ESD-TR-67-517, Vol. III

INVERSE SCATTERING INVESTIGATION
QUARTERLY REPORT NO. 3

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November 1967 8579-3-Q = RL-2185

SPACE DEFENSE SPO (496L/474L/N)
DEPUTY FOR SURVEILLANCE AND CONTROL SYSTEMS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

(Prepared under Contract No. AF 19(628)-67-C-0190 by The University of Michigan, Department of Electrical Engineering, Radiation Laboratory, Ann Arbor, Michigan.)

FOREWORD

This report was prepared by the Radiation Laboratory of the Department of Electrical Engineering of the University of Michigan. The work was performed under Contract No. F 19628-67-C-0190, "Inverse Scattering Investigation" and covers the period 3 July to 3 October 1967. Dr. Vaughan H. Weston is the Principal Investigator and the contract is under the direction of Professor Ralph E. Hiatt, Head of the Radiation Laboratory. The contract is administered under the direction of the Electronic Systems Division, Air Force Systems Command, United States Air Force, Laurence G. Hanscom Field, Bedford, Massachusetts 01730, by Lt. L. E. Nyman, ESSXS. This quarterly report was submitted by the authors 21 November 1967.

This technical report has been reviewed and is approved.

Prior to release of this report to CFSTI (formerly OTS) it must be reviewed by the ESD Office of Public Information, ESTI, Laurence G. Hanscom Field, Bedford, Massachusetts 01730.

Approving Air Force Authority

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ABSTRACT

The problem in question consists of determining means of solving the inverse scattering problem where the transmitted field is given and the received fields are measured, and this data is used to discover the nature of the target. Particular aspects of this overall problem are considered, such as the effect of phase errors upon the determination of the scattering surface, polynomial interpolation of the scattered field measured at a set of discrete points, and the testing of a numerical procedure for finding the surface of a conducting body from the knowledge of the near field. In addition, a review of exact theoretical treatments for the scalar inverse problem is given.

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I

THE EFFECT OF PHASE ERRORS

In any practical approach to the inverse problem, the degradation of the body shape due to phase errors in the far scattered field needs to be considered. These phase errors would arise either from the measurement process or the data processing. In the latter case where the scattered field is measured at a finite set of points, the error would arise from the fact that only an approximate finite polynomial fit can be made to the scattered field.

In order to investigate the effect of phase errors, the high frequency scattering case will be considered. It will be assumed that the scattering surface is perfectly conducting, and smooth except for curves or joins of discontinuity which are many wavelengths apart (i.e., the body is comprised of long smooth sections). In addition it will be assumed that in the cone of observations $0 < \theta < \theta_o$, the high frequency scattered far field can be represented in the form

$$\underline{E}^s = \frac{e^{ikR}}{R} \underline{E}_o(\theta, \phi)$$

$$\underline{E}_o(\theta,\phi) = \sum_{n=1}^{N} \underline{E}_n(\theta,\phi)\, e^{ik\psi_n(\theta,\phi)} \qquad (1.1)$$

where $\underline{E}_n(\theta, \phi)$ are slowly varying functions of the angular coordinates. This representation corresponds to the decomposition of the far scattered field into the components that arise from the various scattering centers (characterized by the subscript n).

The phase errors due to the measurement procedure or the data processing will be given by

$$k\epsilon\,(\theta, \phi) \qquad (1.2)$$

in which case the scattered far field will have the form

$$\tilde{\underline{E}}_o(\theta,\phi) = \sum_{n=1}^{N} \underline{E}_n(\theta,\phi)\ \mathrm{exp}\, ik\left(\psi_n(\theta,\phi) + \epsilon(\theta,\phi)\right) \qquad (1.3)$$

The near-zone scattered field is derivable from the far field by the relation

$$\underline{E}(x) = \frac{ik}{2\pi} \int_0^{\theta_o} \int_0^{2\pi} e^{i\underline{k}\cdot\underline{x}}\ \underline{E}_o(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta \qquad (1.4)$$

where $\underline{k} = k(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$. This is an approximate expression which is employed when the far field is known only over the cone of observation $0 \leqslant \theta < \theta_o$. The errors in using this expression have been discussed in the previous quarterly, where it was pointed out that the expression was quite accurate in the high frequency region for determining certain illuminated portions of the scattering surface.

Taking into account the phase errors in the far field, the near field will be given by

$$\tilde{\underline{E}}(x) = \frac{ik}{2\pi} \sum_{n=1}^{N} \int_0^{\theta_o} \int_0^{2\pi} e^{ikf_n(\alpha,\beta)}\ \underline{E}_n(\alpha,\beta)\sin\alpha\, d\alpha\, d\beta \qquad (1.5)$$

where $f_n(\theta,\phi) = g_n(\theta,\phi) + (x\sin\alpha\cos\beta + y\sin\alpha\sin\beta + z\cos\alpha)$.

Since the integral contains a set of terms, each of which has a rapidly varying phase, asymptotic analysis may be used to obtain an explicit expression for the near field. The dominant contribution of each term will arise from the stationary phase point provided that it is in the region of integration. A particular term which has this property can be easily evaluated as follows:

$$\tilde{\underline{E}}_n(\underline{x}) = \frac{ik}{2\pi}\int_0^{\theta_0}\int_0^{2\pi} e^{ikf_n(\alpha,\beta)}\,\underline{E}_n(\alpha,\beta)\,\sin\alpha\, d\alpha\, d\beta \tag{1.6}$$

$$\sim i\left\{\frac{\eta(f)\,\underline{E}_n(\alpha,\beta)\,e^{ikf_n(\alpha,\beta)}}{\left[Df(\alpha,\beta)\right]^{1/2}}\right\}_{(\alpha_1,\beta_1)} \tag{1.7}$$

where (α_1, β_1) is the stationary phase point given by

$$\frac{\partial f_n}{\partial \alpha_1} = 0 \ , \quad \frac{\partial f_n}{\partial \beta_1} = 0 \quad , \tag{1.8}$$

and

$$Df(\alpha,\beta) = \frac{1}{\sin^2\alpha}\,\frac{\partial^2 f_n}{\partial\alpha^2}\,\frac{\partial^2 f_n}{\partial\beta^2} - \left(\frac{1}{\sin\alpha}\,\frac{\partial^2 f_n}{\partial\alpha\partial\beta}\right)^2 \tag{1.9}$$

The factor $\eta(f)$ is unity unless

$$\frac{\partial^2 f_n}{\partial\alpha_1^2} > 0, \quad \frac{\partial^2 f_n}{\partial\beta_1^2} > 0 \quad \text{and} \quad Df(\alpha_1, \beta_1) > 0$$

in which case $\eta(f) = i$, and if

$$\frac{\partial^2 f_n}{\partial\alpha_1^2} < 0 \ , \quad \frac{\partial^2 f_n}{\partial\beta_1^2} < 0 \quad \text{and} \quad Df(\alpha_1, \beta_1) > 0$$

then $\eta(f) = -i$.

Expression (1.6) may now be used to determine the effect of the phase error $k\epsilon$ upon the near field. First the corresponding expression without the phase error has to be obtained. Set

$$g_n(\theta,\phi) = \psi_n(\theta,\phi) + (\underline{k}\cdot\underline{x})\,/\,k$$

in which case

$$f_n(\theta,\phi) = g_n(\theta,\phi) + \epsilon(\theta,\phi) \tag{1.10}$$

Then the corresponding integral to (1.6), for the near-field term devoid of phase errors, is given by

$$\underline{E}_n(\underline{x}) = \frac{ik}{2\pi}\int_0^{\theta_o}\int_0^{2\pi} e^{ikg_n(\alpha,\beta)}\,\underline{E}_n(\alpha,\beta)\sin\alpha d\alpha d\beta \tag{1.11}$$

If (α_o, β_o) is the stationery phase point given by

$$\frac{\partial g_n}{\partial \alpha_n} = 0\,, \quad \frac{\partial g_n}{\partial \beta_o} = 0 \qquad 0 \leqslant \alpha_o < \theta_o\,, \tag{1.12}$$

then expression (1.11) reduces to the form

$$\underline{E}_n(\underline{x}) \sim i\left\{\frac{\eta(g)\,\underline{E}_n(\alpha,\beta)}{\left[Dg(\alpha,\beta)\right]^{1/2}}\; e^{ikg_n(\alpha,\beta)}\right\}_{(\alpha_o,\,\beta_o)} \tag{1.13}$$

It will be assumed that the angular variation in the phase error $k\epsilon$ is sufficiently small such that the stationary phase point (α_1, β_1) is close to the point (α_o, β_o), thus enabling one to express (α_1, β_1) in terms of (α_o, β_o) by expanding equations (1.8) in terms of a Taylor series about the point (α_o, β_o) as follows:

$$\frac{\partial f_n}{\partial \alpha_1} = 0 = \frac{\partial f_n}{\partial \alpha_o} + (\alpha_1 - \alpha_o)\frac{\partial^2 f_n}{\partial \alpha_o^2} + (\beta_1 - \beta_o)\frac{\partial^2 f_n}{\partial \alpha_o \partial \beta_o} + \ldots \tag{1.14}$$

$$\frac{\partial f_n}{\partial \beta_1} = 0 = \frac{\partial f_n}{\partial \beta_o} + (\alpha_1 - \alpha_o)\frac{\partial^2 f_n}{\partial \alpha_o \partial \beta_o} + (\beta_1 - \beta_o)\frac{\partial^2 f_n}{\partial \beta_o^2} + \ldots \tag{1.15}$$

From equations (1.14) and (1.15), together with the relations

$$\frac{\partial f_n}{\partial \alpha_o} = \frac{\partial \epsilon}{\partial \alpha_o}$$

$$\frac{\partial f_n}{\partial \beta_o} = \frac{\partial \epsilon}{\partial \alpha_o}$$

a first order approximation to α_1 and β_1 can be obtained as follows:

$$\alpha_1 = \alpha_o + \left(\frac{\partial^2 f_n}{\partial \alpha_o \partial \beta_o} \frac{\partial \epsilon}{\partial \beta_o} - \frac{\partial^2 f_n}{\partial \beta_o^2} \frac{\partial \epsilon}{\partial \alpha_o}\right) / \Delta \qquad (1.16)$$

$$\beta_1 = \alpha_o + \left(\frac{\partial^2 f_n}{\partial \alpha_o \partial \beta_o} \frac{\partial \epsilon}{\partial \alpha_o} - \frac{\partial^2 f_n}{\partial \alpha_o^2} \frac{\partial \epsilon}{\partial \beta_o}\right) / \Delta \qquad (1.17)$$

where $\Delta = \sin^2 \alpha_o \, Df(\alpha_o, \beta_o)$.

Keeping only the first derivatives of ϵ. the above expressions reduce to

$$\alpha_1 = \alpha_o + \left(\frac{\partial^2 g_n}{\partial \alpha_o \partial \beta_o} \frac{\partial \epsilon}{\partial \beta_o} - \frac{\partial^2 g_n}{\partial \beta_o^2} \frac{\partial \epsilon}{\partial \alpha_o}\right) / \left[\sin \alpha_o^2 \, Dg(\alpha_o, \beta_o)\right] \qquad (1.18)$$

$$\beta_1 = \beta_o + \left(\frac{\partial^2 g_n}{\partial \alpha_o \partial \beta_o} \frac{\partial \epsilon}{\partial \alpha_o} - \frac{\partial^2 g_n}{\partial \alpha_o^2} \frac{\partial \epsilon}{\partial \beta_o}\right) / \left[\sin \alpha_o^2 \, Dg(\alpha_o, \beta_o)\right] \qquad (1.19)$$

For the general case, only the effect of the phase error ϵ on the phase of the near field will be considered. The phase error induced in the near field is given by

$$kf_n(\alpha_1, \beta_1) - kg_n(\alpha_o, \beta_o)$$

which equals

$$k\epsilon(\alpha_1,\beta_1) + k\left[\frac{(\alpha_1-\alpha_o)^2}{2}\frac{\partial g_n}{\partial\alpha_o^2} + (\alpha_1-\alpha_o)(\beta_1-\beta_o)\frac{\partial^2 g_n}{\partial\alpha_o\partial\beta_o} + (\beta_1-\beta_o)^2\frac{\partial^2 g}{\partial\beta_o^2}\right]$$

$$= \frac{k\epsilon(\alpha_1,\beta_1) + k\left[\frac{\partial^2 g_n}{\partial\alpha_o^2}\left(\frac{\partial\epsilon}{\partial\beta_o}\right)^2 + \frac{\partial^2 g_n}{\partial\beta_o^2}\left(\frac{\partial\epsilon}{\partial\alpha_o}\right)^2 - 2\frac{\partial^2\epsilon}{\partial\alpha_o\partial\beta_o}\frac{\partial^2 g_n}{\partial\alpha_o\partial\beta_o}\right]}{\left[2\sin\alpha_o{}^2 Dg(\alpha_o,\beta_o)\right]} \tag{1.20}$$

In order for the near field phase error to be small, not only must $|k\epsilon| \ll 1$, but

$$\left|\frac{\partial\epsilon}{\partial\alpha_o}\right| \ll \left|\frac{2\sin\alpha_o{}^2 Dg(\alpha_o,\beta_o)}{k\frac{\partial^2 g}{\partial\beta_o^2}}\right|^{1/2} \tag{1.21}$$

$$\left|\frac{\partial\epsilon}{\partial\beta_o}\right| \ll \left|\frac{2\sin\alpha_o{}^2 Dg(\alpha_o,\beta_o)}{k\frac{\partial^2 g}{\partial\alpha_o^2}}\right|^{1/2} \tag{1.22}$$

which implies that the angular variation in the phase error must not be too large.

To obtain a physical insight into the orders of magnitude of the right-hand side of inequalities (1.21) and (1.22), a specific example will be taken. The scattered field $\underline{E}_n(\theta,\phi)$ will be taken to arise from a spherical cap of radius a, centered at the origin, with the incident radiation travelling in the negative z-direction and polarized in the positive x direction. In this case the far field phase is given by

$$k\psi(\theta,\phi) = -2\,ika\cos(\theta/2).$$

For the point (α, θ, ϕ) on the surface of the sphere, the phase factor $g(\alpha, \beta)$ is given by the relation

$$g(\alpha, \beta) = a \left[\sin\alpha \sin\theta \cos(\phi - \beta) + \cos\theta \cos\alpha - 2\cos(\alpha/2)\right]$$

The stationary phase point (α_o, β_o) then satisfies the following relations

$$\cos\alpha_o \sin\theta \cos(\beta_o - \phi) - \cos\theta \sin\alpha_o + \sin(\alpha_o/2) = 0$$

$$-\sin\alpha_o \sin\theta \sin(\beta_o - \phi) = 0$$

which are derivable from equations (1.12). The appropriate solution is

$$\alpha_o = 2\theta, \quad \beta_o = \phi$$

The following relations may then be derived

$$\frac{\partial^2 g}{\partial \alpha_o^2} = \frac{-a}{2}\cos\theta \ ,$$

$$\frac{1}{\sin\alpha_o^2} \frac{\partial^2 g}{\partial \beta_o^2} = \frac{-a}{2} / \cos\theta \ ,$$

$$\frac{1}{\sin\alpha_o} \frac{\partial^2 g}{\partial \alpha_o \partial \beta_o} = 0 \ ,$$

$$Dg(\alpha_o, \beta_o) = a^2/4 \ .$$

For points on the spherical cap, inequalities (1.21) and (1.22) can be expressed in the explicit form

$$\left| k \frac{\partial \epsilon}{\partial \alpha_o} \right| << (ka \cos\theta)^{1/2} \qquad (1.23)$$

$$\left|\frac{k}{\sin\alpha_o}\ \frac{\partial\epsilon}{\partial\beta_o}\right| << \left(\frac{ka}{\cos\theta}\right)^{1/2} \qquad (1.24)$$

For phase errors, that have that property that either inequalities (1.21) and (1.22) are not fulfilled or that $k\epsilon \gtrsim O(1)$, then there will be a significant error in the near field, which will cause a significant degradation of the calculated scattering surface from the actual scattering surface. For perfectly conducting surfaces, the points on the surface are partially determined from the necessary condition

$$(\underline{E}^i + \underline{E}^s) \times (\underline{E}^i + \underline{E}^s)^* = 0 \qquad (1.25)$$

For the special case of high frequency specular scattering where the dominant scattered field in the vicinity of a portion of the surface can be approximated by the geometric optics result, then condition (1.25) can be replaced by the approximate condition

$$|E^s|^2 \simeq |E^i|^2$$

In this case large but slowly varying phase errors will not affect the result.

In the numerical treatment, condition (1.25) is replaced by the condition

$$G \equiv \underline{F} \cdot \underline{F} = 0 \qquad (1.26)$$

where $\underline{F}$ is the real vector given by the relation

$$\underline{F} = i(\underline{E}^i + \underline{E}^s) \times (\underline{E}^i + \underline{E}^s)^*$$

The numerical approach involves the point by point computation of the positive function G along the coordinate ray ($\theta = \theta_o$, $\phi = \phi_o$) of the spherical polar coordinate system, i.e. G is computed as a function of R, at increments of ΔR over a prescribed range. If there are no errors in the computed total field expression, then the point $R = R_o(\theta_o, \phi_o)$ for which $G(R, \theta_o, \phi_o) = 0$ would yield a point which may be on the surface (since condition 1.26 is a necessary but not sufficient condition). To determine the effect that small phase or amplitude far field errors have on condition (1.26), let $\underline{\epsilon}$ be the error in the total field due to such far field errors. In this case, the function that would be computed along the ray (θ_o, ϕ_o) would be

$$\tilde{G} = \underline{\tilde{F}} \cdot \underline{\tilde{F}} \tag{1.27}$$

where $\underline{\tilde{F}} = i(\underline{E}^i + \underline{E}^s + \underline{\epsilon}) \times (\underline{E}^i + \underline{E}^s + \underline{\epsilon})^*$

$$= \underline{F} + \underline{L}$$

where $\underline{L} = i\left[(\underline{E}^i + \underline{E}^s) \times \underline{\epsilon}^* + \underline{\epsilon} \times (\underline{E}^i + \underline{E}^s)^* + \underline{\epsilon} \times \underline{\epsilon}^*\right]$.

The computed expression then becomes

$$\tilde{G} = \underline{F} \cdot (\underline{F} + 2\underline{L}) + \underline{L} \cdot \underline{L} \tag{1.28}$$

and at $R = R_o$ where $G(R_o, \theta_o, \phi_o) = 0$,

$$\tilde{G} = \underline{L} \cdot \underline{L} \tag{1.29}$$

The effect of the errors in the total field, upon the behavior of $\tilde{G}$, will be to produce a non zero minimum or a zero minimum at another point then (R_o, θ_o, ϕ_o). The computing program should search out the minimum of $\tilde{G}$ rather than look for the zeros, to take into account the possibility of a non-zero minimum

To obtain more qualitative results, the case where the errors produce a shift in the zero point of $\tilde{G}$ will be considered. Let $\underline{n}$ be the unit normal to the surface at the point $\underline{R}_o$, E_n the normal component of the total electric field, and $\underline{H}$ the tangential components of the total magnetic field at $\underline{R}_o$. Expanding the total electric field $\underline{E}$ in terms of a Taylor series about the point $\underline{R}_o$ it can be shown that

$$\underline{F}(\underline{R}) = \underline{n} \cdot (\underline{R} - \underline{R}_o) \, \underline{P}(\underline{R}_o) + O(\underline{R} - \underline{R}_o)^2$$

$$\underline{P}(\underline{R}_o) = \left\{ \omega\mu_o \left[E_n^* \underline{H} + E_n \underline{H}^* \right] - i \left[E_n^* \, \underline{n} \times \underline{\nabla} E_n - E_n \underline{n} \times \underline{\nabla} E_n^* \right] \right\}.$$

If $\Delta R = R - R_o$, and γ and the angle between the normal $\underline{n}$ and the ray (θ_o, ϕ_o) then with retention of only linear terms in ΔR, $\tilde{G}$ will vanish at the value ΔR where

$$\Delta R = \frac{-(\underline{L} \cdot \underline{L})_o}{2 \cos\gamma \, \underline{P} \cdot \underline{L}}$$

The expression for $\underline{L}$ simplifies at $\underline{R}_o$, yielding

$$\underline{L} = i \left\{ E_n \underline{n} \times \underline{\epsilon}^* + E_n^* \, \underline{\epsilon} \times \underline{n} + \underline{\epsilon} \times \underline{\epsilon}^* \right\}.$$

Except for the case when the error $\underline{\epsilon}$ in the electric field has only the single component in the normal direction (yielding $\underline{L} = 0$), the shift in the zero point of G has the following order of magnitude

$$\Delta R \sim O \left(\frac{|\epsilon|}{\cos\gamma \left[\omega\mu_o |\underline{H}| + |\underline{\nabla} E_n| \right]} \right) \quad .$$

II

INTERPOLATION OF FAR FIELD MEASUREMENTS

When the far field component $\underline{E}_0(\theta,\phi)$ related to the far scattered field as follows

$$\underline{E}^s = \frac{e^{ikr}}{R}\,\underline{E}_0(\theta,\phi)\,,$$

is known over the complete unit sphere, the near field at a point $\underline{x}$ in the half-space $z > z^*$ is given by the representation

$$\underline{E}^s(\underline{x}) = \frac{ik}{2\pi}\int_0^{\frac{\pi}{2}-i\infty}\int_0^{2\pi} e^{i\underline{k}\cdot\underline{x}}\,\underline{E}_0(\alpha,\beta)\sin\alpha d\alpha d\beta\,.$$

The appropriate region of convergence $z > z^*$, and its relation to the scattering body is given in the final report (Weston, Bowman and Ar).

In practice, measurements of $\underline{E}_0(\theta,\phi)$ will be given at a finite set of points (θ_n,ϕ_n), $n = 1 \ldots N$, and in most cases these points will be confined to a measurement cone of half-angle α such that $0 \leqslant \theta_n \leqslant \alpha$. It was pointed out in previous quarterlies that in this case, it is important not only to obtain a good approximation for $\underline{E}_0(\theta,\phi)$ for θ real, but also for complex values of θ. Thus if $\underline{E}_0(\theta,\phi)$ is some polynomial approximation to $\underline{E}_0(\theta,\phi)$, one would like to find the best polynomial fit which would minimize

$$\left|\underline{E}_0(\theta,\phi) - \underline{\tilde{E}}_0(\theta,\phi)\right|$$

not only for $0 \leqslant \theta \leqslant \alpha$, but also for θ in an extended region in the complex plane. The investigation that follows is an initial effort towards this end.

