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COLLEGE OF ENGINEERING

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Radiation Laboratory

SCATTERING BY A THIN DISK OF LARGE RADIUS

by

Gerard A. Desjardins and Thomas B.A. Senior

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1400 Wilson Boulevard
Arlington, Virginia 22209

Ann Arbor, Michigan 48105

ABSTRACT

Experimental measurements of the surface current on an electrically large, perfectly conducting thin disk indicate that under certain conditions creeping waves can exist on the disk surface. Though theoretical verification of this effect is possible for the electromagnetic problem, the solution of a similar scalar problem is expected to exhibit the same type of behavior but be much simpler to treat mathematically. To this end, the surface field on a soft thin disk of large radius due to the presence of a point source far from the disk is found.

Since the disk is a complete co-ordinate surface in the oblate spheroidal system where the scalar wave equation is separable, the surface field can be expressed as an infinite sum of Resolvent Green's functions. The functions are formed of solutions of the separated differential equations and these solutions are constructed using the theory of differential equations containing a large parameter. Except for edge-on incidence, the series is valid only on the shadow side of the disk.

The series expression for the surface field is evaluated for various angles of incidence, though the results are mathematically simple for only two: broadside and edge-on. In the broadside case, the disk edge field is found to be the same as that of a soft half-plane for normal incidence. Away from the edge, the surface field behavior is more complicated, but can still be characterized as an edge wave behavior. For edge-on incidence, things are markedly different. In this case, the edge field consists of optics and creeping wave terms. The optics term at the specular point is the same as the soft half plane edge field for edge-on incidence. The creeping waves are similar in form to those found on electrically large cylinders. Finally, the expected correspondence between the electromagnetic and scalar analysis of the problem for edge-on incidence is verified.

FOREWORD

This report is the dissertation submitted by the leading author in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Electrical Engineering) in The University of Michigan, 1974.

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Chapter 1

INTRODUCTION

1.1 General Discussion.

The surface fields due to an electromagnetic plane wave incident on bodies such as an electrically large cylinder or sphere are known to contain creeping wave components. For an electrically large but thin disk, creeping waves are also evident (Senior, 1969) when a plane electromagnetic wave whose E-vector lies in the plane of the disk is incident on the disk edge-on. In all three cases, the creeping waves are born in the vicinity of a shadow boundary, but the character of their behavior is different for the disk. The creeping wave found on the cylinder and sphere can be characterized as a slow wave; the energy appears to travel with velocity v (the speed of light) just outside the surface. However, for the disk, the creeping wave is a fast wave and the energy appears concentrated just inside the edge. This fundamental difference in behavior makes a study of the wave motion on the disk of interest.

In the case of the cylinder or sphere, the surface field can be expressed in a form where the creeping wave dependence on electrical size is readily apparent. The series expression for the surface field convergent at low frequencies can be transformed into a series convergent at high frequencies. Through the transformation, explicit expressions for the optics and creeping wave terms as a function of electrical size are obtained. It is desirable to carry out this same type of transformation for a disk in the hope of obtaining explicitly the dependence of the surface wave motion on the disk size.

Such a transformation is possible. However, to avoid the complexity of the vector problem, an analogous scalar problem will be treated. It is expected that the scalar analysis will exhibit a surface field behavior similar to the vector analysis though the decay rates and other parameters may not be the same. The scalar analog of the vector disk problem is the scattering of a scalar plane wave (u^i, where u is a velocity potential) at edge-on incidence on a soft disk for which the boundary condition is the vanishing of the total field u at the surface. However, for mathematical reasons, the scattering problem will be formulated instead in terms of a unit point source far from the disk. In particular, the surface field of an acoustically soft thin disk with $ka \gg 1$ (k is the propagation constant of the medium; a is the disk radius) is to be found.

Since the disk is a complete co-ordinate surface in the oblate spheroidal system and since the scalar wave equation is separable in that system, the surface field can be expressed as a doubly infinite sum of solutions of the separated equations. The solutions are found using the theory of differential equations with a large parameter. The series expression for the surface field convergent for $ka \gg 1$ is constructed for various angles of incidence, including edge-on. For the edge-on case, the creeping wave does exhibit the fast wave behavior and is similar in form to that of the vector case. The results indicate that the disk is one of the basic scattering shapes since it exhibits a surface wave behavior as simple as, yet fundamentally different from, that of the cylinder or sphere.

1.2 Description of the Problem.

A disk of zero thickness and radius a lies in the x-y plane of a Cartesian co-ordinate system with its center at the origin. A point source of unit strength is located in the x-z plane a distance r from the disk center.

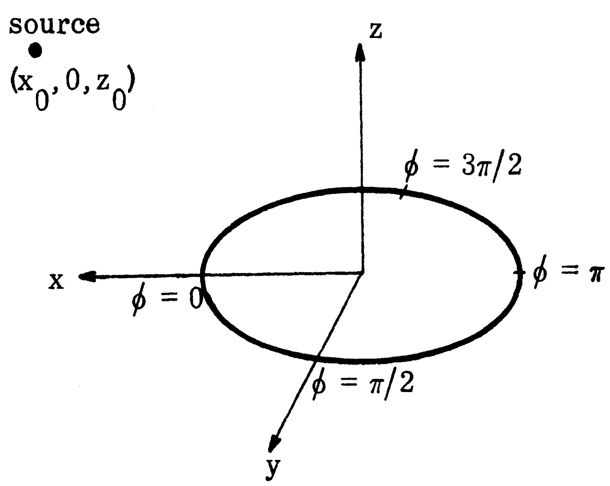

FIG. 1: Geometry of the problem.

The relationship between the Cartesian (x, y, z) and the oblate spheroidal (ξ, η, ϕ) systems is as follows:

$$x = a\left[(\xi^2+1)(1-\eta^2)\right]^{1/2}\cos\phi$$

$$y = a\left[(\xi^2+1)(1-\eta^2)\right]^{1/2}\sin\phi \qquad (1.1)$$

$$z = a\xi\eta$$

where $0 \leq \xi < \infty$, $-1 \leq \eta \leq 1$, $0 \leq \phi < 2\pi$. ξ is the radial variable, η the angular variable, and ϕ the axial.

The disk shown in Fig. 1 is the complete co-ordinate surface, $\xi = 0$, in the oblate spheroidal system defined above. Hence the problem of scalar scattering by a soft thin disk is now easy to formulate. If u is a scalar velocity potential, $(\nabla^2 + k^2)u = \rho$ where ρ is the source density distribution, k is the propagation constant and an $e^{-i\omega t}$ time factor has been suppressed. The boundary condition is that $u = 0$ on the disk.

In particular, for a unit point source located at $(\xi_0, \eta_0, 0)$

$$\rho(\xi_0, \eta_0, 0) \equiv \frac{\delta(\xi-\xi_0)\delta(\eta-\eta_0)\delta(\phi)}{a^3(\xi^2+\eta^2)} \quad . \qquad (1.2)$$

In oblate spheroidal co-ordinates, the inhomogeneous scalar wave equation with the source term given in eq. (1.2) is

$$\left(\frac{\partial}{\partial\xi}\left[(\xi^2+1)\frac{\partial u}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial u}{\partial\eta}\right] + \frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2 u}{\partial\phi^2}\right) + (ka)^2(\xi^2+\eta^2)u$$

$$= \frac{1}{a}\delta(\xi-\xi_0)\delta(\eta-\eta_0)\delta(\phi) \quad . \qquad (1.3)$$

The boundary condition becomes $u(0, \eta, \phi) = 0$.

For $\xi \neq \xi_0$, $\eta \neq \eta_0$, or $\phi \neq 0$, eq. (1.3) can be separated (using 2 separation constants) into three equations. Assume

$$u = H(\xi)X(\eta)\Phi(\phi) \ ,$$

then

$$\frac{d^2\Phi(\phi)}{d\phi^2} + m^2\Phi = 0 , \tag{1.4}$$

$$\frac{d}{d\xi}\left[(1+\xi^2)\frac{dH}{d\xi}\right] + \left[c^2\xi^2 + A_{mn} + \frac{m^2}{1+\xi^2}\right] H = 0 , \tag{1.5}$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dX}{d\eta}\right] + \left[c^2\eta^2 - A_{mn} - \frac{m^2}{1-\eta^2}\right] X = 0 , \tag{1.6}$$

where $c \equiv ka$ and $H(\xi)$ must satisfy the radiation condition. Several considerations can be used to specify m and A_{mn}. If one demands that Φ be single-valued in ϕ, m must be an integer (called the axial eigenvalue). With m specified, A_{mn} is determined by one of two conditions. Conventionally, A_{mn} is selected so that $X(\eta)$ is finite at both $\eta = \pm 1$. Then A_{mn} is an angular eigenvalue. However, the series solution summed over the angular and axial eigenvalues is known to be poorly convergent for large c. If the A_{mn} are selected so that $H_{mn}(0) = 0$, analogous to what Watson did for an electrically large sphere, the series is rapidly convergent for large c and the A_{mn} are radial eigenvalues. With this choice of A_{mn}, no $X(\eta)$ exists which is finite at both $\eta = \pm 1$.

The solution of the inhomogeneous scalar wave equation in oblate spheroidal co-ordinates is

$$u(\xi,\eta,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} H_{mn}(\xi) X_{mn}(\eta) \Phi_m(\phi) \tag{1.7}$$

with the summations over the axial and radial eigenvalues and the B_{mn} chosen so that $u(\xi_0, \eta_0, 0) = \rho$. Because of the disk symmetry

$$\Phi_m(\phi) = \Phi_m(-\phi) = \cos m\phi$$

and eq. (1.7) then becomes

$$u = \sum_{m=0}^{\infty} \cos m\phi \sum_{n=0}^{\infty} B_{mn} X_{mn}(\eta) H_{mn}(\xi) \ . \tag{1.8}$$

Defining

$$V_m(\xi, \eta) = \sum_{n=0}^{\infty} B_{mn} X_{mn}(\eta) H_{mn}(\xi) \ ,$$

eq. (1.8) reduces to

$$u = \sum_{m=0}^{\infty} V_m(\xi, \eta) \cos m\phi \quad . \tag{1.9}$$

Putting the above representation for u into eq. (1.3) and using the orthogonal properties of $\cos m\phi$, one gets

$$\frac{\partial}{\partial \xi}\left[(\xi^2+1)\frac{\partial V_m}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial V_m}{\partial \eta}\right] - \frac{m^2(\xi^2+\eta^2)}{(\xi^2+1)(1-\eta^2)} V_m + c^2(\xi^2+\eta^2)V_m$$

$$= \frac{1}{\pi a \epsilon_m} \delta(\xi-\xi_0)\delta(\eta-\eta_0) \tag{1.10}$$

where

$$\epsilon_m = \begin{cases} 2, & m = 0 \\ 1, & m > 0 \end{cases} \ .$$

The B_{mn} must be selected so that V_m satisfies eq. (1.10).

1.3 Resolvent Green's Functions and the Representation of $\frac{\partial}{\partial \xi} V_m(\xi, \eta)$.

Equation (1.10) can be solved in two equivalent ways. One method, used by Hansen (1962), is to write

$$V_m = \sum_{n=0}^{\infty} B_{mn} X_{mn}(\eta) H_{mn}(\xi) \tag{1.11}$$

as above. Putting eq. (1.11) into eq. (1.10) and using the orthogonal properties of the H_{mn} gives

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}(B_{mn}X_{mn})\right]+\left[c^2\eta^2-A_{mn}-\frac{m^2}{1-\eta^2}\right]B_{mn}X_{mn}$$

$$=\frac{H_{mn}(\xi_0)}{\pi a\epsilon_m}\left(\int_0^\infty\left|H_{mn}(\xi)\right|^2 d\xi\right)^{-1}\delta(\eta-\eta_0) \quad . \tag{1.12}$$

The $B_{mn}X_{mn}(\eta)$ are found using standard Green's function techniques.

A second equivalent method of solution which is available is the method of Resolvent Green's functions as used by Kazarinoff and Ritt (1959). For the source defined in eq. (1.2), V_m can be represented as the contour integral

$$V_m(\xi,\xi_0,\eta,\eta_0)=\frac{1}{\pi a\epsilon_m}\lim_{s\to 0+}\frac{1}{2\pi i}\int_{\Gamma_\nu} G_m(\xi,\xi_0,\nu)\tilde{G}_m(\eta,\eta_0,-\nu)\,d\nu \tag{1.13}$$

where c has been redefined as $c = a(k+is)$ with $s>0$; this is equivalent to a small loss in the medium. $G_m(\xi,\xi_0,\nu)$ and $\tilde{G}_m(\eta,\eta_0,-\nu)$ are the radial and angular Resolvent Green's functions. Γ_ν is a straight line path in the ν-plane a distance proportional to s below the real ν axis.

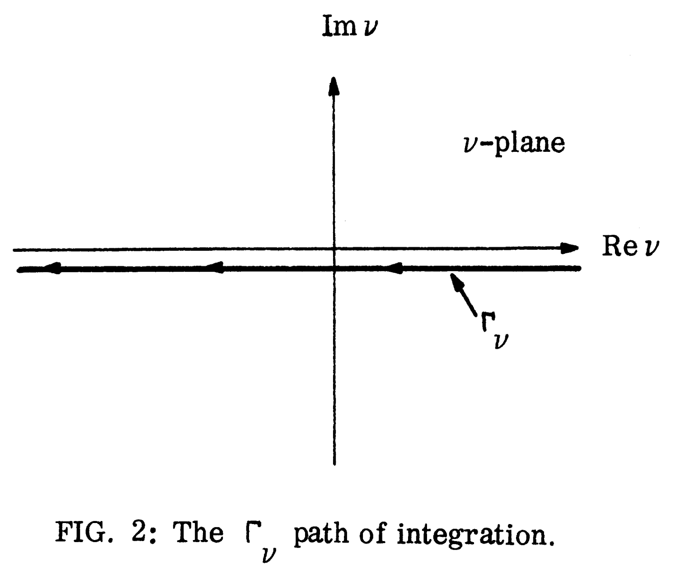

FIG. 2: The Γ_ν path of integration.

The radial Resolvent Green's function is

$$G_m(\xi,\xi_0,\nu) = \frac{1}{(1+\xi^2)W(y_1,y_2,\xi)} \begin{cases} y_1(\xi)y_2(\xi_0)\ , & \xi \le \xi_0 \\ y_1(\xi_0)y_2(\xi)\ , & \xi_0 \le \xi \end{cases} \tag{1.14}$$

where $y_1(0) = 0$ and y_2 is in $\mathcal{L}^2(0,\infty)$. Equation (1.5) has two independent solutions $H(\xi)$ and $\tilde{H}(\xi)$; only one, $H(\xi)$, is in $\mathcal{L}^2(0,\infty)$. Hence y_1 and y_2 are

$$\begin{aligned} y_1(\xi) &= H_{m\nu}(\xi)\tilde{H}_{m\nu}(0) - \tilde{H}_{m\nu}(\xi)H_{m\nu}(0) \\ y_2(\xi) &= H_{m\nu}(\xi) \quad . \end{aligned} \tag{1.15}$$

$W(y_1,y_2,\xi)$ is the Wronskian of the two solutions. The angular Resolvent Green's function is

$$\tilde{G}_m(\eta,\eta_0,-\nu) = \frac{-1}{(1-\eta^2)W(X^1_{m\nu},X^2_{m\nu},\eta)} \begin{cases} X^1_{m\nu}(\eta)X^2_{m\nu}(\eta_0)\ , & \eta \ge \eta_0 \\ X^1_{m\nu}(\eta_0)X^2_{m\nu}(\eta)\ , & \eta_0 \ge \eta \end{cases} \tag{1.16}$$

where $X^2_{m\nu}(\eta)$ is the solution of eq. (1.6) regular at $\eta = -1$ and $X^1_{m\nu}(\eta)$ is the solution regular at $\eta = +1$; from symmetry, $X^2_{m\nu}(\eta) = X^1_{m\nu}(-\eta)$.

The path Γ_ν is completed by a semicircle of radius r in the upper or lower half plane. If the integrand is exponentially small as $r \to \infty$, the contribution of the integration over the semicircle can be neglected. The contour will contain the poles of $\tilde{G}_m$ if Γ_ν is completed in the upper half plane and a Mie series for the disk results; the orthogonal functions in the n summation are the angular eigenfunctions. If the path Γ_ν is completed in the lower half plane, the contour contains the poles of G_m. A series over the radial eigenfunctions results. The expansion over the radial eigenfunctions is chosen for reasons discussed earlier.

To simplify the expression for G_m, only the field on the surface of the disk is found. Since $u = 0$ on the disk, $\partial u/\partial n$ is nonzero and, in particular,

$$\frac{\partial u}{\partial n} = \frac{1}{a\eta}\frac{\partial u}{\partial \xi} = \frac{1}{a\eta}\sum_{m=0}^{\infty} \cos m\phi \frac{\partial V_m}{\partial \xi} \tag{1.17}$$

where $n = \pm z$ depending on which side of the disk is being considered. The presence of $1/\eta$ in the field term is a consequence of the edge condition $\partial u/\partial z \propto 1/r^{1/2}$ where r is the distance from the edge. From eq. (1.13),

$$\frac{\partial}{\partial \xi} V_m(0, \xi_0, \eta, \eta_0) = \frac{1}{\pi a \epsilon_m} \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{\partial}{\partial \xi} G_m(0, \xi_0, \nu) \tilde{G}_m(\eta, \eta_0, -\nu)\, d\nu \tag{1.18}$$

with

$$\frac{\partial}{\partial \xi} G_m(0, \xi_0, \nu) = -\frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} . \tag{1.19}$$

For a hard disk, u is the non-zero field and $\partial G_m/\partial \xi$ is replaced by $G_m = H_{m\nu}(\xi_0)/H'_{m\nu}(0)$.

From eqs. (1.18) and (1.19), it is apparent that the radial eigenfunction expansion of $\partial V_m/\partial \xi$ is the residue series defined by the zeros of $H_{m\nu}(0)$. Since the zeros are simple ones, the residues are given by

$$\frac{1}{\left.\frac{\partial}{\partial \nu} H_{m\nu}(0)\right|_{\nu=\nu_n}} .$$

Finally

$$\frac{\partial}{\partial \xi} V_m(0, \xi_0, \eta, \eta_0) = \frac{1}{\pi a \epsilon_m} \sum_{n=0}^{\infty} \left(- \frac{H_{mn}(\xi_0)}{\left.\frac{\partial}{\partial \nu} H_{m\nu}(0)\right|_{\nu=\nu_n}} \right) \tilde{G}_m(\eta, \eta_0, -\nu_n). \tag{1.20}$$

Equation (1.20) is valid if the contour Γ_ν can be closed. When the behavior of the Resolvent Green's functions is known precisely for all values of ν (as in the cylinder and sphere cases), the conditions for closing are easily established. Unfortunately, as will become evident later, the asymptotic expressions for $H_{m\nu}$ and $X_{m\nu}$ are not valid for all ν and, in particular, are not valid as $\nu \rightarrow \infty$ with c fixed. Hence, mathematical justification for closing the contour is not possible through investigation of the properties of the asymptotic solutions for large ν. For a discussion of some of the problems involved, see Goodrich and Kazarinoff (1963).

1.4 The Mathematical Problem.

To construct the Resolvent Green's functions, solutions of eqs. (1. 5) and (1. 6) having the right properties must be found. Since c is a large parameter, the theory for the solution of differential equations containing a large parameter can be used to obtain asymptotic solutions of (1. 5) and (1. 6) valid as $c \rightarrow \infty$. However, the application of the theory is not straightforward since A_{mn} and m^2 may both become large. Regardless of the size of A_{mn} and m^2, eqs. (1. 5) and (1. 6) can be put in the general form

$$\frac{d^2Y}{dz^2} + \left[\lambda^2 f_0(z) + \lambda f_1(z) + f_2(z)\right] Y = 0$$

where λ is a large parameter.

The first work on this type of equation was done by Birkhoff (1905) for the case $f_1 = 0$ and $f_0(z) > \epsilon > 0$ over the interval of consideration. In 1931, Langer solved the case where $f_0(z)$ was allowed to have a zero, called a turning point. During the next 30 years, Langer's simple turning point theory was extended by Langer himself and several others. Now, asymptotic solutions of equations containing higher order turning points, multiple turning points in an interval, or transition points (points where f_0, f_1, or f_2 have poles) exist. Of particular significance to the present work are the works of Mckelvey (1955) and Olver (1954, 1956).

It is important to remember that solutions obtained using turning point theory are asymptotic in character and thus approach the exact solutions only as $\lambda \rightarrow \infty$. Nevertheless, the use of these solutions for finite but large λ gives valid results.

1.5 Determination of the Surface Field.

As noted in Section 1.3, the determination of the surface field requires that solutions of eqs. (1.5) and (1.6) with the correct properties be found. In Chapter 2, the asymptotic solution of the radial equation which satisfies the radiation condition is found and evaluated at the surface $\xi = 0$ and the source point. Then the zeros of $H_{m\nu}(0)$ as a function of ν are determined. The analysis is broken up into several parts depending on the size of m compared to to c.

In Chapter 3, the angular functions are constructed in a manner paralleling that for the radial functions. The asymptotic solutions of the angular equation are found only in the interval $0 \leq \eta \leq 1$; $\eta = 1$ is a regular point. The specification of the needed angular functions is completed with the determination of the angular Wronskian.

The Resolvent Green's functions are formed in Chapter 4 from the functions constructed in Chapters 2 and 3. The contour integral is evaluated and the expressions for the surface field on the shadow side of the disk are given for various angles of incidence (excluding grazing).

The development of the surface field for edge-on incidence is carried out in Chapter 5. A highly convergent series for the edge field is obtained in terms of the radial and angular eigenfunctions. From this series, an expansion of the edge field in terms of creeping waves and an optics component is possible.

Chapter 6 gives a physical interpretation of the results of Chapters 4 and 5. The surface field in the shadow is constructed of surface waves launched at the disk edge. For edge-on incidence specifically, the structure of the disk edge fields is found to be similar to that measured by Senior (1969) in the electromagnetic case.

Chapter 2

THE RADIAL SOLUTIONS

2.1 Development of the Differential Equations.

As a first step in determining the residue series for the surface field $\partial u/\partial z$, asymptotic expressions for $H_{mn}(\xi)$ and the eigenvalues A_{mn} defined by $H_{mn}(0) = 0$ are found. $H_{mn}(\xi)$ is a solution of the radial equation

$$\frac{d}{d\xi}\left[(1+\xi^2)\frac{dH_{mn}}{d\xi}\right] + \left[c^2\xi^2 + A_{mn} + \frac{m^2}{1+\xi^2}\right] H_{mn} = 0 \quad . \tag{2.1}$$

Equation (2.1) must be transformed to put it in a form suitable for application of the theory of differential equations having a large parameter. Let $H_{mn}(\xi) = (1+\xi^2)^{-1/2} W_{mn}(\xi)$. Then

$$\frac{d^2W_{mn}}{d\xi^2} + \left[\frac{c^2\xi^2 + A_{mn}}{1+\xi^2} + \frac{m^2-1}{(1+\xi^2)^2}\right] W_{mn} = 0 \ . \tag{2.2}$$

The asymptotic theory developed to solve equations such as eq. (2.2) is in terms of only one large parameter, in this case c. To successfully apply the theory, each term in eq. (2.2) must be expanded in powers of c, e.g., $f = \alpha c^2 + \beta c + \gamma + \delta/c + \ldots$, where the coefficients are O(1) and terms of like powers of c grouped together. Depending on the magnitude of m, $m^2 - 1$ can be written as αc^2 or αc. Thus, no one asymptotic solution is valid for all m. In fact, for m in $[0, c^2)$, three asymptotic expressions are necessary, one for m in $[0, \sqrt{c}]$, one for $[\sqrt{c}, c)$ and one for $[c, c^2)$.

We consider the case for m in $[0, \sqrt{c}]$, called Region 1, first. In this case, m^2 is never larger than c and thus is grouped with the c terms. It is assumed that $A_{mn} = c\beta$. Collecting terms having like powers of c, eq. (2.2) reduces to

$$\frac{d^2W_{mn}}{d\xi^2} + \left[c^2\left(\frac{\xi^2}{1+\xi^2}\right) + c\left(\frac{\beta}{1+\xi^2} + \frac{d}{(1+\xi^2)^2}\right)\right] W_{mn} = 0$$

where $d = (m^2 - 1)/c$. In Region 2, m in $[\sqrt{c}\ c)$, m can be as large as c and thus the m^2 term is placed with the c^2 term. In this case, we assume that $A_{mn} = -m^2 + 1 + \tilde{A}_{mn}$ with $\tilde{A}_{mn} = \beta c$. Then

$$\frac{d^2W_{mn}}{d\xi^2} + \left[c^2\left(\frac{\xi^2+1-b}{(1+\xi^2)^2}\right) + c\left(\frac{\beta}{1+\xi^2}\right)\right] W_{mn} = 0$$

where $b = (m^2 - 1)/c^2$. For m in $[c, c^2)$, Region 3, $1-b$ is negative and eq. (2.2) is written as

$$\frac{d^2W_{mn}}{d\xi^2} - \left[c^2\xi^2\left(\frac{1-b-\xi^2}{(1+\xi^2)^2}\right) - c\left(\frac{\beta}{1+\xi^2}\right)\right] W_{mn} = 0 \ .$$

For $m \geq c^2$, no asymptotic solution satisfying the radiation condition and the boundary condition is available.

Asymptotic solutions of the above equations are found. Region 1 results are similar to those of Goodrich et al. (1963) and Hansen (1962). The Region 2 and 3 results are new.

2.2 Region 1 $(0 \leq m \leq \sqrt{c})$.

In Region 1, W_{mn} satisfies the differential equation

$$\frac{d^2W_{mn}}{d\xi^2} + \left[c^2\left(\frac{\xi^2}{1+\xi^2}\right) + c\left(\frac{\beta}{1+\xi^2} + \frac{(m^2-1)/c}{(1+\xi^2)^2}\right)\right] W_{mn} = 0 \qquad (2.3)$$

If β is found to be of O(1) in order that $H_{mn}(0) = 0$, the assumption, $A_{mn} = c\beta$, is verified.

Making the substitutions $\lambda = ic$, $\sigma = i\beta$, eq. (2.3) takes the form

$$\frac{d^2 W_{mn}}{d\xi^2} - \left[\lambda^2 \frac{\xi^2}{1+\xi^2} + \lambda\left(\frac{\sigma}{1+\xi^2} - \frac{(m^2-1)/\lambda}{(1+\xi^2)^2}\right)\right] W_{mn} = 0 \ . \tag{2.4}$$

Since eq. (2.4) has a second order turning point at $\xi = 0$, Mckelvey's (1955) results are used and

$$W_{mn} = \mu_0 V + \frac{\mu_1}{\lambda} V' \tag{2.5}$$

where

$$V = \psi(\xi)\,\eta(x,\lambda) \ ,$$

$$\eta(x,\lambda) = x^{-1/4} W_{-1}(x) \ ,$$

$$\begin{aligned} x &= 2\lambda \int_0^{\xi} \frac{s\,ds}{\sqrt{1+s^2}} = 2ic\left[\sqrt{1+\xi^2} - 1\right] \ , \\ \psi(\xi) &= \left[\sqrt{1+\xi^2} - 1\right]^{1/4} \left[\xi^2/(1+\xi^2)\right]^{-1/4} \ , \\ \mu_0 &= \cosh\left[\theta(\xi)\right] \ , \\ \mu_1 &= \frac{\sqrt{1+\xi^2}}{\xi} \sinh\left[\theta(\xi)\right] \ , \\ \theta(\xi) &= -\frac{1}{4}\left(\sigma - \frac{m^2-1}{\lambda}\right) \log\left[\frac{1+\sqrt{1+\xi^2}}{2}\right] - \frac{m^2-1}{2\lambda}\left[\frac{1}{\sqrt{1+\xi^2}} - 1\right] \ . \end{aligned} \tag{2.6}$$

$W_{-1}(x)$ is related to the Whittaker function (Whittaker and Watson, 1963). In general,

$$W_\nu(x) = \frac{e^{-\nu\pi i/4}\,\Gamma(-1/2)}{i^\nu\,\Gamma(\frac{1}{4} - \gamma k)} M_{k,1/4}(x) + \frac{e^{\nu\pi i/4}\,\Gamma(1/2)}{i^\nu\,\Gamma(\frac{3}{4} - \gamma k)} M_{k,-1/4}(x) \tag{2.7}$$

with $\gamma = (-1)^\nu$ and ν an integer. Hence

$$W_{-1}(x) = \frac{e^{3\pi i/4}\Gamma(-1/2)}{\Gamma(\frac{1}{4}+k)} M_{k,1/4}(x) + \frac{e^{\pi i/4}\Gamma(1/2)}{\Gamma(\frac{3}{4}+k)} M_{k,-1/4}(x) \ . \qquad (2.8)$$

$M_{k,\mu}(x)$ and therefore $W_\nu(x)$ are solutions of Whittaker's equation (see Mckelvey, 1955)

$$\frac{d^2W}{dx^2} + \left[-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4}-\mu^2}{x^2}\right] W = 0 \ . \qquad (2.9)$$

The power series expansion for $M_{k,\mu}(x)$ is easily obtained from eq. (2.9) and is

$$M_{k,\mu}(x) = x^{\frac{1}{2}+\mu} e^{-x/2} \left[1 + \frac{\frac{1}{2}+\mu-k}{2\mu+1} x + \ldots\right] \ . \qquad (2.10)$$

Using eq. (2.10), it can be shown that for $x \ll 1$

$$W_\nu(x) \simeq \frac{e^{\nu\pi i/4}\Gamma(1/2)}{i^\nu \Gamma(\frac{3}{4}-\gamma k)} x^{1/4} + \frac{e^{-\nu\pi i/4}\Gamma(-1/2)}{i^\nu \Gamma(\frac{1}{4}-\gamma k)} x^{3/4}$$

and

$$W_{-1}(x) \simeq \frac{e^{\pi i/4}\Gamma(1/2)}{\Gamma(\frac{3}{4}+k)} x^{1/4} + \frac{e^{3\pi i/4}\Gamma(-1/2)}{\Gamma(\frac{1}{4}+k)} x^{3/4} \ . \qquad (2.11)$$

One other property of $W_\nu(x)$ will be needed. For $|x| \gg 1$ and $(\nu-\frac{3}{2})\pi+\epsilon \leq \arg x \leq (\nu+\frac{3}{2})\pi-\epsilon$, with $0<\epsilon \ll 1$,

$$W_\nu(x) = e^{-\frac{1}{2}\gamma x}(x e^{-\nu\pi i})^{\gamma k}\left[1+\frac{O(1)}{x}\right] \ . \qquad (2.12)$$

Thus in order that $H_{mn}(\xi)$ satisfy the radiation condition, ν is selected to be -1 in eq. (2.6).

To complete the specification of $H_{mn}(\xi)$, σ must be chosen so that $H_{mn}(0) = 0$. Using the small argument expansion of $W_{-1}(x)$,

$$W_{mn}(0) = V(0) = 2^{-1/4}\left[\frac{e^{\pi i/4}\Gamma(1/2)}{\Gamma(\frac{3}{4}+k)}\right] . \tag{2.13}$$

Since $\Gamma(y)$ has simple poles at $y = -n$, $n = 0, 1, 2, 3, \ldots,$

$$W_{mn}(0) = 0 \quad \text{if} \quad k = -\frac{4n+3}{4} ,$$

and it now follows that

$$\sigma = 4n+3+\frac{m^2-1}{\lambda} \qquad \text{(see Appendix)}$$

and

$$A_{mn} = -m^2+1-ic(4n+3) . \tag{2.14}$$

The assumption made earlier concerning the nature of A_{mn} is thereby confirmed.

In summary,

$$H_{mn}(\xi) = (1+\xi^2)^{-1/2} W_{mn}(\xi) ,$$

with

$$W_{mn}(\xi) = \left(\cosh\left[\theta(\xi)\right] - i \sinh\left[\theta(\xi)\right] \frac{\sqrt{1+\xi^2}}{c\xi}\frac{d}{d\xi}\right)\left[(2ic)^{-1/4}\left(\frac{\sqrt{1+\xi^2}}{\xi}\right)^{1/2} W_{-1}(x)\right], \tag{2.15}$$

where

$$x = 2ic\left(\sqrt{1+\xi^2} - 1\right)$$

and

$$\theta(\xi) = -\frac{4n+3}{4}\log\left[\frac{1+\sqrt{1+\xi^2}}{2}\right] + \frac{i(m^2-1)}{2c}\left[\frac{1}{\sqrt{1+\xi^2}} - 1\right] .$$

In Chapters 4 and 5, $H_{mn}(0)$, $\left.\frac{\partial}{\partial\nu}H_{m\nu}(0)\right|_{\nu=\nu_n}$, and $H_{mn}(\xi_0)$, $\xi_0 \gg 1$, are needed. From eqs. (2.13) and (2.15),

$$H_{mn}(0) = 2^{-1/4}\,\frac{e^{\pi i/4}\Gamma(1/2)}{\Gamma(-n)} \quad . \tag{2.16}$$

and

$$\frac{\partial}{\partial\nu} H_{m\nu}(0)\Big|_{\nu=\nu_n} = -e^{3\pi i/4}\, 2^{-9/4}\, c^{-1}\Gamma(1/2)n!\,(-1)^n \tag{2.17}$$

where $\nu = -m^2 + 1 - ic(4n+3)$ and $n = 0, 1, 2, 3, \ldots$. For $\xi_0 \gg 1$,

$x \simeq 2ic(\xi_0 - 1)$,

$V \simeq (2ic)^{-1/4} e^{x/2}(-x)^{-k}$

$$\theta(\xi_0) \simeq -\frac{4n+3}{4}\log\left[\xi_0/2\right] - \frac{i(m^2-1)}{2c} \quad ,$$

and

$$H_{mn}(\xi_0) \simeq \frac{2^{2n+\frac{5}{4}}\, c^{n+\frac{1}{2}}}{\xi_0}\exp\left[ic\xi_0 - ic - \frac{i}{2c}(m^2-1) - (n+2)\frac{\pi}{2}i\right] \quad . \tag{2.18}$$

2.3 Region 2 ($\sqrt{c} \le m < c$).

For m in Region 2, eq. (2.2) is written in the form

$$\frac{d^2W_{mn}}{d\xi^2} + \left[c^2\xi^2\left(\frac{\xi^2+1-b}{(1+\xi^2)^2}\right) + \frac{\tilde{A}_{mn}}{1+\xi^2}\right]W_{mn} = 0 \tag{2.19}$$

where $\tilde{A}_{mn} = A_{mn} + m^2 - 1$ and $b = (m^2-1)/c^2$. Equation (2.19), like eq. (2.3), has a second order turning point at $\xi = 0$, so Mckelvey's (1955) theory can be used here as well. Let $\lambda = ic$ and $\lambda\sigma = -\tilde{A}_{mn}$, then

$$\frac{d^2W_{mn}}{d\xi^2} - \left[\lambda^2\xi^2\left(\frac{\xi^2+(1-b)}{(1+\xi^2)^2}\right) + \lambda\frac{\sigma}{1+\xi^2}\right]W_{mn} = 0 \tag{2.20}$$

and

$$W_{mn} = \mu_0 V + \frac{\mu_1}{\lambda} V' ,$$

$$V = (2ic)^{-1/4}(1+\xi^2)^{1/2}\xi^{-1/2}(\xi^2+1-b)^{-1/4} W_{-1}(x) ,$$

$$x = 2ic\left[\sqrt{\xi^2+1-b} - \sqrt{1-b} - \sqrt{b}\left(\sec^{-1}\sqrt{\frac{1+\xi^2}{b}} - \sec^{-1}\sqrt{1/b}\right)\right] ,$$

$$\mu_0 = \cosh\left[\theta(\xi)\right] , \tag{2.21}$$

$$\mu_1 = \frac{(1+\xi^2)}{\xi}(\xi^2+1-b)^{-1/2}\sinh\left[\theta(\xi)\right] ,$$

$$\theta(\xi) = -\frac{4n+3}{2}\log\left[\frac{\sqrt{1-b}+\sqrt{\xi^2+1-b}}{2\sqrt{1-b}}\right] - \frac{4n+3}{4}\left(\log\left[\frac{x}{2ic}\right] - \log\left[\frac{\sqrt{1-b}}{2}\xi^2\right]\right) .$$

In order that $H_{mn}(0) = 0$, it is necessary that $\sigma = \sqrt{1-b}\,(4n+3)$ and

$$A_{mn} = -m^2+1-ic(4n+3)\sqrt{1-b} . \tag{2.22}$$

Using an analysis similar to that in Section 2.1,

$$H_{mn}(0) = 2^{-1/4}(1-b)^{-1/8}e^{\pi i/4}\frac{\Gamma(1/2)}{\Gamma(-n)} \tag{2.23}$$

and

$$\left.\frac{\partial}{\partial\nu}H_{m\nu}(0)\right|_{\nu=\nu_n} = -2^{-9/4}e^{3\pi i/4}\pi^{1/2}c^{-1}n!\,(-1)^n(1-b)^{-5/8} . \tag{2.24}$$

For $\xi_0 \gg 1$,

$$x \simeq 2ic\left[\xi_0 - \sqrt{1-b} - \sqrt{b}\left(\frac{\pi}{2} - \sec^{-1}\sqrt{1/b}\right)\right],$$

$$V \simeq (2ic)^{-1/4} e^{x/2} (-x)^{n+\frac{3}{4}},$$

$$\theta(\xi_0) \simeq \frac{4n+3}{4}\left(\log\left[2(1-b)^{3/2}\right] - \log\left[\frac{x}{2ic}\right]\right),$$

and

$$H_{mn}(\xi_0) \simeq \frac{c^{n+\frac{1}{2}}}{\xi_0} (1-b)^{\frac{3}{2}n+\frac{9}{8}} 2^{2n+\frac{5}{4}} \exp\left[ic\xi_0 - ic\sqrt{1-b}\right.$$

$$\left. - ic\sqrt{b}\left(\frac{\pi}{2} - \sec^{-1}\sqrt{1/b}\right) - (n+2)\frac{\pi}{2}i\right] . \quad (2.25)$$

Comparison of eqs. (2.6) and (2.21) shows that despite the difference in the forms of the two differential equations the Region 2 results approach the Region 1 results as $m \rightarrow 0$.

A problem develops in the Region 2 results as $m \rightarrow c$. To use Mckelvey's (1955) theory to find a solution of eq. (2.19) asymptotic in c, $1-b$ must be bounded away from zero. This is indeed the case if $m = c^{1/\alpha}$, $\alpha > 1$, but not if $m = c + \beta c^{1/\alpha}$, $\alpha > 1$. In the latter case, the solution is asymptotic in $c^{1/\gamma}$, $\gamma > 1$, rather than c. γ is related to α and the magnitude of $\tilde{A}_{mn}$. For $m = c + \text{constant}$, the use of Mckelvey's (1955) theory cannot be justified.

2.4 Region 3 ($c \leq m < c^2$).

Since $b > 1$ for m in Region 3, eq. (2.2) takes the form

$$\frac{d^2 W_{mn}}{d\xi^2} + \left[c^2\xi^2\left(\frac{\xi^2 - (b-1)}{(1+\xi^2)^2}\right) + \frac{\tilde{A}_{mn}}{1+\xi^2}\right] W_{mn} = 0 . \quad (2.26)$$

There is a first order turning point at $\xi = \sqrt{b-1}$ as well as the second order one at $\xi = 0$. The problem of an isolated first order turning point was initially solved by Langer (1931). His method was generalized by Olver (1954). The case of two first order turning points in an interval was also solved by Langer

(1959), but the problem of first and second order turning points both occurring in an interval has not been solved exactly.

Fortunately, if the turning points are far enough apart, there is a simple method of solution. Asymptotic expansions valid about the two turning points are constructed so that they are identical over some finite region somewhere in the interval between them. Together, the expansions make up an asymptotic solution valid throughout the interval between the turning points.

To find the asymptotic solution valid in Region 3, we must solve two turning point problems. Since the behavior of the solution at infinity is known, the asymptotic expression valid about $\xi = \sqrt{b-1}$ and satisfying the radiation condition is found first. If $c\sqrt{b-1} \gg 1$, the asymptotic expansion valid about $\xi = 0$ can be matched to the one about $\xi = \sqrt{b-1}$ through the use of a constant. The eigenvalues $\tilde{A}_{mn}$ are specified by the condition $H_{mn}(0) = 0$.

Olver's (1954) theory is used to find the asymptotic expression about $\xi = \sqrt{b-1}$; the resulting expression is valid in the interval $(0,\infty)$. However, before the theory can be applied, eq. (2.26) must be transformed. To this end, let

$$S_{mn}(x) = \dot{\xi}^{-1/2} W_{mn}(\xi) \tag{2.27}$$

with $\dot{\xi} = d\xi/dx$. Then

$$\dot{\xi}^2 \left(\frac{\xi^2 + 1 - b}{(1+\xi^2)^2} \right) \xi^2 = x \tag{2.28}$$

and x is found to be

$$\frac{2}{3} x^{3/2} = \begin{cases} \sqrt{\xi^2 + 1 - b} - \sqrt{b}\, \sec^{-1}\sqrt{\dfrac{\xi^2+1}{b}} \ , & \xi \geq \sqrt{b-1} \\ i\left[\sqrt{b-1-\xi^2} - \sqrt{b}\, \log\left(\dfrac{\sqrt{b} + \sqrt{b-1-\xi^2}}{\sqrt{1+\xi^2}}\right)\right] \ , & 0 < \xi \leq \sqrt{b-1} \end{cases} \ . \tag{2.29}$$

In terms of x, eq. (2.26) takes the form

$$\frac{d^2 S_{mn}}{dx^2} + \left[c^2 x + c\,\frac{\sigma x(1+\xi^2)^2}{\xi^2(\xi^2+1-b)} + f_2(x)\right] S_{mn} = 0 \qquad (2.30)$$

where $f_2(x)$ is independent of c and $c\sigma = \widetilde{A}_{mn}$. From Olver (1954),

$$S_{mn}(x) = A_0(x)V + \frac{B_0(x)}{c}\,\frac{dV}{dx} \qquad (2.31)$$

where

$$V = \sqrt{x}\; H^{(1)}_{1/3}\left(\frac{2}{3}\,c x^{3/2}\right)$$

and

$$A_0(x) = \cos\left[\frac{h(x)}{2}\right]$$

$$B_0(x) = x^{-1/2}\sin\left[\frac{h(x)}{2}\right],$$

$$h(\xi) = \begin{cases} \dfrac{\sigma}{\sqrt{b-1}}\sec^{-1}\dfrac{\xi}{\sqrt{b-1}}, & \xi \geq \sqrt{b-1} \\[2ex] \dfrac{-i\sigma}{\sqrt{b-1}}\log\left[\dfrac{\sqrt{b-1}+\sqrt{b-1-\xi^2}}{\xi}\right], & 0 < \xi \leq \sqrt{b-1}. \end{cases} \qquad (2.32)$$

Using eqs. (2.27), (2.31) and (2.32), $W_{mn}(\xi)$ can be written as

$$W_{mn}(\xi) = \left(\frac{\xi^2+1}{\xi\sqrt{\xi^2+1-b}}\right)^{1/2} x^{1/4}\left(\cos\left[\frac{h(\xi)}{2}\right] + \frac{\sin\left[\frac{h(\xi)}{2}\right]}{c\sqrt{x}}\,\frac{d}{dx}\right)V\,. \qquad (2.33)$$

In the far field,

$$H_{mn}(\xi_0) = \frac{e^{ic\xi_0}}{\xi_0}\sqrt{\frac{3}{\pi c}}\,\exp\left[-\frac{\pi}{2}\,i\left(c\sqrt{b} + \frac{5}{6} - \frac{\sigma}{2\sqrt{b-1}}\right)\right]\,. \qquad (2.34)$$

For $0 < \xi < \sqrt{b-1}$ and $\left|c\,\frac{2}{3}x^{3/2}\right| \gg 1$,

$$W_{mn}(\xi) = \left(\frac{\xi^2+1}{\xi\sqrt{\xi^2+1-b}}\right)^{1/2} \left(\frac{3}{\pi c}\right)^{1/2} \left(\frac{\sqrt{b}+\sqrt{b-1-\xi^2}}{\sqrt{1+\xi^2}}\right)^{c\sqrt{b}}$$

$$\times \left(\frac{\sqrt{b-1}+\sqrt{b-1-\xi^2}}{\xi}\right)^{\frac{\sigma}{2\sqrt{b-1}}} \exp\left[-c\sqrt{b-1-\xi^2} - i\frac{5}{12}\pi\right] . \tag{2.35}$$

Since the turning point at $\xi = 0$ is of second order, Mckelvey's (1955) method is used to find the asymptotic expression valid for $0 \leq \xi < \sqrt{b-1}$. Then

$$W_{mn}(\xi) = A\left(\mu_0 V + \frac{\mu_1}{c}\frac{dV}{d\xi}\right) \tag{2.36}$$

where

$$V = (2c)^{-1/4}\left(\frac{\xi^2+1}{\xi\sqrt{b-1-\xi^2}}\right)^{1/2} W_0(x) ,$$

$$x = 2c\left[\sqrt{b-1-\xi^2} - \sqrt{b}\log\left(\frac{\sqrt{b}+\sqrt{b-1-\xi^2}}{\sqrt{1+\xi^2}}\right) + \sqrt{b}\log\left(\frac{\sqrt{b}+\sqrt{b-1}}{\exp\left[\sqrt{(b-1)/b}\right]}\right)\right] ,$$

$$\mu_0 = \cosh\left[\theta(\xi)\right] ,$$

$$\mu_1 = \frac{\xi^2+1}{\xi\sqrt{b-1-\xi^2}} \sinh\left[\theta(\xi)\right] , \tag{2.37}$$

$$\theta(\xi) = \frac{4n+3}{2}\log\left(\frac{\sqrt{b-1}+\sqrt{b-1-\xi^2}}{2\sqrt{b-1}}\right) - \frac{4n+3}{4}\log\left(\frac{\sqrt{b-1}}{2}\xi^2\right) + \frac{4n+3}{4}\log\left[\frac{x}{2c}\right] .$$

In order to satisfy the boundary condition at $\xi = 0$,

$$\tilde{A}_{mn} = c\sqrt{b-1}\,(4n+3) .$$

The constant A in eq. (2.36) is selected to make the asymptotic expression defined in eqs. (2.36) and (2.37) identical to eq. (2.35) for $0 < \xi < \sqrt{b-1}$ and

$|x| \gg 1$, namely

$$A = 2^{-(2n+\frac{5}{4})} c^{-n-1} (b-1)^{-(\frac{3}{2}n+\frac{9}{8})} e^{-i2\pi/3} (3/\pi)^{1/2} \left(\frac{\sqrt{b} + \sqrt{b-1}}{\exp\left[\sqrt{(b-1)/b}\right]} \right)^{c\sqrt{b}} . \tag{2.38}$$

From eqs. (2.36), (2.37) and (2.38),

$$H_{mn}(0) = 2^{-(2n+\frac{3}{2})} c^{-n-1} (b-1)^{-(\frac{3}{2}n+\frac{5}{4})} \times e^{-i2\pi/3} \frac{\Gamma(1/2)}{\Gamma(-n)} (3/\pi)^{1/2} \left(\frac{\sqrt{b} + \sqrt{b-1}}{\exp\left[\sqrt{(b-1)/b}\right]} \right)^{c\sqrt{b}} \tag{2.39}$$

and

$$\frac{\partial}{\partial \nu} H_{m\nu}(0) \Big|_{\nu=\nu_n} = 2^{-(2n+\frac{7}{2})} c^{-n-2} (b-1)^{-(\frac{3}{2}n+\frac{7}{4})} \times e^{-i2\pi/3} 3^{1/2} n! (-1)^n \left(\frac{\sqrt{b} + \sqrt{b-1}}{\exp\left[\sqrt{(b-1)/b}\right]} \right)^{c\sqrt{b}} . \tag{2.40}$$

As was the case for Region 2, the expressions are not truly asymptotic for $m = c+\text{constant}$. However, the validity of the expressions is assumed here and in Region 2 as $m \rightarrow c$.

Finally, the ratio of the radial Resolvent Green's function of Region 3 to that of Region 2 is considered:

$$\frac{\left| \frac{H_{mn}(\xi_0)}{H_{mn}(0)} \right|_{(3)}}{\left| \frac{H_{mn}(\xi_0)}{H_{mn}(0)} \right|_{(2)}} = \frac{(b_3-1)^{\frac{3}{2}n+\frac{5}{4}}}{(1-b_2)^{\frac{3}{2}n+\frac{5}{4}}} \left(\frac{\exp\left[\sqrt{(b_3-1)/b_3}\right]}{\sqrt{b_3} + \sqrt{b_3-1}} \right)^{c\sqrt{b_3}} \tag{2.41}$$

where $0 \leq b_2 < 1$ and $b_3 > 1$. Analysis shows that the above ratio goes to zero as $c \rightarrow \infty$. Thus the contribution of a mode $m > c$ can be made arbitrarily small compared to the contribution of a mode $0 \leq m < c$. Except for the case when the contributions of the $0 \leq m < c$ modes almost cancel, the contributions of the higher modes, $m > c$, need not now be considered.

2.5 Large n.

One of the assumptions made in Sections 2.2 - 2.4 was that

$$\tilde{A}_{mn} = \alpha c \ .$$

In Section 2.2, $\tilde{A}_{mn}$ was found to be

$$\tilde{A}_{mn} = -ic(4n+3)$$

and the assumption is correct as long as n is independent of c. However, for $n = \delta c$ the assumption is no longer valid and a different asymptotic expression must be used.

To simplify the equations involved, m will be taken to be unity. Then eq. (2.2) has the form

$$\frac{d^2 W_{1n}}{d\xi^2} + \left[\frac{c^2\xi^2 + A_{1n}}{1+\xi^2}\right] W_{1n} = 0 \ . \tag{2.42}$$

From eq. (2.14),

$$A_{1n} = -ic(4n+3) \ .$$

For $n = \delta c + \gamma$,

$$A_{1n} = -i\left[4\delta c^2 + (4\gamma+3)c\right] \ . \tag{2.43}$$

On collecting the c^2 terms, eq. (2.43) becomes

$$\frac{d^2 W_{1n}}{d\xi^2} + \left[c^2\left(\frac{\xi^2 - i4\delta}{1+\xi^2}\right) + c\left(\frac{4\gamma+3}{1+\xi^2}\right)\right] W_{1n} = 0 \ . \tag{2.44}$$

An asymptotic solution of eq. (2.44) can be constructed following Olver's (1954) theory using Parabolic Cylinder functions. Unfortunately for m in Regions 2 and 3, no asymptotic solution is available which satisfies the boundary condition at $\xi = 0$ and the radiation condition at infinity.

Chapter 3

THE ANGULAR FUNCTIONS $X^1_{mn}(\eta)$ AND $X^2_{mn}(\eta)$

3.1 Formation of the Asymptotic Solutions.

As the next step in determining the residue series for the surface field $\partial u/\partial z$, asymptotic expressions for $X^1_{mn}(\eta)$, $X^2_{mn}(\eta)$ and the angular Wronskian are found. $X^1_{mn}(\eta)$ and $X^2_{mn}(\eta)$ are solutions of

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dX_{mn}}{d\eta}\right] + \left[c^2\eta^2 - A_{mn} - \frac{m^2}{1-\eta^2}\right] X_{mn} = 0 \ . \tag{3.1}$$

As in the radial case, eq. (3.1) must be transformed to apply the asymptotic theories of Mckelvey (1955) and Olver (1954). To that end, $X_{mn}(\eta) = (1-\eta^2)^{-1/2} S_{mn}(\eta)$ where S_{mn} is a solution of

$$\frac{d^2 S_{mn}}{d\eta^2} + \left[\frac{c^2\eta^2 - A_{mn}}{1-\eta^2} - \frac{m^2-1}{(1-\eta^2)^2}\right] S_{mn} = 0 \ . \tag{3.2}$$

The form that eq. (3.2) will take in each of the m regions defined in Chapter 2 is similar to that of the radial equation. However, because of the presence of the poles at $\eta = \pm 1$ in eq. (3.2), the asymptotic solution is more complicated.

In Section 2.4, the presence of two turning points in the interval of consideration required the use of two asymptotic expressions, one valid about each turning point, to construct a solution valid over the entire interval. These solutions are matched in the interval between them through the use of a constant. This is also true for the angular solution; but because of the presence of the transition point at $\eta = +1$, an additional asymptotic expansion is necessary. The transition point at $\eta = -1$ need not be considered since the interval of consideration is $0 \leq \eta \leq 1$. Knowledge of the angular solution in this interval is sufficient to find the surface field in the shadow of the disk.

As in Section 2.4, convenience determines the order in which the turning and transition points are considered in forming the asymptotic solution. Since

$X^1_{mn}(1)$ must be finite, the asymptotic expression valid about the transition point at $\eta = 1$ is found first. If there is a turning point somewhere in the interval $0 < \eta < 1$, it is considered next, and the asymptotic expression valid about it is matched to the one about $\eta = 1$. The turning point at $\eta = 0$ is considered last and the expression valid about it must be matched to the one closest to it. Any asymptotic expression valid about a turning or transition point is valid only out to the next turning or transition point.

3.2 Region 1 ($0 \leq |m| \leq \sqrt{c}$).

Following Section 2.2, eq. (3.2) is written as

$$\frac{d^2 S^1_{mn}}{d\eta^2} + \left[c^2\left(\frac{\eta^2}{1-\eta^2}\right) - \left(\frac{A_{mn}}{1-\eta^2} + \frac{m^2 - 1}{(1-\eta^2)^2}\right)\right] S^1_{mn} = 0 \;. \tag{3.3}$$

In Region 1, only two asymptotic expressions are needed to construct the asymptotic solution since there is only one turning point (at $\eta = 0$) in addition to the transition point at $\eta = 1$.

For reasons discussed earlier, the transition point at $\eta = 1$ is considered first. The asymptotic expression valid in the interval $(0, 1]$ is found using Olver's (1956) method. To put eq. (3.3) in the appropriate form, let $R^1_{mn} = \dot{\eta}^{-1/2} S^1_{mn}$ with $\dot{\eta} = d\eta/dz$ and

$$\dot{\eta}^2 \frac{\eta^2}{1-\eta^2} = \frac{1}{z} \;. \tag{3.4}$$

The desired solution of eq. (3.4) is $z = \frac{1}{4}(1-\eta^2)$ and eq. (3.3) becomes

$$\frac{d^2 R^1_{mn}}{dz^2} + \left[\frac{c^2}{z} + \frac{1-m^2}{4z^2} - \frac{ca_{mn}}{z(1-4z)} + \frac{3}{(1-4z)^2}\right] R^1_{mn} = 0 \tag{3.5}$$

where $A_{mn} = -m^2 + 1 + a_{mn} c$ and $a_{mn} = -i(4n+3)$. Hence,

$$R^1_{mn} = V \cos\left[\frac{h(z)}{2}\right] + \frac{dV}{dz} \frac{\sin\left[\frac{h(z)}{2}\right]}{cg^{1/2}} \;, \tag{3.6}$$

with

$V = z^{1/2} J_m(2cz^{1/2})$ where $J_m(x)$ is the cylindrical Bessel function of order m;

$g = 1/z$

and

$$h(\eta) = i(4n+3)\log\left[\frac{1+\sqrt{1-\eta^2}}{\eta}\right] \quad .$$

The solution of eq. (3.3) for $2cz^{1/2} \gg m$ (and $\eta > 0$) is

$$S^1_{mn} = (-1/\eta)^{1/2} \frac{(1-\eta^2)^{1/4}}{(\pi c)^{1/2}} \cos\left[c\sqrt{1-\eta^2} - \frac{2m+1}{4}\pi + \frac{h(\eta)}{2}\right] \tag{3.7}$$

To complete the specification of the asymptotic solution in Region 1, an asymptotic expression valid about $\eta = 0$ is found using Mckelvey's (1955) method. The resulting expression is valid in the interval $[0, 1)$. To this end, eq. (3.3) is written as

$$\frac{d^2 S^1_{mn}}{d\eta^2} - \left[\lambda^2\left(\frac{\eta^2}{1-\eta^2}\right) + \sigma\lambda\left(\frac{1}{1-\eta^2} + \frac{B_m}{(1-\eta^2)^2}\right)\right] S^1_{mn} = 0 \tag{3.8}$$

where

$-\sigma\lambda = m^2 - 1 + ic(4n+3)$,

$\sigma\lambda B_m = m^2 - 1$,

$\lambda = ic$.

Then

$$S^1_{mn} = V\cosh\left[\theta(\eta)\right] + \frac{dV}{d\eta}\,\frac{\sinh\left[\theta(\eta)\right]}{ic\phi(\eta)} \tag{3.9}$$

and

$$
\begin{aligned}
V &= A_1 V_1 + A_2 V_0 , \\
V_\nu &= (2ic)^{-1/4}(1-\eta^2)^{1/4}\eta^{-1/2} W_{-\nu}(z), \\
z &= 2ic(1-\sqrt{1-\eta^2}) , \\
k &= \frac{4n+3}{4} , \\
\phi(\eta) &= \eta(1-\eta^2)^{-1/2} , \\
\theta(\eta) &= \frac{4n+3}{4} \log\left[\frac{1+\sqrt{1-\eta^2}}{2}\right] - i\,\frac{m^2-1}{c}\left[(1-\eta^2)^{-1/2} - 1\right] .
\end{aligned}
\tag{3.10}
$$

A_1 and A_2 are specified by the fact that eqs. (3.9) and (3.7) must be identical for $c(1-\sqrt{1-\eta^2}) \gg 0$, in which case

$$
\begin{aligned}
A_1 &= 2^{2n+\frac{3}{4}} c^{n+\frac{1}{2}} \pi^{-1/2} \exp\left[-ic + \frac{2m+1}{4}\pi i + i\frac{\pi}{2}\left(3n+\frac{7}{2}\right)\right] , \\
A_2 &= 2^{-2n-\frac{9}{4}} c^{-n-1} \pi^{-1/2} \exp\left[ic - \frac{2m+1}{4}\pi i - i\frac{\pi}{2}\left(n-\frac{1}{2}\right)\right] .
\end{aligned}
\tag{3.11}
$$

3.3 Region 2 ($\sqrt{c} \le m \le c$).

Writing eq. (3.2) in the form of eq. (2.19), one obtains

$$
\frac{d^2 S^1_{mn}}{d\eta^2} + \left[c^2\eta^2\left(\frac{1-b-\eta^2}{(1-\eta^2)^2}\right) - \frac{\tilde{A}_{mn}}{1-\eta^2}\right] S^1_{mn} = 0 \tag{3.12}
$$

where $b = (m^2-1)/c^2$ and $\tilde{A}_{mn} = A_{mn} + m^2 - 1$. In Region 2, there is a first order turning point at $\eta = \sqrt{1-b}$ in addition to the turning point at $\eta = 0$ and the transition point at $\eta = 1$. Thus, three asymptotic expressions are required to construct the asymptotic solution.

Using Olver's (1954) method, the asymptotic expression valid in $(\sqrt{1-b}, 1]$ is first obtained. For $R^1_{mn} = \dot{\eta}^{-1/2} S^1_{mn}$ with $\dot{\eta} = d\eta/dz$ and

$$\dot{\eta}^2 \eta^2 \left(\frac{\eta^2 + b - 1}{(1-\eta^2)^2} \right) = 1 \ ,$$

z is found to be

$$z = -\sqrt{b + \eta^2 - 1} + \sqrt{b} \log \left[\frac{\sqrt{b} + \sqrt{b - 1 + \eta^2}}{\sqrt{1 - \eta^2}} \right] \ .$$

Since $R^1_{mn}(\eta) = A_0(\eta)\, e^{-\mu z}$,

$$R^1_{mn}(\eta) = A_0(\eta) \left(\frac{\sqrt{1-\eta^2}}{\sqrt{b} + \sqrt{b - 1 + \eta^2}} \right)^{\mu\sqrt{b}} \exp\left[\mu \sqrt{b + \eta^2 - 1} \right] \qquad (3.13)$$

with $\mu^2 = c^2 \left(1 + \frac{1}{m^2 - 1} \right)$ and

$$A_0(\eta) = \exp\left[\frac{i(4n+3)}{2\sqrt{1 + \frac{1}{m^2-1}}} \sec^{-1} \frac{\eta}{\sqrt{1-b}} \right] \ .$$

To find the asymptotic expression valid in (0, 1), Olver's (1954) method is used. In this case, let $R^1_{mn} = \dot{\eta}^{-1/2} S^1_{mn}$ with $\dot{\eta} = d\eta/dz$ and

$$\dot{\eta}^2 \eta^2 \, \frac{(1 - b - \eta^2)}{(1-\eta^2)^2} = z \ .$$

It is easy to show that

$$\frac{2}{3} z^{3/2} = \begin{cases} -\sqrt{1 - b - \eta^2} + \sqrt{b} \sec^{-1} \sqrt{\frac{1-\eta^2}{b}} \ , & 0 < \eta \le \sqrt{1-b} \\ -i\sqrt{b - 1 + \eta^2} + i\sqrt{b} \log \left[\frac{\sqrt{b} + \sqrt{b - 1 + \eta^2}}{\sqrt{1-\eta^2}} \right] , & \sqrt{b-1} \le \eta < 1 \ . \end{cases}$$

Finally,

$$R^{1}_{mn}(\eta) = V\cos\left[\frac{h(\eta)}{2}\right] + \frac{dV}{dz}\,\frac{\sin\left[\frac{h(\eta)}{2}\right]}{cg^{1/2}} \tag{3.14}$$

where

$$V = Az^{1/2}H^{(1)}_{1/3}\left(\frac{2}{3}cz^{3/2}\right), \qquad g = z$$

and

$$h(\eta) = \begin{cases} -i(4n+3)\log\left[\dfrac{\sqrt{1-b}+\sqrt{1-b-\eta^2}}{\eta}\right], & 0 < \eta \le \sqrt{1-b} \\ (4n+3)\sec^{-1}\dfrac{\eta}{\sqrt{1-b}}, & \sqrt{1-b} \le \eta < 1\,. \end{cases}$$

Though the most general expression for V would also contain $H^{(2)}_{1/3}$, only the $H^{(1)}_{1/3}$ term is used since eq. (3.14) must have the same exponential form as eq. (3.13) for $\sqrt{1-b} \le \eta < 1$. A is selected so that eqs. (3.13) and (3.14) are identical for η such that $c\sqrt{b-1-\eta^2} \gg 1$, $\sqrt{b-1} < \eta < 1$:

$$A = \sqrt{\pi c/3}\;\exp\left(i\frac{5}{12}\pi\right).$$

The matching is slightly in error since μ appears in eq. (3.13) and c appears in eq. (3.14). However, for our purposes, the difference can be neglected.

To complete the construction of the asymptotic solution in Region 2, Mckelvey's (1955) theory is used. The resulting expression, valid in $[0, \sqrt{1-b})$, is

$$S^{1}_{mn}(\eta) = V\cosh\left[\theta(\eta)\right] + \frac{dV}{d\eta}\,\frac{\sinh\left[\theta(\eta)\right]}{ic\phi(\eta)} \tag{3.15}$$

where

$$V = A(2ic)^{-1/4}\phi^{-1/2}(\eta)W_{-1}(z) \ ,$$

$$z = 2ic\left[\sqrt{1-b} - \sqrt{1-b-\eta^2} + \sqrt{b}\left(\sec^{-1}\sqrt{\frac{1-\eta^2}{b}} - \sec^{-1}\sqrt{1/b}\right)\right] \ ,$$

$$\phi = \eta\,\frac{\sqrt{1-b-\eta^2}}{1-\eta^2} \ , \qquad (3.16)$$

$$k = \frac{4n+3}{4} \ ,$$

$$\theta(\eta) = \frac{4n+3}{2}\log\left[\frac{\sqrt{1-b}+\sqrt{1-b-\eta^2}}{2\sqrt{1-b}}\right] + \frac{4n+3}{4}\left(\log\left[\sqrt{1-b} - \sqrt{1-b-\eta^2} + \sqrt{b}\left(\sec^{-1}\sqrt{\frac{1-\eta^2}{b}} - \sec^{-1}\sqrt{1/b}\right)\right] - \log\left[\eta^2\,\frac{\sqrt{1-b}}{2}\right]\right).$$

In order that eqs. (3.14) and (3.15) be identical for η such that $c\sqrt{1-b-\eta^2} \gg 1$ and $z \gg 1$, $0 < \eta < \sqrt{1-b}$,

$$A = 2^{2n+\frac{7}{4}}\,c^{n+1}(1-b)^{\frac{3}{2}n+\frac{9}{8}}\exp\left[-ic\sqrt{1-b} + ic\sqrt{b}\,\sec^{-1}\sqrt{1/b} + i\frac{\pi}{2}\left(3n+\frac{5}{2}\right)\right].$$

As in the radial case, the matching can be accomplished only if the turning points at $\eta = 0$ and $\sqrt{1-b}$ are far enough apart, i.e., $c\sqrt{1-b} \gg 1$. Although this is certainly not the case as $m \to c$, the validity of the solutions as $m \to c$ will be assumed.

3.4 Region 3 ($c \le m < c^2$).

For $m > c$, b is greater than unity and eq. (3.2) is written as

$$\frac{d^2 S^1_{mn}}{d\eta^2} - \left[c^2\eta^2\left(\frac{\eta^2+(b-1)}{(1-\eta^2)^2}\right) + \frac{\tilde{A}_{mn}}{1-\eta^2}\right]S^1_{mn} = 0 \ . \qquad (3.17)$$

Only two asymptotic expressions are necessary to construct a solution valid throughout the interval $0 \le \eta \le 1$. The first order turning point present in eq.

(3.12) has moved out of the interval of consideration. The second order turning point at $\eta = 0$ and a transition point at $\eta = 1$ remain.

The asymptotic expression for η in the interval $(0, 1]$ is found using Olver's (1954) theory. For $R^1_{mn} = \dot{\eta}^{-1/2} S^1_{mn}$ with $\dot{\eta} = d\eta/dz$, z is found to be

$$z = -\sqrt{b-1+\eta^2} + \sqrt{b} \log\left[\frac{\sqrt{b} + \sqrt{b-1+\eta^2}}{\sqrt{1-\eta^2}}\right] .$$

The resulting expression for R^1_{mn} is

$$R^1_{mn} = A_0(\eta) \exp[-cz]$$

where

$$A_0(\eta) = -(4n+3) \log\left[\frac{\sqrt{b-1} + \sqrt{\eta^2+b-1}}{\eta}\right] .$$

Finally

$$S^1_{mn}(\eta) = \left(\frac{(1-\eta^2)}{\eta\sqrt{b-1+\eta^2}}\right)^{1/2} \left(\frac{\sqrt{1-\eta^2}}{\sqrt{b} + \sqrt{b-1+\eta^2}}\right)^{c\sqrt{b}} \left(\frac{\sqrt{b-1} + \sqrt{\eta^2+b-1}}{\eta}\right)^{\frac{4n+3}{2}}$$
$$\times \exp\left[c\sqrt{b-1+\eta^2}\right] . \qquad (3.18)$$

Here, no distinction is made between $\mu = c\sqrt{1+1/(m^2-1)}$ and c. For $m > c$, the distinction is unimportant in practice, though necessary for theoretical reasons (see Olver, 1954).

Using Mckelvey's (1955) method, the asymptotic expansion valid in $[0, 1)$ is

$$S^1_{mn}(\eta) = V \cos[\theta(\eta)] + \frac{dV}{d\eta} \frac{\sinh[\theta(\eta)]}{c\phi} . \qquad (3.19)$$

In this case,

$$V = A(2c)^{-1/4}\phi^{-1/2}(\eta)\, W_0(z) \; ,$$

$$z = 2c\left(-\sqrt{\eta^2+b-1} + \sqrt{b-1} + \sqrt{b}\,\log\left[\frac{\sqrt{b}+\sqrt{b-1+\eta^2}}{\sqrt{b}+\sqrt{b-1}}\,(1-\eta^2)^{-1/2}\right]\right) \; ,$$

$$\phi = \eta\,\frac{\sqrt{\eta^2+b-1}}{1-\eta^2} \; ,$$

$$k = -\frac{4n+3}{4} \; ,$$

$$\theta(\eta) = -\frac{4n+3}{2}\log\left[\frac{\sqrt{b-1}+\sqrt{b-1+\eta^2}}{2^{1/2}(b-1)^{3/4}\eta}\right] - \frac{4n+3}{4}\log\left[z/2c\right] \; .$$

A is selected so that eqs. (3.19) and (3.18) are identical for $z \gg 1$:

$$A = 2^{2n+\frac{7}{4}}\, c^{n+1}\, (b-1)^{\frac{3}{2}n+\frac{9}{8}}\, (\sqrt{b}+\sqrt{b-1})^{-c\sqrt{b}}\, \exp\left[c\sqrt{b-1}\right] \; .$$

3.5 Wronskians.

To complete the determination of the angular Resolvent Green's functions, the Wronskian of $X^1_{mn}(\eta)$ and $X^2_{mn}(\eta)$ must be found.

For any two functions V_1 and V_2 of y, their Wronskian is defined as

$$W(V_1, V_2, y) = \begin{vmatrix} V_1 & V_2 \\ V_1' & V_2' \end{vmatrix} = V_1 V_2' - V_1' V_2 \qquad (3.20)$$

If V_1 and V_2 are independent solutions of the Sturm-Liouville differential equation

$$\frac{d}{dy}\left[p(y)\,\frac{dV}{dy}\right] + \left[g(y) + \lambda r(y)\right] V = 0 \; ,$$

then $p(y)W(V_1, V_2, y)$ is a non-zero constant. Thus, the determination of pW at any point in the interval where V_1 and V_2 are defined serves to determine it throughout the interval.

As discussed in Chapter 1, the two independent solutions of eq. (3.1) that we are concerned with are

$$X^1_{mn}(\eta) = (1-\eta^2)^{-1/2} S^1_{mn}(\eta) \quad , \qquad -1 < \eta \leq 1$$

$$X^2_{mn}(\eta) = X^1_{mn}(-\eta) \quad , \qquad -1 \leq \eta < 1 \ .$$

From the above relations, it is easy to show that

$$(1-\eta^2)W(X^1_{mn}, X^2_{mn}, \eta) = W(S^1_{mn}, S^2_{mn}, \eta) \quad . \tag{3.21}$$

The Wronskian of S^1_{mn} and S^2_{mn} is evaluated at $\eta = 0$ using Mckelvey's (1955) results. Suppose

$$S^1_{mn}(\eta) = \left(\mu_0 + \frac{\mu_1}{\lambda}\frac{d}{d\eta}\right)\left(AV_1(\eta) + BV_2(\eta)\right) \ .$$

Hence

$$S^2_{mn}(\eta) = \left(\mu_0 + \frac{\mu_1}{\lambda}\frac{d}{d\eta}\right)\left(AV_1(-\eta) + BV_2(-\eta)\right) \ .$$

In particular,

$$V_1(\eta) = (2\lambda)^{-1/4} \phi^{-1/2}(\eta) M_{k,\,1/4}(z) \ ,$$

$$V_2(\eta) = (2\lambda)^{-1/4} \phi^{-1/2}(\eta) M_{k,\,-1/4}(z) \ ,$$

where λ, ϕ, μ_0, μ_1 and z are defined in the Appendix and $M_{k,\,\pm 1/4}(z)$ in Section 2.2. After some algebra,

$$W\left(S^1_{mn}(\eta), S^1_{mn}(-\eta), \eta\right) = -2ABW(V_1, V_2, \eta) \tag{3.22}$$

and since

$$W(V_1, V_2, \eta) = -(\lambda/2)^{1/2} \quad , \tag{3.23}$$

we have

$$(1-\eta^2)W(X^1_{mn}, X^2_{mn}, \eta) = AB(2\lambda)^{1/2} \quad . \tag{3.24}$$

The values of A and B in Region 1 can be obtained from Sections 2.2 and 3.2 and are

$$A = \frac{A_1 e^{3\pi i/4}\Gamma(-1/2)}{\Gamma(n+1)} + \frac{A_2\Gamma(-1/2)}{\Gamma(-n-\frac{1}{2})}$$

$$B = \frac{A_1 e^{\pi i/4}\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} + \frac{A_2\Gamma(1/2)}{\Gamma(-n)}$$

where A_1 and A_2 are defined in eq. (3.11). Thus the Wronskian is

$$(1-\eta^2)W(X^1_{mn}, X^2_{mn}, \eta) =$$

$$\frac{c^{6n+4}\, c^{2n+\frac{3}{2}}(-1)^n}{\pi^{1/2}(2n+1)!} \exp\left[-2ic - \frac{\pi}{4}i + \frac{2m+1}{2}\pi i\right] + \frac{i}{\pi} \quad . \tag{3.25}$$

Similar results hold for Regions 2 and 3.

Region 2

$$(1-\eta^2)W(X^1_{mn}, X^2_{mn}, \eta) =$$

$$\frac{2^{6n+6}\, c^{2n+\frac{5}{2}}(1-b)^{3n+\frac{9}{4}}}{\pi^{-1/2}(2n+1)!} \exp\Big[-2ic\sqrt{1-b} + 2ic\sqrt{b}\sec^{-1}\sqrt{1/b}$$

$$+ i\pi\left(3n+\frac{11}{4}\right)\Big] \quad . \tag{3.26}$$

Region 3

$$(1-\eta^2)W(X^1_{mn}, X^2_{mn}, \eta) =$$

$$-\frac{2^{6n+6}c^{2n+\frac{5}{2}}(b-1)^{3n+\frac{9}{4}}}{\pi^{-1/2}(2n+1)!}(\sqrt{b}+\sqrt{b-1})^{-2c\sqrt{b}}e^{2c\sqrt{b-1}} . \qquad (3.27)$$

This completes the determination of the angular Resolvent Green's function.

Chapter 4

SURFACE FIELDS FOR NON-GRAZING INCIDENCE

4.1 Summary of Procedure.

Using the results of Chapters 2 and 3, the contour integral representation for the surface field $\partial u/\partial z$ can be evaluated. For convenience, the pertinent equations are repeated here:

$$\frac{\partial u}{\partial z} = \frac{1}{a\eta}\frac{\partial u}{\partial \xi} = \frac{1}{a\eta}\sum_{m=0}^{\infty}\cos m\phi\,\frac{\partial V_m}{\partial \xi} \tag{4.1}$$

where

$$\frac{\partial V_m}{\partial \xi} = \frac{1}{\pi a \epsilon_m}\lim_{s\to 0+}\frac{1}{2\pi i}\int_{\Gamma_\nu}\frac{\partial}{\partial \xi}G_m(0,\xi_0,\nu)\,\tilde{G}_m(\eta,\eta_0,-\nu)\,d\nu \tag{4.2}$$

with

$$\frac{\partial}{\partial \xi}G_m(0,\xi_0,\nu) = -\frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} \tag{4.3}$$

and

$$\tilde{G}_m(\eta,\eta_0,-\nu) = \frac{-1}{(1-\eta^2)W(X^1_{m\nu},X^2_{m\nu},\eta)}\begin{cases} X^1_{m\nu}(\eta)X^2_{m\nu}(\eta_0), & \eta \geq \eta_0 \\ X^1_{m\nu}(\eta_0)X^2_{m\nu}(\eta), & \eta_0 \geq \eta \end{cases}. \tag{4.4}$$

Henceforth, the Wronskian $(1-\eta^2)W(X^1_{m\nu}, X^2_{m\nu}, \eta)$ will be written as $(1-\eta^2)W_{m\nu}$.

As the first step in finding $\partial u/\partial z$, the known expressions for $H_{m\nu}$ and $X_{m\nu}$ are inserted into eq. (4.2) and the integral evaluated. This process is complicated by several factors. As shown in Chapter 3, the $X_{m\nu}$ are made up of several asymptotic expressions each valid in a different η interval. The intervals do overlap and though the basic forms of the asymptotic expressions are different in each interval, in the region common to the intervals, the expressions are the same. Another problem is encountered in the evaluation of the integral itself. For certain values of η and η_0, the integral defined in eq. (4.2) can not be evaluated as a residue series but must be treated as a line integral, and for $\eta = \eta_0 = 0$, even the line integration technique fails.

The final step in determining $\partial u/\partial z$ consists of performing the m summation in eq. (4.1). Unfortunately, for most source and observer positions, the sum cannot be expressed as a simple analytic function. To keep the expression for the surface field in as simple and useful a form as possible, only the fields in certain regions of the disk are found. These regions are the center ($\eta = -1$), the edge ($\eta = 0$), and the annular region where $c\sqrt{1-\eta^2} \gg 1$ and $c\left(1-\sqrt{1-\eta^2}\right) \gg 1$, $\eta < 0$. Determination of the fields elsewhere requires no new mathematical techniques, but is not necessary to an understanding of the surface field behavior.

This chapter is concerned with the determination of the surface field in certain regions of the disk for various source positions. In all cases, the source is far from the disk, i.e., $\xi_0 \gg 1$.

4.2 <u>Point Source at $(\xi_0, 1, 0)$, Broadside Incidence.</u>

4.2.1 <u>The Field at $(0, -1, \phi)$, the Center of the Shadow Side of the Disk.</u>

From eqs. (1.20) and (4.1),

$$\frac{\partial u}{\partial \xi} = \sum_{m=0}^{\infty} {}' \frac{\cos m\phi}{\pi a \epsilon_m} \sum_{n=0}^{\infty} \frac{H_{mn}(\xi_0)}{\frac{\partial}{\partial \nu} H_{m\nu}(0)\Big|_{\nu=\nu_n}} \frac{X^1_{mn}(1) X^2_{mn}(-1)}{(1-\eta^2) W_{mn}} . \tag{4.5}$$

It is easy to show (see eqs. (3.4)-(3.6)) that

$$X^1_{0n}(1) = X^2_{0n}(-1) = (-1/2)^{1/2} \tag{4.6}$$

and

$$X^1_{mn}(1) = X^2_{mn}(-1) = 0 \quad , \quad m > 0 .$$

Well within the shadow of the disk, the residue series is expected to be highly convergent for c large. We therefore assume that only the first $A \ll c$ terms contribute significantly to the sum in eq. (4.5). Then from eqs. (2.17), (2.18) and (3.25)

$$\frac{\partial u}{\partial \xi} \sim \frac{e^{ic\xi_0}}{a\xi_0} \frac{e^{ic}}{2^{5/2}\pi} \sum_{n=0}^{A} \frac{e^{-n\pi i/2}(2n+1)!}{c^n 2^{4n} n!} \tag{4.7}$$

and as $c \longrightarrow \infty$,

$$\frac{\partial u}{\partial \xi} \sim \frac{e^{ic\xi_0}}{a\xi_0} \left(\frac{e^{ic}}{2^{5/2}\pi}\right) \tag{4.8}$$

which is identical to the result obtained by Goodrich et al. (1963).

4.2.2 <u>The Field at $(0,0,\phi)$, the Disk Edge.</u>

As before, we form a residue series for $\partial u/\partial \xi$ hoping that it will be highly convergent. Since $X^1_{mn}(1) = 0$ for $m > 0$, only the $m = 0$ term contributes. From eqs. (1.20) and (4.1),

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a} \sum_{n=0}^{\infty} \frac{H_{0n}(\xi_0)}{\left.\frac{\partial}{\partial \nu} H_{0\nu}(0)\right|_{\nu=\nu_n}} \frac{X^1_{0n}(1) X^2_{0n}(0)}{(1-\eta^2) W_{0n}} . \tag{4.9}$$

It follows from eqs. (3.9) and (3.10) that

$$X^2_{0n}(0) = \frac{2^{2n+\frac{1}{2}} c^{n+\frac{1}{2}}}{\Gamma(n+\frac{3}{2})} \exp\left[-ic + \frac{i\pi}{2}\left(3n+\frac{1}{2}\right)\right] . \tag{4.10}$$

Therefore,

$$\frac{\partial u}{\partial \xi} = \frac{e^{ic\xi_0}}{a\xi_0} \frac{e^{-\pi i/4} c^{1/2}}{\pi^{3/2} 2^{1/2}} \sum_{n=0}^{A} (-1)^n .$$

This summation does not converge for any finite but arbitrary $A \ll c$ and a simple analytical expression for the edge field can not be obtained using the above residue series.

To find the field at the edge, we resort to the integral expression for $\partial u/\partial \xi$:

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a} \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{0\nu}(\xi_0)}{H_{0\nu}(0)} \frac{X^1_{0\nu}(1) X^2_{0\nu}(0)}{(1-\eta^2) W_{0\nu}} \, d\nu \; . \tag{4.11}$$

Two functions f_ν and g_ν are now defined such that

$$X^1_{0\nu}(1) X^2_{0\nu}(0) = g_\nu + f_\nu (1-\eta^2) W_{0\nu} \; .$$

Since

$$(1-\eta^2) W_{0n} = \frac{2^{4n+3} c^{2n+\frac{3}{2}}}{\Gamma(n+1)\Gamma(n+\frac{3}{2})} \exp\left[-2ic + \frac{\pi}{4} i + n\pi i\right] + \frac{i}{\pi} \; , \tag{4.12}$$

where $\nu = -m^2 + 1 - ic(4n+3)$, it follows that

$$f_n = \frac{\Gamma(n+1)}{2^{2n+3} c^{n+1}} \exp\left[ic + \frac{\pi}{2} i(n+1)\right] \tag{4.13}$$

and

$$g_n = \frac{\Gamma(n+1)\pi^{-1}}{2^{2n+3} c^{n+1}} \exp\left[ic + \frac{\pi}{2} ni\right] \; , \tag{4.14}$$

and hence

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a} \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{0\nu}(\xi_0)}{H_{0\nu}(0)} \frac{g_\nu}{(1-\eta^2) W_{0\nu}} \, d\nu$$

$$+ \frac{1}{2\pi a} \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{0\nu}(\xi_0)}{H_{0\nu}(0)} f_\nu \, d\nu \; . \tag{4.15}$$

The first integral, denoted by I_1, can be evaluated by closing the contour and summing over the residues of $H_{0\nu}(0)$. The integral becomes

$$I_1 = \sum_{n=0}^{\infty} \frac{H_{0n}(\xi_0)}{\frac{\partial}{\partial\nu} H_{0\nu}(0)\Big|_{\nu=\nu_n}} \qquad \frac{g_n}{(1-\eta^2)W_{0n}}$$

which is equal to

$$I_1 = \frac{e^{ic\xi_0}}{\xi_0}\left(\frac{-e^{2ic}}{\pi}\right)\sum_{n=0}^{A} \frac{(2n+1)!}{c^{2n+1}2^{6n+\frac{7}{2}}} \quad . \tag{4.16}$$

For $c \gg 1$,

$$I_1 \sim \frac{e^{ic\xi_0}}{\xi_0}\left(\frac{-e^{2ic}}{\pi c\, 2^{7/2}}\right) \quad . \tag{4.17}$$

The second integral, I_2, is evaluated by carrying out the line integration over Γ_ν.

$$I_2 = \lim_{s\to 0+} \frac{1}{2\pi i}\int_{\Gamma_\nu} \frac{H_{0\nu}(\xi_0)}{H_{0\nu}(0)}\, f_\nu d\nu \ .$$

Making the change of variable from ν to n,

$$I_2 = \lim_{s\to 0+}\left(-\frac{2c}{\pi}\right)\int_{\Gamma_n} \frac{H_{0n}(\xi_0)}{H_{0n}(0)}\, f_n\, dn$$

$$= \frac{e^{ic\xi_0}}{\xi_0}\ \frac{e^{\pi i/4}c^{1/2}}{2^{1/2}\pi^{3/2}}\int_{\Gamma_n} \Gamma(n+1)\Gamma(-n)\, dn$$

where Γ_n is a straight line path a small distance $\epsilon \ll 1$ to the left of the imaginary axis in the n plane. Since

$$\int_{-i\infty-\epsilon}^{i\infty-\epsilon} \Gamma(n+1)\Gamma(-n)\,dn = \pi i \ ,$$

we now have

$$I_2 = \frac{e^{ic\xi_0}}{\xi_0}\,\frac{e^{-i\pi/4}c^{1/2}}{2^{1/2}\pi^{1/2}} \tag{4.18}$$

and hence

$$\frac{\partial u}{\partial \xi} = \frac{e^{ic\xi_0}}{a\xi_0}\left(\frac{e^{-i\pi/4}c^{1/2}}{2^{3/2}\pi^{3/2}} - \frac{e^{2ic}}{c\pi^2 2^{9/2}}\right) \quad . \tag{4.19}$$

4.2.3 The Field in the Annulus, $-1 < \eta < 0$.

η must also satisfy the conditions $c\sqrt{1-\eta^2} \gg 1$ and $c\left(1-\sqrt{1-\eta^2}\right) \gg 1$.

From eqs. (4.1) and (4.2),

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a}\sum_{n=0}^{\infty} \frac{H_{0n}(\xi_0)}{\left.\frac{\partial}{\partial\nu} H_{0\nu}(0)\right|_{\nu=\nu_n}}\;\frac{X^1_{0n}(1)X^2_{0n}(\eta)}{(1-\eta^2)W_{0n}} \tag{4.20}$$

For the interval considered (see eq. (3.7)),

$$X^2_{0n}(\eta) = \frac{i(1-\eta^2)^{-1/4}\,|\eta|^{-1/2}}{\pi^{1/2}c^{1/2}}\cos\left[c\sqrt{1-\eta^2} - \frac{\pi}{4} + \frac{h(\eta)}{2}\right] \tag{4.21}$$

with

$$h(\eta) = i(4n+3)\log\left[\frac{1+\sqrt{1-\eta^2}}{|\eta|}\right] \quad .$$

Then

$$\frac{\partial u}{\partial \xi} = \frac{e^{ic\xi_0}}{a\xi_0} \frac{e^{ic} e^{\pi i/4}}{c^{1/2} \pi^{3/2} 2^3} (1-\eta^2)^{-1/4} \left[\exp\left[-ic\sqrt{1-\eta^2}\right] \left(1+\sqrt{1-\eta^2}\right)^{3/2} |\eta|^{-3} - i \exp\left[ic\sqrt{1-\eta^2}\right] \left(1+\sqrt{1-\eta^2}\right)^{-3/2} \right] . \quad (4.22)$$

This result is identical to that of Goodrich et al. (1963).

4.3 Point Source at $(\xi_0, \eta_0, 0)$, $0 < \eta_0 < 1$, Oblique Incidence.

To simplify the expressions for the surface field, η_0 will be further constrained to satisfy $c\sqrt{1-\eta_0^2} \gg 1$ and $c\left(1-\sqrt{1-\eta_0^2}\right) \gg 1$.

4.3.1 The Field at $(0, -1, \phi)$.

Only the $m = 0$ term contributes, giving

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a} \sum_{n=0}^{\infty} \frac{H_{0n}(\xi_0)}{\left.\frac{\partial}{\partial \nu} H_{0\nu}(0)\right|_{\nu=\nu_n}} \frac{X^1_{0n}(\eta_0) X^2_{0n}(-1)}{(1-\eta^2) W_{0n}} . \quad (4.23)$$

Equation (4.23) is the same as eq. (4.20) since $X^2_{0n}(-1) = X^1_{0n}(1)$. Thus $\partial u/\partial\xi$ is given by eq. (4.22) with η replaced by η_0.

4.3.2 The Field at $(0, 0, \phi)$.

Because of the complexity of the expressions for the surface field, the result will be left in the form of a summation over m.

From eqs. (4.1) - (4.4),

$$\frac{\partial u}{\partial \xi} = \sum_{m=0}^{\infty} \frac{\cos m\phi}{\pi a \epsilon_m} \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} \frac{X^1_{m\nu}(\eta_0) X^2_{m\nu}(0)}{(1-\eta^2) W_{m\nu}} d\nu .$$

Since the asymptotic expressions for $H_{m\nu}(\xi)$ and $X_{m\nu}(\eta)$ are different in two overlapping regions of m, the m summation is divided into two parts, a sum over m in the interval $[0, \sqrt{c}]$ and a sum over m in $(\sqrt{c}, c]$. For reasons discussed in Section 2.4, the contribution of modes with $m > c$ can be neglected.

The expression for the surface field is broken up into two terms:

$$\frac{\partial u}{\partial \xi} = \frac{\partial u^{I}}{\partial \xi} + \frac{\partial u^{II}}{\partial \xi} \tag{4.25}$$

where

$$\frac{\partial u^{I}}{\partial \xi} = \sum_{m=0}^{[\sqrt{c}]} \frac{\cos m\phi}{\pi a \epsilon_m} \frac{\partial V_m}{\partial \xi}$$

and

$$\frac{\partial u^{II}}{\partial \xi} = \sum_{m=[\sqrt{c}+1]}^{[c]} \frac{\cos m\phi}{\pi a} \frac{\partial V_m}{\partial \xi} .$$

$[x]$ is the greatest integer n such that $n \leq x$.

I) Region 1 $(0 \leq m \leq \sqrt{c}\,)$.

As was done in Section 4.2.2, the integral in eq. (4.24) is expanded into two terms. If $X^2_{m\nu}(0)$ is written as

$$X^2_{m\nu}(0) = g_{m\nu} + f_{m\nu}(1-\eta^2)W_{m\nu} ,$$

the expression for $\partial u^{I}/\partial \xi$ becomes

$$\frac{\partial u^{I}}{\partial \xi} = \sum_{m=0}^{[\sqrt{c}]} \frac{\cos m\phi}{\pi a \epsilon_m} (I_{m1} + I_{m2})$$

where

$$I_{m1} = \sum_{n=0}^{\infty} \frac{H_{mn}(\xi_0)}{\frac{\partial}{\partial \nu} H_{m\nu}(0)\Big|_{\nu=\nu_n}} \frac{X^1_{mn}(\eta_0) g_{mn}}{(1-\eta^2)W_{mn}} ,$$

$$I_{m2} = \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} f_{m\nu} X^1_{m\nu}(\eta_0)\, d\nu .$$

After some algebra,

$$I_{m1} = i\,\frac{e^{ic\xi_0}}{\xi_0}\,\frac{X^1_{m0}(\eta_0)}{2^3\pi c}\,\exp\left[2ic - \frac{i}{2c}(m^2-1) - \frac{3\pi}{2}mi\right] \tag{4.26}$$

with

$$X^1_{m0} = i(1-\eta_0^2)^{-1/4}\,\eta_0^{-1/2}(\pi c)^{-1/2}\cos\left[c\sqrt{1-\eta_0^2} - \frac{2m+1}{4}\pi + \frac{h(\eta_0)}{2}\right] \tag{4.27}$$

and $h(\eta_0)$ defined in eq. (4.21). As before, only the first residue contributes significantly.

For I_{m2}, one gets

$$I_{m2} = \frac{e^{ic\xi_0}}{\xi_0}\,\frac{e^{-\pi i/4}c^{1/2}}{\pi^{3/2}}\,\exp\left[-\frac{i}{2c}(m^2-1) - \frac{\pi}{2}mi\right]$$

$$\times \int_{\Gamma_n} \Gamma(-n)\Gamma(n+1)X^1_{mn}(\eta_0)\,dn$$

and

$$I_{m2} = \frac{e^{ic\xi_0}}{\xi_0}\left(\frac{-(1-\eta_0^2)^{-1/4}}{\pi\eta_0^{1/2}}\right)\exp\left[-\frac{i}{2c}(m^2-1) - \frac{2m+1}{4}\pi i\right]$$

$$\times\left(\frac{\alpha^{3/2}}{1+\alpha^2}\exp\left[ic\sqrt{1-\eta_0^2} - i\,\frac{2m+1}{4}\pi\right] + \frac{\alpha^{-3/2}}{1+\alpha^{-2}}\exp\left[-ic\sqrt{1-\eta_0^2} + i\,\frac{2m+1}{4}\pi\right]\right) \tag{4.28}$$

where

$$\alpha = \frac{\eta_0}{1+\sqrt{1-\eta_0^2}} \quad .$$

II) Region 2 ($\sqrt{c} < m \leq c$)

In this case, because of the form of the Wronskian, the integrand is treated as a single function.

$$\frac{\partial u^{II}}{\partial \xi} = \sum_{m=[\sqrt{c}+1]}^{[c]} \frac{\cos m\phi}{\pi a} \lim_{s\to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} \frac{X^1_{m\nu}(\eta_0) X^2_{m\nu}(0)}{(1-\eta^2)W_{m\nu}} d\nu \ . \tag{4.29}$$

Since

$$X^2_{mn}(0) = \frac{2^{2n+\frac{3}{2}} c^{n+1} (1-b)^{\frac{3}{2}n+1} \pi^{1/2}}{\Gamma\left(n+\frac{3}{2}\right)} \exp\left[-ic\sqrt{1-b} + ic\sqrt{b}\sec^{-1}\sqrt{1/b} + i\frac{3\pi}{2}(n+1)\right] ,$$

it is easy to show that

$$\lim_{s\to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} \frac{X^1_{m\nu}(\eta_0) X^2_{m\nu}(0)}{(1-\eta^2)W_{m\nu}} d\nu$$

$$= i\frac{e^{ic\xi_0}}{\xi_0} \frac{(1-b)^{1/2}}{2} e^{-ic\sqrt{b}\frac{\pi}{2}} \int_{\Gamma_n} \Gamma(n+1)\Gamma(-n) X^1_{mn}(\eta_0) dn \tag{4.30}$$

where $\nu = -ic(4n+3)\sqrt{1-b}$. For m in $[\sqrt{c}\ , c]$ (see eq. (3.14)),

$$X^1_{mn}(\eta_0) = (1-\eta_0^2)^{-1/2} \eta_0^{1/2} \left(V\cos\left[\frac{h(\eta_0)}{2}\right] + \frac{dV}{dz} \frac{\sin\left[\frac{h(\eta_0)}{2}\right]}{cg^{1/2}} \right) . \tag{4.31}$$

Only $h(\eta_0)$ depends on n, and the n integral is evaluated for three different regions.

i) $\sqrt{1-b} = \eta_0$

$$\int_{\Gamma_n} \Gamma(n+1)\Gamma(-n) X^1_{mn}(\eta_0) dn = -\pi i (1-\eta_0^2)^{-1/2} \eta_0^{1/2} V \ . \tag{4.32}$$

ii) $\sqrt{1-b} \geq \eta_0$

$$\int_{\Gamma_n} \Gamma(n+1)\Gamma(-n) X^1_{mn}(\eta_0) dn =$$

$$= -(1-\eta_0^2)^{-1/2}\, \eta_0^{1/2} \left[\left(\frac{\pi i \alpha^{3/2}}{1+\alpha^2} + \frac{\pi i \alpha^{-3/2}}{1+\alpha^{-2}} \right) V \right.$$

$$\left. + \left(\frac{\pi \alpha^{3/2}}{1+\alpha^2} - \frac{\pi \alpha^{-3/2}}{1+\alpha^{-2}} \right) \frac{dV/dz}{cg^{1/2}} \right] \qquad (4.33)$$

where

$$\alpha = \frac{\sqrt{1-b} + \sqrt{1-b-\eta_0^2}}{\eta_0} \quad .$$

iii) $\sqrt{1-b} \leq \eta_0$

$$\int_{\Gamma_n} \Gamma(n+1)\Gamma(-n) X^1_{mn}(\eta_0) dn$$

$$= -(1-\eta_0^2)^{-1/2}\, \eta_0^{1/2} \left[\left(\frac{\pi i \beta^{3/2}}{1+\beta^2} + \frac{\pi i \beta^{-3/2}}{1+\beta^{-2}} \right) V \right.$$

$$\left. + \left(\frac{\pi \beta^{3/2}}{1+\beta^2} - \frac{\pi \beta^{-3/2}}{1+\beta^{-2}} \right) \frac{dV/dz}{cg^{1/2}} \right] \qquad (4.34)$$

where

$$\beta = \exp\left[i \sec^{-1}\left(\frac{\eta_0}{\sqrt{1-b}} \right) \right] \quad .$$

4.3.3 Field in the Annulus, $-1 < \eta < 0$.

The residue series for the surface field is highly convergent in n. However, interest in the field here is not sufficient to warrant the complicated development that would be required because of the m dependence. Thus the surface field in $-1 < \eta < 0$ is not found.

Chapter 5

SURFACE FIELD FOR GRAZING INCIDENCE

5.1 Discussion of the Analysis.

A point source is located in the plane of the disk at $(\xi_0, 0, 0)$ where $\xi_0 \gg 1$. From Section 4.1, the surface field of the disk is then

$$\frac{\partial u}{\partial z} = \frac{1}{a\eta} \frac{\partial u}{\partial \xi} \tag{5.1}$$

where

$$\frac{\partial u}{\partial \xi} = \sum_{m=0}^{\infty} \frac{\cos m\phi}{\pi a \epsilon_m} \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} \frac{X^1_{m\nu}(0) X^2_{m\nu}(\eta)}{(1-\eta^2) W_{m\nu}} d\nu . \tag{5.2}$$

For edge-on incidence, $\partial u / \partial \xi$ is the same on both sides of the disk.

As in Chapter 4, the disk surface is divided up into 3 regions, the center $(\eta = -1)$, the edge $(\eta = 0)$ and an annulus such that $c\left(1 - \sqrt{1-\eta^2}\right) \gg 1$ and $c\sqrt{1-\eta^2} \gg 1$. When $\eta = -1$, $\partial u / \partial \xi$ is the same as that found in Section 4.2.2 and when η is in the annulus, $\partial u / \partial \xi$ is the same as that found in Section 4.3.2 with η_0 replaced by η. Hence, further computation is required only for the field at the disk edge.

At the edge, $\eta = 0$ and

$$\frac{\partial u}{\partial \xi} = \sum_{m=0}^{\infty} \frac{\cos m\phi}{\pi a \epsilon_m} \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_\nu} \frac{H_{m\nu}(\xi_0)}{H_{m\nu}(0)} \frac{X^1_{m\nu}(0) X^2_{m\nu}(0)}{(1-\eta^2) W_{m\nu}} d\nu . \tag{5.3}$$

Using the results of Sections 2.3 and 3.3, the ν integral is found to be

$$I_m = A_m \lim_{s \to 0+} \int_{\Gamma_n} \frac{\Gamma(n+1)\Gamma(-n)}{\Gamma(n+\frac{3}{2})} \left[(1-b)^{3/2} 2^2 c e^{-\pi i/2}\right]^n dn$$

where A_m is a constant independent of n and Γ_n is described in Chapter 4.

The integrand is such that the path of integration can be closed by a semicircle to the right. The integral can then be expressed as a sum of residues of the poles of $\Gamma(-n)$ and

$$I_m = A_m \lim_{s \to 0+} 4\pi i\, M\left(1,\ 3/2,\ ic4(1-b)^{3/2}\right)$$

where M(a, b, z) is the Kummer function (NBS, 1964). However, the fact that the asymptotic expressions for H_{mn} and X_{mn} are valid only for $n \ll c$ must be considered. In Chapter 4, we found that the first $n \ll c$ terms alone contribute significantly to the residue series and the breakdown in the asympotic expressions for large n caused no problem. Unfortunately, the contribution to the residue series for I_m of terms of order $n \sim c$ cannot now be neglected and the method which was used in Chapter 4 fails here.

To get a highly convergent series for the surface field at high frequencies, the radial and axial or angular eigenfunctions are used. For the source and observer positions considered so far, expansion over the radial and axial eigenvalues has resulted in useful residue series. To get similar convergence for the edge fields for edge-on incidence requires an expansion over the radial and angular eigenvalues.

Because of the unknown nature of the asymptotic expressions for H_{mn} and X_{mn} for large complex m and n, certain assumptions are unavoidable in the analysis that follows.

5.2 The Watson Transformation and the Angular Eigenvalues.

Equation (5.3) is in the form of a summation over the axial and radial eigenvalues and must be transformed into a sum over the radial and angular eigenvalues.

From eq. (5.3),

$$\frac{\partial u}{\partial \xi} = \sum_{m=0}^{\infty} \frac{\cos m\phi}{\pi a \epsilon_m} I_m \tag{5.4}$$

where

$$I_m = \lim_{s \to 0+} \frac{1}{2\pi i} \int_{\Gamma_n} \frac{H_{mn}(\xi_0)}{H_{mn}(0)} \frac{X^1_{mn}(0) X^2_{mn}(0)}{(1-\eta^2) W_{mn}} \left(\frac{d\nu}{dn}\right) dn \quad . \tag{5.5}$$

A transformation equivalent to that of Watson (1914) is performed on the m variable in eq. (5.4). For this to be valid, I_m must be exponentially small for $m = a + i\epsilon$ as $a \to \pm\infty$ with $0 < |\epsilon| \ll 1$. An assumption to this effect is made.

Equation (5.4) can be written as

$$\frac{\partial u}{\partial \xi} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{\cos m\phi}{\pi a} I_m$$

since $I_m = I_{-m}$ for m an integer. Then

$$\frac{\partial u}{\partial \xi} = \frac{1}{4\pi a i} \int_{\Gamma_\beta} \frac{e^{-\pi i \beta}}{\sin \pi\beta} \cos \beta\phi \, I_\beta \, d\beta \tag{5.6}$$

where Γ_β is shown in Figure 3.

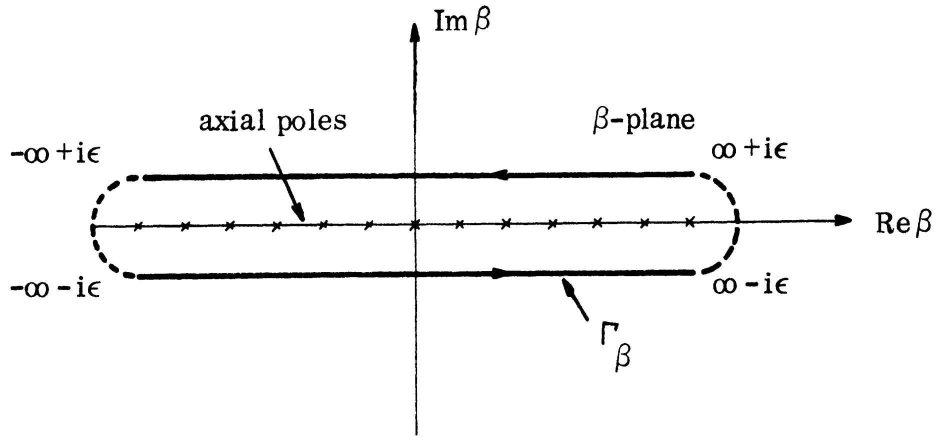

FIG. 3: The Γ_β path of integration.

Using the properties of the trigonometric functions, eq. (5.6) can be put in the form

$$\frac{\partial u}{\partial \xi} = \frac{1}{4\pi a i} \int_{\infty + i\epsilon}^{-\infty + i\epsilon} \frac{\cos \beta\phi}{\sin \beta\pi} \left[e^{-i\pi\beta} I_{\beta} + e^{i\pi\beta} I_{-\beta} \right] d\beta \,. \tag{5.7}$$

The integral is to be expanded over the radial and angular eigenvalues which are specified by the zeros of $H^2_{\beta n}(0)$ and $(1-\eta^2)W(X^1_{\beta n}, X^2_{\beta n}, \eta)$, respectively.

The Wronskian is zero if and only if X^1_{mn} and X^2_{mn} are not independent. If m is selected so that $X^1_{mn}(\eta)$ is regular at both $\eta = \pm 1$, $X^1_{mn}(\eta) = X^2_{mn}(\eta)$ and $(1-\eta^2)W(X^1_{mn}, X^2_{mn}, \eta) = 0$. Thus the solution of eq. (3.1) regular at $\eta = \pm 1$ must be found.

To this end, let $R_{mn}(z) = (1-4z)^{1/4} z^{1/2} X_{mn}(\eta)$ with $z = (1-\eta^2)/4$. Equation (3.1) then becomes

$$\frac{d^2 R_{mn}}{dz^2} + \left[\frac{c^2}{z} + \frac{1-m^2}{4z^2} - \frac{A_{mn} + m^2 - 1}{z(1-4z)} + \frac{3}{(1-4z)^2} \right] R_{mn} = 0 \tag{5.8}$$

where $R_{mn}(z)$ must be zero at $z = 0$ $(\eta = \pm 1)$. A solution of eq. (5.8) valid near $z = 0$ is $R_{mn} = z^{1/2} J_m(2\gamma z^{1/2})$ where $\gamma = \sqrt{c^2 - \tilde{A}_{mn}}$ and $J_m(x)$ is the cylindrical Bessel function (Watson, 1922). This solution is regular at $\eta = \pm 1$ and using it as a guide, a function $f_{mn}(z)$ is defined such that $R_{mn}(z) = z^{1/2} J_m(2\gamma z^{1/2}) f_{mn}$ is a solution of eq. (5.8) for $0 \le z \le 1/4$. If $z^{1/2} J_m(2\gamma z^{1/2}) \neq 0$, f_{mn} must satisfy

$$f''_{mn} + 2 \frac{\left[z^{1/2} J_m(2\gamma z^{1/2}) \right]'}{\left[z^{1/2} J_m(2\gamma z^{1/2}) \right]} f'_{mn} + \left[-\frac{4\tilde{A}_{mn}}{1-4z} + \frac{3}{(1-4z)^2} \right] f_{mn} = 0 \,. \tag{5.9}$$

No elementary solution of eq. (5.9) is possible for all z, $0 \le z \le 1/4$. However, near $z = 1/4$ $(\eta = 0)$, eq. (5.9) can be approximated by

$$f''_{mn} + 4\left[1 + \gamma \frac{J'_m(\gamma)}{J_m(\gamma)}\right] f'_{mn} + \left[-\frac{4\tilde{A}_{mn}}{1-4z} + \frac{3}{(1-4z)^2}\right] f_{mn} = 0 \ . \tag{5.10}$$

Making the substitutions $x = 1-4z$ and $y = \alpha x$ where $\alpha = 1+\gamma\ J'_m(\gamma)/J_m(\gamma)$, eq. (5.10) transforms to

$$f''_{mn} - f'_{mn} + \left[-\frac{\tilde{A}_{mn}}{4\alpha y} + \frac{3/16}{y^2}\right] f_{mn} = 0 \ . \tag{5.11}$$

The solutions of eq. (5.11) are well known and are

$$f_{mn} = e^{y/2}\left[AM_{k,-1/4}(y) + BM_{k,1/4}(y)\right] \tag{5.12}$$

where $M_{k,\pm 1/4}(y)$ are the Whittaker functions defined in Section 2.2, $k = -\tilde{A}_{mn}/4\alpha$, $y = \alpha\eta^2$, and A and B are arbitrary constants.

Assuming for the moment that the angular eigenvalues are such that $\sqrt{c} \le |m| < c$, the asymptotic solution of eq. (3.1) near $\eta = 0$ is (see eq. (3.16))

$$X^1_{mn}(\eta) = C\eta^{-1/2} W_{-1}(z) \tag{5.13}$$

with $z = ic\sqrt{1-b}\,\eta^2$ for $\eta \ll |1-b|$. The form of the asymptotic expression is independent of m and for that reason cannot be used to find the angular eigenvalues. However, from eqs. (5.8), (5.9) and (5.12), $X^1_{mn}(\eta)$ can also be approximated by

$$X^1_{mn}(\eta) = \eta^{-1/2}\left[AM_{k,-1/4}(y) + BM_{k,1/4}(y)\right] \tag{5.14}$$

near $\eta = 0$. For eqs. (5.13) and (5.14) to be the same, the constants A and B must be

$$A = C\left[\frac{e^{\pi i/4}\Gamma(1/2)}{\Gamma(\frac{3}{4}+k)}\right] \ , \qquad B = C\left[\frac{e^{3\pi i/4}\Gamma(-1/2)}{\Gamma(\frac{1}{4}+k)}\right]$$

with

$$ic\sqrt{1-b} = 1+\gamma \frac{J'_m(\gamma)}{J_m(\gamma)} \quad . \tag{5.15}$$

Equation (5. 15) is the defining equation for the angular eigenvalues and since $\gamma \simeq c$ for $c \gg 1$, eq. (5. 15) can be approximated by

$$1+c \frac{J'_m(c)}{J_m(c)} = ic\sqrt{1-b} \quad . \tag{5.16}$$

For convenience, another assumption is now made: $m = c - \alpha c^{1/3}$. Then, eq. (5. 16) takes the form

$$1+c \frac{J'_m(m+\alpha m^{1/3})}{J_m(m+\alpha m^{1/3})} = i\sqrt{2\alpha}\, c^{2/3} \quad . \tag{5.17}$$

For $|\nu| \gg 1$ and $|\arg \nu| < \pi/2$,

$$J_\nu(\nu+z\nu^{1/3}) \simeq \frac{2^{1/3}}{\nu^{1/3}} \mathrm{Ai}(-2^{1/3}z)$$

$$J'_\nu(\nu+z\nu^{1/3}) \simeq -\frac{2^{2/3}}{\nu^{2/3}} \mathrm{Ai}'(-2^{1/3}z)$$

where Ai is the Airy function (NBS, 1964). Using the above approximations for the Bessel functions, eq. (5. 17) becomes

$$\frac{\mathrm{Ai}'(-2^{1/3}\alpha)}{\mathrm{Ai}(-2^{1/3}\alpha)} = -i\sqrt{2^{1/3}\alpha} \quad . \tag{5.18}$$

The roots of eq. (5. 18) were first found graphically using Fig. 4 (see Logan, 1965). The approximate locations of the first two are indicated by stars. In general, there is a root associated with each zero of $\mathrm{Ai}(-2^{1/3}\alpha)$. Using the computer, the first root was found to be $\alpha_0 = 2.21-0.87i$. The second root is located at $\alpha_1 = 3.53-0.82i$ and the third at $\alpha_2 = 4.64-0.77i$.

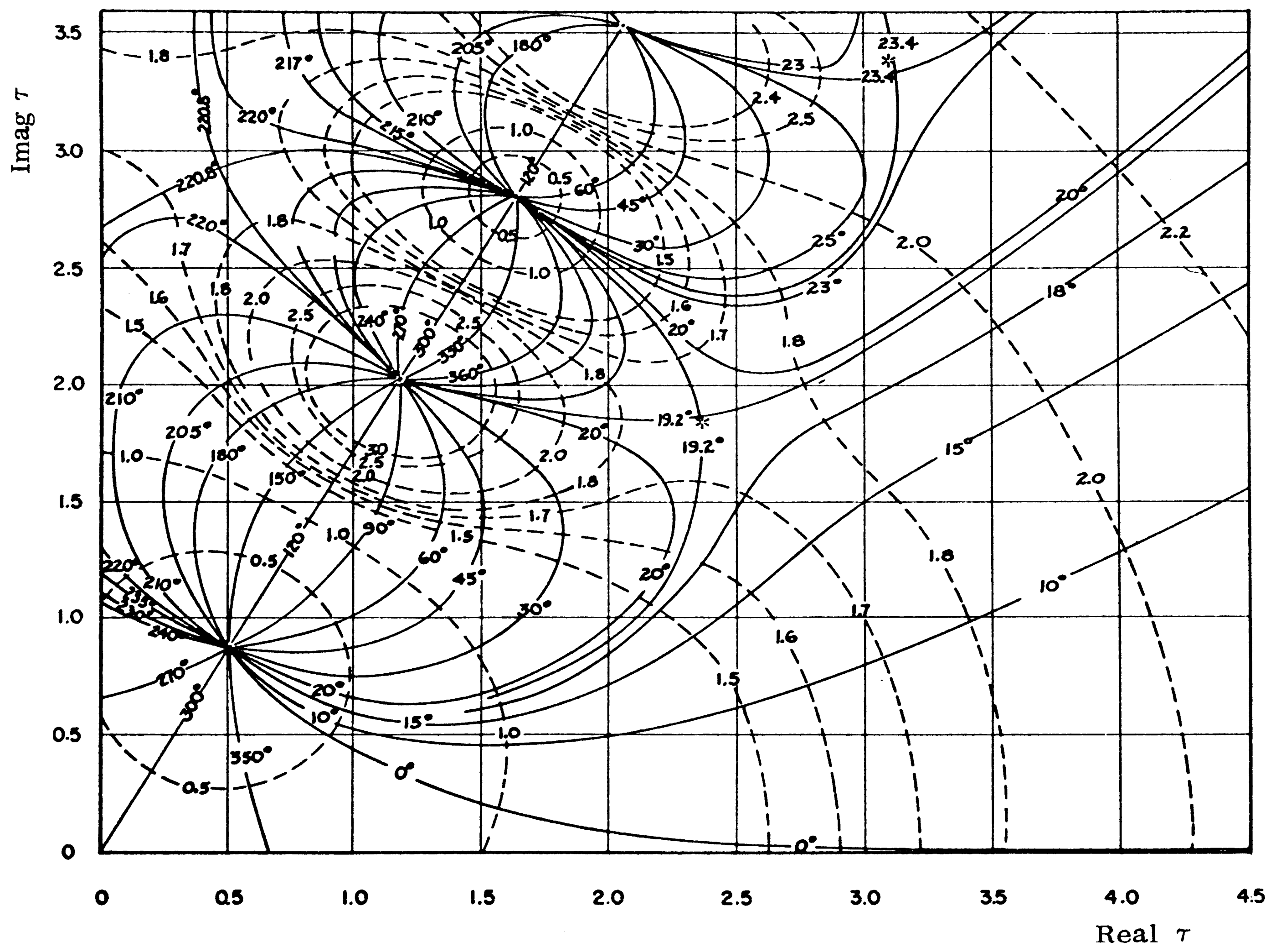

FIG. 4: The logarithmic derivative $q = w_1'(\tau)/w_1(\tau)$ (Logan, 1965).
(—— phase; - - - modulus)

Unfortunately, the imaginary parts of the roots get smaller as the order increases. This effect is surprising. From previous experience with the cylinder and sphere, the imaginary part of the roots was expected to get larger with increasing order. Although it is impossible to say whether the unusual behavior is physically valid or is a consequence of the fact that eq. (5.18) is constructed of approximate solutions of differential equations, we assume the latter since the results obtained in that case agree rather well with theory and experiment (Senior, 1969).

The location of the first root is approximately correct since small changes in eq. (5.18) will not shift it much. However, the higher order roots are much more sensitive. For example, if $\alpha^{1/2}$ is replaced by $\alpha^{1/2+\epsilon}$ with $\epsilon = -1/c$, then $\alpha^{1/2+\epsilon} \rightarrow \alpha^{1/2}$ as $c \rightarrow \infty$; but the roots, for $|\alpha| \gg 1$, of eq. (5.18) for $\epsilon = 0$ and $\epsilon = -1/c$ are not the same.

With this knowledge of the first angular eigenvalue, the wave motion on the disk can be determined. As in the case of the cylinder, it is convenient to break the analysis up into two distinct regions: the front half of the disk $(-\pi/2 < \phi < \pi/2)$ and the back half $(\pi/2 < \phi < 3\pi/2)$.

5.3 Back Half Analysis $(\pi/2 < \phi < 3\pi/2)$.

Using the fact that $I_\beta = I_{-\beta}$, eq. (5.7) can be written as

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a i} \int_{\infty + i\epsilon}^{-\infty + i\epsilon} I_\beta \left[e^{-i\beta\pi/2} \left(\frac{\exp\left[i\beta(\phi - \frac{\pi}{2})\right] + \exp\left[i\beta(\frac{3\pi}{2} - \phi)\right]}{4 \sin \beta\pi} \right) + \frac{1}{2} \frac{\cos\left[\beta(\pi + \phi)\right]}{\sin \beta\pi} \right] d\beta \tag{5.19}$$

which after some manipulation becomes

$$\frac{\partial u}{\partial \xi} = \frac{1}{4\pi a i} \int_{\infty + i\epsilon}^{-\infty + i\epsilon} I_\beta e^{-i\beta\pi/2} \left(\frac{\exp\left[i\beta(\phi - \frac{\pi}{2})\right] + \exp\left[i\beta(\phi + \frac{\pi}{2})\right]}{\sin \beta\pi} \right) d\beta \ . \tag{5.20}$$

Inserting the expression for I_β and reversing the order of integration (the integrand is assumed absolutely convergent), we have

$$\frac{\partial u}{\partial \xi} = \frac{ic}{2\pi^2 a} \int_{\Gamma_n} \int_{\infty + i\epsilon}^{-\infty + i\epsilon} \sqrt{1-b}\, \frac{H_{\beta n}(\xi_0)}{H_{\beta n}(0)} \frac{X^1_{\beta n}(0) X^2_{\beta n}(0)}{(1-\eta^2) W_{\beta n}} e^{-i\beta\pi/2}$$
$$\times \left(\frac{\exp\left[i\beta(\phi - \frac{\pi}{2})\right] + \exp\left[i\beta(\frac{3\pi}{2} - \phi)\right]}{\sin \beta\pi} \right) d\beta dn \qquad (5.21)$$

where $b = (\beta^2 - 1)/c^2$. Assuming that Γ_β can be closed in the upper half plane, the β integral can be expressed as a sum over the residues of the poles within the contour. As discussed in the previous section, the poles are located where $(1-\eta^2) W_{\beta n} = 0$.

The angular eigenvalue with the smallest imaginary part will dominate the field behavior well away from the shadow boundary ($\phi = \pm\pi/2$). Summing over the residues,

$$\frac{\partial u'}{\partial \xi} = -\frac{1}{2a} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{H_{jn}(\xi_0)}{\frac{\partial}{\partial \nu} H_{j\nu}(0)\Big|_{\nu=\nu_n}} \frac{X^1_{jn}(0) X^2_{jn}(0)}{\frac{\partial}{\partial \beta}\left[(1-\eta^2) W_{\beta n}\right]\Big|_{\beta=\beta_{jn}}} \exp(-i\beta_{jn}\pi/2)$$
$$\times \left(\frac{\exp\left[i\beta_{jn}(\phi - \frac{\pi}{2})\right] + \exp\left[i\beta_{jn}(\frac{3\pi}{2} - \phi)\right]}{\sin \beta_{jn}\pi} \right) \qquad (5.22)$$

which, far enough into the shadow, simplifies to

$$\frac{\partial u}{\partial \xi} = -\frac{1}{2a} \frac{H_{00}(\xi_0)}{\frac{\partial}{\partial \nu} H_{0\nu}(0)\Big|_{\nu=\nu_0}} \frac{X^1_{00}(0) X^2_{00}(0)}{\frac{\partial}{\partial \beta}\left[(1-\eta^2) W_{\beta 0}\right]\Big|_{\beta=\beta_{00}}} \exp(-i\beta_{00}\pi/2)$$
$$\times \left(\frac{\exp\left[i\beta_{00}(\phi - \frac{\pi}{2})\right] + \exp\left[i\beta_{00}(\frac{3\pi}{2} - \phi)\right]}{\sin \beta_{00}\pi} \right) \qquad (5.23)$$

where the first eigenvalue has been denoted by β_{00}. For convenience, $Z_{\beta_{jn} n}$ has been written as Z_{jn}.

Since

$$\frac{\partial}{\partial\beta}\left[(1-\eta^2)W_{\beta n}\right] = \left[-\frac{3n+\frac{9}{4}}{1-b} + \frac{ic}{\sqrt{b}}\sec^{-1}\sqrt{1/b}\right]\frac{2\beta}{c^2}(1-\eta^2)W_{\beta n} \; . \tag{5.24}$$

and since $\beta_{00} = c - \alpha_0 c^{1/3}$, eq. (5.23) becomes

$$\frac{\partial u}{\partial \xi} = -\frac{e^{ic\xi_0}}{a\xi_0}\left[\frac{(2\alpha_0)^{3/2}2^{3/2}c^{1/3}}{\pi\left(\frac{9}{8\alpha_0} - i\sqrt{2\alpha_0}\right)}\right]\frac{\exp\left[i\beta_{00}(\phi-\frac{\pi}{2})\right] + \exp\left[i\beta_{00}(\frac{3\pi}{2}-\phi)\right]}{1-\exp(2\pi i\beta_{00})} \; . \tag{5.25}$$

With $\alpha_0 = 2.21 - 0.87i$, eq. (5.25) is

$$\frac{\partial u}{\partial \xi} \sim \frac{e^{ic\xi_0}}{a\xi_0}(4.9e^{-0.69\pi i})c^{1/3}\left(\frac{\exp\left[i\beta_{00}(\phi-\frac{\pi}{2})\right]+\exp\left[i\beta_{00}(\frac{3\pi}{2}-\phi)\right]}{1-\exp(2\pi i\beta_{00})}\right). \tag{5.26}$$

Equation (5.26) is valid only well away from the shadow boundary.

5.4 Front Half Analysis ($-\pi/2 < \phi < \pi/2$).

Equation (5.7) is now written as

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a i}\int_{\infty+i\epsilon}^{-\infty+i\epsilon} I_\beta \frac{\cos\beta\pi}{\sin\beta\pi}\cos\beta\phi \, d\beta \; . \tag{5.27}$$

In a manner similar to what is done for a cylinder, an integral representing an "optics" term is separated out of the integral in eq. (5.27). After some manipulation, eq. (5.27) can be written as

$$\frac{\partial u}{\partial \xi} = \frac{1}{2\pi a}\int_{-\infty}^{\infty} I_\beta e^{i\beta\phi}d\beta + \frac{1}{2\pi a i}\int_{\infty+i\epsilon}^{-\infty+i\epsilon} I_\beta e^{i\beta\pi}\frac{\cos\beta\phi}{\sin\beta\pi}d\beta \; . \tag{5.28}$$

The second integral can be evaluated using the techniques of the previous section. Well away from the shadow boundary,

$$\frac{1}{2\pi a i}\int_{\infty+i\epsilon}^{-\infty+i\epsilon} I_{\beta} e^{i\beta\pi} \frac{\cos\beta\phi}{\sin\beta\pi}\, d\beta$$

$$\sim \frac{e^{ic\xi_0}}{a\xi_0}(4.9e^{-0.69\pi i})c^{1/3}\left(\frac{\exp\left[i\beta_{00}(\phi+\frac{3\pi}{2})\right]+\exp\left[i\beta_{00}(\frac{3\pi}{2}-\phi)\right]}{1-\exp(2\pi i\beta_{00})}\right) \tag{5.29}$$

where again only the β_{00} residue contributes significantly.

