \xi_2, \eta) \, .$$

Inserting (E.13), we find this integral becomes

$$C(k, \xi_1, \xi_2, \eta)\, e^{-ik\xi_1} e^{-ik(\xi_2+\eta)} e^{\pi i/2} \int_{C_3} d\lambda\, \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) \cdot$$

$$\cdot\, (2ik\xi_1)^{-\lambda/2ik} (2ik\xi_2)^{-\lambda/2ik} (2ik\eta)^{\lambda/2ik} e^{\pi\lambda/2k} \; .$$

We estimate it by using the relation (similar to (E.14))

$$\int_{C_3} d\lambda\, \underline{\underline{I}}^{-}(\lambda, \xi_1, \xi_2, \eta) = \sum_{j=1}^{5} \int_{C'_j} d\lambda\, \underline{\underline{I}}^{-}(\lambda, \xi_1, \xi_2, \eta) - \sum_{j=1,2,4,5} \int_{C'_j} d\lambda\, \underline{\underline{I}}^{-}(\lambda, \xi_1, \xi_2, \eta) \; , \tag{E.23}$$

where

C'_1 is a straight line path from $|\lambda| = \infty$, $\pi < \arg\lambda < 5\pi/4$, to a point where $|\lambda/k| \gg O(k\eta)$, $\arg\lambda = \pi$,

$C'_2 = C_2$, $C'_3 = C_3$, $C'_4 = C_4$,

C'_5 is a straight line path from a point where $|\lambda/k| \gg O(k\eta)$, $\arg\lambda = 0$, to $|\lambda| = \infty$, $-\pi/4 < \arg\lambda < 0$,

and evaluating the sum

$$\sum_{j=1}^{5} \int_{C'_j} d\lambda\, \underline{\underline{I}}^{-}(\lambda, \xi_1, \xi_2, \eta)$$

as a single contour integral

$$\int_{C'} d\lambda\, \underline{\underline{I}}^{-}(\lambda, \xi_1, \xi_2, \eta) \qquad \left(C' = \sum_{j=1}^{5} C'_j\right) \; .$$

In order to approximate the integral over C_1', we note that $\pi < \arg\lambda \leqslant 5\pi/4$ $(-3\pi/4 < \arg\lambda \leqslant -\pi/2)$ implies $-\lambda = e^{-i\pi}\lambda$ $(-\lambda = e^{i\pi}\lambda)$; thus $\arg(-\lambda/2ik) = -\gamma \in (-\pi/2, -\pi/4]$ $((-\pi/4, 0])$. Hence we use (C.9) to approximate $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, and the dependence of the integrand on λ can be written as

$$I_{\bar{=}}(\lambda, \xi_1, \xi_2, \eta) \sim \frac{1}{e^{-\lambda\pi/2k} \exp\left\{-\frac{\lambda}{2ik} \log -\frac{\lambda}{2ike}\right\}} \exp\left\{-\frac{\lambda}{2ik} \log \frac{2k\xi_1\xi_2}{\eta}\right\} e^{\lambda\pi/4k}$$

$$\sim \exp\left\{\frac{\lambda}{2ik} \log\left|\frac{\lambda}{2ike}\right|\right\} \exp\left\{-\frac{\lambda}{2ik} \log \frac{2k\xi_1\xi_2}{\eta}\right\} \cdot \frac{e^{-\lambda\gamma/2k}}{e^{\lambda 3\pi/4k}} .$$

(E.24)

If the end point of C_1' does not satisfy the condition

$$\left|\frac{\lambda}{k}\right| \cdot \frac{\eta}{4ek\xi_1\xi_2} > 1 ,$$

then we chose the original definition of $C_1, C_2, \ldots, C_5$ so that it does. Accordingly, the integrand is exponentially small over C_1' and C_1' does not contribute materially to the second sum of (E.23).

To estimate the integral over C_5', we note that $-\pi/2 < \arg\lambda < 0$ implies $\arg\lambda/2ik = -\delta \in (-\pi, -\pi/2)$. Hence (C.8) applies to $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$, and the dependence of the integrand on λ can be written as

$$I_{\bar{=}}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{\frac{\lambda}{2ik} \log \frac{\lambda}{2ike}\right\} \exp\left\{-\frac{\lambda}{2ik} \log \frac{2k\xi_1\xi_2}{\eta}\right\} e^{\lambda\pi/4k} ,$$

$$\sim \exp\left\{\frac{\lambda}{2ik} \log\left|\frac{\lambda}{2ike}\right|\right\} \exp\left\{-\frac{\lambda}{2ik} \log \frac{2k\xi_1\xi_2}{\eta}\right\} e^{\lambda\pi/4k} e^{-\delta\lambda/2k} .$$

(E.25)

If the end point of C_5' does not satisfy the condition $|\lambda/k| \cdot \frac{\eta}{4ek\xi_1\xi_2} > 1$, then we choose the original definition of $C_1, C_2, \ldots, C_5$ so that it does. Thus the integrand is exponentially small over C_5' and C_5' does not contribute materially to the second sum of (E.23).

The last two paragraphs also show that $I_{\mp}^{-}(\lambda, \xi_1, \xi_2, \eta)$ is exponentially small for all values of ξ_1, ξ_2, η on a large arc from C_1' to C_5' in the lower half plane. Hence we can evaluate

$$\int_{C'} d\lambda\, I_{\mp}^{-}(\lambda, \xi_1, \xi_2, \eta)$$

as a sum of residues of $I_{\mp}^{-}(\lambda, \xi_1, \xi_2, \eta)$ at the poles of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$. The residues of $\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ at the poles $\lambda = -ik(2n+1)$, $n = 0, 1, 2, \ldots,$ are $\frac{2k}{(-i)} \frac{(-1)^n}{n!}$. Therefore, the residue theorem yields

$$\int_{C'} d\lambda\, I_{\mp}^{-}(\lambda, \xi_1, \xi_2, \eta) \sim -\frac{e^{-ik\xi_1} e^{-ik(\xi_2+\eta)}}{2k\eta} \sum_{n=0}^{\infty} \frac{2k}{(-i)} \frac{(-1)^n}{n!} \cdot$$

$$\cdot (2ik\xi_1)^n (2ik\xi_2)^n (2ik\eta)^{-n} (-1)^n (-i) ,$$

$$\sim -\frac{e^{-ik\xi_1} e^{-ik(\xi_2+\eta)}}{\eta} e^{2ik(\xi_1\xi_2/\eta)} . \qquad \text{(E.26)}$$

Inserting (E.26) into (E.23) we obtain

$$\int_{C_3} d\lambda\, I_{\mp}^{-}(\lambda, \xi_1, \xi_2, \eta) \sim -\frac{e^{-ik\xi_1} e^{-ik(\xi_2+\eta)}}{\eta} e^{2ik(\xi_1\xi_2/\eta)} -$$

$$- \sum_{j=2,4} \int_{C_j'} d\lambda\, I_{\mp}^{-}(\lambda, \xi_1, \xi_2, \eta) , \qquad \text{(E.27)}$$

since the integrals over C_1' and C_5' do not contribute.

We consider now the integral over C_2'. Since $\arg\lambda = \pi$ implies $\arg\lambda/2ik = \pi/2$,

$$I_{\bar{=}}^{-}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{\frac{\lambda}{2ik}\log\left|\frac{\lambda}{2ike}\right|\right\} e^{\lambda\pi/4k} \exp\left\{-\frac{\lambda}{2ik}\log \xi_1/\eta\right\} e^{\lambda\pi/2k} , \tag{E.28}$$

which is exponentially small. Therefore, the integral over C_2' does not contribute materially to the second sum of (E.23).

To estimate the integral over C_4' we observe that for $\arg\lambda = 0$, $-\lambda = e^{i\pi}\lambda$. Hence,

$$I_{\bar{=}}^{-}(\lambda, \xi_1, \xi_2, \eta) \sim \exp\left\{-\frac{\lambda}{2ik}\log 4s_{\xi_2} e\right\} \exp\left\{-\frac{\lambda}{2ik}\log \xi_1/\eta\right\}. \tag{E.29}$$

If we define

$$\psi_o^{-}(\lambda) = \frac{\lambda}{k}\cdot\frac{1}{2}\log \xi_1/\eta + \frac{\lambda}{k}\cdot\frac{1}{2}\log 4s_{\xi_2} e ,$$

then substituting in (E.28) yields

$$I_{\bar{=}}^{-}(\lambda, \xi_1, \xi_2, \eta) \sim e^{i\psi_o^{-}(\lambda)} . \tag{E.30}$$

For $\arg\lambda = 0$, $\psi_o^{-}(\lambda)$ is real valued, and since $\lambda/k = O(k\eta)$, we again study the integral over C_4' by using the method of stationary phase. Clearly

$$\frac{d\psi_o^{-}}{d\lambda} = \frac{1}{k}\cdot\frac{1}{2}\log \xi_1/\eta + \frac{1}{k}\cdot\frac{1}{2}\log 4s_{\xi_2} . \tag{E.31}$$

Therefore, the stationary point equation $d\psi_o^{-}(\lambda)/d\lambda = 0$ has a solution given by $\lambda_o^{-}/k = 4k\xi_1\xi_2/\eta$. This value is contained in C_4' because of the above choice of end points. Since

$$\frac{d^2\psi_o^{-}}{d\lambda^2} = -\frac{1}{2k\lambda} , \qquad \psi_o^{-}(\lambda_o^{-}) = \frac{2k\xi_1\xi_2}{\eta} ,$$

the method of stationary phase gives

$$\int_{C_4'} d\lambda\, I_{\Xi}^{-}(\lambda, \xi_1, \xi_2, \eta) = -\frac{1}{\eta} e^{-ik\xi_1} e^{-ik(\xi_2+\eta)} e^{2ik(\xi_1\xi_2/\eta)} + \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right). \tag{E.32}$$