The choice of best approximation will depend upon the asymptotic behavior of $\underline{E}_0(\theta,\phi)$ for $|\mathrm{Im}\,\theta| \to \infty$. This can be obtained from the following representation for $\underline{E}_o$

$$\underline{E}_o(\theta,\phi) = \frac{-i}{\epsilon_o \omega 4\pi} \iiint_V \underline{k} \times \underline{k} \times \underline{J}(\underline{x})\, e^{-i\underline{k}\cdot\underline{x}}\, d\underline{x}'$$

where $\underline{k} = k(\sin\theta\cos\phi,\ \sin\theta\sin\phi, \cos\theta)$

and V is a bounded region. If we set

$$\zeta = e^{i\theta}$$

so that

$$2\cos\theta = \zeta + \frac{1}{\zeta},\quad 2i\sin\theta = \zeta - \frac{1}{\zeta}$$

then each component of $\underline{E}_o(\theta,\phi)$ rise to a function

$$\begin{aligned} f(\zeta) &= f(\phi,\zeta) \\ &= \int_V \tilde{J}(\underline{x}, \zeta, \frac{1}{\zeta}) \exp\left\{A(\underline{x})(\zeta - \frac{1}{\zeta}) + B(\underline{x})(\zeta + \frac{1}{\zeta})\right\} d\underline{x} \end{aligned}$$

for certain linear functions A, B, and $\tilde{J}$, which is a polynomial in ζ and $\frac{1}{\zeta}$.

It immediately follows that the behavior for $\operatorname{Im}\theta \to \infty$ is equivalent to the behavior for $|\zeta| \to \infty$; and thus the function $f(\zeta)$ which is single-valued and analytic in the finite plane punctured at the origin, satisfies the growth condition

$$\lambda(R) < A e^{BR}$$

where

$$\lambda(R) = \operatorname{Max} |f(\zeta)|$$

$$\frac{1}{R} = |\zeta| \leqslant R \ .$$

The mathematical problem that will be considered is the following. Given functions $f(\zeta)$ which satisfy the above analytic and growth conditions, and which are measured at points on the arc T_α, given by

$$T_\alpha ;\ |\zeta| = 1 ;\ |\arg\zeta| \leqslant \alpha < \pi ,$$

we wish to find a numerically feasible way to approximate $f(\zeta)$ in any bounded region, using measurements of $f(\zeta)$ on T_α.

Now let us make the transformation

$$w = -(i/\alpha)\log\zeta, \quad \zeta = e^{\alpha i w} .$$

The arc T_α is mapped onto the real segment $-1 \leqslant w \leqslant 1$; as ζ tends to zero, Im w tends to infinity, and as ζ tends to infinity, Im w tends to negative infinity. The ζ-plane is represented on the strip $-\pi/\alpha < \mathrm{Re}\, w < \pi/\alpha$. If we define F by

$$F(w) = f\left(\zeta(w)\right),$$

then F(w) is periodic, satisfying

$$F\left(w + \frac{2\pi}{\alpha}\right) = F(w),$$

and F(w) is entire. Furthermore,

$$\left|F(R\,e^{i\theta})\right| \leqslant \lambda(e^{\alpha R}) < A e^{B e^{\alpha R}} \tag{2.1}$$

We shall show how to approximate F(w), to within an arbitrary error $\epsilon > \theta$, in the disc $|w| < (\log S)/\alpha$. This will yield an approximation, with the same tolerance, ϵ, for $f(\zeta)$ in the annulus

$$e^{\pi}/S < |\zeta| < S/e^{\pi}, \tag{2.2}$$

for if we let $|(\arg\zeta)| \leqslant \pi$ we have

$$|w| = \left|\frac{\log\zeta}{\alpha}\right| \leqslant \frac{1}{\alpha}\left|\log|\zeta|\right| + \frac{1}{\alpha}|\arg\zeta|$$

$$< \left(\frac{\log S}{\alpha} - \frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}$$

$$= \frac{\log S}{\alpha} .$$

2. The Legendre Coefficients of F(w)

The Legendre polynomial of degree n is

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2 - 1)^n .$$

The Legendre polynomials are orthogonal over $[-1, 1]$, with

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases} .$$

The Legendre coefficients of a function h(x), assumed integrable over $[-1, 1]$, are

$$a_j = \frac{2j+1}{2} \int_{-1}^{1} h(x)\, P_j(x) dx$$

and the formal Legendre series of h(x) is

$$h(x) \sim \sum_{j=0}^{\infty} a_j P_j(x) .$$

We state the following theorem, which is a special case of Walsh (1935).

Theorem 2.1: Let h(z) be an entire function. Then

$$h(z) = \sum_{j=0}^{\infty} a_j P_j(z) , \qquad (2.3)$$

$\{a_j\}$ the Legendre coefficients of h. The series converges uniformly in any bounded set.

We shall need to be able to estimate the coefficients a_j in (2.3) in terms of the growth of h(z), as one would for an ordinary power series.

The proof of the following lemma is suggested by the proof in Indritz (1963, Theorem 5.3E). We shall need the identities

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)\, P_n(x) , \qquad n \geqslant 1 . \qquad (2.4)$$

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n . \tag{2.5}$$

<u>Lemma</u> 2.1. Let h(x) be infinitely differentiable on $[-1, 1]$, with a_j the Legendre coefficient of h(x).

Then

$$|2a_j| \leqslant \frac{2^k}{(2j-1)(2j-3)\ldots(2j-2k+3)} \max_{|r| \leqslant h} \left| \int_{-1}^{1} h^{(k)}(t) P_{j+r}(t)\, dt \right| \tag{2.6}$$

for $2 \leqslant k \leqslant j-1$. (Here, $h^{(k)}$ is the k^{th} derivative of h.)

<u>Proof</u>: We have

$$2a_j = (2j+1) \int_{-1}^{1} h(t) P_j(t) dt$$

$$= \int_{-1}^{1} h(t)\, d\left\{P_{j+1}(t) - P_{j-1}(t)\right\}$$

$$= - \int_{-1}^{1} h'(t) \left\{P_{j+1}(t) - P_{j-1}(t)\right\} dt,$$

using (2.4) and (2.5), integrating by parts. Iterating this operation again gives us

$$2a_j = \int_{-1}^{1} h''(t) \left\{ \frac{P_{j+2}(t) - P_j(t)}{2j+3} - \frac{P_j(t) - P_{j-2}(t)}{2j-1} \right\} dt ,$$

then

$$|2a_j| \leqslant \frac{4}{2j-1} \max_{|r| \leqslant 2} \left| \int_{-1}^{1} h''(t) P_{j+r}(t) dt \right|$$

which is (2.6) with k=2. To complete the verification of (2.6) we use induction. Assuming (2.6) with $k<j-1$, we have

$$|2a_j| \leqslant \frac{2^k}{(2j-1)\dots(2j-2h+3)} \operatorname*{Max}_{|r|\leqslant k} \left| \int_{-1}^{1} h^{(k)}(t) \frac{d\left(P_{j+r+1}(t)-P_{j+r-1}(t)\right)}{2j+2r+1} \right|$$

$$\leqslant \frac{2' 2^k}{(2j-1)\dots(2j-2k+3)(2j-2k+1)} \operatorname*{Max}_{|r|\leqslant k+1} \left| \int_{-1}^{1} h^{9k+1)}(t)P_{j+r}(t)dt \right|$$

and the proof is complete.

Now setting $k=j-1$ in (2.6) we get

$$|2a_j| \leqslant \frac{2^{j-1}}{(2j-1)(2j-3)\dots7\cdot5\cdot3} \operatorname*{Max}_{1\leqslant r\leqslant 2j-1} \left| \int_{-1}^{1} h^{(j-1)}(t)P_r(t)dt \right|$$

$$\leqslant \frac{2^j}{(2j-1)(2j-3)\dots7\cdot5\cdot3} \operatorname*{Max}_{x\in[-1,1]} \left| h^{(j-1)}(x) \right|$$

since $|P_r(t)| \leqslant 1$ for $-1\leqslant t\leqslant 1$ (Walsh, 1935). Thus

$$|2a_j| < \frac{2^j}{(2j-1)\dots7\cdot5\cdot3} \frac{(2j-2)\dots4\cdot2}{(2j-2)\dots4\cdot2} \operatorname*{Max}_{x\in[-1,1]} \left| h^{(j-1)}(x) \right|$$

$$= \frac{2^{2j}(j-1)!}{(2j-1)!} \operatorname*{Max}_{x\in[-1,1]} \left| h^{(j-1)}(x) \right| . \qquad (2.7)$$

Now we wish to apply (2.7) to the specific function F(w) described above. We need to estimate

$$\operatorname*{Max}_{x\in[-1,1]} \left| F^{(k)}(x) \right| .$$

Now, assuming $R > 2$,

$$\frac{|F^{(k)}(x)|}{k!} = \left| \frac{1}{2\pi i} \int_{|w|=R} \frac{F(w)}{(w-x)^{k+1}} \, dw \right|$$

$$< \frac{R}{(R-1)^{k+1}} \max_{|w|=r} |F(w)|$$

$$< 2R^{-k} \max_{|w|=R} |F(w)|$$

$$< 2AR^{-k} e^{Be^{\alpha R}} \quad , \qquad (2.8)$$

using (2.1) in the last step.

Of course, (2.8) remains true if we choose R to minimize the right-hand member, subject to the constraint $R > 2$. Let

$$\phi(R) = 2AR^{-k} e^{Be^{\alpha R}} \quad .$$

Then

$$\log \phi(R) = \log 2A - k \log R + Be^{\alpha R}$$

and

$$\frac{\phi'(R)}{\phi(R)} = -\frac{k}{R} + B\alpha e^{\alpha R} \quad .$$

We have

$$\frac{d^2}{dR^2} \log \phi(R) = \frac{k}{R^2} + B\alpha^2 e^{\alpha R} \quad ,$$

which is positive, thus $\phi'(R)/\phi(R)$ increases, as R increases from θ to ∞, from negative to positive values, and there is a unique number R_k such that $\phi'(R_k) = 0$.

If

$$k > 2B\alpha e^{2\alpha}$$

then $R_k > 2$, so that R_k satisfies the constraint which was necessary to impose on R for (2.8) to be valid.

The equation $\phi'(R_k) = 0$ can be put into the form

$$\alpha R_k + \log R_k = \log (k/B\alpha). \tag{2.9}$$

From (2.9) we immediately get

$$R_k < \frac{1}{\alpha} \log (k/B\alpha), \tag{2.10}$$

which is an asymptotic quality as k tends to infinity. Now let $k_o(\alpha)$ be the first integer k such that

$$\log(k/B\alpha) > \alpha e + 1. \tag{2.11}$$

Then for $k > k_o(\alpha)$ we must have $R_k > e$. Now by elementary calculus

$$\frac{x}{\log x} > e, \qquad x > e \tag{2.12}$$

since the derivative of the left hand side of (2.12) is positive for $x > e$. Thus

$$\alpha R + \log R < R \left[\alpha + \frac{1}{e}\right], \qquad R > e$$

so that for $k \geq k_o(\alpha)$,

$$R_k \left[\alpha + \frac{1}{e}\right] > \alpha R_k + \log R_k = \log(k/B\alpha),$$

and solving this inequality,

$$R_k > \frac{e}{e\alpha + 1} \log(k/B\alpha) . \tag{2.13}$$

We now use both (2.10) and (2.13) to calculate $\phi(R_k)$, which is an upper bound for $|F^{(k)}(x)|$ /k! on $[-1,1]$, recall, (2.8). We have, assuming of course $k>k_o(\alpha)$,

$$\phi(R_k) < 2A\left(\frac{e^{\alpha+1}}{e}\right)^k \frac{1}{[\log(k/B\alpha)]^k} e^{Be^{\log(k/B\alpha)}}$$

$$< \frac{2(\pi+1)^k A e^{k/\alpha}}{[\log(k/B\alpha)]^k}$$

$$< 2A\left\{5e^{1/\alpha}/\log(k/B\alpha)\right\}^k. \qquad (2.14)$$

The right-hand side of (2.14) is an upper bound for $F^{(k)}(w)/k!$ on $[-1,1]$. Combining this fact with (2.7), we have for the Legendre coefficients a_j of F the upper bound

$$|a_j| < \frac{2^{2j}[(j-1)!]^2}{(2j-1)!} A_j\left\{\frac{5e^{1/\alpha}}{\log\frac{j-1}{B\alpha}}\right\}^{j-1}, \quad j > k_o(\alpha)+1. \qquad (2.15)$$

We wish to simplify (2.15) as muchas possible. We have

$$\frac{j[(j-1)!]^2}{(2j-1)!} = \frac{j!(j-1)!}{(2j-1)!}$$

$$= \frac{j-1}{2j-1}\,\frac{j-2}{2j-2}\cdots\frac{1}{j+1}$$

$$< 1/2^{j-1}$$

since $(j-k)/(2j-k) < 1/2$. Thus

$$|a_j| < 2^j A\left\{\frac{5e^{1/\alpha}}{\log\frac{j-1}{B\alpha}}\right\}^{j-1} < 2^j A\,\frac{5^{j-1}}{\left(\log\frac{j-1}{B\pi}\right)^{j-1}}\,e^{(j-1)/\alpha} = 2A\left(\frac{10e^{1/\alpha}}{\log(j-1)/B\pi}\right)^{j-1},$$

since $\alpha<\pi$, and if

$$j-1>(B\pi)^2$$

so that

$$\log(j-1)/B\pi > \frac{1}{2}\log(j-1)$$

we get the more convenient estimate

$$|a_j| < 2A\left(\frac{20e^{1/\alpha}}{\log(j-1)}\right)^{j-1}, \tag{2.16}$$

provided

$$j>(B\pi)^2+1; \qquad j>k_0(\alpha)+1. \tag{2.16'}$$

Equations (2.16) and (2.16') comprise the objective of this selection; bounds for the Legendre coefficients of the function $F(w)$.

3. Error in Approximating $F(w)$ by a Partial Sum of the Legendre Series

We wish to approximate $F(w)$ in the disc

$$|w|<\frac{1}{\alpha}\log S\ . \tag{2.17}$$

Let $S_N(w)$ be the partial sum

$$S_N(w)=\sum_{j=0}^{N} a_jP_j(w)$$

of the Legendre series for F. By Theorem 2.3,

$$\left|F(w)-S_N(w)\right|=\left|\sum_{j=N+1}^{\infty} a_jP_j(w)\right|<2A\sum_{j=N+1}^{\infty}\left(\frac{20e^{1/\alpha}}{\log(j-1)}\right)^{j-1}\left|P_j(w)\right|, \tag{2.18}$$

provided

$$N > (B\pi)^2 + 1, \qquad N > k_o(\alpha) + 1\ , \tag{2.18'}$$

by (2.16) and (2.16'). Now we use the well-known inequality (Indritz, 1963, p. 269)

$$\left|P_j(w)\right| \leqslant \left|2w\right|^j\ , \qquad |w| > 1\ . \tag{2.19}$$

Substituting in (2.18), we have

$$\left|F(w) - S_N(w)\right| < 4A \sum_{j=N+1}^{\infty} \left(\frac{40e^{1/\alpha}}{\log(j-1)}\right)^{j-1} |w|^j$$

$$< \frac{4A}{\alpha} \log S \sum_{j=N+1}^{\infty} \left(\frac{40\,|w|\,e^{1/\alpha}}{\log(j-1)}\right)^{j-1}$$

$$< \frac{4A\log S}{\alpha} \sum_{j=N}^{\infty} \left(\frac{40e^{1/\alpha}\log S}{\alpha \log j}\right)^{j} \tag{2.20}$$

for $|w| < \frac{1}{\alpha}\log S$, N satisfying (2.18').

If N is so large that

$$\log N > \frac{80\,e^{1/\alpha}\log S}{\alpha}$$

then from (2.20) we have

$$\left|F(w) - S_N(w)\right| < \frac{4Ae^{1/\alpha}\log S}{\alpha} \sum_{j=0}^{\infty} \left(\frac{40\,e^{1/\alpha}\log S}{\alpha\log N}\right)^N \frac{1}{2^j}$$

$$< \frac{8A\,e^{1/\alpha}\log S}{\alpha} \left(\frac{40\,e^{1/\alpha}\log S}{\alpha\log N}\right)^N .$$

We have

Lemma 2.2. Let N satisfy the inequalities

(i) $N > (B\pi)^2 + 1$

(ii) $N > k_o(\alpha) + 1$, where

$$k_o(\alpha) = \text{Min}\left\{\frac{k}{\log(k/B\alpha)} > \alpha e + 1\right\}$$

(iii) $\log N > \dfrac{80\, e^{1/\alpha} \log S}{\alpha}$.

Then

$$\left|F(w) - S_N(w)\right| < \frac{8A\, e^{1/\alpha} \log S}{\alpha} \left(\frac{40\, e^{1/\alpha} \log S}{\alpha \log N}\right)^N . \qquad (2.21)$$

4. <u>The Beginning Coefficients</u>

Let $N_1 = N_1(S,\epsilon)$ be the first integer N which satisfies the hypothesis of Lemma 2.17, and such that

$$\frac{8A\, e^{1/\alpha} \log S}{\alpha} \left(\frac{40\, e^{1/\alpha} \log S}{\alpha \log N}\right)^N < \epsilon/2 .$$

Then

$$\left|F(w) - S_{N_1}(w)\right| < \epsilon/2.$$

To approximate F(w) to within an error ϵ in the disc $|w| < (\log S)/\alpha$, we need only approximate a_j, for $j < N_1$, closely enough.

Let $F^*(w)$ be a piecewise continuous approximation, on $-1 \leqslant w \leqslant 1$, to F(w), and let a_j^* be the Legendre coefficients of $F^*(w)$. Let

$$S_N^*(w) = \sum_{j=0}^{N_1} a_j^* P_n(w) .$$

Then, since $|P(w)| < |2w|^n$ for $|w| > 1$,

$$\left|S_N(w) - S_N^*(w)\right| < \sum_{j=0}^{N_1} \left|a_j - a_j^*\right| \left(\frac{2 \log S}{\alpha}\right)^j$$

$$< N_1 \left(\frac{2 \log S}{\alpha}\right)^{N_1} \max_{j \leqslant N_1} \left|a_j - a_j^*\right| , \qquad (2.22)$$

provided

$$\frac{2 \log S}{\alpha} > 1 .$$

Now

$$\left|a_j - a_j^*\right| = \left| \int_{-1}^{1} \left\{F(x) - F^*(x)\right\} P_j(x)dx \right|$$

$$\leqslant \sqrt{\int_{-1}^{1} \left[F(x) - F^*(x)\right]^2 dx \cdot \int_{-1}^{1} \left[P_j(x)\right]^2 dx}$$

by Schwarz's inequality, and the norm of P_j, as mentioned in Section 2, is $2/(2j + 1)$, thus

$$\left|a_j - a_j^*\right| < \frac{2}{2j+1} \left\| F - F^*\right\| < 2\left\| F - F^*\right\| . \qquad (2.23)$$

(Here, $\|h\|$ is the inner-product norm over $[-1, 1]$.)

From (2.22) and (2.23) we see that

$$\left|S_N(w) - S_N^*(w)\right| < \epsilon/2, \qquad |w| < \frac{\log S}{\alpha}$$

provided

$$\left\|F - F^*\right\| < \frac{1}{4N_1} \left(\frac{\alpha}{2 \log S}\right)^{N_1} \epsilon . \qquad (2.24)$$

Now let us take interpolation points

$$x_j = -1 + \frac{1}{m} j, \qquad j = 1 - m,$$

and define $F^*(x)$ by

$$F^*(x) = F(x_j) + \eta_j, \qquad x_1 \leqslant x < x_{j+1}.$$

Then

$$\left|F(x) - F^*(x)\right| < \eta_j + \frac{1}{m} \max_{-1 \leqslant t \leqslant 1} \left|F'(t)\right| . \tag{2.25}$$

To estimate the derivation in (2.25), we have

$$\left|F'(t)\right| = \left|\frac{1}{\pi i} \int_{|w| = R} \frac{F(w)}{(w-t)^2} dt\right| < \frac{R}{(R-1)^2} \max_{|w| = R} \left|F(w)\right|$$

and choosing the convenient value $R = 2$, making use of (2.1), we get

$$\begin{aligned} \left|F'(t)\right| &< 2A\, e^{Be^{2\alpha}} \\ &= C \quad . \end{aligned} \tag{2.26}$$

Substituting in (2.26) we get

$$\left|F(x) - F^*(x)\right| < \eta_j + \frac{C}{m}, \qquad -1 \leqslant x \leqslant 1 , \tag{2.27}$$

with C defined by (2.25). Therefore

$$\left\|F - F^*\right\|^2 \leqslant \int_{-1}^{1} (\eta_j + \frac{C}{m})^2 dt$$

$$= 2 \left[\max_j (\eta_j + \frac{C}{m})\right]^2 = 2(\eta + \frac{C}{m})^2$$

where

$$\eta = \operatorname*{Max}_{j} \eta_j \quad .$$

(η is to be interpreted as the maximal error in calculating F at the data-bearing points x_j .)

From (2.24) and (2.27), we see that

$$\left| S_N(w) - S_N^*(w) \right| < \epsilon/2 \;, \qquad |w| < \frac{\log S}{\alpha}$$

provided

$$\eta + \frac{C}{m} < \frac{1}{\sqrt{8N_1}} \left(\frac{\alpha}{2 \log S} \right)^{N_1/2} \sqrt{\epsilon} \quad .$$

5. Statement of the Result

We bring together the above considerations by stating a formal theorem:

Theorem 2.1: Suppose that

(i) $x_j = -1 + \frac{2}{m} j \;, \qquad j = 1 - m$

(ii) $F^*(x) = F(x_j) + \eta_j \;, \qquad x_j < x < x_{j+1}$

(iii) $S_N^*(x) = \sum_{j=0}^{N} a_j^* P_j(x) \;,$

where

$$a_j^* = \int_{-1}^{1} F(x)\, P_j(x) dx$$

(iv) $k_o(\alpha) = \operatorname{Min} \left\{ \frac{k}{\log kB\alpha} > \alpha e + 1 \right\} \; .$

Then, for $\epsilon > \theta$, we have

$$\left| F(w) - S_N^*(w) \right| < \epsilon , \qquad |w| < \frac{\log S}{\alpha} \tag{2.28}$$

provided

$$\text{(v)} \quad N > \text{Max} \left\{ (B\pi)^2 + 1, \qquad k_o(\alpha) + 1 \right\}$$

and

$$\log N > \frac{80\, e^{1/\alpha} \log S}{\alpha}$$

$$\text{(vi)} \quad \text{If } \eta = \underset{j}{\text{Max}} \left| \eta_j \right| \quad \text{then}$$

$$\eta + \frac{C}{m} < \frac{1}{\sqrt{8N_1}} \left(\frac{\alpha}{2 \log S} \right)^{N_1/2} \sqrt{\epsilon} \ ,$$

N_1 the minimal N satisfying conditions (v), C defined by (2.26).

