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MEMO TO: T. B. A. Senior
FROM: Sharad R. Laxpati
SUBJECT: Integral equations of scattering by a thin sheet:
          Impedance boundary condition on a sheet vs. an
          impedance sheet.

The formulation of the boundary value problem associated with a thin sheet
employing the impedance boundary condition has some definite advantages. In
particular, it provides a simultaneous treatment of E and H polarization through
duality. However, the physical materials are described through their permeability
and permittivity. It would be of considerable interest, from a practical standpoint,
to relate these two types of boundary conditions and the associated integral equa-
tions. This would then provide a relatively simple means of arriving at the
diffraction coefficient for a resistive (and, in general, an impedance) sheet
from that derived from an impedance boundary condition.

In the following, the characteristics of the two formulations are briefly
discussed. Based on this it is apparent that even for two dimensional problems
there exists no simple relationships for the general impedance sheet (with both
permeability and permittivity different from free space). However, the striking
similarity between the integral equations for the electric current for scattering
by a two dimensional scatterer, based on these two forms of boundary conditions,
does warrant closer examination. It is shown that for edge on incidence, the
diffraction coefficient for impedance boundary condition and a resistive sheet,
of semi-infinite extent, are related.
I. Impedance boundary condition on a sheet

Note: Throughout this memo, \( \hat{n} \) is a unit vector, directed from \( s_- \) to \( s_+ \) surface.

(1) Let \( \overline{\eta}_+ \) a \( (2 \times 2) \) matrix represent normalized surface impedance on \( s_+ \),
\[
\overline{\eta}_+ = \begin{pmatrix} \eta_{11} & \eta_{12} \\
\eta_{21} & \eta_{22} \end{pmatrix}
\]

Leontovich form of the boundary condition is
\[
E^- (\hat{n} \cdot E^-) \hat{n} = \overline{\eta}_+ z_o (\hat{n} \cdot H^-) .
\]

(2) Both tangential electric and magnetic fields are discontinuous across the sheet.
\[
\overline{K}_m = \hat{n} \times (E^+ - E^-)
\]
\[
\overline{K}_e = \hat{n} \times (H^+ - H^-)
\]

\( K \) - surface currents

superscripts \( \pm \) represents values on \( s_\pm \).

(3) Through the boundary condition equation (1), \( \overline{K}_m \) and \( \overline{K}_e \) are related.

Assuming \( \overline{\eta}_+ = \overline{\eta}_- = \overline{\eta} \),
\[
\overline{K}_m = z_o \hat{n} \times (\overline{\eta} \cdot \overline{K}_e) .
\]

(4) The integral equation for the electric current for the case of a two dimensional scatterer with \( E_z^{\text{inc}} \) incident field is (Knott and Senior, 1973)

\[
Y E_z^{\text{inc}}(s) = \frac{1}{2} \eta_s K_z(s) + \frac{1}{4} \int_C K_z(s') H_0^{(1)}(kr) \, d(ks') +
\]
\[
+ \frac{1}{4} \int \eta_s(s') K_z(s') (\hat{n} \cdot \hat{z}) H_1^{(1)}(kr) \, d(ks')
\]

(4a)
where
\[ \xi = \rho - \rho^t \]
and
\[ \eta = \eta_s \frac{1}{1} \]

Equation (4a) can be rewritten as follows, if the contribution of \( s = s^t \) point is included in the second integral.

\[
\begin{align*}
Y_e^{\text{inc}}(s) &= \eta_s K_z(s) + \frac{1}{4} \int_C K_z(s') H_0^{(1)}(kr) \, d(ks') + \\
&\quad + \frac{i}{4} \int_C \eta_s(s') K_z(s') (\hat{n} \cdot \hat{r}) H_1^{(1)}(kr) \, d(ks') . \quad (4b)
\end{align*}
\]

II. **Impedance Sheet**

![Diagram](image)

The material of thickness \( \Delta \) is described through

\[ x_e \] — electric susceptibility (volume)

and

\[ x_m \] — magnetic susceptibility (volume).