Substituting (E.32) into (E.27), we find

$$\int_{C_3} d\lambda\, I_{\Xi}^{-}(\lambda, \xi_1, \xi_2, \eta) = \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right);$$

consequently, by (E.11) and (E.22a),

$$\int_{C_3} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) = \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right).$$

If we now substitute the result of the previous paragraphs into equation (E.4a), we obtain

$$-\frac{e^{-ikR_{\Xi}}}{R_{\Xi}} = \int_{C_4} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) + \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right). \tag{E.33}$$

This equation shows that if $k\xi_1 \gg 1$, $k\xi_2 \gg 1$, $k\eta \gg 1$, the derivation of the asymptotic representation of $-e^{-ikR_{\Xi}}/R_{\Xi}$ (from its integral representation) depends on the behavior of the Whittaker functions in the region C_4. If the point $\lambda/k = k\xi_1$ is not already contained within C_4, we choose the original definition of $C_1, C_2, \ldots, C_5$ so that it is. In this range $\arg\lambda = 0$; thus $\arg s_{\xi_j} = 0$, and for $s_{\xi_j} = O(1)$ (we discuss later the case where this is not true for s_{ξ_2}) equations (D.21) and (D.20) yield, respectively,

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} C(-\lambda/2ik)\, e^{\pi i/4}\, e^{\lambda\pi/2k} \bar{\psi}(s_{\xi_1})\, Ai(-\bar{\sigma}_{\xi_1}) , \qquad (E.34)$$

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} C(-\lambda/2ik)\, e^{\pi i/12} \exp\left\{-\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike}\right\} \bar{\psi}(s_{\xi_2}) \cdot$$

$$\cdot Ai(-\bar{\sigma}_{\xi_2} e^{-2\pi i/3}), \qquad (E.35)$$

with

$$C(-\lambda/2ik) = 2(3)^{1/6} \pi^{1/2} (\lambda/2k)^{1/6} ,$$

$$\bar{\phi}^2(s_{\xi_j}) = \frac{1}{s_{\xi_j}} (s_{\xi_j} - 1) , \quad \bar{\Phi}(s_{\xi_j}) = \int_1^{s_{\xi_j}} \bar{\phi}(\bar{s})\, d\bar{s} ,$$

$$\bar{\zeta}_{\xi_j} = \frac{\lambda}{k} \bar{\Phi}(s_{\xi_j}) , \qquad \bar{\sigma}_{\xi_j} = \left(\frac{3}{2} \bar{\zeta}_{\xi_j}\right)^{2/3} ,$$

$$\bar{\psi}(s_{\xi_j}) = \left[\bar{\Phi}(s_{\xi_j})\right]^{1/6} \cdot \left[\bar{\phi}(s_{\xi_j})\right]^{-1/2} .$$

Also, $\arg\lambda = 0$ implies $\arg s_\eta = \pi$, and since $s_\eta = O(1)$ implies $\zeta_\eta = O(k\eta)$ (see equation (3.32)), equation (D.42b) gives

$$v_2(\eta, \lambda) \sim (2ik\eta)^{\mp 1/2} (-i)^{1/2} \exp\left\{\frac{\lambda}{2ik} \log - \frac{\lambda}{2ike}\right\} \cdot \frac{e^{-i\zeta_\eta}}{\left(\phi(s_\eta)\right)^{1/2}} , \qquad (E.36)$$

with

$$\phi^2(s_\eta) = \frac{1}{s_\eta}(1 - s_\eta) ,$$

$$\zeta_\eta = -\frac{\lambda}{k}\int_0^{s_\eta}\left(\frac{a-1}{a}\right)^{1/2} da = \frac{\lambda}{k}\left[\sqrt{-s_\eta(1-s_\eta)} - \log\left(\sqrt{1-s_\eta} - \sqrt{-s_\eta}\right)\right]. \quad \text{(E.37)}$$

As discussed in Sections 3.1 and 3.2, the behavior of the Airy functions in equations (E.34) and (E.35) depends strongly on the value of λ with respect to the turning point value given by $\lambda/k = k\xi_j$, $j = 1,2$. If $\lambda < k^2\xi_1$ and

$$\frac{k^2\xi_1}{\lambda} - 1 \gg \frac{1}{(k\eta)^{2/3}} \qquad \left(s_{\xi_1} - 1 \gg \frac{1}{(k\eta)^{2/3}}\right) ,$$

then equations (E.34) and (E.35) become, since by definition $\lambda < k^2\xi_2$ and

$$\frac{k^2\xi_2}{\lambda} - 1 \gg \frac{1}{((k\eta)^{2/3}} \qquad \left(s_{\xi_2} - 1 \gg \frac{1}{(k\eta)^{2/3}}\right) ,$$

$$v_1(\xi_1, -\lambda) \sim \frac{(2ik\xi_1)^{-1/2}}{\sqrt{2\pi}} e^{\lambda\pi/2k} \cdot \frac{\left[e^{i\bar{\zeta}_{\xi_1}} + ie^{-i\bar{\zeta}_{\xi_1}}\right]}{\left(\bar{\phi}(s_{\xi_1})\right)^{1/2}} , \quad \text{(E.38)}$$

$$v_2(\xi_2, -\lambda) \sim (2ik\xi_2)^{-1/2} \exp\left\{-\frac{\lambda}{2ik}\log - \frac{\lambda}{2ike}\right\} \frac{e^{-i\bar{\zeta}_{\xi_2}}}{\left(\bar{\phi}(s_{\xi_2})\right)^{1/2}} . \quad \text{(E.39)}$$

If λ is in a neighborhood of $k^2\xi_1$ governed by

$$\frac{k^2\xi_1}{\lambda} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right) ,$$

then we use (E.34) for $v_1(\xi_1, -\lambda)$, with

$$\bar{\Phi}(s_{\xi_1}) = i\int_1^{s_{\xi_1}} \left(\frac{1-b}{b}\right)^{1/2} db \qquad \text{for} \qquad s_{\xi_1} < 1 \ .$$

Since both in this appendix and in Chapter III we are studying a Green's function, we do not consider ξ_2 in a neighborhood of ξ_1. Thus for these latter values of λ, (E.39) applies to $v_2(\xi_2, -\lambda)$. For $\lambda > k^2\xi_1$ ($s_{\xi_1} < 1$) we see that $\arg \bar{\zeta}_{\xi_1} = 3\pi/2$. If, in addition,

$$1 - \frac{k^2\xi_1}{\lambda} \gg \frac{1}{(k\eta)^{2/3}} \ ,$$

the Airy function $Ai(-\bar{\sigma}_{\xi_1})$ is exponentially decreasing:

$$Ai(-\bar{\sigma}_{\xi_1}) \sim \exp\left\{-\frac{\lambda}{k}\int_{s_{\xi_1}}^{1} \left(\frac{1-t}{t}\right)^{1/2} dt\right\}. \tag{E.40}$$

If $\lambda < k^2\xi_2$ ($s_{\xi_2} > 1$) and

$$\frac{k^2\xi_2}{\lambda} - 1 \gg \frac{1}{(k\eta)^{2/3}} \ ,$$

(E.39) applies to $v_2(\xi_2, -\lambda)$. If λ is in a neighborhood of $k^2\xi_2$ governed by

$$\frac{k^2\xi_2}{\lambda} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right) \ ,$$

then we use (E.35) for $v_2(\xi_2, -\lambda)$, with

$$\bar{\Phi}(s_{\xi_2}) = i\int_1^{s_{\xi_2}} \left(\frac{1-s}{s}\right)^{1/2} ds \qquad \text{for} \qquad s_{\xi_2} < 1 \ .$$

Finally, for $\lambda > k^2\xi_2$ ($s_{\xi_2} < 1$) we note that $\arg \bar{\zeta}_{\xi_2} = 3\pi/2$. If, in addition,

$$1 - \frac{k^2\xi_2}{\lambda} \gg \frac{1}{(k\eta)^{2/3}} ,$$

the Airy function $\mathrm{Ai}(-\bar{\sigma}_{\xi_2} e^{-2\pi i/3})$ is exponentially increasing: we denote this by

$$\mathrm{Ai}(-\bar{\sigma}_{\xi_2} e^{-2\pi i/3}) \sim \exp\left\{\frac{\lambda}{k}\int_{s_{\xi_2}}^{1} \left(\frac{1-s}{s}\right)^{1/2} ds\right\}. \tag{E.41}$$

We now consider equation (E.33). For values of λ satisfying $\lambda > k^2\xi_1$ and

$$1 - \frac{k^2\xi_1}{\lambda} \gg \frac{1}{(k\eta)^{2/3}} ,$$

we assert that $I_{\pm}(\lambda, \xi_1, \xi_2, \eta)$ is exponentially small. If the values of λ also satisfy the condition $\lambda < k^2\xi_2$ or

$$\frac{k^2\xi_2}{\lambda} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right) ,$$

this follows immediately from equation (E.34) together with (E.40) and equations (E.36), (E.39) or (E.35). However, for $\lambda > k^2\xi_2$ and

$$1 - \frac{k^2\xi_2}{\lambda} \gg \frac{1}{(k\eta)^{2/3}} ,$$

we must show that the product of the exponential factors (E.40) and (E.41) remains exponentially small. This is true since $\xi_1 < \xi_2$ implies $s_{\xi_1} < s_{\xi_2}$ which in turn implies

$$\int_{s_{\xi_2}}^{1} \left(\frac{1-t}{t}\right)^{1/2} dt < \int_{s_{\xi_1}}^{1} \left(\frac{1-t}{t}\right)^{1/2} dt ,$$

or that the product is always less than one. But we do not consider ξ_2 in a neighborhood of ξ_1; thus the product is always exponentially very small. Therefore, we can write equation (E.33) as

$$-\frac{e^{-ikR_{\Xi}}}{R_{\Xi}} = \int_{C_4''} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) + \int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) + \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right) , \tag{E.42}$$

where $C_4'' = \left\{\lambda \in C_4 \,\middle|\, \lambda < (k^2\xi_1)^-\right\}$ and $(k^2\xi_1)^- \leqslant \lambda \leqslant (k^2\xi_1)^+$ denotes the neighborhood of $k^2\xi_1$ defined by

$$\frac{k^2\xi_1}{\lambda} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right) ,$$

since the integral over the remainder of C_4 is negligible.

To estimate the integral over C_4'', we begin by using equations (E.36), (E.38) and (E.39) to find that in this range

$$I_{\Xi} \sim \frac{C(k, \xi_1, \xi_2, \eta)}{\sqrt{2\pi}}\, \Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right) e^{\lambda\pi/2k} (-i)^{1/2} \cdot$$

$$\cdot \frac{\left[e^{i\bar{\zeta}_{\xi_1}} + ie^{-i\bar{\zeta}_{\xi_1}}\right] e^{-i\bar{\zeta}_{\xi_2}}\, e^{-i\zeta_\eta}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} ,$$

where $C(k, \xi_1, \xi_2, \eta)$ was defined earlier in this appendix. Inserting (C.9) for $\Gamma\left(-\frac{\lambda}{2ik} + \frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2ik} + \frac{1}{2}\right)$ together with the exponential form of the cosine, this becomes

$$I_{\Xi} \sim \sqrt{2\pi}\, C(k, \xi_1, \xi_2, \eta)(-i)^{1/2}\, \frac{\left[e^{i\bar{\zeta}_{\xi_1}} + ie^{-i\bar{\zeta}_{\xi_1}}\right] e^{-i\bar{\zeta}_{\xi_2}}\, e^{-i\zeta_\eta}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} , \tag{E.43}$$

with ζ_η given by (E.37). Since $s_{\xi_j} > 1$, $j = 1, 2$, equation (D.5) yields

$$\bar{\zeta}_{\xi_j} = \frac{\lambda}{k}\left[\sqrt{s_{\xi_j}(s_{\xi_j} - 1)} - \log\left(\sqrt{s_{\xi_j} - 1} + \sqrt{s_{\xi_j}}\right)\right], \quad j = 1, 2. \qquad \text{(E.44)}$$

Thus we can write (E.43) as

$$I_{\bar{\bar{}}} \sim \frac{\sqrt{2\pi}\, C(k, \xi_1, \xi_2, \eta)(-i)^{1/2}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}}\left[e^{i\Phi_o^+(\lambda)} + ie^{i\Phi_o^-(\lambda)}\right], \qquad \text{(E.45)}$$

with

$$\Phi_o^+(\lambda) = \frac{\lambda}{k}\bar{\Phi}_o(s_{\xi_1}) - \frac{\lambda}{k}\bar{\Phi}_o(s_{\xi_2}) - \frac{\lambda}{k}\Phi_o(s_\eta), \qquad \text{(E.46)}$$

$$\Phi_o^-(\lambda) = -\frac{\lambda}{k}\bar{\Phi}_o(s_{\xi_1}) - \frac{\lambda}{k}\bar{\Phi}_o(s_{\xi_2}) - \frac{\lambda}{k}\Phi_o(s_\eta), \qquad \text{(E.47)}$$

$$\bar{\Phi}_o(s) = \sqrt{s(s-1)} - \log\left(\sqrt{s-1} + \sqrt{s}\right), \qquad \text{(E.48)}$$

$$\Phi_o(s) = \sqrt{-s(1-s)} - \log\left(\sqrt{1-s} - \sqrt{-s}\right). \qquad \text{(E.49)}$$

From these last equations we see that over C_4'' both $\Phi_o^+(\lambda)$ and $\Phi_o^-(\lambda)$ are real valued. Since, in addition $\lambda/k = O(k\eta)$, we estimate the integral over C_4'' by the method of stationary phase.

Let us first consider the term containing $e^{i\Phi_o^+(\lambda)}$. We note that

$$\frac{d\bar{\Phi}_o(s)}{ds} = \frac{s - \frac{1}{2}}{\sqrt{s(s-1)}} - \frac{1}{\sqrt{s-1} + \sqrt{s}}\left[\frac{1}{2}\,\frac{(\sqrt{s} + \sqrt{s-1})}{\sqrt{s(s-1)}}\right] = \frac{s-1}{\sqrt{s(s-1)}}.$$

Since in $\bar{\Phi}_o(s)$ both $\arg s = 0$ and $s > 1$, this becomes

$$\frac{d\bar{\Phi}_o(s)}{ds} = \sqrt{\frac{s-1}{s}} \quad . \tag{E.50}$$

Similarly

$$\frac{d\Phi_o(s)}{ds} = \frac{s-\frac{1}{2}}{\sqrt{-s(1-s)}} - \frac{1}{\sqrt{1-s}-\sqrt{-s}}\left[\frac{1}{2}\,\frac{\left(-\sqrt{s}+\sqrt{1-s}\right)}{\sqrt{-s(1-s)}}\right] = -\frac{(1-s)}{\sqrt{-s(1-s)}} \quad .$$

Since in $\Phi_o(s)$ $\arg s = \pi$ and therefore $\arg(1-s) = 0$, this becomes

$$\frac{d\Phi_o(s)}{ds} = -\sqrt{\frac{1-s}{-s}} \quad . \tag{E.51}$$

Then differentiating $\overset{+}{\Phi}_o(\lambda)$ and using (E.50), (E.51), we obtain

$$\frac{d\overset{+}{\Phi}_o(\lambda)}{d\lambda} = \frac{1}{k}\bar{\Phi}(s_{\xi_1}) - \frac{1}{k}\bar{\Phi}(s_{\xi_2}) - \frac{1}{k}\Phi(s_\eta) +$$

$$+\frac{\lambda}{k}\left(\frac{ds_{\xi_1}}{d\lambda}\right)\sqrt{\frac{s_{\xi_1}-1}{s_{\xi_1}}} - \frac{\lambda}{k}\left(\frac{ds_{\xi_2}}{d\lambda}\right)\sqrt{\frac{s_{\xi_2}-1}{s_{\xi_2}}} + \frac{\lambda}{k}\left(\frac{ds_\eta}{d\lambda}\right)\sqrt{\frac{1-s_\eta}{-s_\eta}} \quad .$$

But since

$$s_{\xi_j} = \frac{k^2\xi_j}{\lambda} \;,\; \arg s_{\xi_j} = 0 \quad (j = 1,2), \qquad s_\eta = -\frac{k^2\eta}{\lambda} \;,\; \text{and } \arg s_\eta = \pi$$

(which implies $\arg -s_\eta = 0$), this becomes

$$\frac{d\overset{+}{\Phi}_o(\lambda)}{d\lambda} = \frac{1}{k}\bar{\Phi}(s_{\xi_1}) - \frac{1}{k}\bar{\Phi}(s_{\xi_2}) - \frac{1}{k}\Phi(s_\eta) -$$

$$-\frac{1}{k}\sqrt{s_{\xi_1}(s_{\xi_1}-1)} + \frac{1}{k}\sqrt{s_{\xi_2}(s_{\xi_2}-1)} + \frac{1}{k}\sqrt{-s_\eta(1-s_\eta)} \; . \tag{E.52}$$

Substituting (E.48) and (E.49) into (E.52), we observe that the stationary point equation $d\Phi_0^+(\lambda)/d\lambda = 0$ is

$$0 = -\log\left(\sqrt{s_{\xi_1} - 1} + \sqrt{s_{\xi_1}}\right) + \log\left(\sqrt{s_{\xi_2} - 1} + \sqrt{s_{\xi_2}}\right) + \log\left(\sqrt{1 - s_\eta} - \sqrt{-s_\eta}\right),$$

or

$$\frac{\left(\sqrt{s_{\xi_2} - 1} + \sqrt{s_{\xi_2}}\right)\left(\sqrt{1 - s_\eta} - \sqrt{-s_\eta}\right)}{\left(\sqrt{s_{\xi_1} - 1} + \sqrt{s_{\xi_1}}\right)} = 1 . \tag{E.53}$$

If we make the substitution $\lambda = zk^2$, where z has the dimensions of η, and $0 < z < \xi_1$, then (E.53) can be written as

$$\frac{\sqrt{\xi_1 - z} + \sqrt{\xi_1}}{\sqrt{\xi_2 - z} + \sqrt{\xi_2}} = \frac{\sqrt{\eta + z} - \sqrt{\eta}}{\sqrt{z}} . \tag{E.54}$$

Prior to finding an explicit solution of (E.54), we investigate under what conditions a solution exists. Since $z < \xi_1$, the functions $w(z)$, $u(z)$ defined by

$$w(z) = \frac{\sqrt{\xi_1 - z} + \sqrt{\xi_1}}{\sqrt{\xi_2 - z} + \sqrt{\xi_1}} , \qquad u(z) = \frac{\sqrt{\eta + z} - \sqrt{\eta}}{\sqrt{z}} ,$$

are real valued functions of z. Differentiating $w(z)$ we obtain

$$w'(z) = \frac{-1}{2\sqrt{\xi_1 - z}\,\sqrt{\xi_2 - z}\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right)^2}\left[\sqrt{\xi_2 - z}\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right) - \sqrt{\xi_1 - z}\left(\sqrt{\xi_1 - z} + \sqrt{\xi_1}\right)\right].$$

But $\xi_2 > \xi_1$, thus $\sqrt{\xi_2 - z} > \sqrt{\xi_1 - z}$ and $\sqrt{\xi_2 - z} + \sqrt{\xi_2} > \sqrt{\xi_1 - z} + \sqrt{\xi_1}$. Hence $w'(z) < 0$, and $w(z)$ is a decreasing function of z with minimum value approaching

$$w(\xi_1) = \frac{\sqrt{\xi_1}}{\sqrt{\xi_2 - \xi_1} + \sqrt{\xi_2}} .$$

Differentiating u(z) we obtain

$$u'(z) = \frac{1}{2z\sqrt{z(\eta+z)}}\left[z - \sqrt{\eta+z}\left(\sqrt{\eta+z} - \sqrt{\eta}\right)\right],$$

or

$$u'(z) = \frac{1}{2z\sqrt{z(\eta+z)}}\left[\sqrt{\eta+z}\cdot\sqrt{\eta} - \eta\right] .$$

Thus $u'(z) > 0$, and $u(z)$ is an increasing function of z with maximum value approaching

$$u(\xi_1) = \frac{\sqrt{\eta+\xi_1} - \sqrt{\eta}}{\sqrt{\xi_1}} .$$

Since $u(0^+) \to 0$, equation (E.54) has exactly one real solution z_o^+ in the considered range of z if and only if $w(\xi_1) < u(\xi_1)$ or if and only if

$$\xi_1 < \left(\sqrt{\xi_2 - \xi_1} + \sqrt{\xi_2}\right)\left(\sqrt{\eta+\xi_1} - \sqrt{\eta}\right). \qquad \text{(E.55)}$$

A direct solution of (E.54) entails quite lengthy and cumbersome calculations. However, Ivanov (1962) encounters a similar equation in his paper considering diffraction by the convex paraboloid of revolution. Using his solution as a guide, we produce a solution of (E.54) which is verifiable by direct substitution. This solution is

$$z_o^+ = \frac{4\xi_1\xi_2\eta}{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2} . \qquad \text{(E.56)}$$

Before indicating the substitutions to show that it satisfies (E.54), we note that

$$(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2 = (\xi_2 - \xi_1)^2 + 2\eta(\xi_1+\xi_2) + \eta^2 > 0 ,$$

which implies that z_o^+ is indeed positive. This can also be written

$$(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2 = \left[(\xi_2-\xi_1)-\eta\right]^2 + 4\xi_2\eta > 4\xi_2\eta \ .$$

Thus

$$z_o^+ < \frac{4\xi_1\xi_2\eta}{4\xi_2\eta} = \xi_1 \ ;$$

the choice of z_o^+ does not contradict the assumptions on z. That (E.56) satisfies (E.54) follows by inserting it together with

$$\eta+z_o^+ = \frac{\eta(\xi_1+\xi_2+\eta)^2}{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2} \ , \tag{E.57}$$

$$\xi_1-z_o^+ = \frac{\xi_1(\xi_2-\xi_1-\eta)^2}{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2} \ , \tag{E.58}$$

$$\xi_2-z_o^+ = \frac{\xi_2(\xi_2-\xi_1+\eta)^2}{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2} \ , \tag{E.59}$$

into (E.54) and reducing the equation to an equality. Since (E.54) contains the square roots of (E.57), (E.58), (E.59), we demand that the terms being squared are positive. This is evident for (E.57) and (E.59). However, for (E.58) to hold we need $\xi_2 > \xi_1+\eta$. This follows since $\xi_2 \leq \xi_1+\eta$ implies $\sqrt{\xi_2-\xi_1} + \sqrt{\xi_2} \leq \sqrt{\eta} + \sqrt{\eta+\xi_1}$, which in turn implies $\left(\sqrt{\xi_2-\xi_1} + \sqrt{\xi_2}\right)\left(\sqrt{\eta+\xi_1} - \sqrt{\eta}\right) \leq \xi_1$, or that no solution of (E.54) exists.

We now estimate the term containing $e^{i\Phi_o^+(\lambda)}$ using the solution (E.56). We observed that equation (E.52) implied

$$\frac{d\Phi_o^+(\lambda)}{d\lambda} = -\frac{1}{k}\log\left(\sqrt{s_{\xi_1}-1} + \sqrt{s_{\xi_1}}\right) + \frac{1}{k}\log\left(\sqrt{s_{\xi_2}-1} + \sqrt{s_{\xi_2}}\right) + \frac{1}{k}\log\left(\sqrt{1-s_\eta} - \sqrt{-s_\eta}\right) .$$

Thus using the derivatives of $\log\left(\sqrt{s-1} + \sqrt{s}\right)$, $\log\left(\sqrt{1-s} - \sqrt{-s}\right)$ previously obtained, we find

$$\frac{d^2\Phi_o^+(\lambda)}{d\lambda^2} = \frac{-1}{2k\sqrt{s_{\xi_1}(s_{\xi_1}-1)}}\left(\frac{-k^2\xi_1}{\lambda^2}\right) + \frac{1}{2k\sqrt{s_{\xi_2}(s_{\xi_2}-1)}}\left(\frac{-k^2\xi_2}{\lambda^2}\right) + \frac{1}{2k\sqrt{-s_\eta(1-s_\eta)}}\left(\frac{k^2\eta}{\lambda^2}\right).$$

This can be written as

$$\frac{d^2\Phi_o^+(\lambda)}{d\lambda^2} = \frac{1}{2ik\lambda\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]}\left[\bar{\phi}(s_{\xi_2})i\phi(s_\eta) - \bar{\phi}(s_{\xi_1})i\phi(s_\eta) + \bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\right].$$

Since

$$\left(\bar{\phi}(s_{\xi_1})\right)_{z=z_o^+} = \sqrt{\frac{\xi_1 - z_o^+}{\xi_1}} = \frac{\xi_2 - \xi_1 - \eta}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}},$$

$$\left(\bar{\phi}(s_{\xi_2})\right)_{z=z_o^+} = \sqrt{\frac{\xi_2 - z_o^+}{\xi_2}} = \frac{\xi_2 - \xi_1 + \eta}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}},$$

$$\left(i\phi(s_\eta)\right)_{z=z_o^+} = \sqrt{\frac{\eta + z_o^+}{\eta}} = \frac{\xi_1 + \xi_2 + \eta}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}},$$

we see that

$$\left(\frac{d^2\Phi_o^+(\lambda)}{d\lambda^2}\right)_{\lambda=\lambda_o^+} = \frac{1}{2ik\lambda_o^+\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]_{\lambda=\lambda_o^+}}. \qquad \text{(E.60)}$$

In addition,

$$\Phi_o^+(\lambda_o^+) = kz_o^+\left[\sqrt{s_{\xi_1}(s_{\xi_1}-1)} - \sqrt{s_{\xi_2}(s_{\xi_2}-1)} - \sqrt{-s_\eta(1-s_\eta)}\right]_{\lambda=\lambda_o^+},$$

or

$$\Phi_o^+(\lambda_o^+) = k\left[\sqrt{\xi_1(\xi_1 - z_o^+)} - \sqrt{\xi_2(\xi_2 - z_o^+)} - \sqrt{\eta(\eta + z_o^+)}\right],$$

which upon using (E.57), (E.58) and (E.59) becomes

$$\Phi_o^+(\lambda_o^+) = -k\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2} \quad . \tag{E.61}$$

Then using the method of stationary phase, we obtain

$$\int_{C_4''} \frac{d\lambda\sqrt{2\pi}\, C(k,\xi_1,\xi_2,\eta)(-i)^{1/2} e^{i\Phi_o^+(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \sim$$

$$\sim \frac{2\pi C(k,\xi_1,\xi_2,\eta)(-i)^{1/2}}{\left(\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}\right)_{\lambda=\lambda_o^+}} \cdot \frac{e^{i\Phi_o^+(\lambda_o^+)} e^{i\pi/4}}{\sqrt{\left(\dfrac{d^2\Phi_o^+(\lambda)}{d\lambda^2}\right)_{\lambda=\lambda_o^+}}} \quad .$$

By (E.60), (E.61) and the definition of $C(k,\xi_1,\xi_2,\eta)$, this reduces to

$$\int_{C_4''} \frac{d\lambda\sqrt{2\pi}\, C(k,\xi_1,\xi_2,\eta)(-i)^{1/2} e^{i\Phi_o^+(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \sim \frac{-e^{-ik\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}}}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} \quad . \tag{E.62}$$

Before attempting to interpret this result, let us consider the term containing $e^{i\Phi_o^-(\lambda)}$. It follows immediately from the calculations already performed that the stationary point equation $d\Phi_o^-(\lambda)/d\lambda = 0$ is

$$0 = \log\left(\sqrt{s_{\xi_1}-1} + \sqrt{s_{\xi_1}}\right) + \log\left(\sqrt{s_{\xi_2}-1} + \sqrt{s_{\xi_2}}\right) + \log\left(\sqrt{1-s_\eta} - \sqrt{-s_\eta}\right)$$

or

$$\left(\sqrt{s_{\xi_1}-1} + \sqrt{s_{\xi_1}}\right)\left(\sqrt{s_{\xi_2}-1} + \sqrt{s_{\xi_2}}\right)\left(\sqrt{1-s_\eta} - \sqrt{-s_\eta}\right) = 1 \quad . \tag{E.63}$$

If we make the same substitution as previously, then (E.63) can be written as

$$\left(\sqrt{\xi_1 - z} + \sqrt{\xi_1}\right)\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right) = \frac{z^{3/2}}{\sqrt{\eta + z} - \sqrt{\eta}} \ . \tag{E.64}$$

Prior to finding an explicit solution of (E.64), we again investigate under what conditions a solution exists. Since $z < \xi_1$, the functions f(z), g(z) defined by

$$f(z) = \left(\sqrt{\xi_1 - z} + \sqrt{\xi_1}\right)\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right), \quad g(z) = \frac{z^{3/2}}{\sqrt{\eta + z} - \sqrt{\eta}}$$

are real valued functions of z. It is obvious that f(z) is a decreasing function of z with minimum value approaching

$$f(\xi_1) = \sqrt{\xi_1}\left(\sqrt{\xi_2 - \xi_1} + \sqrt{\xi_2}\right).$$

Differentiating g(z) we obtain

$$g'(z) = \frac{z^{1/2}}{2\sqrt{\eta + z}\left(\sqrt{\eta + z} - \sqrt{\eta}\right)^2}\left[3\sqrt{\eta + z}\left(\sqrt{\eta + z} - \sqrt{\eta}\right) - z\right],$$

or

$$g'(z) = \frac{z^{1/2}}{2\sqrt{\eta + z}\left(\sqrt{\eta + z} - \sqrt{\eta}\right)^2}\left[3\eta + 2z - 3\sqrt{\eta(\eta + z)}\right] \ .$$

Then the sign of g'(z) follows that of $h(z) = 3\eta + 2z - 3\sqrt{\eta(\eta + z)}$. But $h(0) = 0$ and

$$h'(z) = 2 - \frac{3\eta}{2\sqrt{\eta(\eta + z)}} > 0 \ ;$$

thus $h(z) > 0$ and $g'(z) > 0$. Therefore, g(z) is an increasing function of z with maximum value approaching

$$g(\xi_1) = \frac{\xi_1^{3/2}}{\sqrt{\eta + \xi_1} - \sqrt{\eta}} \ .$$

Since $g(0^+) \to 0$, equation (E.64) has exactly one real solution z_o^- in the considered range of z if and only if $f(\xi_1) < g(\xi_1)$ or

$$\xi_1 > \left(\sqrt{\xi_2 - \xi_1} + \sqrt{\xi_2}\right)\left(\sqrt{\eta + \xi_1} - \sqrt{\eta}\right) . \tag{E.65}$$

We see at once that this condition is complementary to (E.55). Either one or the other may be satisfied, but not both.

We continue to defer any interpretation and consider the solution of (E.64). As above, a direct solution entails quite lengthy and cumbersome calculations. We will again produce a solution which is verifiable by direct substitution. This solution is

$$z_o^- = \frac{4\xi_1 \xi_2 \eta}{(\xi_1 + \xi_2 + \eta)^2 - 4\xi_1 \xi_2} , \tag{E.66}$$

which is, of course, equal to z_o^+ (the use of the different symbol corresponds to the notation in Section 3.4) and thus does not contradict the assumptions on z. That (E.66) satisfies (E.64) follows by inserting it together with

$$\eta + z_o^- = \frac{\eta(\xi_1 + \xi_2 + \eta)^2}{(\xi_1 + \xi_2 + \eta)^2 - 4\xi_1 \xi_2} , \tag{E.67}$$

$$\xi_1 - z_o^- = \frac{\xi_1(\xi_1 + \eta - \xi_2)^2}{(\xi_1 + \xi_2 + \eta)^2 - 4\xi_1 \xi_2} , \tag{E.68}$$

$$\xi_2 - z_o^- = \frac{\xi_2(\xi_2 - \xi_1 + \eta)^2}{(\xi_1 + \xi_2 + \eta)^2 - 4\xi_1 \xi_2} , \tag{E.69}$$

into (E.64) and reducing the equation to an equality. Since (E.64) contains the square roots of (E.67), (E.68), (E.69), we demand that the terms being squared are positive. This is evident for (E.67) and (E.69). However, for (E.68) we need $\xi_2 < \xi_1 + \eta$. This follows since $\xi_2 \geqslant \xi_1 + \eta$ implies $\sqrt{\xi_2 - \xi_1} + \sqrt{\xi_2} \geqslant \sqrt{\eta} + \sqrt{\eta + \xi_1}$, which in turn implies $\left(\sqrt{\xi_2 - \xi_1} + \sqrt{\xi_2}\right)\left(\sqrt{\eta + \xi_1} - \sqrt{\eta}\right) \geqslant \xi_1$, or that no solution of (E.64) exists.

We now estimate the term containing $e^{i\bar{\Phi}_o^-(\lambda)}$ using the solution (E.66). We immediately find

$$\frac{d^2\bar{\Phi}_o^-(\lambda)}{d\lambda^2} = \frac{1}{2ik\lambda\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]}\left[-\bar{\phi}(s_{\xi_2})i\phi(s_\eta) - \bar{\phi}(s_{\xi_1})i\phi(s_\eta) + \bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\right] .$$

Since

$$\left(\bar{\phi}(s_{\xi_1})\right)_{z=z_o^-} = \sqrt{\frac{\xi_1 - z_o^-}{\xi_1}} = \frac{\xi_1 + \eta - \xi_2}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} ,$$

$$\left(\bar{\phi}(s_{\xi_2})\right)_{z=z_o^-} = \sqrt{\frac{\xi_2 - z_o^-}{\xi_2}} = \frac{\xi_2 - \xi_1 + \eta}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} ,$$

$$\left(i\phi(s_\eta)\right)_{z=z_o^-} = \sqrt{\frac{\eta + z_o^-}{\eta}} = \frac{\xi_1 + \xi_2 + \eta}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} ,$$

we see that

$$\left(\frac{d^2\bar{\Phi}_o^-(\lambda)}{d\lambda^2}\right)_{\lambda=\lambda_o^-} = \frac{-1}{2ik\lambda_o^-\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]_{\lambda=\lambda_o^-}} . \qquad \text{(E.70)}$$

In addition,

$$\bar{\Phi}_o^-(\lambda_o^-) = k\left[-\sqrt{\xi_1(\xi_1 - z_o^-)} - \sqrt{\xi_2(\xi_2 - z_o^-)} - \sqrt{\eta(\eta + z_o^-)}\right] ,$$

which upon using (E.67), (E.68) and (E.69) becomes

$$\Phi_o^-(\lambda_o^-) = -k\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2} \; . \tag{E.71}$$

Then using the method of stationary phase we obtain

$$\int_{C''_4} \frac{d\lambda\sqrt{2\pi}\, C(k,\xi_1,\xi_2,\eta)(-i)^{1/2} e^{i\pi/2} e^{i\Phi_o^-(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \sim$$

$$\sim \frac{2\pi\, C(k,\xi_1,\xi_2,\eta)(-i)^{1/2} e^{i\pi/2}}{\left(\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}\right)_{\lambda=\lambda_o^-}} \cdot \frac{e^{i\Phi_o^-(\lambda_o^-)} e^{-i\pi/4}}{\sqrt{-\left(\frac{d^2\Phi_o^-(\lambda)}{d\lambda^2}\right)_{\lambda=\lambda_o^-}}} \; .$$

Inserting (E.70), (E.71) and the definition of $C(k,\xi_1,\xi_2,\eta)$, this reduces to

$$\int_{C''_4} \frac{d\lambda\sqrt{2\pi}\, C(k,\xi_1,\xi_2,\eta)(-i)^{1/2} e^{i\pi/2} e^{i\Phi_o^-(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \sim \frac{-e^{-ik\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}}}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} \; . \tag{E.72}$$

Since the conditions (E.65) and (E.55) are complementary, both (E.72) and (E.62) cannot be true simultaneously. If there is a solution of (E.54) ((E.64)), then $e^{i\Phi_o^-(\lambda)}$ $(e^{i\Phi_o^+(\lambda)})$ does not have a stationary point in the interval C''_4. Therefore, (E.62) ((E.72)) is valid while (E.72) and (E.62) are replaced by

$$\int_{C''_4} \frac{d\lambda\sqrt{2\pi}\, C(k,\xi_1,\xi_2,\eta)(-i)^{1/2} e^{i\pi/2} e^{i\Phi_o^{\pm}(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \sim \frac{1}{\sqrt{\xi_1\xi_2}}\, O\left(\frac{1}{(k\eta)^{1/2}}\right) , \tag{E.73}$$

which is obtained by integration by parts (here the upper sign in $\Phi_o^{\pm}(\lambda)$ corresponds to (E.72) and the lower sign to (E.62).

In deriving equation (E.72), as well as (E.62), we tacitly assumed that the stationary value $\lambda_o^+ = k^2 z_o^+ = k^2 z_o^- = \lambda_o^-$ lies within the range C_4''. We now investigate the necessary and sufficient condition that this be true. We also complete the derivation of the asymptotic representation of $-e^{-ikR_\mp}/R_\mp$ by considering the second integral in equation (E.42). Finally, we interpret the result. We first observe that it is indeed possible for the stationary value to lie outside C_4''. To show this, we recall that $\xi_2 < \xi_1+\eta$ implies $\left(\sqrt{\xi_2-\xi_1} + \sqrt{\xi_2}\right)\left(\sqrt{\eta+\xi_1} - \sqrt{\eta}\right) < \xi_1$ or that there exists a solution of (E.64). Similarly $\xi_2 > \xi_1+\eta$ implies $\left(\sqrt{\xi_2-\xi_1} + \sqrt{\xi_2}\right)\left(\sqrt{\eta+\xi_1} - \sqrt{\eta}\right) > \xi_1$ or that there exists a solution of (E.54). Thus if we set $\xi_2 = \xi_1+\eta+x$ with x small, then x positive implies we can solve (E.54) and x negative implies we can solve (E.64). If we use (E.58) for x positive or (E.68) for x negative together with the value of $z_o = z_o^+ = z_o^-$ ((E.56) or (E.66)), we find

$$\frac{\xi_1 - z_o}{z_o} = \frac{1}{4\xi_2\eta}\left|\xi_2-(\xi_1+\eta)\right|^2 = \frac{1}{4\eta(\xi_1+\eta)}|x|^2\left[1+\frac{x}{(\xi_1+\eta)}\right]^{-1} . \qquad \text{(E.74)}$$

Using the Taylor expansion for $\left[1+x/(\xi_1+\eta)\right]^{-1}$, we obtain

$$\frac{\xi_1 - z_o}{z_o} = \frac{\xi_1}{z_o} - 1 = \frac{|x|^2}{4\eta(\xi_1+\eta)}\left[1 - \frac{x}{(\xi_1+\eta)} + O(x^2)\right] ,$$

or

$$z_o = \xi_1\left[1 - \frac{|x|^2}{4\eta(\xi_1+\eta)} + O(x^3)\right] . \qquad \text{(E.75)}$$

Consequently if x is small enough, then z_o is outside C_4''. The necessary and sufficient condition that z_o lie inside C_4'' follows immediately from (E.74) together with the definition of C_4''. It is

$$\frac{1}{4\xi_2\eta}\left|\xi_2-(\xi_1+\eta)\right|^2 \gg \frac{1}{(k\eta)^{2/3}} . \qquad \text{(E.76)}$$

We now complete the study of equation (E.42) if the condition (E.76) is satisfied. If it is,

$$\int_{C_4''} d\lambda\, I_{\Xi}(\lambda, \xi_1, \xi_2, \eta) \sim \frac{-e^{-ik\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}}}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} \quad . \tag{E.77}$$

It remains to estimate the second integral in (E.42). As in C_4'', equation (E.36) applies to $v_2(\eta, \lambda)$. Since we do not consider ξ_2 in a neighborhood of ξ_1, equation (E.39) is still valid for $v_2(\xi_2, -\lambda)$. However, for $v_1(\xi_1, -\lambda)$ we need to use (E.34). Then (E.43) becomes

$$I_{\Xi} \sim \frac{\sqrt{2\pi}\, C(k, \xi_1, \xi_2, \eta)}{\left[\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \cdot C(-\lambda/2ik)\bar{\psi}(s_{\xi_1})Ai(-\bar{\sigma}_{\xi_1})e^{-i\bar{\zeta}_{\xi_2}} e^{-i\zeta_\eta} \,. \tag{E.78}$$

Using (E.37) together with (E.49), (E.44) together with (E.48), and the definition of $\bar{\psi}(s_{\xi_1})$ together with (D.7) and (D.8), we find

$$I_{\Xi} \sim \frac{2\sqrt{2\pi}\, C(k, \xi_1, \xi_2, \eta)}{\left[\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \cdot (\lambda/k)^{1/6} Ai(-\bar{\sigma}_{\xi_1})e^{i\bar{\Phi}_o^{(0)}(\lambda)} \quad , \tag{E.79}$$

with

$$\bar{\Phi}_o^{(0)}(\lambda) = -\frac{\lambda}{k}\bar{\Phi}(s_{\xi_2}) - \frac{\lambda}{k}\Phi_o(s_\eta) \ . \tag{E.80}$$

But in this range

$$\frac{k^2\xi_1}{\lambda} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right) \qquad \left(s_{\xi_1} - 1 = O\left(\frac{1}{(k\eta)^{2/3}}\right)\right) \quad ;$$

thus $\left|\bar{\zeta}_{\xi_1}\right| = O(1)$, which indicates that $Ai(-\bar{\sigma}_{\xi_1})$ is bounded independently of $k\eta$ and does not oscillate rapidly. Therefore, the integral

$$\int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda\, I_{\overline{+}}(\lambda, \xi_1, \xi_2, \eta)$$

can be estimated by the method of stationary phase. It follows immediately from the previous calculations that the stationary point equation $d\Phi_0^{(0)}(\lambda)/d\lambda = 0$ is

$$0 = \log\left(\sqrt{s_{\xi_2} - 1} + \sqrt{s_{\xi_2}}\right) + \log\left(\sqrt{1 - s_\eta} - \sqrt{-s_\eta}\right),$$

or

$$\left(\sqrt{s_{\xi_2} - 1} + \sqrt{s_{\xi_2}}\right)\left(\sqrt{1 - s_\eta} - \sqrt{-s_\eta}\right) = 1. \tag{E.81}$$

If we make the substitution $\lambda = zk^2$, where z has the dimensions of η, and $0 < z < \xi_2$, then (E.81) becomes

$$\left(\sqrt{\xi_2 - z} + \sqrt{\xi_2}\right)\left(\sqrt{\eta + z} - \sqrt{\eta}\right) = z\ . \tag{E.82}$$

Hence, the integrand $I_{\overline{+}}(\lambda, \xi_1, \xi_2, \eta)$ (given by (E.79)) has a stationary point within the interval of integration if and only if (E.82) has a solution in the z interval defined by

$$\xi_1 - O\left(\frac{1}{(k\eta)^{2/3}}\right) \leqslant z \leqslant \xi_1 + O\left(\frac{1}{(k\eta)^{2/3}}\right)\ .$$

Before investigating an explicit solution of (E.82), we first consider under what conditions a solution exists. Since $z < \xi_2$, the functions $W(z)$, $U(z)$ defined by

$$W(z) = \sqrt{\xi_2 - z} + \sqrt{\xi_2}\ , \qquad U(z) = \frac{z}{\sqrt{\eta + z} - \sqrt{\eta}}\ ,$$

are real valued functions of z. $W(z)$ is obviously a decreasing function of z with minumum value approaching $W(\xi_2) = \sqrt{\xi_2}$. Differentiating $U(z)$, we obtain

$$U'(z) = \frac{2\sqrt{\eta+z}\left(\sqrt{n+z} - \sqrt{\eta}\right) - z}{2\sqrt{\eta+z}\left(\sqrt{n+z} - \sqrt{\eta}\right)^2} \quad ,$$

or

$$U'(z) = \frac{U_1(z)}{2\sqrt{\eta+z}\left(\sqrt{\eta+z} - \sqrt{\eta}\right)^2} \quad ,$$

with

$$U_1(z) = 2\eta + z - 2\sqrt{\eta(\eta+z)} \quad .$$

But

$$U_1'(z) = 1 - \frac{\eta}{\sqrt{\eta(\eta+z)}} > 0 \ ;$$

thus $U_1(z)$ is an increasing function of z. Then $U_1(0) = 0$ implies that $U_1(z) > 0$ for $z > 0$, or that $U'(z) > 0$ for $z > 0$. Therefore, $U(z)$ is an increasing function of z on $0 < z < \xi_2$ with maximum value approaching

$$U(\xi_2) = \frac{\xi_2}{\sqrt{\eta+\xi_2} - \sqrt{\eta}} \quad .$$

In addition, $U(0^+) \rightarrow 2\sqrt{\eta}$ while $W(0^+) \rightarrow 2\sqrt{\xi_2}$. Hence if $\eta > \xi_2$, there is no solution of (E.82) on $0 < z < \xi_2$. If $\eta < \xi_2$, there is one solution of (E.82) in $0 < z < \xi_2$ if and only if $W(\xi_2) < U(\xi_2)$ or if and only if

$$\sqrt{\xi_2} < \frac{\xi_2}{\sqrt{\eta+\xi_2} - \sqrt{\eta}} \quad .$$

But $\sqrt{\eta+\xi_2} < \sqrt{\eta} + \sqrt{\xi_2}$ implies $\sqrt{\eta+\xi_2} - \sqrt{\eta} < \sqrt{\xi_2}$, which in turn implies

$$\frac{\xi_2}{\sqrt{\eta+\xi_2} - \sqrt{\eta}} > \frac{\xi_2}{\sqrt{\xi_2}} > \sqrt{\xi_2} \quad ,$$

or that the inequality is valid for all values of ξ_2 and η. Consequently, $\eta < \xi_2$ implies there exists one solution of (E. 82) on $0 < z < \xi_2$. By direct substitution we observe that this solution is $z = \xi_2 - \eta$.

We now assert that if (E. 76) is true, the integrand $I_{\pm}(\lambda, \xi_1, \xi_2, \eta)$ (given by (E. 79)) has no stationary point within the range of integration. For if the integrand did have such a stationary point, then it would necessarily follow that

$$\xi_1 - O\left(\frac{1}{(k\eta)^{2/3}}\right) \lesssim \xi_2 - \eta \lesssim \xi_1 + O\left(\frac{1}{(k\eta)^{2/3}}\right) ,$$

or

$$\left| \xi_2 - (\xi_1 + \eta) \right| = \xi_1 O\left(\frac{1}{(k\eta)^{2/3}}\right) .$$

However, this implies

$$\frac{1}{4\xi_2\eta} \left| \xi_2 - (\xi_1 + \eta) \right|^2 = \frac{(\xi_2 - \eta)^2}{4\xi_2\eta} O\left(\frac{1}{(k\eta)^{4/3}}\right) ,$$

which violates (E. 76). Therefore in this case, the integral

$$\int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda \, I_{\pm}(\lambda, \xi_1, \xi_2, \eta)$$

can be estimated by integration by parts. Since $(\lambda/k)^{1/6} = O\left((k\xi_1)^{1/6}\right)$ in (E. 79), this yields

$$\int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda \, I_{\pm}(\lambda, \xi_1, \xi_2, \eta) = \frac{1}{\sqrt{\xi_1 \xi_2}} O\left(\frac{1}{(k\eta)^{1/3}}\right) . \qquad \text{(E. 83)}$$

The case where the condition (E. 76) is not fulfilled and the integrand has a stationary point in the range of integration, is discussed further in Section 3. 4. Thus if

(E.76) is fulfilled, we can substitute (E.83) and (E.77) into (E.42) to obtain

$$\frac{e^{-ikR_{\Xi}}}{R_{\Xi}} \sim \frac{e^{-ik\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}}}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} \quad . \tag{E.84}$$

But according to Buchholz (1953, Chapter 4, equation 3)

$$R_{\Xi}^2 = \left[(\xi - \eta) - \Xi\right]^2 + 4\xi\eta \ ,$$

or

$$R_{\Xi}^2 = \xi^2 + \eta^2 + \Xi^2 + 2\xi\eta + 2\eta\Xi - 2\xi\Xi \ .$$

This can be written as

$$R_{\Xi}^2 = \xi^2 + \eta^2 + \Xi^2 + 2\xi\eta + 2\eta\Xi + 2\xi\Xi - 4\xi \ ,$$

or

$$R_{\Xi}^2 = (\xi + \Xi + \eta)^2 - 4\xi\Xi \quad .$$

Hence $R_{\Xi}^2 = (\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2$, which shows for the configuration of ξ, Ξ, and η that satisfies (E.76), the term of largest order ~~reproduces~~ $e^{-ikR_{\Xi}}/R_{\Xi}$.

We conclude the study of $e^{-ikR_{\Xi}}/R_{\Xi}$ with some observations concerning the relative magnitudes of ξ_1, ξ_2, η. In the above derivation, we made no explicit assumptions regarding order of magnitude relationships between these variables. We tacitly assumed that $\xi_1/\eta = O(1)$, $\xi_1/\xi_2 = O(1)$. We assert that this derivation can be repeated (with minor modifications) for any other order of magnitude relationships, provided these relationships are independent of the large parameters of the asymptotic analysis. This is to ensure that we are not considering a limiting case, and is best illustrated by an example. We consider $\xi_1/\eta = O(1)$, $\xi_1/\xi_2 \ll 1$; these conditions represent a possibility of considerable physical interest. Then provided we can choose $k\eta$ such that $\xi_1/\xi_2 \gg 1/(k\eta)^{1/3}$, the above derivation can be repeated, with the appropriate modifications, as will be shown below. If we could not make this choice of $k\eta$, then we would, in reality, be considering a plane wave which has a different integral representation than (E.1).

In the case $\xi_1/\eta = O(1)$, $\xi_1/\xi_2 \ll 1$, $\xi_1/\xi_2 \gg 1/k\eta)^{1/3}$, we begin by writing (E.4) as

$$\frac{-e^{-ikR_{\mp}}}{R_{\mp}} = \sum_{j=1}^{7} \int_{C_j} d\lambda\, I_{\mp}(\lambda, \xi_1, \xi_2, \eta) \quad , \tag{E.4b}$$

where for some $M_i > 0$ $(i = 1, 2, \ldots 11)$ with $M_5 \gg 1$ and $M_7 \gg 1$

$$C_1 = \left\{\lambda \,\middle|\, -\infty < \lambda < -M_1 k^2 \xi_2 < 0\right\} ,$$

$$C_2 = \left\{\lambda \,\middle|\, -(M_1+M_2)k^2\xi_2 < \lambda < -M_3 k^2 \eta\right\} ,$$

$$C_3 = \left\{\lambda \,\middle|\, -(M_3+M_4)k^2\eta < \lambda < -M_5 k\right\} ,$$

$$C_4 = \left\{\lambda \,\middle|\, -(M_5+M_6)k < \lambda < (M_6+M_7)k\right\} ,$$

$$C_5 = \left\{\lambda \,\middle|\, M_7 k < \lambda < (M_8+M_9)k^2\eta\right\} ,$$

$$C_6 = \left\{\lambda \,\middle|\, M_9 k^2\eta < \lambda < (M_{10}+M_{11})k^2\xi_2\right\} ,$$

$$C_7 = \left\{\lambda \,\middle|\, M_{11}k^2\xi_2 < \lambda < \infty\right\} .$$

The argument for the outer intervals C_1 and C_7 remains the same as the one above for the outer intervals. The argument for C_4 must be modified only to the extent of redefining the intervals C_1' through C_5' so that the stationary points of $\psi_o^+(\lambda)$ and $\psi_o^-(\lambda)$ lie within C_4'. The difference between the argument for C_2 and the one previously is in the changing of s_{ξ_1} from $|s_{\xi_1}| = O(1)$ to

$$|s_{\xi_1}| = \frac{k\xi_1}{|\lambda/k|} = O(\xi_1/\xi_2) \ll 1 ,$$

and the changing of s_η from $|s_\eta| = O(1)$ to

$$|s_\eta| = \frac{k\eta}{|\lambda/k|} = O(\eta/\xi_2) \ll 1 .$$

However, (E.5) is not affected by this change. In addition,

$$|s_\eta| = O(\eta/\xi_2) = O(\xi_1/\xi_2) \gg \frac{1}{(k\eta)^{1/3}} > \frac{1}{(k\eta)^{2/3}}$$

implies that (E.7) together with the previous argument for $s_\eta < 1$ can still be used. Therefore, the derivation for C_2 remains the same. This is also true for C_3, since the only difference is the changing of s_{ξ_2}, from $|s_{\xi_2}| = O(1)$ to

$$|s_{\xi_2}| = \frac{k\xi_2}{|\lambda/k|} = O(\xi_2/\eta) \gg 1,$$

which does not affect the validity of (E.6).

We can use similar reasoning for the intervals C_5 and C_6. The only difference for C_5 is the changing from $|s_{\xi_2}| = O(1)$ to

$$|s_{\xi_2}| = \frac{k\xi_2}{|\lambda/k|} = O(\xi_2/\eta) \gg 1 .$$

Then (E.39) is valid for $v_2(\xi_2, -\lambda)$ throughout all of C_5 and the derivation proceeds as before. Finally, for C_6 we see that s_η changes from $|s_\eta| = O(1)$ to

$$|s_\eta| = \frac{k\eta}{|\lambda/k|} = O(\eta/\xi_2) \ll 1 ,$$

and s_{ξ_1} changes from $|s_{\xi_1}| = O(1)$ to

$$\left|s_{\xi_1}\right| = \frac{k\xi_1}{|\lambda/k|} = O(\xi_1/\xi_2) \ll 1 \ .$$

But (E.36) is not affected by this change, while

$$\left|s_{\xi_1}\right| = O(\xi_1/\xi_2) \gg \frac{1}{(k\eta)^{1/3}} > \frac{1}{(k\eta)^{2/3}} \gg \frac{1}{(k\xi_2)^{2/3}}$$

implies that (E.34) and the previous argument for $s_{\xi_1} < 1$ can be used for $v_1(\xi_1, -\lambda)$. Consequently, the derivation of the stationary point equations remains the same.

One of the reasons why this case is of interest is that in similar problems an approximate solution of the stationary point equations (or equation) can be readily found. If we examine equation (E.54) we observe that provided a solution exists, an approximation can be found by making the appropriate expansions for small values of z. Moreover, there is no possibility of obtaining an approximate solution to (E.64). Therefore, in order that this case produce an approximate solution of the stationary point equations it must imply that (E.54) has a solution or, equivalently, that (E.55) be valid. We can write (E.55) as

$$1 < \left(\sqrt{\frac{\xi_2}{\xi_1} - 1} + \sqrt{\frac{\xi_2}{\xi_1}}\right)\left(\sqrt{\frac{\eta}{\xi_1} + 1} - \sqrt{\frac{\eta}{\xi_1}}\right). \qquad \text{(E.55a)}$$

But $\eta/\xi_1 = O(1)$ implies

$$\sqrt{\frac{\eta}{\xi_1} + 1} - \sqrt{\frac{\eta}{\xi_1}} = O(1) \ ,$$

and since $\xi_2/\xi_1 \gg 1$, (E.55a) can be approximated by

$$1 < 2\sqrt{\xi_2/\xi_1}\ O(1) \ . \qquad \text{(E.55b)}$$

This shows that if we consider the condition $\sqrt{\xi_1/\xi_2} \ll 1$ added to the defining relations, then (E.55) will be fulfilled, (E.54) will have a solution, and there will be an approximate solution of the stationary point equations. We find the approximation by writing (E.54) as

$$\sqrt{\xi_1/\xi_2}\left[1+O(z)\right] = \frac{\sqrt{z}}{2\sqrt{\eta}}\left[1+O(z)\right] , \tag{E.54a}$$

which has the solution

$$\hat{z}_o^+ = 4\eta\,\frac{\xi_1}{\xi_2}\left[1+O(\xi_1/\xi_2)\right] . \tag{E.56a}$$

We now evaluate the result of the stationary phase integration if we use the stationary point given by (E.56a). Since

$$\left(\bar{\phi}(s_{\xi_1})\right)_{z=\hat{z}_o^+} = \sqrt{\frac{\xi_1-\hat{z}_o^+}{\xi_1}} = \sqrt{1-\frac{4\eta}{\xi_2}+O(\xi_1^2/\xi_2^2)} = 1-\frac{2\eta}{\xi_2}+O(\xi_1^2/\xi_2^2) ,$$

$$\left(\bar{\phi}(s_{\xi_2})\right)_{z=\hat{z}_o^+} = \sqrt{\frac{\xi_2-\hat{z}_o^+}{\xi_2}} = \sqrt{1-\frac{4\xi_1\eta}{\xi_2^2}+O(\xi_1^3/\xi_2^3)} = 1+O(\xi_1^2/\xi_2^2) ,$$

$$\left(i\phi(s_\eta)\right)_{z=\hat{z}_o^+} = \sqrt{\frac{\eta+\hat{z}_o^+}{\eta}} = \sqrt{1+\frac{4\xi_1}{\xi_2}+O(\xi_1^2/\xi_2^2)} = 1+\frac{2\xi_1}{\xi_2}+O(\xi_1^2/\xi_2^2) ,$$

we see that (E.60) becomes

$$\left(\frac{d^2\phi_o^+(\lambda)}{d\lambda^2}\right)_{\lambda=\hat{\lambda}_o^+} = \frac{\xi_2}{8k^3\xi_1\eta}\left[1+O(\xi_1/\xi_2)\right] . \tag{E.60a}$$

In addition,

$$\phi_o^+(\hat{\lambda}_o^+) = k\left[\sqrt{\xi_1(\xi_1-\hat{z}_o^+)} - \sqrt{\xi_2(\xi_2-\hat{z}_o^+)} - \sqrt{\eta(\eta+\hat{z}_o^+)}\right] ,$$

or

$$\Phi_o^+(\hat{\lambda}_o^+) = k\left[\xi_1\left(1-\frac{2\eta}{\xi_2}+O(\xi_1^2/\xi_2^2)\right)-\xi_2\left(1+O(\xi_1^2/\xi_2^2)\right)-\eta\left(1+\frac{2\xi_1}{\xi_2}+O(\xi_1^2/\xi_2^2)\right)\right] ,$$

which reduces to

$$\Phi_o^+(\hat{\lambda}_o^+) = -k\xi_2 + k(\xi_1-\eta) + k\eta\, O(\xi_1/\xi_2) . \tag{E.61a}$$

Then using the method of stationary phase, we obtain

$$\int_{C_4''} \frac{d\lambda\, C(k,\xi_1,\xi_2,\eta)(-i)^{1/2} e^{i\Phi_o^+(\lambda)}}{\left[\bar{\phi}(s_{\xi_1})\bar{\phi}(s_{\xi_2})\phi(s_\eta)\right]^{1/2}} \sim \frac{-e^{-ik\xi_2} e^{ik(\xi_1-\eta)}}{\xi_2} . \tag{E.62a}$$

This immediately implies

$$\int_{C_4''} d\lambda\, I_{\pm}(\lambda,\xi_1,\xi_2,\eta) \sim \frac{-e^{-ik\xi_2} e^{ik(\xi_1-\eta)}}{\xi_2} . \tag{E.77a}$$

But in this case (E.76) is true. Hence, (E.83) is also true. We write (E.83) as

$$\int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda\, I_{\pm}(\lambda,\xi_1,\xi_2,\eta) = \frac{1}{\xi_2}\, O\left(\frac{1}{(k\eta)^{1/3}}\cdot(\xi_2/\xi_1)^{1/2}\right) .$$

Since $\xi_2/\xi_1 \ll (k\eta)^{1/3}$ implies $(\xi_2/\xi_1)^{1/2} < (k\eta)^{1/6}$, this becomes

$$\int_{(k^2\xi_1)^-}^{(k^2\xi_1)^+} d\lambda\, I_{\pm}(\lambda,\xi_1,\xi_2,\eta) = \frac{1}{\xi_2}\, O\left(\frac{1}{(k\eta)^{1/6}}\right) . \tag{E.83a}$$

Substituting (E.83a) and (E.77a) into (E.42), we obtain

$$\frac{e^{-ikR_{\Xi}}}{R_{\Xi}} = \frac{e^{-ik\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}}}{\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2}} \sim \frac{e^{-ik\xi_2}\, e^{ik(\xi_1-\eta)}}{\xi_2} \quad . \tag{E.84a}$$

Observing that we can write

$$\sqrt{(\xi_1+\xi_2+\eta)^2 - 4\xi_1\xi_2} = \xi_2\left[1+\frac{\eta-\xi_1}{\xi_2} + O(\xi_1^2/\xi_2^2)\right] \quad ,$$

we see that the term of largest order reproduces the first two terms of the phase and the first term of the amplitude of $e^{-ikR_{\Xi}}/R_{\Xi}$.

E.2 Asymptotic Representation of $-e^{-ikR}/R$ for the Source at (0, H)

We again assume $\sigma = 0$; thus (E.2) becomes

$$\frac{-e^{-ikR_H}}{R_H} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda\, \Gamma\left(\frac{\lambda}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{-\lambda}{2ik}+\frac{1}{2}\right) v_2(\xi,-\lambda)v_1(\eta_1,\lambda)v_2(\eta_2,\lambda) \quad . \tag{E.2a}$$

If we make the substitution $\lambda = -\mu$ in the integrand, then equation (E.2a) can be written as

$$\frac{-e^{-ikR_H}}{R_H} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu\, \Gamma\left(\frac{\mu}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{-\mu}{2ik}+\frac{1}{2}\right) v_1(\eta_1,-\mu)v_2(\eta_2,-\mu)v_2(\xi,\mu) \, . \tag{E.85}$$

Moreover, we can define

$$I_H(\mu,\eta_1,\eta_2,\xi) = \frac{1}{2\pi i}\Gamma\left(\frac{\mu}{2ik}+\frac{1}{2}\right)\Gamma\left(\frac{-\mu}{2ik}+\frac{1}{2}\right) v_1(\eta_1,-\mu)v_2(\eta_2,-\mu)v_2(\xi,\mu) \quad . \tag{E.86}$$

Substituting (E.86) in (E.85) yields

$$\frac{-e^{-ikR_H}}{R_H} = \int_{-\infty}^{\infty} d\mu I_H(\mu, \eta_1, \eta_2, \xi) \; . \tag{E.87}$$

Upon comparing (E.86) with (E.3) and (E.87) with (E.4), we see that the derivation of the asymptotic representation of $-e^{-ikR_H}/R_H$ is entirely equivalent to that of $-e^{-ikR_{\mp}}/R_{\mp}$. We need only make the appropriate change of parameters to go from one to the other.

The same is not quite true of the remarks concerning the relative magnitudes of η_1, η_2, ξ. The difference arises in that for the source at (0, H), conditions that represent a possibility of considerable physical interest are $\eta_1/\eta_2 = O(1)$, $\eta_2/\xi \ll 1$. However, virtually the same argument can be employed to show that for these conditions, the derivation of the asymptotic representation proceeds as previously. We again find that the additional condition $\sqrt{\eta_2/\xi} \ll 1$ will produce an approximate solution to the stationary point equations. It is interesting to note that in this case the analog of equation (E.64)

$$\left(\sqrt{\eta_1 - z} + \sqrt{\eta_1}\right)\left(\sqrt{\eta_2 - z} + \sqrt{\eta_2}\right) = \frac{z^{3/2}}{\sqrt{\xi + z} - \sqrt{\xi}} \; , \tag{E.88}$$

with the condition for a solution given by the analog of (E.65)

$$\eta_1 > \left(\sqrt{\eta_2 - \eta_1} + \sqrt{\eta_2}\right)\left(\sqrt{\xi + \eta_1} - \sqrt{\xi}\right), \tag{E.89}$$

produces the approximate solution by making the appropriate expansions for small values of z. The analog of equation (E.64)

$$\frac{\sqrt{\eta_1 - z} + \sqrt{\eta_1}}{\sqrt{\eta_2 - z} + \sqrt{\eta_2}} = \frac{\sqrt{\xi + z} - \sqrt{\xi}}{\sqrt{z}} \; , \tag{E.90}$$

with the condition for a solution given by the analog of (E.55)

$$\eta_1 < \left(\sqrt{\eta_2 - \eta_1} + \sqrt{\eta_2}\right)\left(\sqrt{\xi + \eta_1} - \sqrt{\xi}\right), \tag{E.91}$$

is now the one with no possibility of producing an approximate solution.

APPENDIX F

COMPARISON OF CONE WITH PARABOLOID OF REVOLUTION

We consider here a cone with vertex at the origin of our coordinate system (Section 1.3) extending along the positive z-axis. The half cone angle θ_o is defined to be less than $\pi/2$. Then the domain described by the interior of the cone is given by $0 \leqslant \theta < \theta_o$. Let ϕ represent a possible solution to the Neumann potential problem for an interior point source on the cone axis. Then Gauss' theorem for the bounded volume defined by the intersection of the cone with a sphere (center at the origin) of radius r leads to the equation

$$\int_0^{2\pi}\int_0^{\theta_o} \frac{\partial\phi}{\partial r}\, r^2 \sin\theta\, d\theta d\phi = \int_0^{2\pi}\int_0^{\theta_o}\int_0^{r} \nabla^2\phi\, r^2 \sin\theta\, dr d\theta d\phi \;, \qquad (F.1)$$

since the outward normal over the sphere is the unit vector $\hat{r}$. Taking the limit as $r \to \infty$ of both sides of equation (F.1) we obtain the condition

$$\lim_{r\to\infty} r^2 \int_0^{\theta_o} \frac{\partial\phi}{\partial r} \sin\theta d\theta = 2 \;. \qquad (F.2)$$

Therefore, choosing a solution of the type chosen for the Dirichlet potential (Section 2.3) does not violate this condition.

The difference between the cone of half angle θ_o and paraboloid of revolution $\eta = \eta_o$ can best be illustrated by considering the area intersected by both on a sphere (center at the origin) of radius r. For the cone this area is given by

$$dA_r^c = r^2 \int_0^{2\pi}\int_0^{\theta_o} \sin\theta d\theta d\phi$$

or $\qquad (F.3)$

$$dA_r^c = 2\pi r^2 \left[1 - \cos\theta_o\right] \;,$$

while for the paraboloid

$$dA_r^p = r^2 \int_0^{2\pi} \int_0^{\theta(r)} \sin\theta d\theta d\phi \ ,$$

where $\theta(r)$ is defined by $\cos\theta(r) = \xi - \eta_o / \xi + \eta_o$. But

$$\frac{\xi - \eta_o}{\xi + \eta_o} = 1 - \frac{2\eta_o}{\xi + \eta_o} = 1 - \frac{2\eta_o}{r} \ ,$$

and hence

$$dA_r^p = 2\pi r^2 \left[-\cos\theta(r) + 1 \right]$$

or (F.4)

$$dA_r^p = 4\pi \eta_o r \ .$$

Consequently

$$\lim_{r \to \infty} \frac{dA_r^c}{r^2} = \text{constant},$$

while

$$\lim_{r \to \infty} \frac{dA_r^p}{r^2} = 0 \ .$$

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1. ORIGINATING ACTIVITY (Corporate author) The Radiation Laboratory Department of Electrical Engineering The University of Michigan	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Diffraction by the Concave Surface of the Paraboloid of Revolution

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report No. 3

5. AUTHOR(S) (Last name, first name, initial)

Stone, Stephen E.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 1967	272	31

8a. CONTRACT OR GRANT NO. F 04-694-67 C-0055	9a. ORIGINATOR'S REPORT NUMBER(S) 8525-3-T
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) BSD-TR-67-143
d.	

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13. ABSTRACT

Let η_o denote the focal length of a paraboloid of revolution, and let D be the closure of the domain bounded by its concave surface. Then for a point source, with wave number k, located in D and on the axis of the paraboloid, the diffraction by the boundary of D is considered not only if $k\eta_o \gg 1$ but also if $k\eta_o \ll 1$. If $k\eta_o \gg 1$, an asymptotic representation of the total field on the boundary of D is derived for the Neumann boundary condition provided the source is far (with respect to wavelength) from the focus and the field point is far from the tip of the paraboloid. This representation is interpreted in terms of geometric optics.

If $k\eta_o \ll 1$, an asymptotic representation of the total field anywhere in D is derived for both Dirichlet and Neumann boundary conditions and for the source (field) point in the near field and the field (source) point in the far field as well as for both source and field points in the near field. The near field result is compared with the solution of the corresponding potential problem. A necessary and sufficient condition for the existence of a solution to the corresponding Neumann potential problem is also derived.

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