When (2.28) is satisfied, we have

$$\left| f(z) - s_N(\zeta) \right| < \epsilon , \qquad 1/S < \left| \zeta \right| < S , \tag{2.28'}$$

where

$$\begin{cases} s_N(\zeta) = S_N(w) , \\ w = -(i/\alpha) \log \zeta \end{cases} \tag{2.28''}$$

III

ANALYSIS OF THE COMPUTATIONAL RESULTS OF DETAIL SPECIFICATION (PART I) - SUBROUTINE F6: $\underline{E}_T \times \underline{E}^*_T = 0$.

In the present CPCEI Inverse Scattering, which is restricted to the identification of perfectly conducting targets, the boundary condition $\underline{E}_T \times \underline{E}_T^* = 0$ was implied where $\underline{E}_T$ denotes the sum of the incident and scattered electric field vectors, i.e., $\underline{E}_T = \underline{E}_i + \underline{E}_s$. The near field representation of $\underline{E}_i$ and $\underline{E}_s$ is obtained from an expansion into proper vector wave functions, in particular for finite convex-shaped bodies the approximation representation in vector spherical harmonics is employed.

Using Stratton's notation (Stratton, 1941, Ch. 9. 25) it can be shown that the total field $\underline{E}_T$ for the case of a perfectly conducting body can be given with exp $(-i\omega t)$ time dependence for

$$\underline{E}_i = \hat{x} E_o e^{ikz} = \hat{x}\, 1\, e^{ikz} = (\hat{R}_o \sin\theta \cos\phi + \hat{\theta}_o \cos\theta \cos\phi - \hat{\phi}_o \sin\phi)\, e^{i(kR\cos\theta)} \tag{3.1}$$

as

$$\underline{E}_T = \hat{R}_o E_R^T + \hat{\theta}_o E_\theta^T + \hat{\phi}_o E_\phi^T \tag{3.2}$$

where

$$E_R^T = \sum_{n=1}^{\infty} \frac{2(2n+1)}{n(n+1)} \left[g_{ein}^{i} j_n(\rho) + g_{ein}^{s} h_n^{(1)}(\rho) \right] \frac{P_n(\cos\theta)}{\rho} \cos\phi$$

$$= \sum_{n=1}^{\infty} \frac{(2n+1)}{\rho} (i)^{n+1} \mathcal{H}_n^{(1)} P_n(\cos\theta) \cos\phi \tag{3.2a}$$

$$E_\theta^T = \sum_{n=1}^{\infty} \frac{2(2n+1)}{\left[n(n+1)\right]^2} \left\{ \left[f_{o1n}^i \, j_n(\rho) + f_{o1n}^s \, h_n^{(1)}(\rho) \right] \frac{P_n(\cos\theta)}{\sin\theta} \right.$$

$$\left. + \left[g_{e1n}^i \left[\rho \, j_n(\rho)\right]' + g_{e1n}^s \left[\rho h_n^{(1)}(\rho)\right]' \right] \frac{\partial P_n(\cos\theta)}{\rho \partial\theta} \right\} \cos\phi$$

$$= \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} (i)^n \left[\mathcal{F}_n^{(1)} \frac{P_n(\cos\theta)}{\sin\theta} + \frac{i}{\rho} \mathcal{G}_n^{(1)} \frac{\partial P_n(\cos\theta)}{\partial\theta} \right] \cos\phi$$

(3.2b)

$$E_\phi^T = - \sum_{n=1}^{\infty} \frac{2(2n+1)}{\left[n(n+1)\right]^2} \left\{ \left[f_{o1n}^i \, j_n(\rho) + f_{o1n}^s \, h_n^{(1)}(\rho) \right] \frac{\partial P_n(\cos\theta)}{\partial\theta} \right.$$

$$\left. + \left[g_{e1n}^i \left[\rho \, j_n(\rho)\right]' + g_{e1n}^s \left[\rho \, h_n^{(1)}(\rho)\right]' \right] \frac{P_n(\cos\theta)}{\rho \sin\theta} \right\} \sin\phi$$

$$= - \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} (i)^n \left[\mathcal{F}_n^{(1)} \frac{\partial P_n(\cos\theta)}{\partial\theta} \right.$$

$$\left. + \frac{i}{\rho} \mathcal{G}_n^{(1)} \frac{P_n(\cos\theta)}{\sin\theta} \right] \sin\phi \qquad (3.2c)$$

where for a perfectly conducting sphere of $\sigma = ka$:

$$g_{e1n}^{i,s} = -(i)^{n+1} \frac{n(n+1)}{2} a_n^{i,s} \qquad a_n^i = 1, \qquad a_n^s = - \frac{\left[\sigma j_n(\sigma)\right]'}{\left[\sigma h_n^{(1)}(\sigma)\right]'} \qquad (3.2d)$$

$$f^{i,s}_{oln} = (i)^n \frac{n(n+1)}{2} b^{i,s}_n \qquad \begin{aligned} b^i_n &= 1 \\ b^s_n &= -\frac{j_n(\sigma)}{h^{(1)}_n(\sigma)} \end{aligned} \tag{3.2e}$$

$$\mathcal{H}^{(1)}_n(\rho,\sigma) = - \left. \frac{\left[j_n(\rho) \left[\sigma h^{(1)}_n(\sigma)\right]' - h^{(1)}_n(\rho) \left[\sigma j_n(\sigma)'\right] \right]}{\left[\sigma h^{(1)}_n(\sigma)\right]'} \right|_{\rho=\sigma}$$

$$= \frac{-i2}{\pi \left[\sigma h^{(1)}_n(\sigma)\right]'} \tag{3,2f}$$

$$\mathcal{J}^{(1)}_n(\rho,\sigma) = \left. \frac{\left[j_n(\rho)\, h^{(1)}_n(\sigma) - h^{(1)}_n(\rho)\, j_n(\sigma) \right]}{h^{(1)}_n(\sigma)} \right|_{\rho=\sigma} = 0 \tag{3.2g}$$

$$\mathcal{G}^{(1)}_n(\rho,\sigma) = \left. \frac{\left\{ \left[\rho\, j_n(\rho)\right]' \left[\sigma h^{(1)}_n(\sigma)\right]' - \left[\rho\, h_n(\rho)\right]' \left[\sigma\, j_n(\sigma)\right]' \right\}}{\left[\sigma\, h^{(1)}_n(\sigma)\right]'} \right|_{\rho=\sigma} = 0 \tag{3.2h}$$

It was decided to work with Stratton's representation only and to consider one test sample exlusively, namely a perfectly conducting sphere of $\sigma = ka = 2$ where the associated expansion coefficients f^s_{oln}, and g^s_{eln} are given in Table III-1.

For these values the total field $\underline{E}_T$ was derived from equations 2 and the boundary condition $\underline{E}_T \times \underline{E}^*_T = 0$ subsequently applied. Since an absolute zero of this condition cannot be found a minimum searching routine was employed which searches for that $X_{min}=kR_{min}$ for which

$$\left\{ - \left| \underline{E}_T x \, \underline{E}_T^* \right|^2 \right\}$$

becomes a minimum along a particular aspect angle. In Table III-2 the results for a perfectly conducting sphere of $\sigma=2$ are presented for $T=0^o(22.5^o)180^o$, $\phi= 0^o(45^o)315^o$ where the X_{min_2} were obtained for a searching increment of $\Delta X_1=.01$ over the range $1.6 \leq X_1 \leq 2.5$, employing a subsequent researching subroutine over the angle $X_{min_1} - \Delta X_1 \leq X_2 \; (\Delta X_2=.001) \leq X_{min} + \Delta X_1$ yielding X_{min_2} with Min $\left\{ - \left| \underline{E}_T x \, \underline{E}_T^* \right| \right\}$. The results obtained must be considered excellent, since aside from some isolated critical points, the deviation of X_{min} from the exact value is less than 1 percent. Table III-2 furthermore presents those values X_{min} for which the boundary condition $\left\{ \left| |\underline{E}_i| - |\underline{E}_s| \right| \right\}$ becomes a minimum. In Figs. 3-2a, b, c and d, the surface loci are plotted. It can be seen that the condition $|\underline{E}_i| - |\underline{E}_s| = 0$ fails to yield the exact result in the shadow region for particular aspect angles, whereas the condition $\underline{E}_T x \, \underline{E}_T^* = 0$ is applicable far into the shadow region.

Those critical points on the sphere for which the boundary condition $\underline{E}_T \, x \, \underline{E}_T^* = 0$ fails to yield the proper results for a given incident field, will be determined from Eq. (3.2a - h).

In Weston et al (1966) it has explicitly been stated that the boundary condition $\underline{E}_T \, x \, \underline{E}_T^* = 0$ is a necessary but not sufficient condition. For example, it can be shown that for a parallel polarized plane wave $H_i=x \, E_o/\eta_o \cdot \exp[i(y \sin\alpha - Z \cos\alpha)]$ as indicated in Fig. 3-1, $(\underline{E}_T x \, \underline{E}_T^*)_{//} = \mp \hat{x} \, i \sin 2\alpha \cdot \sin(2kz \cos\alpha)$. Hence for normal incidence ($\alpha = 0$) the behavior is identical with that of a normally polarized plane wave where $\underline{E}_T = 0$ for $Z = 0$ and $\underline{E}_T \, x \, \underline{E}_T^* = 0$ is identically zero and actually not applicable. If $\alpha \neq 0, \; \pi/2 \; (\underline{E}_T \, x \, \underline{E}_T^*) = 0$, yields

$$Z = \frac{n\pi}{2k\cos\alpha} \; , \quad n = 1, 2, \ldots ,$$

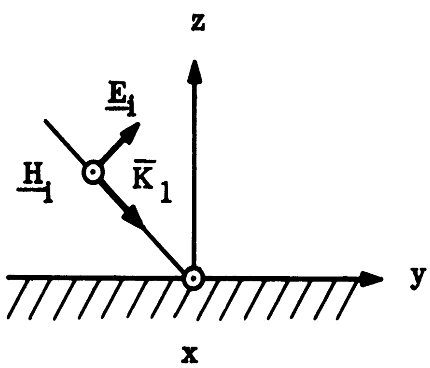


FIG3-1: PLANE SURFACE CASE

resulting in an infinite number of planes for which the condition $(\underline{E}_T \times \underline{E}_T^*) = 0$ is satisfied, and thus the proper solution is given for n = 0 only. In the case of grazing incidence $\alpha = \pi/2$, $\underline{E}_T \neq 0$ at 2 = 0 and the nodal point of $\underline{E}_T$ is off the conducting surface, however $\underline{E}_T \times \underline{E}_T^* = 0$. To determine those isolated points for which the condition $\underline{E}_T \times \underline{E}_T^* = 0$ may fail, the following analysis is helpful. Let $\rho = \sigma$ in (3.2a, b, c) thus

$$E_R^T\,(\rho = \sigma) = \sum_{n=1}^{\infty} \frac{2(2n+1)\,P_n(\cos\theta)\,\cos\phi}{\sigma\pi\left[\sigma\,h_n^{(1)}(\sigma)\right]'}\,(i)^n \qquad (3.2a')$$

$$E_\theta^T(\rho = \sigma) = E_\phi^T\,(\rho = \sigma) = 0$$

For $\phi = 0,\ \pi$

$$E_R^T \pm \sum_{n=1}^{\infty} \frac{2(2n+1)\,P_n\,(\cos\theta)}{\sigma\left[\sigma\,h_n^{(1)}(\sigma)\right]'\,\pi}\,(i)^n$$

and since for $\theta = \pi/2$, $P_n(0) = P_{n=2m}(0) = (-1)^m \dfrac{(2m-1)!}{2^m m!}$ (3.2a) becomes:

$$E_R^T = \pm \sum_{m=1}^{\infty} \frac{2(4m+1)(2m-1)!}{\pi\,2^m m!\,\sigma\left[\sigma\,h_n^{(1)}(\sigma)\right]'}$$

which is a complex nonzero constant corresponding to the case of grazing incidence of a parallel polarized plane wave onto a planar surface. For this case $\underline{E}_T \times \underline{E}_T^* = 0$, although $\underline{E}_T \neq 0$. In Fig. 3-6a both $\left\{|\underline{E}_T \times \underline{E}_T^*|\right\} = f\,(X_{min})$ and $\left\{|\underline{E}_i| - |\underline{E}_s|\right\} = f\,(Y_{min})$ are plotted versus $\rho = kR$, indicating that the first zero of $(\underline{E}_T \times \underline{E}_T^*) = 0$ is not identical with that of $\left\{|\underline{E}_i| - |\underline{E}_s|\right\} = 0$, and the Y_{min} ($\phi = 0$, $\theta \leq 90^o$) has a large deviation as can be seen from Figs. 3-2a, b, c, d and Figs. 3-6a and 3-7a . The condition $\left\{|\underline{E}_i| - |\underline{E}_s|\right\}$ then only yields the correct result for an exceedingly large number of expansion terms.

For $\theta = \pi/2$, $\phi = \pm \pi/2$, the boundary condition for a truncated series expansion of the scattered field $\underline{E}_s$ may work, since E_ϕ^T $(\rho=\sigma, n=N \neq \infty) \neq 0$, although E_R^T $(\rho=\sigma, n=N\neq \infty, \phi = \pm \pi/2) = 0$, corresponding to the case of normal incidence of a parallel polarized or an oblique normally polarized plane wave onto a planar surface. Thus the condition $\underline{E}_T x \underline{E}_T^* = 0$ for $n=N\neq \infty$ does work indeed in this case, however, the minimum searching subroutine will determine those points for which $E_R^T(\rho=\sigma, n=N\neq \infty, \theta\neq 0, \pi, \phi= \pm \pi/2) = 0$. The computational results verify this point to its best, since for a searching increment of X = .0001, ρ_{min}= 2.0000 for all values at $\phi = \pm \pi/2$. This property is also verified by the fact that the values of $\underline{E}_T x \underline{E}_T^* = 0$ are by a factor 10^{-15} smaller. In Figs. 3-5c, 3-6c and 3-7c the corresponding $\left\{-|\underline{E}_T x \underline{E}_T^*|^2\right\}$ and $\left\{|\underline{E}_i| - |\underline{E}_s|\right\}$ values are presented, also indicating that the first minimum of $\underline{E}_T$ x $\underline{E}_T^*$ is identical with that of $\left\{|\underline{E}_i| - |\underline{E}_s|\right\}$, and precisely at ρ_{min} = 2.0000. For this particular a spect angle ($\phi = \pm 90^o$) the condition $\left\{|\underline{E}_i| - |\underline{E}_s|\right\}$ holds also in the shadow region (see Figs. 3-2a, b, c and d).

The distinct singular point for which $\underline{E}_T$ x $\underline{E}_T^*$ = 0 fails entirely is the focal point of the shadow region or the zenith (θ=0), at this particular point the number of expansion terms as well as the number of digits to which the computation is correct must be a maximum, otherwise the corresponding minimum may be at random between $0 \leq \rho \leq \infty$ due to the slow convergence of the vector spherical harmonics of this singular point.

The boundary condition $\underline{E}_T \times \underline{E}_T^* = 0$ is correctly applicable for $\theta = 180^o$ and in fact for all ϕ, yielding identical minima as shown in Fig. 3-4. In Figs 3-5b, 3-6b, and 3-7b, the values of $\left\{ -| \underline{E}_T \times \underline{E}_T^* |^2 \right\}$ and $\left\{ |\underline{E}_i| - |\underline{E}_s| \right\}$ are plotted versus ρ=kR for $\phi = 45^o, \theta = 135^o; 90^o; 67.5^o$, indicating again that the condition $\underline{E}_T \times \underline{E}_T^* = 0$ is superior to $\left\{ |\underline{E}_i| - |\underline{E}_s| \right\}$ since it is applicable far into the shadow region. Inspecting Figs. 3-2 to 3-7 will show that the boundary condition $\underline{E}_T \times \underline{E}_T^* = 0$, aside from $\theta = 0$, can be employed to its best. For the particular value $\sigma = ka = 2$ the computed values in both the illuminated as well as the shadow region lie within one percent of the exact value. This result may not be obtained if $\sigma \gg 1$ in the shadow region. Since, however, in practical cases the points in the illuminated region will be considered only, this matter is of less concern. However, the deviation of $\rho_{min} \geq s$ in the illuminated region or shadow region respectively needs further interpretation which will not be presented here.

An entirely different question of interest must be answered; namely how to find the range for which $\rho = \sigma = ka$, corresponding to the proper locus, for which the $\min \left\{ -| \underline{E}_T \times \underline{E}_T^* |^2 \right\}$ may yield the distinct point on the surface of the unknown target. To do so, Figs. 3-3a, b, c and d will be interpreted, which present the loci of successive minima over the range $1.8 \leq \rho \leq 15.5$, resulting from the boundary condition $\underline{E}_T \times \underline{E}_T^* = 0$. It can be seen that the locus of the first minimum describes exactly the sphere of radius $\sigma = ka = 2$. All loci associated with the minima of higher order describe concentrical, asymmetrical hyperboloids, separated from one another by almost equal spacing. In fact for $\theta = 180$, and $\rho > 2\sigma$,

$\Delta\rho_{min}$ = const = .86. This particular phenomena, due to the fact that the condition $\underline{E}_T \times \underline{E}_T^* = 0$ is a necessary but not sufficient condition, may be employed to find the approximate range of the scale size σ = ka of the target in question. Namely, applying the condition $\underline{E}_T \times \underline{E}_T^* = 0$ at the backscatter-aspect angle at fairly large values of ρ over a few periods should yield the approximate range of σ, since $\Delta\rho_{min} \leq \sigma$. In addition the properties that $|\underline{E}_i| - |\underline{E}_s|$ = const for $\rho > \sigma$ and varies only near ρ σ, with a minimum at approximately the first minimum of of $\underline{E}_T \times \underline{E}_T^*$, may be used as an ultimative check.

Another question of interest is related with the necessary number of expansion terms as well as the number of necessary digits to which the expansion coefficients $g_{\substack{e\\o}\, m,n}$ $f_{\substack{e\\o}\, m,n}$ are correct. It has been shown already that proper number varies with the particular aspect angle. However it can be shown that application of the condition $\underline{E}_T \times \underline{E}_T^* = 0$ requires a smaller number of expansion terms as well as digits. This is indicated in Tables III-3 and III-4 for the particular aspect angle $\phi = 45$, $\theta = 135^o$.

It may be concluded from the presented results that the boundary condition $\underline{E}_T \times \underline{E}_T^* = 0$ although not restricted to perfectly conducting bodies, is an extremely helpful tool in the problems of inverse scattering.

TABLE III-1: Expansion Coefficients f_{oin}, g_{ein} for a Perfectly Conducting Sphere of $\sigma = ka = 2.$

n	$Re\{f_{oin}\}$	$Im\{f_{oin}\}$	$Re\{g_{ein}\}$	$Im\{g_{ein}\}$
1	.4884982	-.6066279	-.2764022	.4472206
2	.2043585	.7558526	-1.341662	-.8292217
3	-2.450365	.01002389	.0308867	-.4293786
4	$-.994961 \cdot 10^{-4}$	$-.3155869 \cdot 10^{-1}$	.04377755	.000191651
5	$.2126115 \cdot 10^{-2}$	$-.3013576 \cdot 10^{-6}$	$-.4768353 \cdot 10^{-6}$	$.267442 \cdot 10^{-2}$
6	$.3764457 \cdot 10^{-9}$	$.8891209 \cdot 10^{-9}$	$-.1064619 \cdot 10^{-3}$	$-.539721 \cdot 10^{-9}$

TABLE III-2:

SURFACE POIUBS DETERMINED BY THE BOUNDARY CONDITIONS $\underline{E}_T \times \underline{E}_T^* = 0$ AND $\left\{|\underline{E}_i| - |\underline{E}_s|\right\} = 0$ FOR A PERFECTLY CONDUCTING SPHERE OF $\sigma = ka = 2$, AND N = 8 EXPANSION TERMS

θ^o	ϕ^o	$X_{min}(\underline{E}_T \times \underline{E}_T^*)$	$\left\{-\left\|(\underline{E}_T \times \underline{E}_T^*)\right\|^2\right\}$	$Y_{min}(\|\underline{E}_i\|-\|\underline{E}_s\|)$	$(\underline{E}_i - \underline{E}_s)$
0	0				
22.5°	0	1.979	$7.3 \cdot 10^{-6}$	2.7	$1.1 \cdot 10^{-2}$
45	0	2.000	$5.6 \cdot 10^{-6}$	2.8	$2.6 \cdot 10^{-2}$
67.5	0	1.999	$2.81 \cdot 10^{-4}$	2.3	$7.0 \cdot 10^{-4}$
90	0	2.000	$7.2 \cdot 10^{-2}$	1.9	$7.5 \cdot 10^{-3}$
112.5	0	2.09	$5.27 \cdot 10^{-6}$	2.02	$2.3 \cdot 10^{-4}$
135	0	2.01	$2.19 \cdot 10^{-5}$	2.1	$4.13 \cdot 10^{-3}$
157.5	0	2.012	$1.09 \cdot 10^{-4}$	2.05	$9.8 \cdot 10^{-4}$
180	0	2.0000	$1.38 \cdot 10^{-19}$	2.00	$4.11 \cdot 10^{-5}$
0	45				
22.5	45	1.985	$1.67 \cdot 10^{-4}$	2.5	$9.7 \cdot 10^{-3}$
45	45	2.001	$4.49 \cdot 10^{-7}$	2.6	$2.1 \cdot 10^{-2}$
67.5	45	1.988	$9.66 \cdot 10^{-5}$	2.2	$8.4 \cdot 10^{-3}$
90	45	2.0000	$3.6 \cdot 10^{-8}$	1.94	$5.5 \cdot 10^{-3}$
112.5	45	1.984	$2.36 \cdot 10^{-3}$	2.04	$5.1 \cdot 10^{-3}$
135	45	2.009	$7.32 \cdot 10^{-3}$	2.04	$3.5 \cdot 10^{-3}$
157.5	45	2.001	$7.27 \cdot 10^{-4}$	2.0	$2.1 \cdot 10^{-3}$
180	45	2.000	$8.25 \cdot 10^{-20}$	2.00	$4.11 \cdot 10^{-5}$
0	90				
22.5	90	1.99	$1.21 \cdot 10^{-22}$	2.000	$9.6 \cdot 10^{-3}$
45	90	2.0000	$5.1 \cdot 10^{-23}$	2.00	$9.14 \cdot 10^{-5}$
67.5	90	2.004	$8.03 \cdot 10^{-23}$	2.00	$6.39 \cdot 10^{-3}$
90	90	2.0000	$5.65 \cdot 10^{-24}$	2.00	$1.60 \cdot 10^{-4}$
112.5	90	1.989	$2.26 \cdot 10^{-22}$	1.99	$4.7 \cdot 10^{-3}$
135	90	1.987	$3.31 \cdot 10^{-21}$	1.76	$4.3 \cdot 10^{-4}$
157.5	90	1.997	$1.15 \cdot 10^{-21}$	1.92	$2.3 \cdot 10^{-3}$
180	90	2.5	$4.8 \cdot 10^{-20}$	2.000	$4.11 \cdot 10^{-5}$

TABLE III-2:
(Continued)

θ^o	ϕ^o	$X_{min}(\underline{E}_T \times \underline{E}_T^*)$	$\left\{-\lvert(\underline{E}_T \times \underline{E}_T^*)\rvert^2\right\}$	$Y_{min}(\lvert E_i\rvert - \lvert E_s\rvert)$	$(\lvert E_i\rvert - \lvert E_s\rvert)$
22.5	135	1.985	$1.67 \cdot 10^{-4}$	2.5	$9.2 \cdot 10^{-3}$
45	135	2.001	$4.49 \cdot 10^{-7}$	2.6	$2.1 \cdot 10^{-2}$
67.5	135	1.988	$9.66 \cdot 10^{-5}$	2.2	$8.45 \cdot 10^{-3}$
90	135	2.0000	$3.6 \cdot 10^{-8}$	1.94	$5.5 \cdot 10^{-3}$
112.5	135	1.984	$2.36 \cdot 10^{-3}$	2.04	$5.1 \cdot 10^{-3}$
135	135	2.009	$7.32 \cdot 10^{-3}$	2.05	$3.5 \cdot 10^{-3}$
157.5	135	2.01	$7.2 \cdot 10^{-4}$	2.00	$2.1 \cdot 10^{-5}$
0	180				
22.5	180	1.968	$2.76 \cdot 10^{-10}$	2.7	$1.1 \cdot 10^{-2}$
45	180	2.001	$1.26 \cdot 10^{-6}$	2.8	$2.6 \cdot 10^{-2}$
67.5	180	1.987	$6.62 \cdot 10^{-8}$	2.3	$7.0 \cdot 10^{-4}$
90	180	2.000	$1.12 \cdot 10^{-7}$	1.9	$7.5 \cdot 10^{-3}$
112.5	180	2.088	$5.03 \cdot 10^{-8}$	2.07	$2.3 \cdot 10^{-4}$
135	180	2.051	$2.22 \cdot 10^{-6}$	2.1	$4.15 \cdot 10^{-3}$
157.5	180	2.024	$1.06 \cdot 10^{-7}$	2.05	$9.8 \cdot 10^{-4}$
180	180	1.999	$1.99 \cdot 10^{-20}$	2.00	$4.1 \cdot 10^{-5}$
0	225				
22.5	225	1.985	$1.67 \cdot 10^{-4}$	2.5	$9.2 \cdot 10^{-3}$
45	225	2.001	$4.48 \cdot 10^{-7}$	2.6	$2.1 \cdot 10^{-2}$
67.5	225	1.988	$9.66 \cdot 10^{-5}$	2.2	$8.4 \cdot 10^{-3}$
90	225	2.0000	$3.60 \cdot 10^{-8}$	1.94	$5.5 \cdot 10^{-3}$
112.5	225	1.984	$2.36 \cdot 10^{-3}$	2.04	$5.1 \cdot 10^{-3}$
135	225	2.009	$7.32 \cdot 10^{-3}$	2.04	$3.5 \cdot 10^{-3}$
157.5	225	2.001	$7.26 \cdot 10^{-4}$	2.00	$2.1 \cdot 10^{-3}$
180	225	2.000	$8.26 \cdot 10^{-20}$	2.00	$4.1 \cdot 10^{-5}$

TABLE III-2:
(Continued)

θ^o	ϕ^o	$X_{min}(\underline{E}_T \times \underline{E}_T^*)$	$\left\{-\left\|(\underline{E}_T \times \underline{E}_T^*)\right\|^2\right\}$	$Y_{min}(\|E_i\|-\|E_s\|)$	$(\|E_i\|-\|E_s\|)$
0	270				
22.5	270	1.999	$1,94 \cdot 10^{-21}$	2.00	$9.6 \cdot 10^{-3}$
45	270	2.000	$8.19 \cdot 10^{-22}$	2.00	$9.14 \cdot 10^{-3}$
67.5	270	2.001	$1.28 \cdot 10^{-21}$	2.01	$6.4 \cdot 10^{-3}$
90	270	2.0000	$9.03 \cdot 10^{-23}$	2.00	$1.6 \cdot 10^{-4}$
112.5	270	1.989	$5.3 \cdot 10^{-20}$	1.99	$4.7 \cdot 10^{-4}$
135	270	1.997	$1.84 \cdot 10^{-20}$	1.96	$4.2 \cdot 10^{-3}$
157.5	270	1.997	$1.86 \cdot 10^{-20}$	1.97	$2.3 \cdot 10^{-3}$
180	270	2.008	$4.0 \cdot 10^{-30}$	2.00	$4.1 \cdot 10^{-5}$
0	315				
22.5	315	1.985	$1.67 \cdot 10^{-4}$	2.5	$9.2 \cdot 10^{-3}$
45	315	2,001	$4.49 \cdot 10^{-7}$	2.6	$2.1 \cdot 10^{-2}$
67.5	315	1.988	$9.66 \cdot 10^{-5}$	2.2	$8.4 \cdot 10^{-3}$
90	315	2.000	$3.6 \cdot 10^{-8}$	1.94	$5.5 \cdot 10^{-3}$
112.5	315	1.984	$2.36 \cdot 10^{-3}$	2.04	$5.1 \cdot 10^{-3}$
135	315	2.001	$7.32 \cdot 10^{-4}$	2.04	$3.5 \cdot 10^{-3}$
157.5	315	2.009	$7.26 \cdot 10^{-4}$	2.00	$2.1 \cdot 10^{-3}$
180	315	2.002	$6.2 \cdot 10^{-22}$	2.00	$4.1 \cdot 10^{-5}$

TABLE III-3:

SURFACE POIUBS DETERMINED BY THE BOUNDARY CONDITIONS
$E_T \times E_T^* = 0$ AND $\left\{ |\underline{E}_i| - |\underline{E}_s| \right\} = 0$ FOR $\sigma = ka = 2$ AND
$\theta = 135^o$, $\phi = 45^o$ AND N EXPANSION TERMS

N	X_{min}	$\left\{ - \left| \underline{E}_T \times \underline{E}_T^* \right|^2 \right\}$	Y_{min}	$\left\{ |E_i| - |E_s| \right\}$
2	2.05	.36	2.22	$.18 \cdot 10^{-1}$
3	2.03	$.284 \cdot 10^{-2}$	2.18	.68
4	2.00	$.18 \cdot 10^{-3}$	2.09	.32
5	2.00	$.68 \cdot 10^{-4}$	2.06	$.42 \cdot 10^{-2}$
6	2.00	$.78 \cdot 10^{-6}$	2.02	$.71 \cdot 10^{-3}$

TABLE III-4:

SURFACE POIUBS DETERMINED BY THE BOUNDARY CONDITIONS

$$\underline{E}_T \times \underline{E}_T^* = 0 \text{ AND } \left\{ |\underline{E}_i| - |\underline{E}_s| \right\} = 0 \text{ FOR}$$

$\sigma = ka = 2$, AND $\theta = 135^o$, $\phi = 45^o$, $N = 6$

AND M DIGITS.

M	$X_{min}(\underline{E}_T \times \underline{E}_T^*)$	$\left\{ - \lvert(\underline{E}_T \times \underline{E}_T^*)\rvert^2 \right\}$	$Y_{min}(\lvert\underline{E}_i\rvert - \lvert\underline{E}_s\rvert)$	$(\lvert\underline{E}_i\rvert - \lvert\underline{E}_s\rvert)$
2	2.09	$.28 \cdot 10^{-3}$	2.22	$.33 \cdot 10^{-2}$
3	2.07	$.33 \cdot 10^{-3}$	2.18	$.13 \cdot 10^{-3}$
4	2.02	$.22 \cdot 10^{-4}$	2.15	$.50 \cdot 10^{-2}$
5	2.03	$.21 \cdot 10^{-3}$	2.09	$.42 \cdot 10^{-2}$
6	2.009	$.36 \cdot 10^{-5}$	2.03	$.71 \cdot 10^{-3}$
7	2.001	$.78 \cdot 10^{-6}$	2.02	$.34 \cdot 10^{-4}$

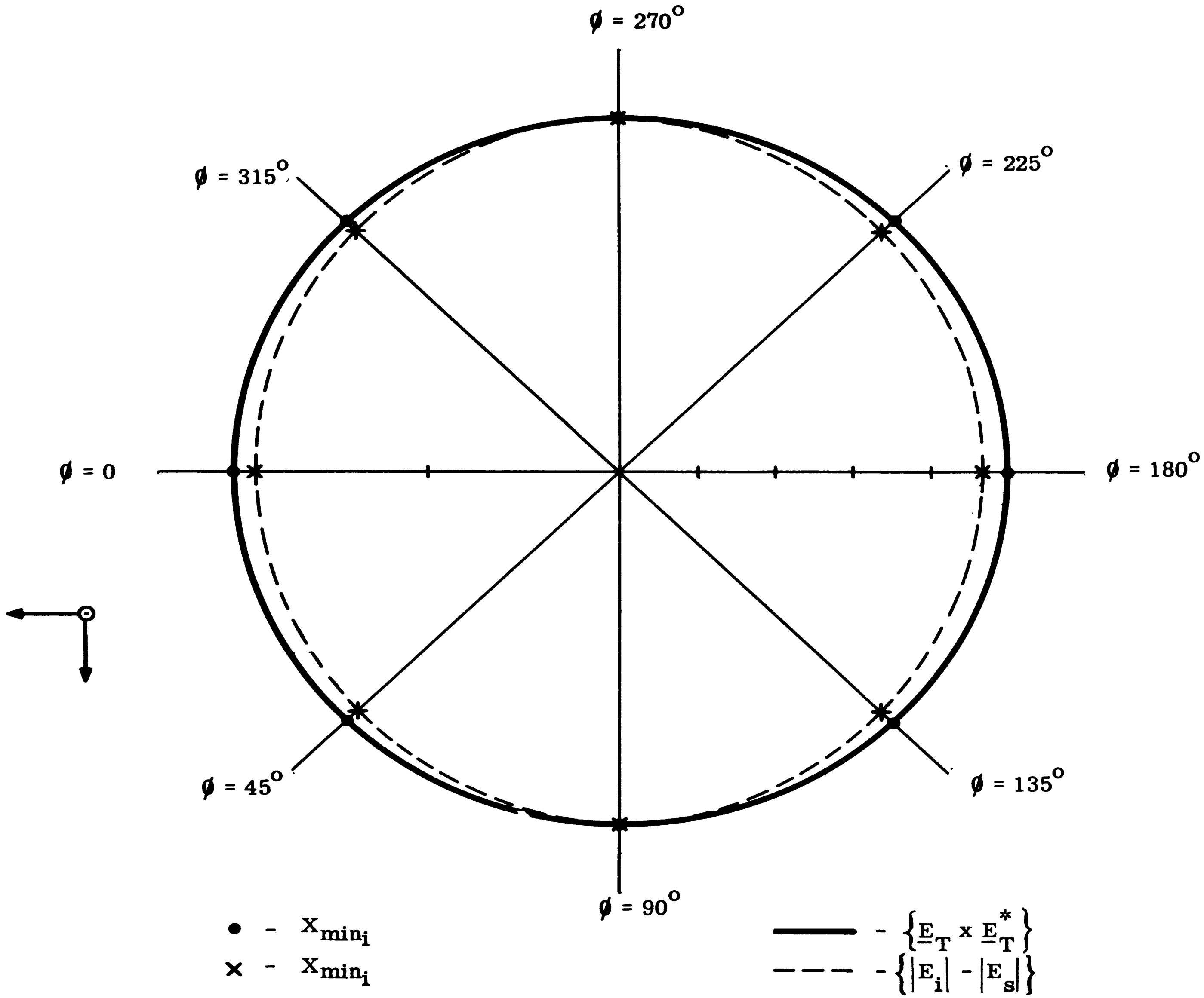


FIG. 3-2a: RESULTING SURFACE LOCI FOR $\left\{\underline{E}_T \times \underline{E}_T^* = 0\right\}$ AND $\left\{|E_i| - |E_s|\right\} = 0$ (θ = 90° - Plane).

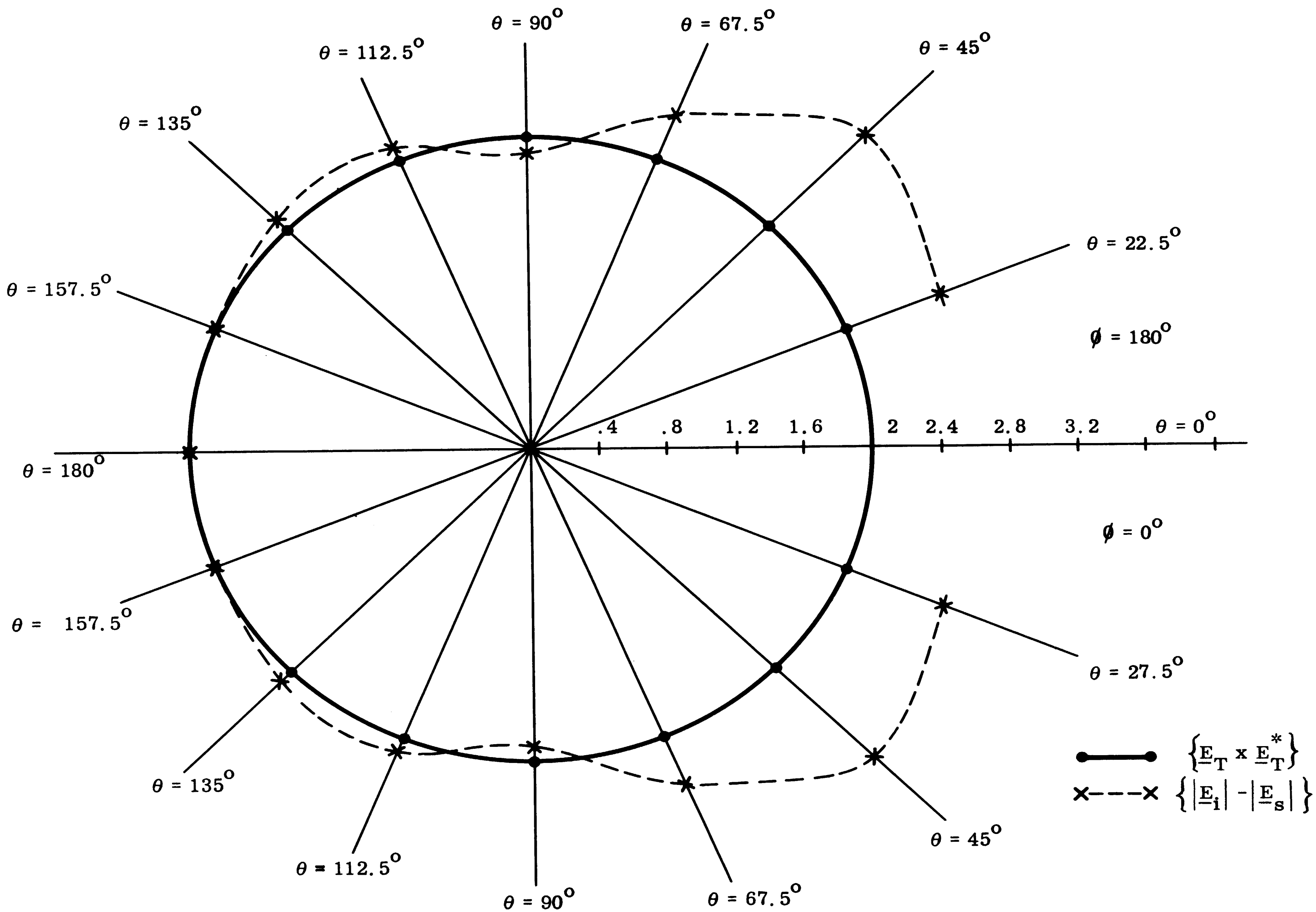


FIG. 3-2b: RESULTING SURFACE LOCI FOR $\{\underline{E}_T \times \underline{E}_T^* = \theta\}$ AND $\{|\underline{E}_i| - |\underline{E}_s|\} = 0$ ($\phi = 0 - \phi = 180^\circ$ - Plane).

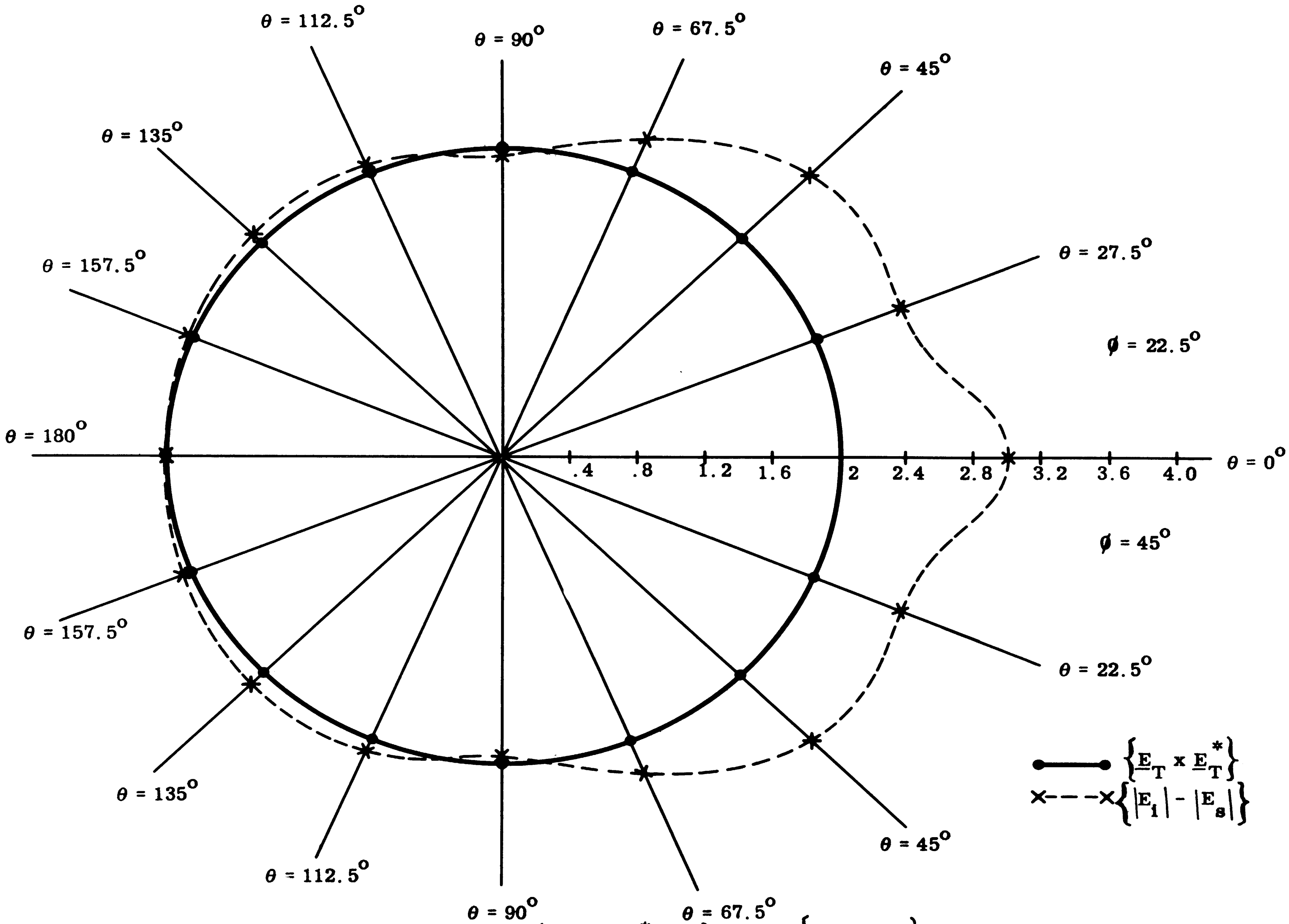


FIG. 3-2c: RESULTING SURFACE LOCI FOR $\{\underline{E}_T \times \underline{E}_T^* = 0\}$ AND $\{|E_i| - |E_s|\} = 0$ ($\phi = 45^\circ$ - $\phi = 225^\circ$ Plane).

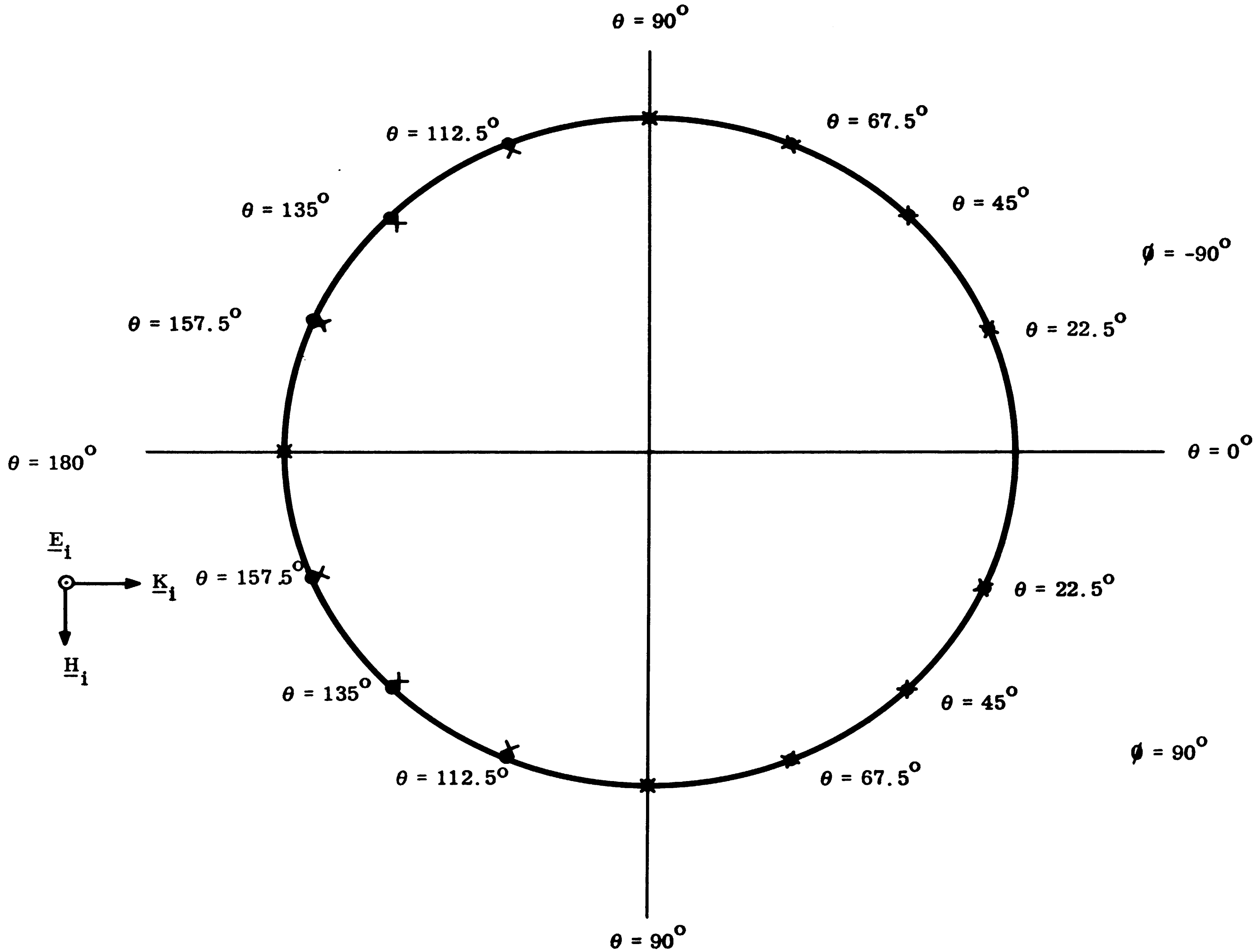


FIG. 3-2d: RESULTING SURFACE LOCI FOR $\left\{\underline{E}_T \times \underline{E}_T = 0\right\}$ AND $\left\{\left|\underline{E}_i\right| - \left|\underline{E}_s\right|\right\} = 0$ ($\theta = 90°$, $\phi = 90°$ Plane).

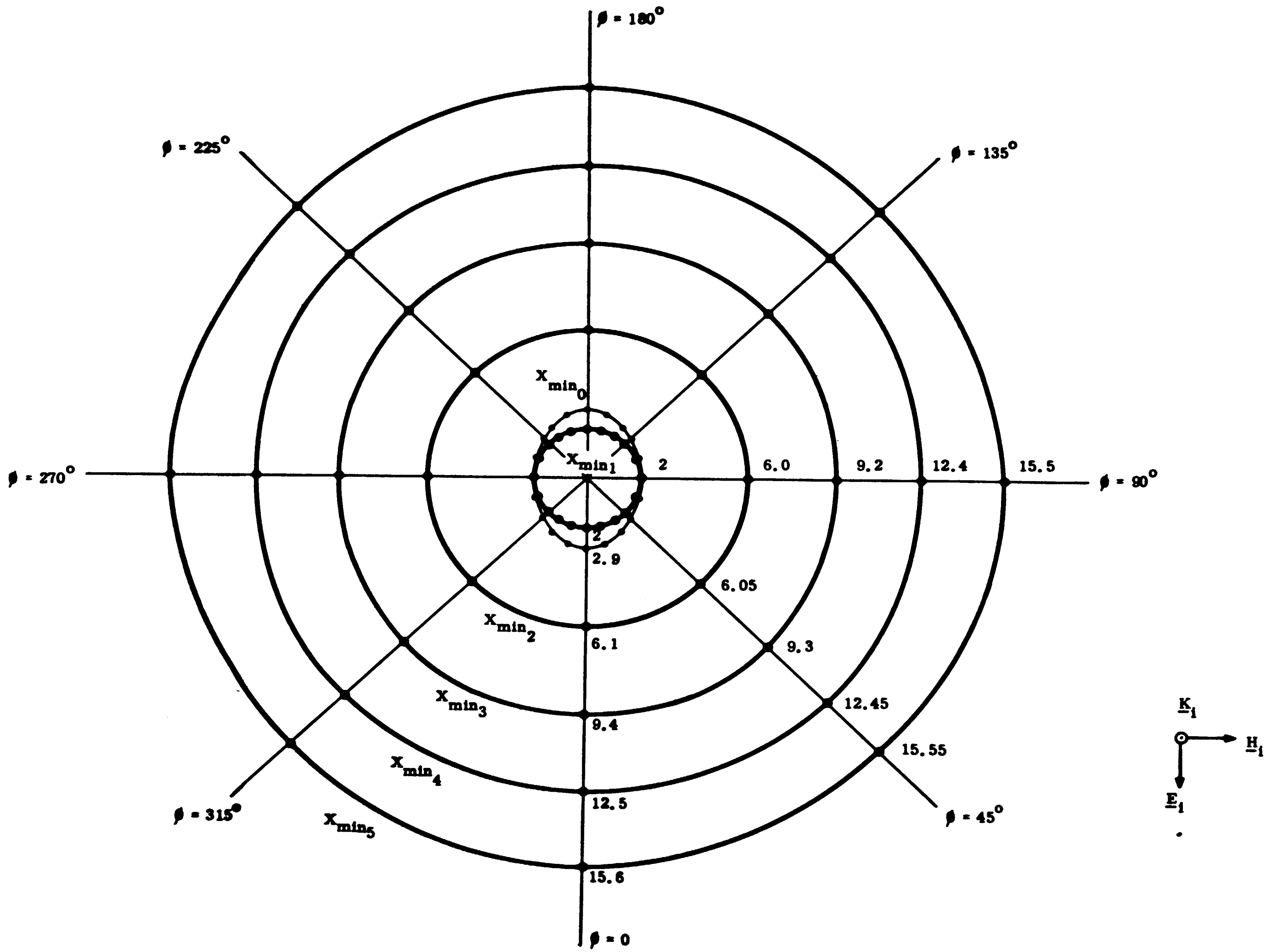


FIG. 3-3a: RESULTING SURFACE LOCI OF SUCCESSIVE MINIMA OF $\underline{E}_T \times \underline{E}_T^*$ ($\theta = 90°$ - Plane).

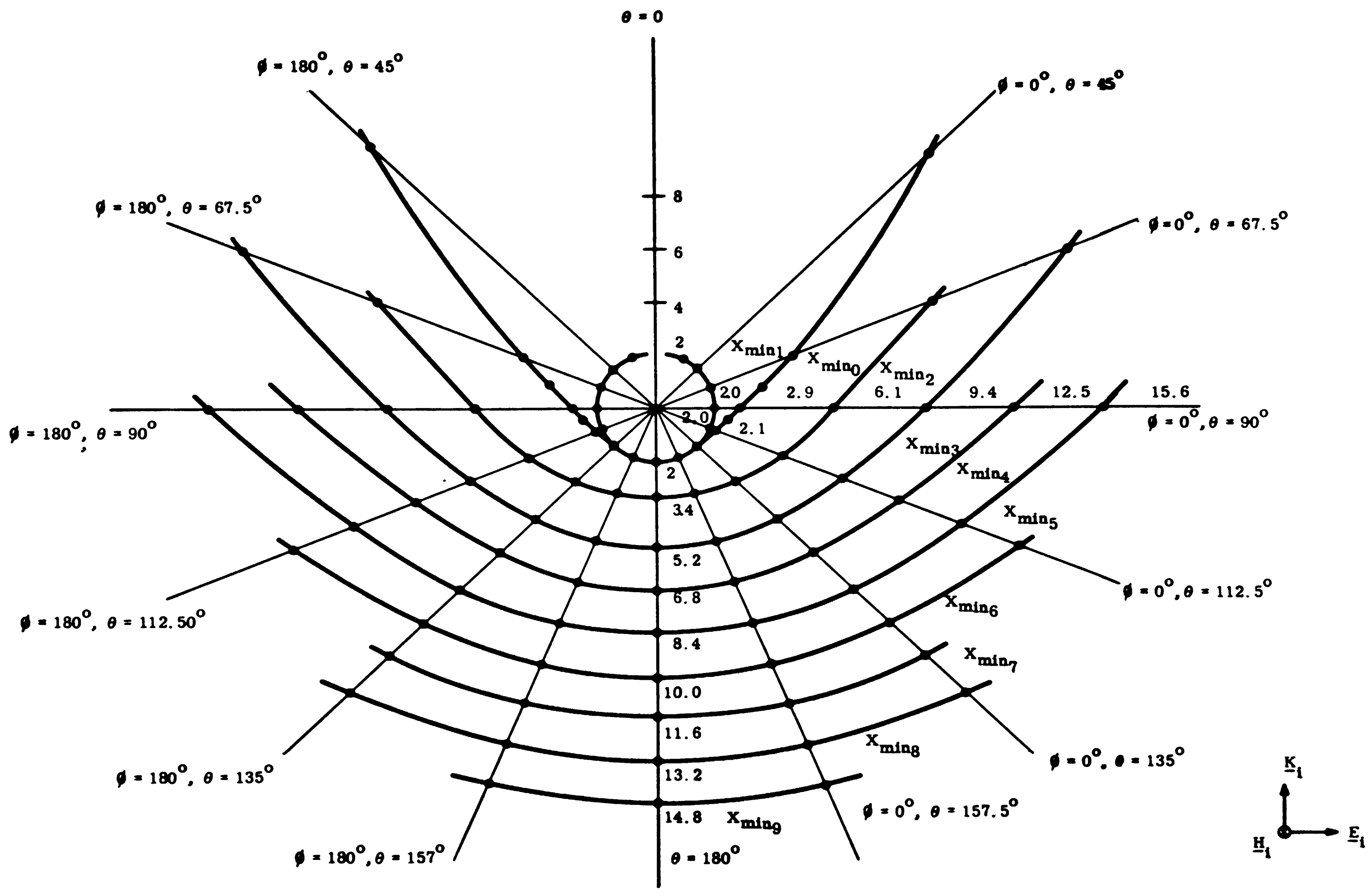


FIG. 3-3b: RESULTING SURFACE LOCI OF SUCCESSIVE MINIMA OF $\underline{E}_T = \underline{E}_T^*$ ($\phi = 0^\circ$ - $\phi = 180^\circ$ Plane).

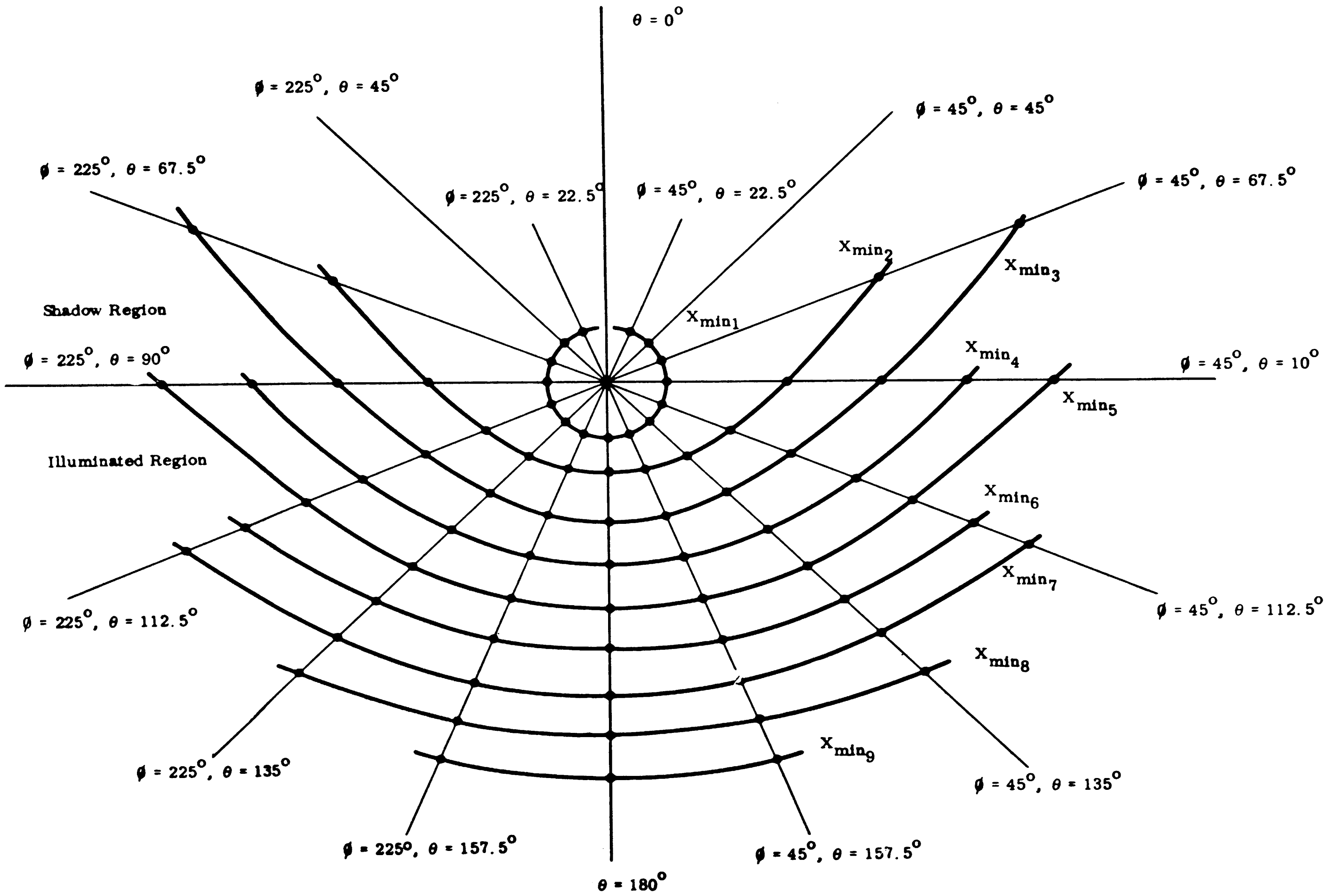


FIG. 3-3c: RESULTING SURFACE LOCI OF SUCCESSIVE MINIMA OF $\underline{E}_T \times \underline{E}_T^*$ (ø = 45 - ø = 225 Plane).

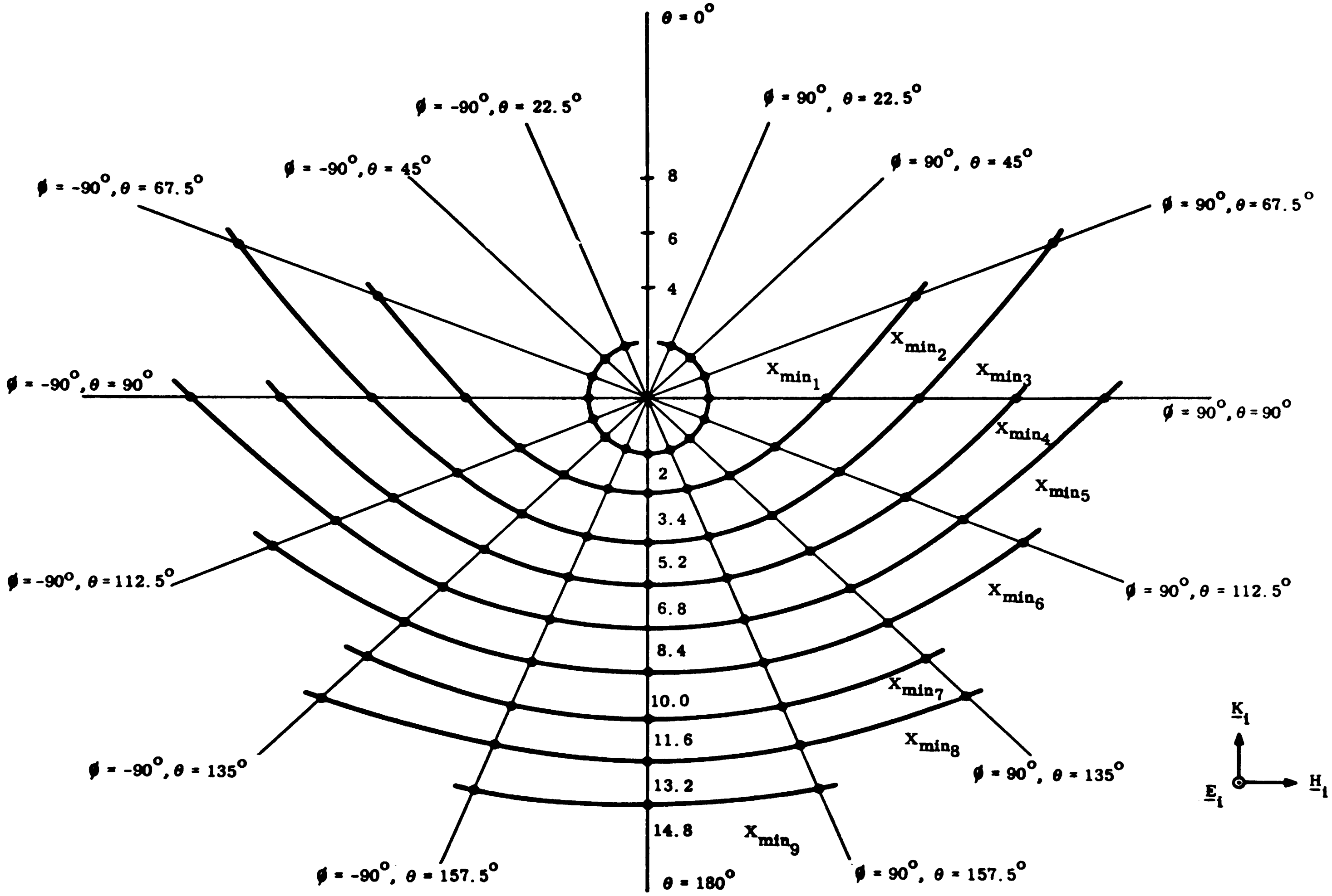


FIG. 3-3d: RESULTING SURFACE LOCI OF SUCCESSIVE MINIMA OF $\underline{E}_T \times \underline{E}_T^*$ ($\phi = 90^\circ - \phi - 90^\circ$ Plane).

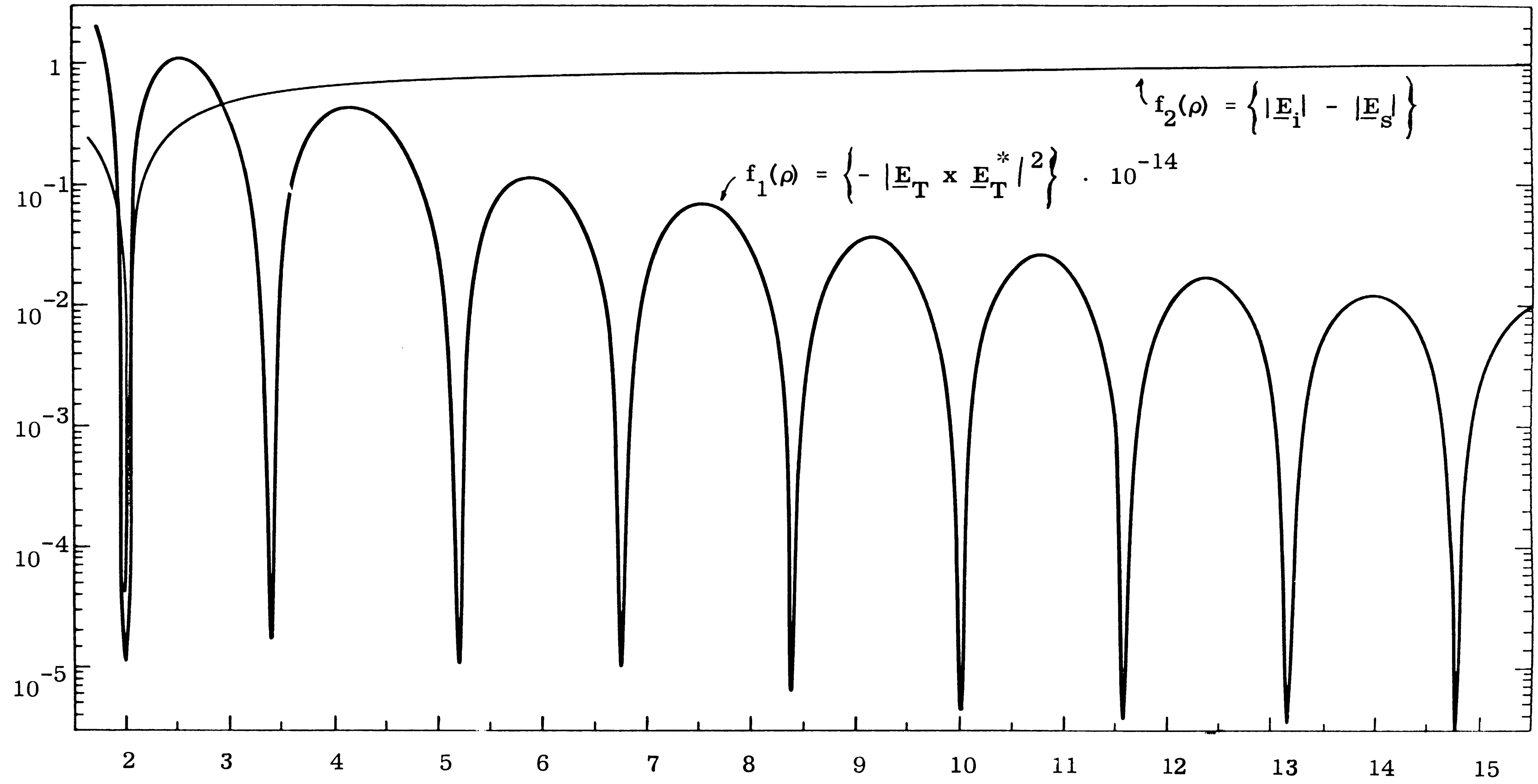


FIG. 3-4: ILLUMINATED REGION - SPECULAR POINT, $\theta = 180^{\circ}$, (ϕ = 0, 180).

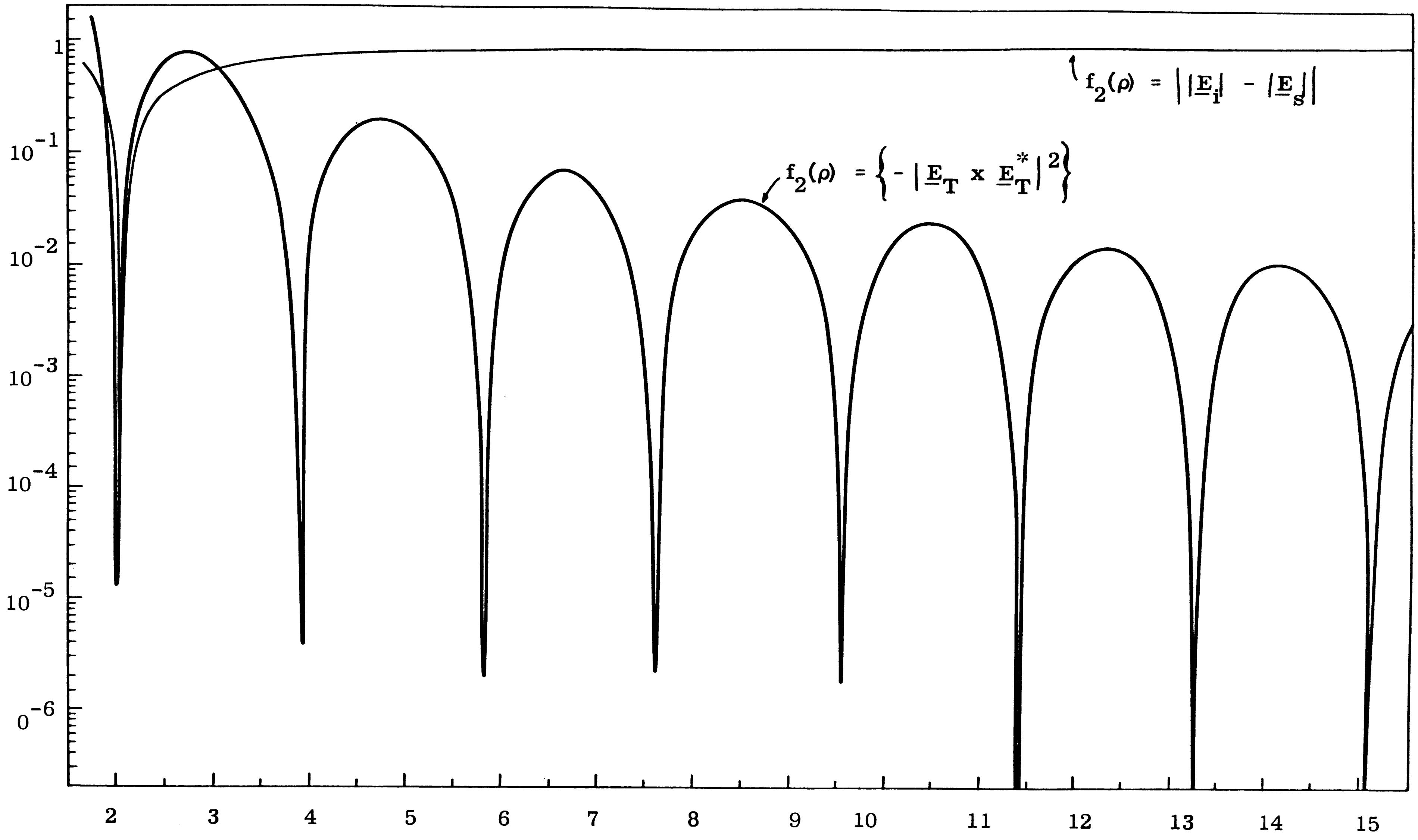


FIG. 3.5a: ILLUMINATED REGION, $\theta = 135^o$, $\phi = 0^o$.

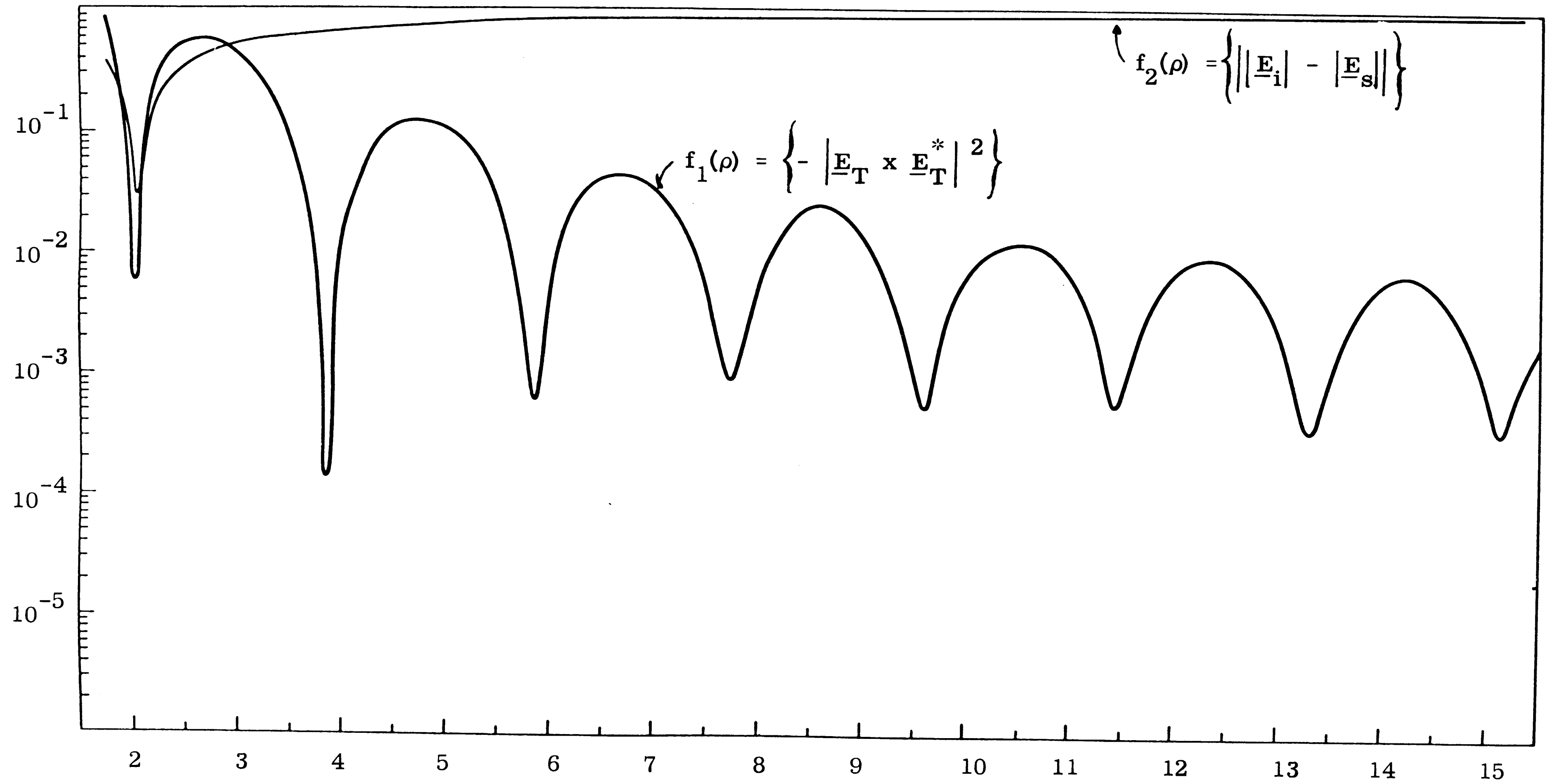


FIG. 3-5b: ILLUMINATED REGION, $\theta = 135^{\circ}$, $\phi = 45^{\circ}$.

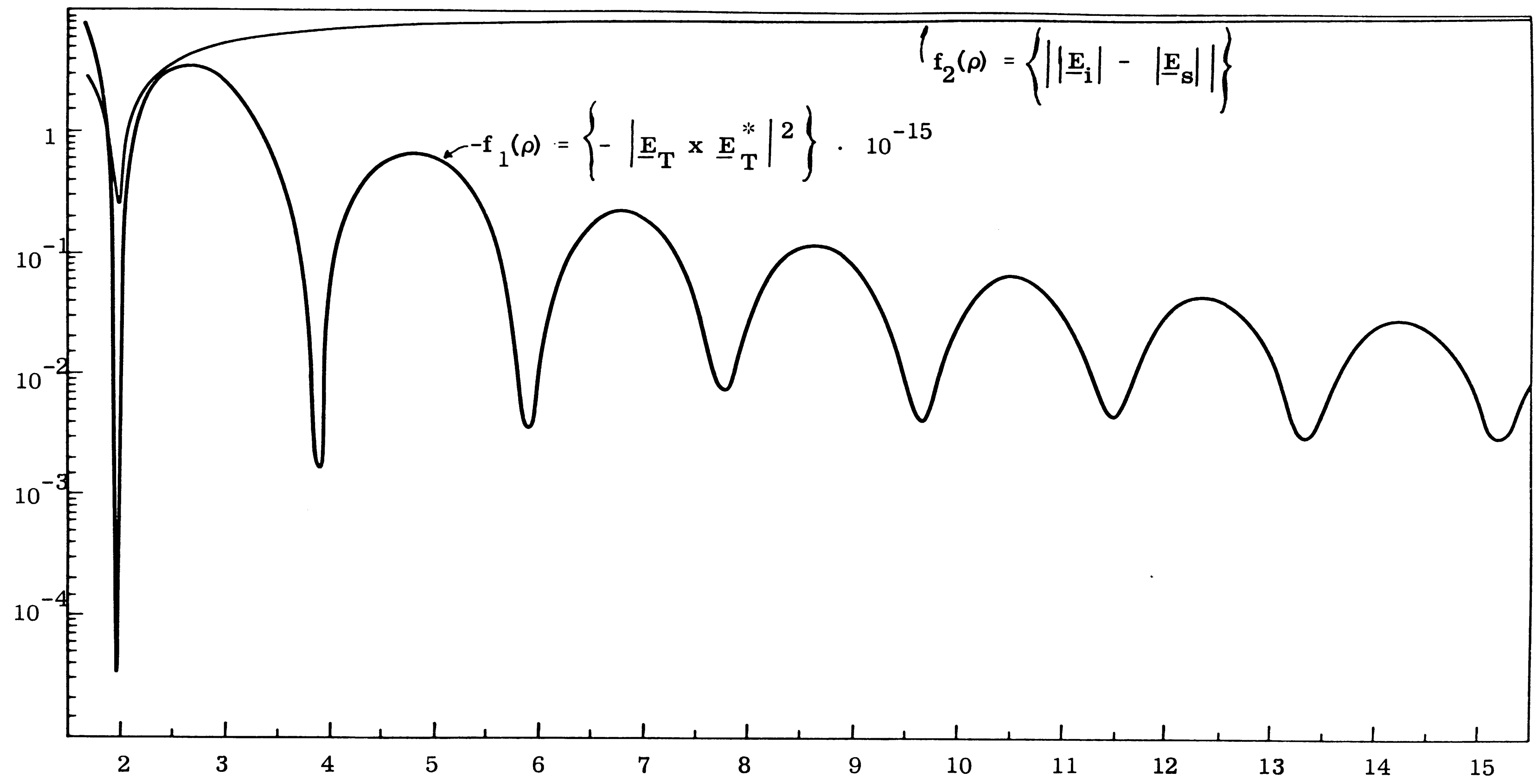


FIG. 3-5c: ILLUMINATED REGION, θ = 135°, ϕ = 90°.

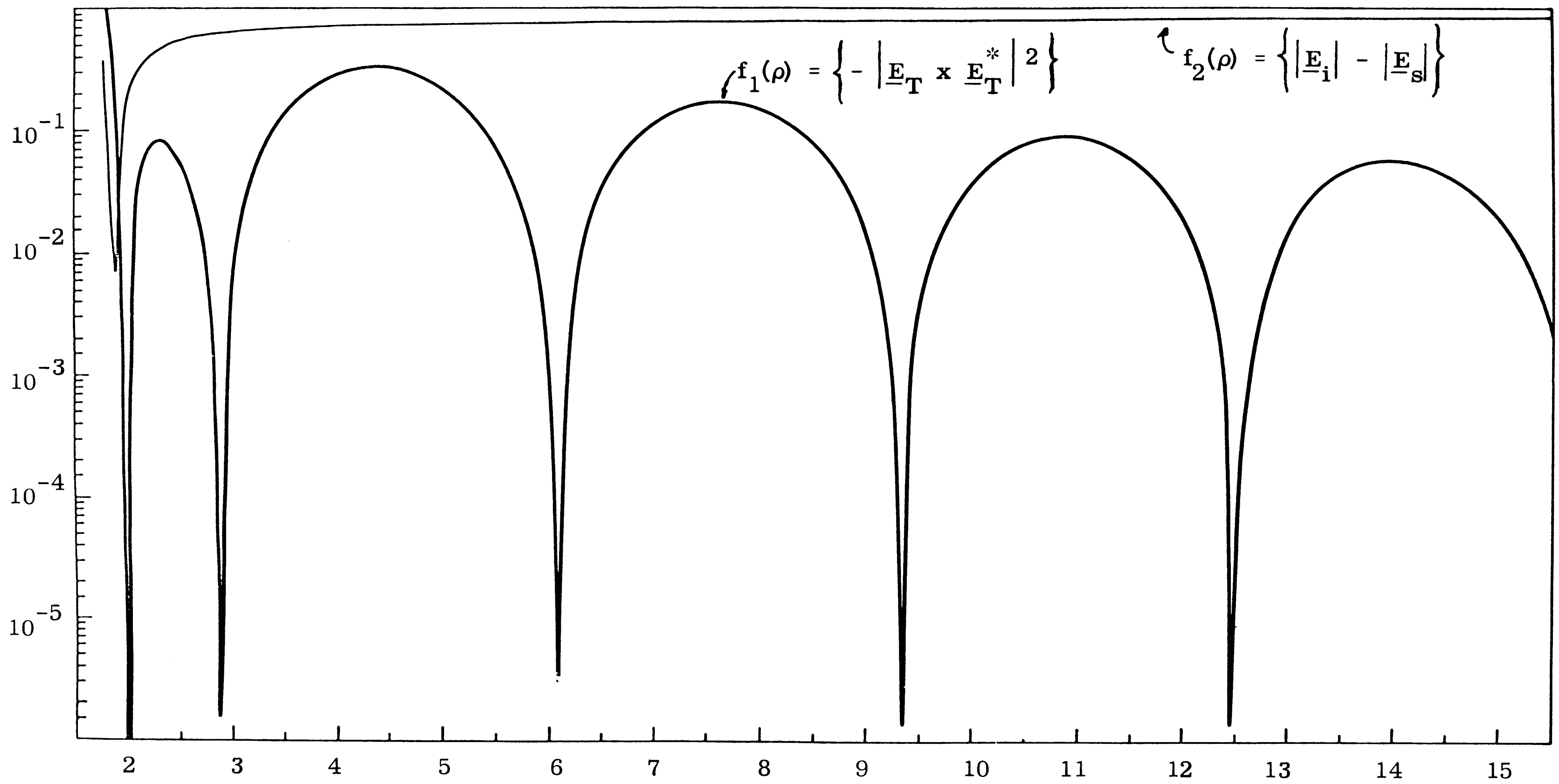


FIG. 3-6a: SHADOW BOUNDARY $\theta = 90^o$, $\phi = 0^o$.

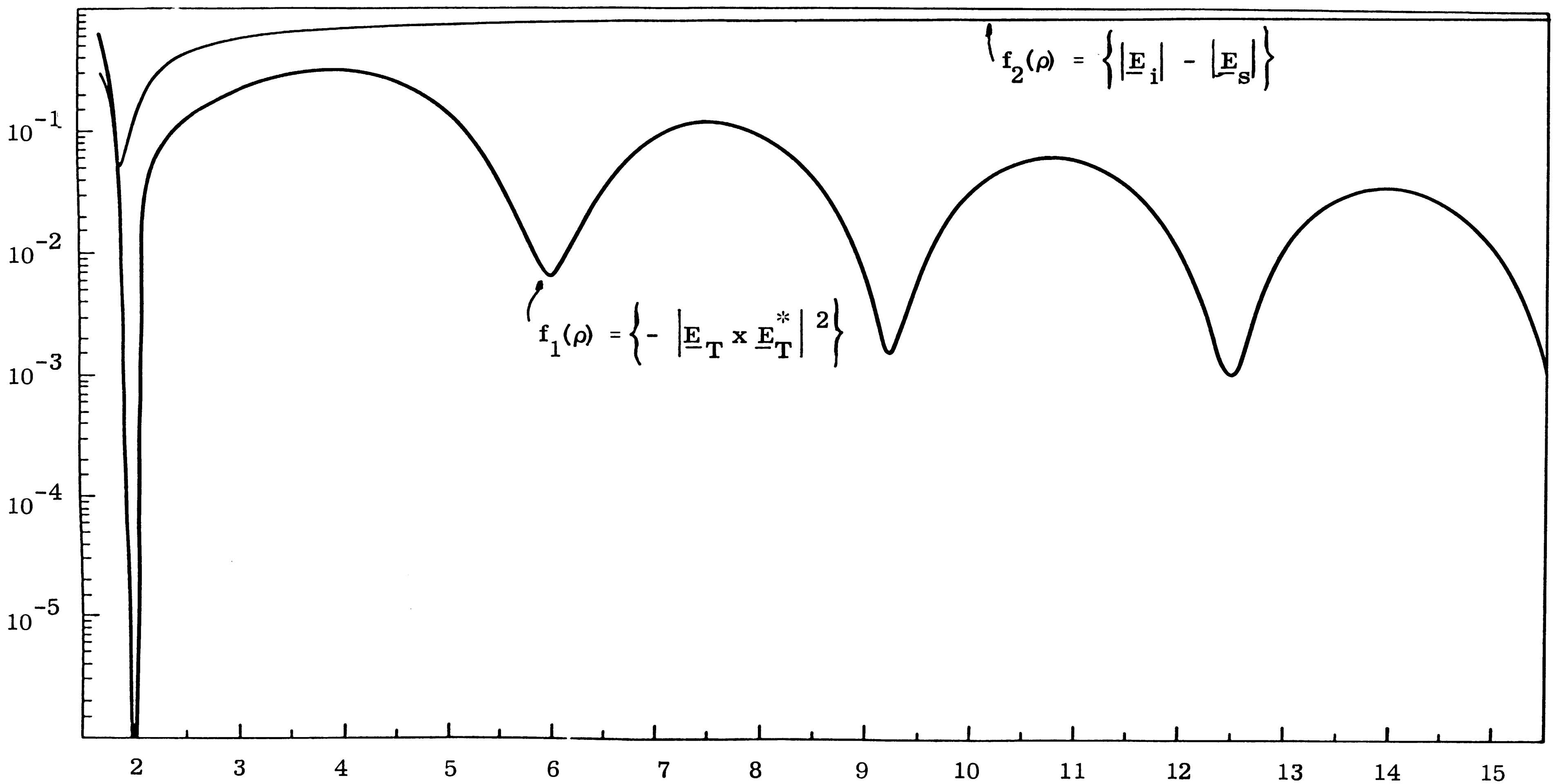


FIG. 3-6b: **Shadow Boundary**, $\theta = 90^o$, $\phi = 45^o$.

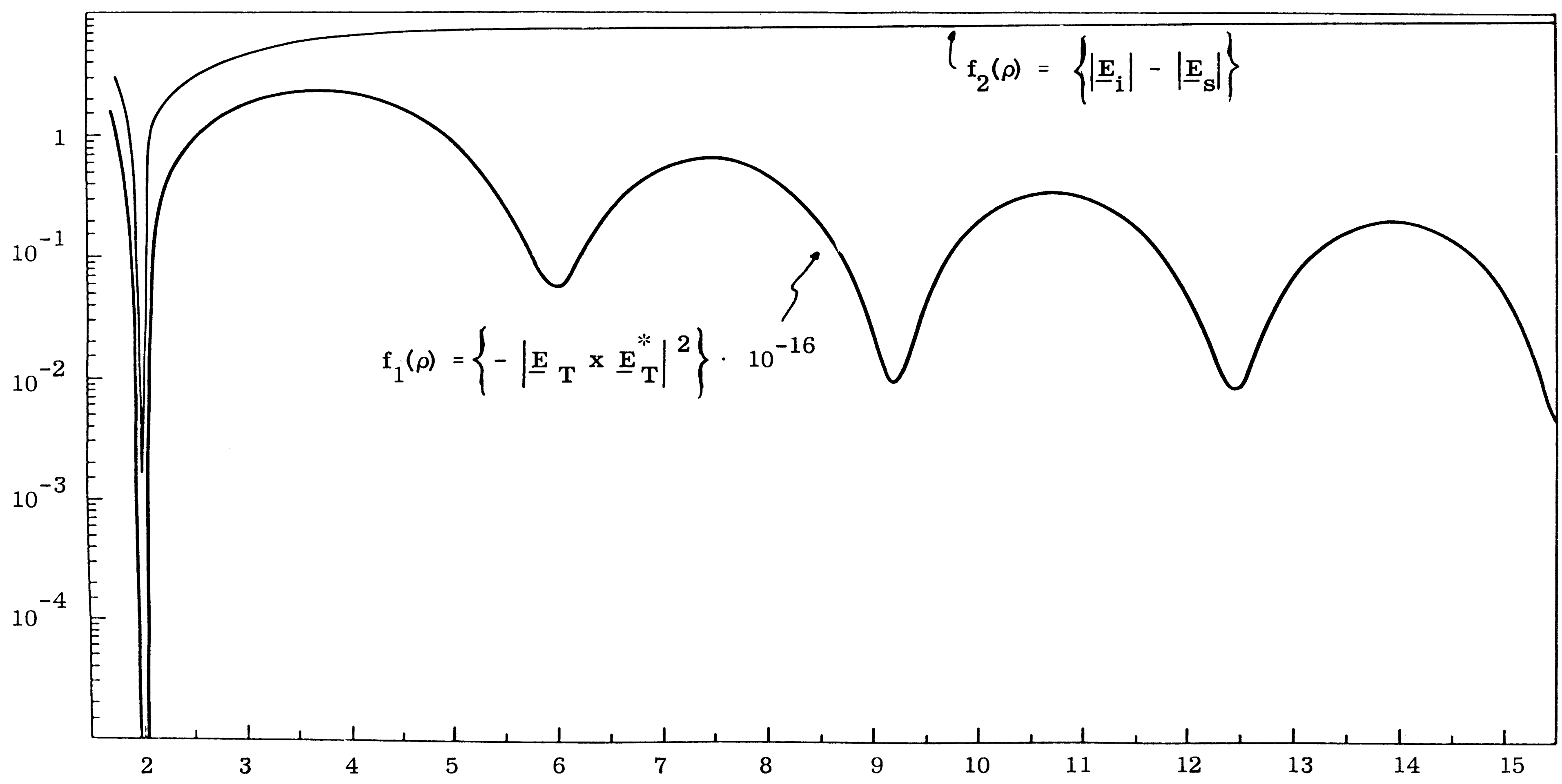


FIG. 3-6c: SHADOW BOUNDARY, θ = 90, ϕ = 90^{o}.

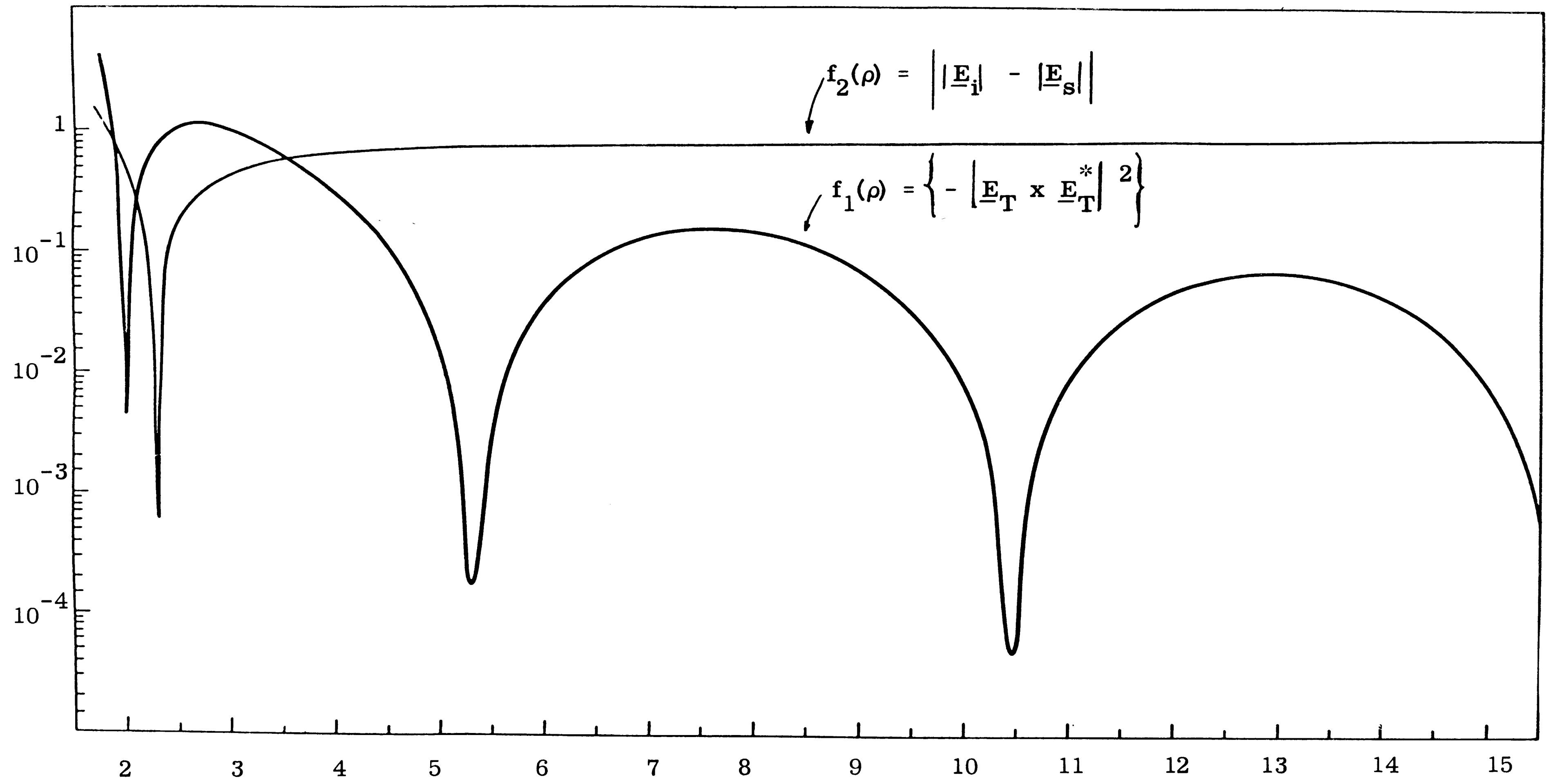


FIG. 3-7a: SHADOW REGION, $\theta = 67.5^\circ$, $\phi = 0^\circ$.

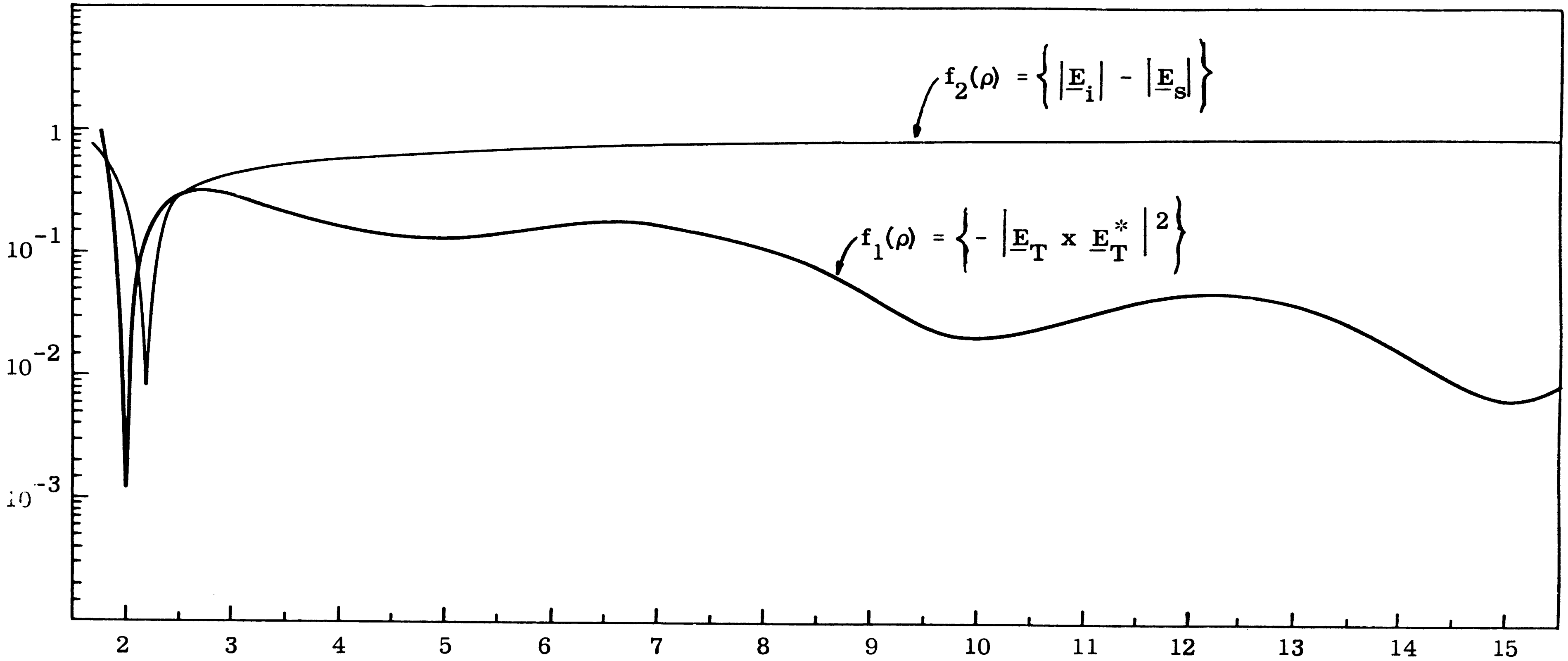


FIG. 3-7b: SHADOW REGION $\theta = 67.5^o$, $\phi = 45^o$,

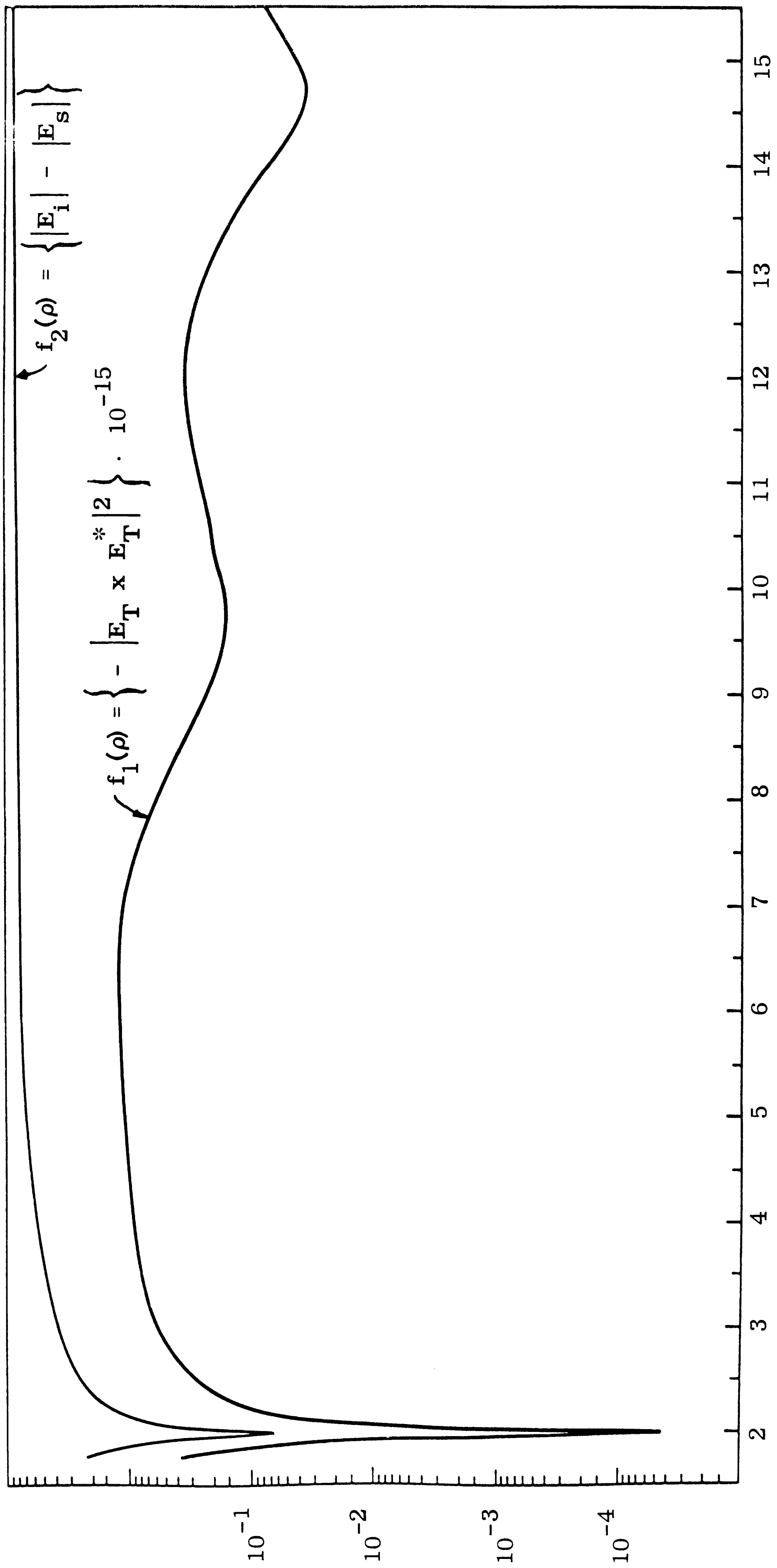


FIG. 3-7c: SHADOW REGION. $\theta = 67.5^o$, $\phi = 90^o$

IV

MOTIVATION AND HEURISTIC DEVELOPMENT OF INVERSE SCATTERING THEORY

Several different methods have been developed for the inverse scattering problem as is evident from the review article by Faddeyev (1963). The best known attack is originally due to Gelfand and Levitan (1951). For the quantum mechanical case, Levinson (1953) has motivated this attack. A very similar method was developed by Kay (1955) and Kay and Moses (1955 - 1961) for not only the quantum mechanical problem but also for the one-dimensional wave-equation. For this latter case, it is possible to give a more transparent motivation and development in a manner similar to that used by Kay (1960) in a little known paper. In addition, Moses (1956) has given an entirely different method which appears more natural to people acquainted with diffraction theory and which in addition is valid in three spatial dimensions.

Both of these methods use ideas from perturbation theory as their starting point and both are limited to the case of perturbation by a real potential. The reality of the potential implies an analytic continuation of the reflection coefficient which appears essential for anything like a practical application. It appears that the difficulties associated with surmounting this are the chief ones preventing the extension of either method to the case of a complex potential.

As is customary, we will denote the self-adjoint extension of the operator $-\Delta$ by H_o and that of the operator $-\Delta + q(\underline{x})$ by H so that $H = H_o + V$ where V denotes multiplication by the real potential $q(\underline{x})$. Of oourse for H to exist, q must satisfy certain conditions--for our purposes it will be sufficient to assume that q is locally Hoelder continuous except for a finite number of singularities and that is in $L_1 \cap L_2$ over E_1 or E_3. With these assumptions the results of Ikebe (1960) are valid so that the interested reader can refer to this paper for detailed proofs of the arguments we are about to give in an attempt to motivate the perturbation theories underlying the two afore-mentioned approaches to scattering theory. For simplicity we will also assume that q has no bound states--for example, q could be a repulsive potential or barrier. Then if $\phi(\underline{x}, \underline{k})$ satisfies the scattering integral equation

1) $$\phi(\underline{x},\underline{k}) = e^{i\underline{k}\cdot\underline{x}} - \frac{1}{4\pi}\int_E \frac{e^{ik|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|} q|\underline{y}|\, \phi(\underline{y},\underline{k})\, d\underline{y}$$

Ikebe has established the following results:

(Part of his Theorem 5)

Let $f(\underline{x})$ be an arbitrary L_2 function. i) Then the generalized Fourier transform

$$\hat{f}(\underline{k}) = (2\pi)^{-3/2} \text{l.i.m.} \int_E \overline{\phi(\underline{x},\underline{k})}\, f(\underline{x})\, d\underline{x}$$

of $f(\underline{x})$ exists and belongs to $L_2(M)$ where M is the 3-dimensional space formed by vectors $\underline{k}$.

ii) The following expansion formula is valid:

$$f(\underline{x}) = (2\pi)^{-3/2} \text{l.i.m.} \int_M \phi(\underline{x},\underline{k})\, \hat{f}(\underline{k})\, d\underline{k}$$

iii) f is in the domain of H which is equal to the domain of H_o if and only if $|k|^2 \hat{f}(\underline{k}) \in L_2(M)$ and under these circumstances we have the following representation of H:

$$H f(\underline{x}) = (2\pi)^{-3/2} \text{ l.i.m.} \int_M |k|^2 \, \phi(\underline{x}, \underline{k}) \, \hat{f}(\underline{k}) \, dk \, .$$

Of course, in terms of the ordinary Fourier transform

$$2) \qquad \hat{f}_o(\underline{k}) = (2\pi)^{-3/2} \text{l.i.m.} \int_E e^{\overline{i\underline{k}\cdot\underline{x}}} \, f(\underline{x}) \, dx$$

H_o admits the representation

$$3) \qquad H_o f = (2\pi)^{-3/2} \text{ l.i.m.} \int_M |k|^2 \, e^{i\underline{k}\cdot\underline{x}} \, f_o(\underline{k}) \, dk$$

and as Ikebe shows the continuous spectra of H and H_o are unitarily equivalent. It is this last remark which underlies the approach of Kay and Moses (1955) and Faddeyev (1963) for it is the basis for Friedrich's (1948) method based on spectral representers. On the other hand, the Fourier transform of the scattering integral equation (1) form the basis for the alternate approach of Moses (1956). By means of its solution, the "eigenfunctions" $\phi(\underline{x}, \underline{k})$ of H are expressed in terms of the distorted plane wave "eigenfunctions" $\exp(i\,\underline{k}\cdot\underline{x})$ of H_o. To see that this is not the only possible linear transformation relating these two let us introduce the linear operator U defined by the relation

$$4) \qquad U f(\underline{x}) = (2\pi)^{-3/2} \text{l.i.m.} \int_M \phi(\underline{x}, \underline{k}) \, \hat{f}_o(\underline{k}) \, dk$$

Then it follows that

$$5) \qquad U H_o f = (2\pi)^{-3/2} \text{l.i.m.} \int \phi(\underline{x}, \underline{k}) \, \widehat{(H_o f)}_o \, dk = (2\pi)^{-3/2} \text{ l.i.m.} \int \phi(\underline{x}, \underline{k}) \, |k|^2 \, f_o(\underline{k}) \, dk \, ,$$

since it is clear from (3) that

$$6) \qquad \widehat{(H_o f)}_o = |k|^2 \, \hat{f}_o(\underline{k}) \, .$$

On the other hand, it follows from (4) and (ii) above that

7) $$H(Uf) = (2\pi)^{-3/2} \text{ l.i.m.} \int |\underline{k}|^2 \phi(\underline{x}, \underline{k}) \hat{f}_o(\underline{k})\, dk \quad .$$

Thus for any f in the common domain of H, H_o

8) $$H\,Uf = U H_o f \quad .$$

It is clear here that U^{-1} exists so that (8) can be written as

9) $$HU = UH$$

or as

10) $$H_o = U^{-1} HU \quad .$$

Since

11) $$H_o \phi_o = |k|^2 \phi_o$$

it follows that

12) $$U^{-1} HU \phi_o = |k|^2 \phi_o$$

or that

13) $$HU\phi_o = |k|^2 U\phi_o \quad .$$

Comparison with $H\phi = k^2 \phi$ shows that the operator U takes ϕ_o into ϕ, i.e. $\phi = U\phi_o$ and that $\phi_o = U^{-1}\phi$. In the first approach to the inverse scattering problem the existence of such an operator U is postulated but since the eigenfunctions $\phi(x, k)$ are not known it is sought in still a different form, namely as $I + K$ where K is an integral operator with a kernel $k(x, y)$ in the one-dimensional case. Fredholm operators of this kind are convenient since the existence of an inverse is necessary for the above argument.

We proceed to develop this approach in detail for the one-dimensional wave equation

14) $$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} - q(x)\, U(x, t) = 0$$

under the assumptions on q noted above. For simplicity we will also assume $q(x) = 0$ for $x < 0$ and that initially there are no bound states. The associated time independent equation is

15) $$u_{xx}(x,k) + \left[k^2 - q(x)\right] u(x,k) = 0$$

which, in view of the above, is to be considered as a perturbation of

16) $$\frac{d^2}{dx^2} u_o(x,k) + k^2 u_o(x,k) = 0$$

Thus we seek a kernel in the relation

17) $$u(x,k) = u_o(x,k) + \int_{-\infty}^{\infty} k(x,y)\, u_o(y,k)\, dy$$

or, equivalent in the relation

18) $$U(x,t) = U_o(x,t) + \int_{-\infty}^{\infty} K(x,y)\, U_o(y,t)\, dy$$

Now if $U(x,t)$ is a right moving transient in the sense that $U(x,t) = 0$ for $x > t$, it would seem reasonable to conjecture that it would depend on $U_o(x,t)$ only through those values of x which the non-zero travelling wave would have had time to reach at t. That is that $K(x,y) = 0$ for $y > x$. The reasonableness of this is less evident for the quantum case since there is no causility but the conclusion is still true. Thus (18) can be tentatively replaced by

19) $$U(x,t) = U_o(x,t) + \int_{-\infty}^{x} K(x,y)\, U_o(y,t)\, dy$$

Now it is known that under the above hypotheses on q(x) that asymptotically

20) $$u(x,k) \sim e^{ikx} + r(k)\, e^{-ikx} \qquad x \to -\infty$$

21) $$\sim t(k)\, e^{ikx} \qquad x \to +\infty$$

These conditions correspond to a wave of unit amplitude incident from the left and a transmitted wave of amplitude t(k). We shall suppose that the reflection coefficient r(k) is given for all real positive values of k. In addition we shall suppose

i) $\overline{r(k)} = r(-k)$ Im $k = 0$

ii) $r(k) = O(1/k)$ Im $k > 0$

iii) $|r(k)| \leq 1$, Im $k = 0$

iv) r(k) is analytic and its singularities lying strictly above the real axis in the k-plane consist of a finite number of simple poles on the imaginary axis having residues with positive imaginary parts and zero real parts. It will be assumed that there are no singularities on the real axis except possibly at $k = 0$.

v) The Cauchy principal value of the Fourier transform of r(k) is coninuous with piecewise--continuous first and second derivatives for $-\infty < x < \infty$.

These conditions are necessary and sufficient for the solution of the inverse scattering problem. The proof of their sufficiency is due to Kay (1955), that of their necessity to Sims (1957).

A few additional comments are in order. (i) will hold automatically if q(x) is real while (ii) will hold if q(x) has a zero and first order moments. Finally, we shall initially assume that r(k) is analytic in the half-plane Im $k > 0$, generalizing our results to the case of (iv) subsequently.

Since the differential equations (15) and (16) agree for $x < 0$ it is natural to set $u(x,k) = u_0(x,k)$ for $x<0$ and hence $U(x,t) = U_0(x,t)$ for $x < 0$. Now the generalized function $U_{inc} = \delta(x-t)$ is the Fourier transform of the first term of (20) and

$$R(x+t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k) \; e^{-ik(x+t)} \, dk \tag{23}$$

is the Fourier transform of the second term. It is necessary to use (i) for the construction of this transform. In view of the (temporarily) assumed analytic properties of r(k), it follows from the Paley-Wiener-Titchmarsch theorem that $r(y) = 0$ for $y < 0$. Inserting these expressions in for $U(x,t) = U_0(x,t) = \delta(w-t) + R(x+t)$, $x < 0$ into (18) yields

$$24)\qquad U(x,t) = \delta(x-t) + R(x+t) + \int_{-\infty}^{x} K(x,y)\left[\delta(y-t) + R(y+t)\right] dy$$

$$= \delta(x-t) + R(x+t) + \int_{-\infty}^{\infty} K(x,y)\,\eta(x-y)\left[\delta(y-t) + R(y+t)\right] dy$$

$$= \delta(x-t) + R(x+t) + K(x,t)\,\eta(x-t) + \int_{-t}^{x} K(x,y)\,\eta(x-y)\,R(\eta+t)\, dy$$

where $\eta(z) = 0$ if $z < 0$

$= 1$ if $z > 0$

and where the lower limit follows from the fact that $R(y+t) = 0$ if $y + t < 0$.

If we assume that $U(x,t)$ is a right-moving wave that vanishes for $x > t$, from the above we obtain the following integral equation for $K(x,t)$:

$$25)\qquad 0 = R(x+t) + K(x,t) + \int_{-t}^{x} K(x,y)\, R(y+t)\, dy$$

To simplify this equation still further we will note that without appeal to any special properties of R(y) that it follows from (25) that if $R(x) = 0$ for $x < -2a$ then $K(x,t) = 0$ for $x < -a$. That is if $t < x < -a$ then $t + x < -2a$ so that $R(x+t) = 0$ by hypothesis. Moreover in the integral above $y < x < -a$ and this coupled with the fact that $t < x < -a$ implies that $y + t < -2a$ so that the factor $R(y+t) = 0$ and thus under these conditions (25) reduces to the statement $K(x,t) = 0$ for $t < x < -a$. Applying this to $R(y) = 0$ for $y < 0$ yields the fact that R(x + t) = 0 for $x + t < 0$ or $x < -t$ and thus the fact that K(x, t) = 0 for $x < -t$. Thus (25) can be written as the Fredholm integral equation

$$26) \qquad 0 = R(x+t) + K(x,t) + \int_{-x}^{x} K(x, y)\, R(y+t)\, dy$$

By the Fredholm alternative this will have a unique solution if the corresponding homogeneous equation

$$27) \qquad \Delta(x,t) + \int_{-x}^{x} R(y+t)\, \Delta(x,y)\, dy = 0$$

can be shown to possess only the trivial solution. To see that this is the case we rewrite (27) successively as

$$28) \qquad 0 = \int_{-x}^{x} \Big[R(y+t) + \delta(t-y) \Big] \Delta(x, y)\, dy$$

$$= \int_{-x}^{x} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k)\, e^{-ik(y+t)}\, dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(y-t)}\, dk \right\} \Delta(x,y)\, dy$$

Now it is readily verified that

$$29) \qquad \int_{-\infty}^{\infty} \left\{ r(k)\, e^{-ik(y+t)} + e^{-ik(y-t)} \right\} dk =$$

$$= \int_{0}^{\infty} (1 - |r(k)|^2)\, e^{ik(t-y)}\, dk + \int_{-\infty}^{0} (e^{ikt} + r(k)\, e^{-ikt})\, \overline{(e^{iky} + r(k)\, e^{-iky})}\, dk$$

so that (28) implies that

$$30)\int_0^\infty \left\{1 - |r(k)|^2\right\} \left\{\int_{-x}^{x} \overline{\Delta(x,t)\, e^{-ikt}}\, dt\right\}\left\{\int_{-x}^{x} \Delta(x,y)\, e^{-iky}\, dy\right\} dk \; +$$

$$+ \int_{-\infty}^{0} \left\{ \int_{-x}^{x} \left[e^{ikt} + r(k)\, e^{-ikt}\right] \overline{\Delta(x,t)}\, dt\right\} \overline{\left\{\int_{-x}^{x} \left[e^{iky} + r(k)\, e^{-iky}\right] \Delta(x,y) dy\right\}} dk = 0$$

However for real k, conservation of energy demands that

$$\left| r\,(k) \right|^2 \leq 1$$

so that each term on the left above is positive for any function

$$\int_{-x}^{x} \Delta(x,t)\;\; e^{-ikt}\, dt$$

unless it is identically zero when $x < 0$ so that (30) implies that $\Delta(x, t) = 0$ if $x + t = 0$. Thus only the trivial solution exists and thus (26) will have a unique solution and the representation (19) will exist.

In order to find the relation between K (x, y) and the potential it is merely necessary to apply the differential operator (14) to (19) and to make use of the fact that

$$31) \qquad dK(x,x)\,/\,dx \;=\; K_x\,(x,x) \,\pm\, K_y\,(x,x) \;.$$

Thus *

$$32) \qquad 0 = U_{xx} - U_{tt} - qU = -qU_o + \frac{\partial}{\partial x}\left\{\int_{-x}^{x} K_x(x,y)\, U_o(y,t) \; + \right.$$

$$\left. + K(x,x)\, U_o\,(x,t) \;-\; K\,(x,-x)\;\; U_o\,(x,t) \right\} -$$

$$- \int_{x}^{x} K\,(x,y)\;\; U^o_{tt}\,(y,t)\;\, dy \quad =$$

$$= -\, q\, U_o + \int_{-x}^{x} K_{xx}\;\; U_o(y,t) \;+\; K_x\,(x,x)\;\; U_o\,(x,t) \;-$$

$$- K_x\,(x,-x)\;\; U_o(x,t) \;+\; \frac{d\,K(x,x)}{dx}\; U_o(x,t) + K(x,x)\, \frac{d}{dx}\, U_o(x,t) \;\; -$$

* For clarity, the subscript o will sometimes appear as a superscript o in the following.

$$- \frac{d\,K(x,-x)}{dx} U_o(x,t) - K(x,-x) \frac{d}{dx} U_o(x,t) -$$

$$- \int_{-x}^{x} K\, U^o_{yy}\, dy \quad =$$

$$= -q\,U_o + \int_{-x}^{x} K_{xx} U_o + K_x(-x,x)\, U_o(x,t) - K_x(x,-x)\, U_o(x,t) +$$

$$+ \frac{dK}{dx}(x,x)\, U_o(x,t) + K(x,x) \frac{d}{dx} U_o(x,t) -$$

$$- \frac{dK}{dx}(x,-x)\, U_o(x,t) - K(x,-x) \frac{d}{dx} U_o(x,t) -$$

$$- K(x,x)\, U^o_y(x,t) + K(x,-x)\, U^o_y(x,t) +$$

$$+ K_y(x,x)\, U^o(x,t) - K_y(x,-x)\, U^o(x,t) - \int K_{yy}\, U^o\, dy =$$

$$= \int_{-x}^{x} (K_{xx} - K_{yy} - q\,K)\, U_o(y,t)\, dy +$$

$$+ \left[2 \frac{dK(x,x)}{dx} - q(x) \right] U_o(x,t) + 2 \frac{dK(x,-x)}{dx} U_o(x,t)$$

That is, (14) will be satisfied if K(x, y) satisfies the partial differential equation

33) $$K_{xx} - K_{yy} - qK = 0$$

subject to the conditions

34) $$\frac{dK(x,-x)}{dx} = 0 \qquad \frac{dK(x,x)}{dx} = \frac{q(x)}{2}$$

This Cauchy problem of the second order linear hyperbolic equation has a solution so that such a K can be found.

N.B. $\left[K(x,-x) = 0 \text{ implies } \frac{d}{dx} K(x,-x) \right] = 0$

Conversely suppose that q(x) is defined by (34) where $K(x, -x) = 0$ then applying the differential operator

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - q(x)$$

to (26) yields

$$35) \quad 0 = \left[\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - q(x) K\right] + \int_{-x}^{x} \left[\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - q(x) K\right] R(y+t)\, dy$$

which is the same as (27) if one now sets

$$\Delta = \frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - q K .$$

It follows from the previous uniqueness argument that

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - q K = 0$$

so that any solution of (26) will also satisfy (33).

All of the above can be generalized to the case where the assumption of analyticity in the upper-half plane is replaced by the condition (iv) provided that the function R(x) is appropriately defined. Assume in accordance with this assumption that r(k) has poles at $k_j = i\tau_j$ where $\tau_j > 0$. Taking for convenience r(k) in the form

$$r(k) = g(k)\, e^{-2i\alpha K}$$

where $g(-k) = \overline{g(k)}$ for k real and where the residue of g(k) at the above poles is r_j we will now define R(x) by the expression

$$R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)\, e^{-ik(x+2\alpha)}\, dK + \sum \frac{e^{i\tau_j x}}{A_j}$$

where the normalization constants A_j will be chosen so as to make $R(x) = 0$ for $x < -2a$. Using (ii) we can close the contour in the upper half-plane by Jordan's lemma and thus obtain for $x < -2a$ that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g(K)\, e^{-ik(x + 2\alpha)}\, dK = i \sum x_j \; e^{i\alpha\tau_j + \tau_j x}$$

Thus if A_i is chosen so that

$$\frac{1}{A_j} = -i\, r_j \; e^{i\alpha\tau_j}$$

Then $R(x) = 0$ for $x < -2a$. By our previous observation this will imply that $K(x, t) = 0$ for $x < -a$ and hence that

$$q(x) = 2\, dK(x, x) / dx \mp 0 \quad \text{for} \quad x < -a \; .$$

There is actually no loss of generality in now setting $a = 0$ so that the potentials constructed above even in the presence of bound states will all vanish for $x < 0$. There is also no difficulty in extending the uniqueness argument for the homogeneous equation (27) to this case--it is carried out in detail by Kay in (1955). This completes our description of Kay's adaption of the Gelfand-Levitan approach to the one-dimensional wave equation. To actually construct $q(x)$ it is of course necessary to solve (26) by some process such as successive iterations.

The method described above does not appear to offer much promise for the solution of a large class of electromagnetic problems although as demonstrated by Moses (1967) elementary transformations will sometimes permit an electro-magnetic problem to be rephrased in such a way as to make the above applicable.

In contrast the method developed also by Moses in (1956) seems more straightforward and perhaps capable of generalizations at least to the electro-magnetic problems which can be formulated in terms of vector integral equations analogous to (1). Cf. Dolph and Barrar (1954) and Miller (1957) . Moreover there is little difference between the cases of one and three dimensions so that one may as well develop the theory for the latter. From (1) it follows that for back scattering, one has asymptotically that *

$$36) \qquad \phi(\underline{x},\underline{k}) \sim \frac{e^{i\underline{k}\cdot\underline{x}}}{(2\pi)^{3/2}} + \frac{e^{i|\underline{k}||\underline{x}|}}{|\underline{x}|} \; r(\underline{k})$$

where

$$37) \qquad r(\underline{k}) = -(\pi/2)^{1/2} \int e^{i\underline{k}\cdot\underline{x}'} q(\underline{x}')\, \phi \frac{(\underline{x}'\,\underline{k})}{(2\pi)^{3/2}} \, d\mathbf{x}', \quad k > 0$$

Again, although $r(\underline{k})$ is defined only for real $k > 0$ in the event that $q(\underline{x})$ is real it can be shown that

$$38) \qquad r(-\underline{k}) = \overline{r(\underline{k})} \qquad k > 0$$

or that

$$39) \qquad r(-k,\theta,\phi) = \overline{r(k,\theta,\phi)} \qquad k > 0 \,.$$

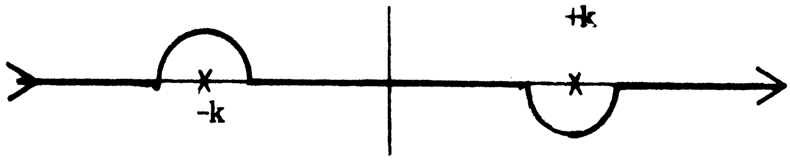
It will now be shown how the scattering potential can be obtained from $r(\underline{k})$ where $\underline{k}$ is such that it makes an angle less than $\pi/2$ radians with the positive z-axis. This restriction takes care of the over-determinancy mentioned by Faddeyev (1963).

The first basic equation of this theory is just (1) written in the momentum representation. More explicitly using the known fact that

$$40) \qquad \frac{e^{ik|\underline{x}-\underline{y}|}}{4\pi|\underline{x}-\underline{y}|} = \frac{1}{8\pi^3} \int \frac{e^{i\underline{k}'\cdot\underline{x}} \; e^{-i\underline{k}'\cdot\underline{y}}}{k'^2 - k^2} \, dk'$$

* To conform with Moses normalization, $e^{i\underline{k}\cdot\underline{x}}$ in (1) has been replaced by $e^{i\underline{k}\cdot\underline{x}}(2\pi)^{-3/2}$.

where the path in the k-plane is the one shown directly below:



one multiplies (1) through by $q(\underline{x})$ and introduces the definitions

41) $$T(\underline{k},\underline{k}') = \frac{1}{(2\pi)^{3/2}} \int e^{-i\underline{k}\cdot\underline{x}}\, q(\underline{x})\, \phi(\underline{x},\underline{k}')\, d\underline{x}$$

42) $$V(\underline{k},\underline{k}') = \frac{1}{(2\pi)^{3}} \int e^{i(\underline{k}'-\underline{k})\cdot x}\, q(\underline{x})\, d\underline{x}$$

so that (1) becomes (Cf. Morse and Feshbach II, p. 1077, 1954)

43) $$T(\underline{k},\underline{k}') = V(\underline{k},\underline{k}') + \int \frac{V(\underline{k},\underline{k}'')\, T(\underline{k}'',\underline{k}')}{k'^2 - k^2}\, dk''$$

The second equation of this theory consists in essentially solving this equation for $V(\underline{k},\underline{k}')$. We first observe that if

44) $$\phi_0(\underline{k}',\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i(\underline{k}'\cdot\underline{x})}\, \phi(\underline{x},\underline{k})\, dx$$

Then equation (1) can be written after Fourier Transform as

45) $$\phi_0(\underline{k}',\underline{k}) = \delta(\underline{k}-\underline{k}') + \underset{e \to 0}{\mathrm{l.i.m.}} \frac{1}{k^2 - k'^2 + ie}\, T(\underline{k}',\underline{k})$$

If for simplicity we again assume that there are no bound states then the completeness relation

46) $$\int_{-\infty}^{\infty} \phi(\underline{x},\underline{k})\, \overline{\phi(\underline{x},\underline{k}')}\, d\underline{x} = \delta(\underline{k}-\underline{k}')$$

will imply the completeness relation

47) $$\int_{-\infty}^{\infty} \phi_o(\underline{k}'', \underline{k}')\ \overline{\phi_o(\underline{k}'', \underline{k})}\ dk'' =$$

$$= \delta(\underline{k}-\underline{k}') \int_{-\infty}^{\infty} \phi_o(\underline{k}', \underline{k}'')\ \overline{\phi_o(\underline{k}, \underline{k}'')}\ d\,\underline{k}'' = \delta(\underline{k}-\underline{k}') .$$

Now by definition we have that

48) $$V(\underline{k}, \underline{k}') = \frac{1}{(2\pi)^3} \int e^{i(\underline{k}-\underline{k}')\cdot\underline{x}}\ q(\underline{x})\ dx$$

from which it follows that

49) $$V(\underline{k}', \underline{k}'')\ \phi_o(\underline{k}'', \underline{k})\ d\,k'' = \frac{1}{(2\pi)^3} \iint q(\underline{x})\ e^{-i(\underline{k}'-\underline{k}'')\cdot\underline{x}}\ \phi_o(\underline{k}'', \underline{k})\ dx\ dk'' =$$

$$= \int \frac{e^{-i\underline{k}',\underline{x}}\ q(\underline{x})dx}{(2\pi)^{3/2}} \int \frac{e^{i\underline{k}'',\underline{x}}\ \phi_o(\underline{k}'', \underline{k})\ dk''}{(2\pi)^{3/2}} =$$

$$= \int \frac{e^{-i\underline{k}',\underline{x}}\ q(\underline{x})}{(2\pi)^{3/2}}\ \phi(\underline{x}, \underline{k})\ dx = T(\underline{k}', \underline{k})$$

in view of (47) and the definition of $T(\underline{k}, \underline{k}')$. Therefore we have at once that

50) $$V(\underline{k}, \underline{k}') = \int T(\underline{k}, \underline{k}'')\ \overline{\phi_o(\underline{k}', \underline{k}'')}\ dk'' .$$

Inserting the expression for ϕ_o given by (45) yields the solution to (43) in the form

51) $$V(\underline{k}, \underline{k}') = T(\underline{k}, \underline{k}') + \int T \frac{(\underline{k}, \underline{k}'')}{\overline{k''}^2 - \overline{k'}^2}\ \overline{T(\underline{k}', \underline{k}'')}\ dk''$$

[The correspond expression in the presence of bound state is eq. 7.52a in Newton (1966), page 189] .

Defining $W(\underline{k})$ by the relation

52) $$W(\underline{k}) = V(-\underline{k}, \underline{k}) = \frac{1}{8\pi^3} \int q(\underline{x})\ e^{i(2\underline{k}\ \underline{x})}\ dx$$

and noting that from this it follows that

53) $$q(\underline{x}) = V \int e^{-i2\underline{k}\cdot\underline{x}} W(\underline{k})\, d\underline{k}$$

we see that (51) can be written as

54) $$W(\underline{k}) = -\left(\frac{2}{\pi}\right)^{1/2} r(\underline{k}) + \int \frac{T(-\underline{k}, \underline{k}')}{k'^2 - k^2} \overline{T(\underline{k}, \underline{k}')}\, dk'$$

This last equation must be extended to $k < 0$. We first note that the property of the potential being real implies that

55) $$W(-\underline{k}) = \overline{W(\underline{k})} \qquad k > 0$$

so that (54) implies that

56) $$W(\underline{k}) = \overline{W(-\underline{k})} = -\left(\frac{2}{\pi}\right)^{1/2} r(\underline{k}) + \int T \frac{(-\underline{k}, \underline{k}')\ \overline{T(\underline{k}, \underline{k}')}}{\bar{k}'^2 - \bar{k}^2}\, d\underline{k}'$$

Using the definition of η, this can be written as

57) $$W(\underline{k}) = -\left(\frac{2}{\pi}\right)^{1/2} r(\underline{k}) + \int T(-\underline{k}, \underline{k}') \left\{ \frac{\eta(\underline{k})}{\overline{k'^2 - k^2}} + \frac{\eta(-\underline{k})}{k'^2 - k^2} \right\} \overline{T(\underline{k}, \underline{k}')}\, dk'$$

Equations (43), (52), (53) and (57) form the basis for this new theory. They must be solved simultaneously and as in the previous case the method of iterations suggests itself. One replaces $r(\underline{k})$ by $\epsilon\, r(k)$ and makes the Ansatz that

58) $$T(\underline{k}, \underline{k}') = \sum_{1}^{\infty} \epsilon^n T_n(\underline{k}, \underline{k}')$$

$$W(\underline{k}) = \sum_{1}^{\infty} \epsilon^n W_n(\underline{k})$$

$$V(\underline{k}, \underline{k}') = \sum_{1}^{\infty} \epsilon^n W_n\left(\frac{\underline{k}' - \underline{k}}{2}\right)$$

Upon substitution, one sees that T_n and W_n can be obtained from a knowledge of $r(\underline{k})$ alone while one uses (53) to obtain $q(\underline{x})$.

Alternately, one can use the expression for $W(\underline{k})$ as given by (57) and substitute it into (53) to obtain

59) $$q(\underline{x}) = \left(\frac{2}{\pi}\right)^{1/2} \int r(\underline{k})\, e^{-2i\underline{k}\cdot\underline{x}}\, d\underline{k} \; + $$
$$+ 8 \int d\underline{k} \int q(\underline{x}')\, \phi(\underline{x}',\underline{k})\, d\underline{x}' \int q(\underline{x}'')\, \overline{\phi(\underline{x}'',\underline{k})}\, d\underline{x}''\, g_k(\underline{x}' + \underline{x}'' - 2\underline{x})$$

where

60) $$g_k(\underline{x}) = \frac{1}{(2\pi)^3} \int e^{i\underline{k}'\cdot\underline{x}} \left\{ \frac{\eta(\underline{k}')}{\overline{k}^2 - \overline{k'}^2} + \frac{\eta(-\underline{k}')}{k^2 - k'^2} \right\} dk'$$

equations (59) and (1) are the basic equations to be solved. One writes

61) $$q(\underline{x}) = \sum_1^{\infty} \epsilon^n q_n(\underline{x})$$

and

62) $$\phi(\underline{x},\underline{k}) = \frac{e^{i\underline{k}\cdot\underline{x}}}{(2\pi)^{3/2}} + \sum_1^{\infty} \epsilon^n \phi_n(\underline{x},\underline{k})$$

which upon substitution leads to a perturbation series. It can be shown that to any order of approximation, ϕ reproduces the reflection coefficient $r(\underline{k})$. Faddeyev (1963) expresses the view that it is quite probable that this method of Moses converges for sufficient small $r(\underline{k})$. To support this opinion it should be noted that this procedure is so reminiscent of the Born series approach to (1) that much of the recent work giving sufficient conditions for the convergence of this series could probably be extended to equations (61) and (62).

We note the crucial role played by the relations (38) and (i) in these theories respectively. Without something like them it seems difficult to see how a theory could be developed but it is perhaps worthwhile to consider some of the work on the optical model where a complex potential is employed to give a model of nuclear scattering.

As a concluding remark that Lax and Phillips (1967) give two proofs of the fact that the scattering operator for the wave equation without potential does in its time independent form in fact determine the obstacle under Dirichlet conditions. The first of these due to Schiffer proceeds along classical lines and uses the Green's representation theorem of the exterior problem and also Rellich's uniqueness theorem. An attempt to extract it in detail will not be attempted here since it involves much of the previous notation and results of previous chapters of the book.

Since this was written, some recent Russian work has been noted based on A. N. Tihonov's 1943 paper on incorrectly posed problems. This should be pursued. See for example:

A.N. Tihonov, 1. Dokl. Akad. Nauk, S.S.S.R. 39, 1943, p. 176.
2. Sov. Math 4, 1963, p. 1035.
3. Sov. Math 4, 1963, p. 1624.

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
The University of Michigan Radiation Laboratory, Dept. of Electrical Engineering, 201 Catherine Street, Ann Arbor, Michigan 48108	Unclassified
	2b. GROUP N/A

3. REPORT TITLE

INVERSE SCATTERING INVESTIGATION
Quarterly Report No. 3

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Third Quarterly Report (3 July to 3 October 1967)

5. AUTHOR(S) *(First name, middle initial, last name)*

Vaughan H. Weston, Wolfgang M. Boerner, and Charles L. Dolph

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November 1967	79	25

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
F 19628-67-C-0190	ESD-TR-67-517, Vol. III
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	8579-3-Q

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited. [redacted]

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Deputy for Surveillance and Control Systems Electronic Systems Division, AFSC, USAF, L.G. Hanscom Field, Bedford, Mass. 01730

13. ABSTRACT

The problem in question consists of determining means of solving the inverse scattering problem where the transmitted field is given and the received fields are measured, and this data is used to discover the nature of the target. Particular aspects of this overall problem are considered, such as the effect of phase errors upon the determination of the scattering surface, polynomial interpolation of the scattered field measured at a set of discrete points, and the testing of a numerical procedure for finding the surface of a conducting body from the knowledge of the near field. In addition, a review of exact theoretical treatments for the scalar inverse problem is given.

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Inverse Scattering						
Electromagnetic Theory						

Unclassified

Security Classification