Let

\[ \Sigma_e = \Delta x_e \] — surface electric susceptibility

\[ \Sigma_m = \Delta x_m \] — surface magnetic susceptibility.

Before considering the general case, consider the two special cases: (a) electric resistive sheet and (b) magnetic resistive sheet.
(a) **Electric resistive sheet:** \( x_e \neq 0, x_m = 0 \).

1. Let

\[
\frac{1}{z_o} \Delta \frac{k}{\Delta} \tan \frac{r}{E^+ e} x e \left[ E^+ - (\hat{n} \cdot E^+) \hat{n} \right].
\]

(5)

2. The boundary conditions are:

\[
\hat{n} x (E^+ - E^-) = 0
\]

\[
\hat{n} x (H^+ - H^-) = (K^+_e + K^-_e)
\]

(6)

3. From (5) and (6) we have

Since \( \hat{n} x E^+ = \hat{n} x E^- \), \( K^+_e = K^-_e \),

let \( E = E^+ = E^- \), then

\[
\frac{1}{z_o} \Delta \frac{k}{\Delta} x e \left[ E - (\hat{n} \cdot E) \hat{n} \right] = \hat{n} x (H^+ - H^-).
\]

(7)

4. An equivalent of (3) above, viz. a relationship between the electric and magnetic currents, is not possible. However, we note that \( K_m = 0 \), hence, (1) and (3) are satisfied if

\[
\frac{1}{\eta} \neq 0.
\]

(8)

5. The integral equation for the electric current for the case of a two dimensional scatterer with \( E_{z}^{inc} \) incident field is (Oshiro and Cross, 1966)

\[
Y_0 E_{z}^{inc} (s) = Y_0 R_k K_{z} (s) + \frac{1}{4} \int_C K_{z} (s') H_{o}^{(1)} (k r) d (k s')
\]

(9)

\[
R_s = -\frac{z_o}{1k \Delta x_e} \times \frac{1}{-i \omega \epsilon \Delta x_e}
\]

(10)
(b) Magnetic resistive sheet: \( x_e = 0, \ x_m \neq 0 \).

1. Let \( K_m^\pm = -\frac{ikz}{o} \overline{A_x} m H_{xtan}^\pm \)

\[
= -\frac{ikz}{o} \Delta x_m^\pm \left( H_{xtan}^+ - (\hat{n} \cdot H_{xtan}) \hat{n} \right). \tag{11}
\]

2. The boundary conditions are

\[
\hat{n} \times (E^+ - E^-) = (K_m^+ + K_m^-)
\]

\[
\hat{n} \times (H^+ - H^-) = 0 \tag{12}
\]

3. From (11) and (12) we have,

\[
\text{Since } \hat{n} \times H^+ = \hat{n} \times H^-, \ K_m^+ = K_m^-,
\]

\[
-ikz \Delta x_m \left( H - (\hat{n} \cdot H) \hat{n} \right) = \hat{n} \times (E^+ - E^-) \tag{13}
\]

4. Once again, an attempt to develop an equivalent of (3) leads to some difficulty. However, we note,

\[ K_e \neq 0, \]

hence (3) is satisfied if \( \overline{n} \rightarrow \infty \).

5. A dual of integral equation (9) may be derived for a magnetic resistive sheet, if \( H_z \)-polarized incident field is assumed.

(c) Now we consider the general case of an impedance sheet.

1. Let \( K_e^\pm = -\frac{1}{2} \frac{ik}{z_o} \Delta x_e \left( E^\pm - (\hat{n} \cdot E^\pm) \hat{n} \right) \),

\[
K_m^\pm = -\frac{1}{2} \frac{ikz}{o} \Delta x_m \left( H^\pm - (\hat{n} \cdot H^\pm) \hat{n} \right), \tag{14}
\]

2. Both tangential electric and magnetic fields are discontinuous across both surfaces \( s^\pm \). We assume the fields inside the material are non-zero and
equal to $E_1$ and $H_1$.