The first integral can be written as

$$I = \frac{1}{4\pi^2 ai}\int_{-\infty}^{\infty}\int_{\Gamma_\nu} \frac{H_{\beta\nu}(\xi_0)}{H_{\beta\nu}(0)} \frac{X^1_{\beta\nu}(0)X^2_{\beta\nu}(0)}{(1-\eta^2)W_{\beta\nu}}\, e^{i\beta\phi}\, d\beta\, d\nu \quad .$$

For β in Region 1 ($|\beta| \leq \sqrt{c}$), a reversal of the order of integration and a transformation from ν to n gives

$$I = \frac{e^{ic\xi_0}}{a\xi_0}\,\frac{c}{2^{1/2}\pi^{5/2}}\int_{\Gamma_n} \frac{\Gamma(n+1)\Gamma(-n)}{\Gamma(n+\frac{3}{2})}(2^2 c\, e^{-\pi i/2})^n$$

$$\times \int_{-\infty}^{\infty} \exp\left[-ic\left(1+\frac{\beta^2-1}{2c^2}-\frac{\beta\phi}{c}\right)\right] d\beta\, dn \ . \tag{5.30}$$

Since c is large, an approximate value for the β integral can be obtained using stationary phase methods. In this case, the stationary phase point is at $\beta = c\phi$ and thus

$$\int_{-\infty}^{\infty} \exp\left[-ic\left(1+\frac{\beta^2-1}{2c^2}-\frac{\beta\phi}{c}\right)\right] d\beta = \exp\left[-ic\left(1-\frac{\phi^2}{2}-\frac{1}{2c^2}\right)\right]\sqrt{2\pi c}\; e^{-\pi i/4} \ ,$$

valid for $\phi^2 \leq 1/c$. Using the above expression, eq. (5.30) becomes

$$I = \frac{e^{ic\xi_0}}{a\xi_0} \frac{e^{-i\pi/4} c^{3/2}}{\pi^2} \exp\left[-ic\left(1 - \frac{\phi^2}{2} - \frac{1}{2c^2}\right)\right] \int_{\Gamma_n} \frac{\Gamma(n+1)\Gamma(-n)}{\Gamma(n+\frac{3}{2})} (2^2 c e^{-\pi i/2})^n \, dn \; .$$

Evaluation of the n integral (see Gradshteyn and Ryzhik, 1965) gives for $\phi^2 \leq 1/c$

$$I = \frac{e^{ic\xi_0}}{a\xi_0} \frac{e^{-i\pi/4} c^{1/2}}{2\pi^{3/2}} \exp\left[-ic\left(1 - \frac{\phi^2}{2} - \frac{1}{2c^2}\right)\right] \; . \tag{5.31}$$

A similar analysis can be carried out for the ranges $1/\sqrt{c} \leq \phi < \pi/2$ and $-\pi/2 < \phi \leq -1/\sqrt{c}$ using Region 2 expressions. We have

$$\frac{1}{2\pi a} \int_{-\infty}^{\infty} I_\beta e^{i\beta\phi} \, d\beta = \frac{e^{ic\xi_0}}{a\xi_0} \frac{e^{-\pi i/4} \cos^{1/2}\phi \, c^{1/2}}{2\pi^{3/2}} \exp[-ic\cos\phi] \; . \tag{5.32}$$

which is valid for $1/\sqrt{c} \leq \phi < \frac{\pi}{2} - \epsilon$ and $-\frac{\pi}{2} + \epsilon < \phi \leq -1/\sqrt{c}$ where ϵ is some positive real quantity which is a function of c only and $\epsilon \to 0$ as $c \to \infty$. Comparison of eqs. (5.31) and (5.32) indicates that eq. (5.32) is also valid for $\phi^2 \leq 1/c$ with negligible error. We can therefore combine eqs. (5.29) and (5.32) to obtain

$$\frac{\partial u}{\partial \xi} = \frac{e^{ic\xi_0}}{a\xi_0} \left[\frac{e^{-i\pi/4} \cos^{1/2}\phi \, c^{1/2}}{2\pi^{3/2}} \exp[-ic\cos\phi] + 4.9 e^{-0.69\pi i} c^{1/3} \left(\frac{\exp\left[i\beta_{00}(\phi + \frac{3\pi}{2})\right] + \exp\left[i\beta_{00}(\frac{3\pi}{2} - \phi)\right]}{1 - \exp(2\pi i \beta_{00})} \right) \right] \tag{5.33}$$

for $-\frac{\pi}{2} + \epsilon < \phi < \frac{\pi}{2} - \epsilon$.

Chapter 6

DISCUSSION OF RESULTS

6.1 Edge and Creeping Waves.

As is apparent from Chapters 4 and 5, the surface field behavior is very different for different source positions. The simplest results are obtained for broadside incidence; the problem is mathematically two-dimensional since there is no m dependence. For edge-on incidence, the surface field at the disk edge is also of simple form. Here, it is the n dependence that drops out, the β_{0n} being independent of n to first order in c. In both cases, the surface field can be accurately approximated by only the first term of the residue series.

The mathematical simplicity evident in the two cases discussed above has a physical basis. For broadside incidence, the surface fields are made up solely of waves launched at the disk edge. These waves are radially directed and similar to the edge waves found on an infinite half plane due to a plane wave incident normal to the edge. However, for grazing incidence, the edge field exhibits a creeping wave behavior similar to what is found on a cylinder. In other cases, the surface field behavior is a combination of the above effects and for that reason is much more complicated.

6.2 Broadside Incidence.

For an electrically large ($c \gg 1$) disk, the Geometrical Theory of Diffraction (Keller, 1962) predicts that the field on the shadow side of the disk is due to waves launched at the disk edge. The field incident on the disk edge excites waves which carry energy across the disk. Further, the waves that arrive at any point on the surface were launched at the two points where the diameter containing the observation point intersects the disk rim.

For a point away from the edge, Goodrich et al. (1963) show that the surface field can be made up of a sum of waves. Those arriving directly have the greatest strength, but there are also contributions due to waves reflected back from the opposite rim and so on. Thus, the surface field is in the form of a series in increasing inverse powers of c. Such is also true at the edge. The first two terms of the series expression for the edge field are

$$a\eta \frac{\partial u}{\partial z}\bigg|_{\eta=0} = \frac{e^{ic\xi_0}}{4\pi a\xi_0}\left[(2c/\pi)^{1/2} e^{-\pi i/4} - \frac{e^{2ic}}{c\pi 2^{5/2}}\right] . \qquad (6.1)$$

The first is an optics term; this is the only term where c has a positive power. Since the disk edge is locally plane, the edge field should look similar to that of an acoustically soft half plane for a normally incident plane wave. (The plane wave solution for the disk can be obtained by removing the $e^{ic\xi_0}/4\pi a\xi_0$ factor.) For the half plane, a local transformation from a Cartesian co-ordinate system to an oblate spheroidal one gives

$$a\eta \frac{\partial u}{\partial z}\bigg|_{\eta=0} = (2c/\pi)^{1/2} e^{-\pi i/4} \qquad (6.2)$$

at the edge and the leading terms of the disk and half plane edge fields are seen to be the same.

The second term is due to a wave launched at the opposite point on the rim. It travels across the disk and is incident on the rim at the observation point. As is evident from its negative power in c dependence, the second term is a diffraction term.

6.3 Edge-on Incidence.

Whereas the surface fields for broadside incidence are characterized by an edge wave behavior, for grazing incidence a creeping wave behavior is evident, particularly on the back half of the disk. Creeping wave effects are also found in the front half of the disk though their magnitude is greatly decreased. These waves are of primary importance in determining the field scattered by the disk. Since most of the energy associated with creeping waves is found near the disk edge and since the fields in the interior are too complicated for simple analysis, only the creeping wave behavior at the disk edge is analyzed.

Though the results obtained in this work are for scalar scattering, we expect a great deal of similarity between these and Senior's (1969) results for electromagnetic scattering. Senior's eq. (1) is

$$J_\phi = A\exp\left[i(c+\frac{\pi}{3})\right]\left(\exp\left[i\beta(\frac{3\pi}{2}-\phi)\right]+\exp\left[i\beta(\phi-\frac{\pi}{2})\right]\right) \tag{6.3}$$

with $\beta \simeq c-(e^{-\pi i/3}/2)c^{1/3}$. Our result for the same region of the disk $\left(\frac{3\pi}{2}-\epsilon<\phi<\frac{\pi}{2}+\epsilon\right)$ is

$$\frac{\partial u}{\partial \xi} \sim \frac{e^{ic\xi_0}}{4\pi a\xi_0}(61.6\ e^{-0.7\pi i})c^{1/3}\left(\frac{\exp\left[i\beta_{00}(\frac{3\pi}{2}-\phi)\right]+\exp\left[i\beta_{00}(\phi-\frac{\pi}{2})\right]}{1-\exp\left[2\pi i\beta_{00}\right]}\right) \tag{6.4}$$

Equations (6.3) and (6.4) are similar, as expected. However, it is important to note that $\beta \neq \beta_{00}$. Though both are of the form $\beta = c-\alpha c^{1/3}$, the α's are markedly different: $\alpha = 0.25-0.43i$ and $\alpha_{00} = 2.21-0.87i$. Nevertheless, our hope that the scalar analysis would give results similar to Senior's (1969) is fulfilled.

As in the broadside case, a physical interpretation of the disk results is possible in terms of simpler geometries, e.g., the half plane and cylinder. Geometrical Theory of Diffraction is usable only in a limited way since it cannot predict the creeping wave behavior present on both the front and back halves of the disk. On the front half, the surface field is

$$a\eta\left.\frac{\partial u}{\partial z}\right|_{\eta=0} = \frac{e^{ic\xi_0}}{4\pi a\xi_0}\Bigg[(2c/\pi)^{1/2}e^{-\pi i/4}\cos^{1/2}\phi\,\exp\left[-ic\cos\phi\right]$$
$$+61.6e^{-0.7\pi i}c^{1/3}\left(\frac{\exp\left[i\beta_{00}(\phi+\frac{3\pi}{2})\right]+\exp\left[i\beta_{00}(\frac{3\pi}{2}-\phi)\right]}{1-\exp\left[2\pi i\beta_{00}\right]}\right)\Bigg]\ . \tag{6.5}$$

The second term is a creeping wave term and its magnitude decreases with increasing c. However, the magnitude of the first term increases with c since it is an optics term. Dividing the optics term of eq. (6.5) ($\phi = 0$) by the optics term in the edge field for normal incidence, eq. (4.19), one obtains $\sqrt{2}$, the same ratio exhibited by the half plane edge fields for edge-on and normal incidence. Thus the similarity with the half plane behavior which was found in the broadside case is also valid in the edge-on case for the front half of the disk.

A shape similar to the disk is a cylinder. Though the cylinder fields are nowhere infinite, they are of the same form as the disk fields. The surface fields for both hard and soft cylinders due to plane wave incidence can be found in Bowman et al. (1969). The E-polarization result is mathematically the same as that for a soft cylinder. The H-polarization case is equivalent to the hard cylinder. A comparison of eqs. (6.4) and (6.5) for the disk edge fields with the surface field of the cylinders indicates that the disk field is not the same as that for either the hard or soft cylinder, but lies between them. In all three cases, the expressions for the field are of the form

$$f = Ac^n \cos^m\phi \exp\left[-ic\cos\phi\right] + Bc^p \frac{\exp\left[i\beta(\frac{3\pi}{2} - \phi)\right] + \exp\left[i\beta(\frac{3\pi}{2} + \phi)\right]}{1 - \exp\left[2\pi i\beta\right]} \tag{6.6}$$

for the front half of the bodies $(|\phi| < \pi/2)$ and

$$f = Bc^p \frac{\exp\left[i\beta(\phi - \frac{\pi}{2})\right] + \exp\left[i\beta(\frac{3\pi}{2} - \phi)\right]}{1 - \exp\left[2\pi i\beta\right]} \tag{6.7}$$

for the back half of the bodies $(|\phi - \pi| < \pi/2)$. Here, $\beta = c + \alpha c^{1/3}$. Assigning the subscripts h, s, and d to the hard and soft cylinders and disk respectively, it is easy to show that $n_h < n_d < n_s$, $m_h < m_d < m_s$, $p_h < p_d < p_s$, and $\mathrm{Im}(\alpha_h) < \mathrm{Im}(\alpha_d) < \mathrm{Im}(\alpha_s)$. However, the creeping wave is a fast wave for the disk, $\mathrm{Re}(\alpha_d) < 0$, but is a slow wave for the cylinders, $\mathrm{Re}(\alpha_{h,s}) > 0$.

The similarity and difference between the cylinder and disk surface fields can be further illuminated. For the cylinder, the α are related to the zeros of Ai or Ai' for E (soft) and H (hard) polarizations respectively. If, in the H-polarization case, an impedance (mixed) boundary condition is imposed at the cylinder surface, the equation for α is

$$\frac{Ai'(2^{1/3}e^{2\pi i/3}\alpha)}{Ai(2^{1/3}e^{2\pi i/3}\alpha)} = Dc^{-1/3} \tag{6.8}$$

where D is a constant that depends only on the surface impedance Z. Equation (6.8) is very similar to eq. (5.18). In fact, a realizable Z can be selected so that at a given frequency, the attenuation constants, Im(α), for the cylinder and disk are the same. However, the creeping wave on the cylinder is still a slow wave. If Z is selected to make α the same for the cylinder and disk, it is found to have a negative real part; the surface is an active one and imparts energy to the creeping wave. Thus the creeping wave behavior on the cylinder and disk, though similar in form, is as fundamentally different as positive and negative resistances.

6.4 Final Comments.

Since the results of Chapters 4 and 5 were rather complex, a comprehensive explanation of the surface field behavior was not possible in any simple way. Except at grazing incidence, we had to be content to consider only the field in the disk shadow. Nevertheless, from a study of the broadside and edge-on incidence cases, we were able to get some physical understanding of the interaction between the incident field and the disk. Our analysis showed that the surface field could be made up of edge and creeping wave terms.

Most of the assumptions in our analysis were a result of the unknown nature of the asymptotic solutions of the radial and angular differential equations for certain ranges of parameters. Some of the assumptions made could have been removed through a more extensive analysis, but some are inherent in the use of asymptotic theory and would remain as long as we considered only the first term in the asymptotic expressions. Yet the use of an asymptotic analysis did not appear to greatly affect the accuracy of our results. Only in the case of the angular eigenvalues where we are concerned with second order effects did our asymptotic analysis cause problems.

APPENDIX

McKELVEY'S AND OLVER'S ASYMPTOTIC THEORIES

In this appendix, the works of Mckelvey (1955) and Olver (1954) on which the asymptotic theory used in this work is based are briefly discussed.

Mckelvey's (1955) theory is concerned with the asymptotic solution of the equation

$$\frac{d^2W}{dx^2} - \left[\lambda^2 p_0(x) + \lambda p_1(x) + Q(x,\lambda)\right] W = 0 \tag{A. 1}$$

for large values of λ. Here, x is a real variable and $p_0(x)$ has a second order zero in the interval where x is defined. $Q(x,\lambda)$ is of the form

$$Q(x,\lambda) = \sum_{j=0}^{\infty} \frac{q_j(x)}{\lambda^j} \quad .$$

The first term in the asymptotic series for W is

$$W = \mu_0 V + \frac{\mu_1}{\lambda} V' \tag{A. 2}$$

where

$$V = \psi(x)\xi^{-1/4}\left[AM_{k,\,1/4}(\xi) + BM_{k,\,-1/4}(\xi)\right]$$

with A and B arbitrary and $M_{k,\mu}(\xi)$ defined in Section 2.2. Also

$$\phi(x) = p_0^{1/2}(x) \ ,$$

$$\xi = 2\lambda \int_0^x \phi(s)ds \ ,$$

$$\psi(x) = \left[\int_0^x \phi(s)ds\right]^{1/4} \phi^{-1/2}(x) \ ,$$

and

$$k = -\frac{1}{2} p_1(0)\psi^4(0) \quad .$$

Olver's (1954) theory was not directly applicable to the equations in this work. He was concerned with equations of the form

$$\frac{d^2W}{dx^2} + \left[\lambda p_0(x) + Q(x,\lambda)\right] W = 0 \quad . \tag{A. 3}$$

However, only a slight modification of the theory is necessary to make it suitable for equations such as

$$\frac{d^2W}{dx^2} + \left[\lambda^2 p_0(x) + \lambda p_1(x) + Q(x,\lambda)\right] W = 0 \tag{A. 4}$$

where $p_0(x)$ has a pole or zero at x_0. Following Olver (1954), eq. (A. 4) is first transformed. Letting $W = \dot{x}^{1/2} V$ with $\dot{x} = dx/d\xi$, eq. (A. 4) becomes

$$\frac{d^2V}{d\xi^2} + \left(\lambda^2 \dot{x}^2 p_0(x) + \lambda \dot{x}^2 p_1(x) + \left[\dot{x}^2 Q(x,\lambda) - \frac{3\ddot{x}^2 - 2\dot{x}\dddot{x}}{4\dot{x}^2}\right]\right) V = 0 \tag{A. 5}$$

with ξ related to x through the equation $\dot{x}^2 p_0(x) = g(\xi)$. The particular $g(\xi)$ used depends on $p_0(x)$, but in general it must satisfy two criteria:

1) $g(\xi)$ must have the same order pole or zero at $\xi = 0$ as $p_0(x)$ does at $x = x_0$;
2) $g(\xi)$ must be simple enough in form so that

$$\frac{d^2J}{d\xi^2} + \lambda^2 g(\xi) J = 0$$

is solvable in terms of known functions.

The leading term of the asymptotic series for V is

$$V = J(\xi)A(\xi) + \frac{B(\xi)}{\lambda} \frac{dJ(\xi)}{d\xi}$$

where

$$\int_{x_0}^{x} p_0(s)ds = \int_0^{\xi} g(s)ds \quad .$$

Then

$$W(x) = \dot{x}^{1/2}\left[J(\xi)A(x) + \frac{B(x)}{\lambda}\,\frac{dJ(\xi)}{d\xi}\right]$$

with

$$A(x) = \cos\left[\frac{h(x)}{2}\right] \quad ,$$

$$B(x) = g^{-1/2}\sin\left[\frac{h(x)}{2}\right]$$

and

$$h(x) = \int \frac{p_1(x)}{p_0^{1/2}(x)}\,dx \quad .$$

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