Let

$$K^+_{m} = \hat{n} \times (E^+ - E^-)$$

and

$$K^+_{e} = \hat{n} \times (H^+ - H^-) .$$

(15)

(3) From (15), since for the case of $\Delta \rightarrow 0$, $E_1^+ \rightarrow E_1^-$ and $H_1^+ \rightarrow H_1^-$,

we have

$$\hat{n} \times (E^+ - E^-) = K^+_{m} + K^+_{e}$$

and

$$\hat{n} \times (H^+ - H^-) = K^+_{e} + K^-_{e}$$

(16)

using (14) in (16) above.

$$- \frac{1}{2} \frac{ik}{z_o} \Delta x_e \left[ \left( E^+ - (\hat{n} \cdot E^+) \hat{n} \right) + \left( E^- - (\hat{n} \cdot E^-) \hat{n} \right) \right] = \hat{n} \times (H^+ - H^-)$$

and

$$- \frac{1}{2} \frac{ikz_o}{\Delta x_m} \left[ \left( H^+ - (\hat{n} \cdot H^+) \hat{n} \right) + \left( H^- - (\hat{n} \cdot H^-) \hat{n} \right) \right] = \hat{n} \times (E^+ - E^-) .$$

(17)

(4) In general, the quantities on the $s_+$ surfaces are not equal and are not necessarily related through a constant multiplier. A relationship between the electric and magnetic currents cannot be obtained.

(5) Integral equations for the currents may be obtained as a special case of the more general problem discussed in a separate memo (Laxpati).

III. Conclusion

(a) The inability to arrive at an equation similar to equation (3) for the case of an impedance sheet demonstrates that, in general, the diffraction coefficients for the case of an impedance boundary condition cannot be related to those for an impedance sheet. This lack of similarity arises due to the fundamental difference in the 'physical' property of the two surfaces. In the case
of an impedance surface, there exists a direct coupling between the tangential components of the electric and magnetic fields (see eq. (1)). The discontinuity of the magnetic field is defined through the boundary conditions as the electric current on the surface. In direct contrast, in the case of an impedance sheet, the tangential electric and magnetic fields determine the electric and magnetic currents, respectively (see eq. (14)). There is no direct coupling between the electric and magnetic fields through the boundary conditions.

(b) Although the physics is dissimilar, the similarity between the integral equations (4) and (9) deserves a second look. If the second integral in equation (4b), viz.

\[ \int \eta_s (s') K_z (s') (\hat{n} \cdot \hat{r}) H_1^{(1)} (kr) \ d (ks') = 0 \]  

then equation (9) will be identical to equation (4b) with \( \eta_s \) replaced by \( Y_{R_s} \). This identity is a mathematical one only.

For the case of magnetic resistance sheet, the dual of that for electric resistance sheet will hold. For the impedance sheet, it should be possible to arrive at a similar conclusion. However, two complexities occur. The first one is that for a scalar formulation, the appropriate integral equations for impedance boundary conditions must be those for an anisotropic surface impedance. The second difficulty may occur since a vector formulation of the problem may be necessary. This particular aspect will be dealt with at a later time.

(c) The implications of the above conclusion (b) are quite important for the scattering by a semi-infinite sheet. Senior (1952) has derived analytical expressions for the diffracted field employing the impedance boundary condition. Equations (32) and (33) define the scattered field for E-polarization. The dependence of the second term in the expression for \( F(\tau) \) (equation 33) on the angle of incidence \( \alpha \) shows that for \( \alpha = \pi \) (this corresponds to edge-on incidence),
the contribution of this term (specifically this corresponds to the left hand side of our equation (18)) is zero. Hence, we conclude:

For edge-on incidence \((\alpha = \pi)\), for the case of a semi-infinite sheet with impedance boundary conditions, the diffraction coefficient for E-polarization is identical to that for the case of a semi-infinite resistive sheet if one replaces \(\eta\) by \(Y_0 R_s\).

A dual of this will hold for a magnetic resistive sheet.

References:

