THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
Radiation Laboratory

THE SPHERICAL CAVITY PROBLEM

T.B.A. Senior

January 1975

Prepared for

The Dikewood Corporation
Albuquerque, New Mexico

12643-1-T = RL-2258

Ann Arbor, Michigan

THE SPHERICAL CAVITY PROBLEM

T. B. A. Senior
Radiation Laboratory
Department of Electrical and Computer Engineering
The University of Michigan
Ann Arbor, Michigan 48104

January 1975

Abstract

For a time-harmonic electromagnetic field incident on a thin, perfectly conducting spherical shell with a circular aperture, methods for the exact determination of the interior fields are discussed. The advantages of the direct E field integral equation are pointed out. The unknowns are then the tangential components of the currents induced in the shell, and two alternative versions of the resulting coupled integral equations are developed. The one found most convenient is particularized to the case of a plane wave incident symmetrically upon the aperture.

This study was performed under subcontract to

The Dikewood Corporation
1009 Bradbury Drive, S. E.
University Research Park
Albuquerque, New Mexico 87106

CONTENTS

SECTION I

INTRODUCTION

One of the more difficult problems in electromagnetic theory is the penetration of an electromagnetic field through an aperture into a finite cavity beyond. These difficulties are present even in the idealized case of a spherical shell with a circular aperture illuminated by a plane wave at symmetrical incidence, but in a previous report (ref. 1) a method was used based on the expansion of the interior and exterior fields in spherical wave functions.

The method is an extension of one developed by Chang and Senior (ref. 2). The conditions that the tangential components of the electric field are continuous across the entire surface $r = a$ are imposed explicitly, and relate the coefficients of the interior and exterior expansions. The remaining conditions that $E_{tan} = 0$ on the shell and H_{tan} is continuous in the aperture produce four infinite series relations which are solved for (say) the exterior mode coefficients by applying weighting factors to each relation and using the method of least square error. To improve the numerical convergence of the scheme, the known field behavior close to the edge of the aperture is analytically extracted, and as shown in reference 1, the method is then capable of providing data for the fields at points within the cavity.

Even with this modification, however, the method is still a poor one from a numerical standpoint. The program is expensive to run and the data are sensitive to the number of terms retained in the expansions. This sensitivity was particularly apparent when we came to use the complex frequencies necessary for the calculation of the eigenvalues. Moreover, the scheme itself is not well suited to the singularity expansion method because of the spherical mode expansions employed and the <u>intrinsic</u> involvement of the incident field throughout. Since one of the main purposes of our renewed study was the determination of the eigenvalues and their dependence on aperture size, it was felt desirable (if not necessary) to explore other approaches.

One such approach is to construct integral equations for the tangential components of the electric field in the aperture. If

$$E_\theta = f(\theta)\cos\phi \quad , \qquad\qquad E_\phi = g(\theta)\sin\phi$$

for $r = a$, $0 \leq \theta \leq \theta_0$ where r, θ, ϕ are spherical polar coordinates with origin at the center of the shell, two coupled integral equations can be derived (ref. 1) having the general form

$$\int_0^{\theta_0} \left\{ f(\alpha)K_1(\alpha,\theta) + g(\alpha)K_2(\alpha,\theta) \right\} d\alpha = T(\theta) \ , \qquad 0 \leq \theta \leq \theta_0$$

where $T(\theta)$ is known and, for example,

$$K_1(\alpha,\theta) = -\sin\alpha \sum_{n=1}^{\infty} \frac{2n+1}{2n^2 (n+1)^2} \left\{ \frac{1}{\psi_n \xi_n} \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \frac{P_n^1(\cos\alpha)}{\sin\alpha} \right.$$

$$\left. + \frac{1}{\psi'_n \xi'_n} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{\partial}{\partial\alpha} P_n^1(\cos\alpha) \right\} .$$

For large n, the nth term in the first part of the series is $O(n^0)$, and the series for $K_2(\alpha,\theta)$ is still more divergent. Even the analytical subtraction of the edge behavior does not produce convergence, so that the interchange of integration and summation which was assumed in the derivation of the integral equations is mathematically unjustifiable. Perhaps more to the point, the method is numerically worthless. This was one of the reasons why Chang and Senior adopted the circuitous approach that they did, and hints at the convergence difficulties which even their method has.

Although it is by no means impossible that the divergent portions of the kernels K_i could be analytically summed, the only real advantage of this type of integral equation is that the integration is limited to the aperture rather than to the larger shell. On a physical basis it is convenient to think of the interior fields as generated by the field in, or 'passing through', the aperture, but mathematically it is more natural to attribute them to the currents which are induced in the shell. From the resulting expressions for the scattered field, integral equations are obtained by allowing the observation point to lie on the shell, and the relative simplicity of the kernels more than compensates for the larger region of integration. In addition, the basic format is now conducive to the calculation of the interior and exterior resonances.

The effectiveness of the direct integral equation approach to the cavity problem has been verified (ref. 3) in the analogous two-dimensional problem of a cylindrical shell with a slit aperture, and with the confidence that this has brought, we here apply it to the three-dimensional problem of an infinitesimally thin, perfectly conducting spherical shell with a circular aperture. Because of the necessity for using the E field integral equation, the formulation is not trivial, and two alternative versions of the coupled integral equations are explored. The one found most convenient is then particularized to the case of a plane wave at symmetrical incidence.

SECTION II

GENERAL FORMULATION

Consider an infinitesimally thin, perfectly conducting shell constituting the open surface S. This is illuminated by an electromagnetic field $\underline{E}^i(\underline{r})$, $\underline{H}^i(\underline{r})$, and if $\underline{\pi}(\underline{r})$, $\underline{\pi}^*(\underline{r})$ are the electric and magnetic Hertz vectors respectively of the scattered field,

$$\underline{\pi}^*(\underline{r}) = 0 \tag{1}$$

since the surface is perfectly conducting, and

$$\underline{\pi}(\underline{r}) = \frac{iZ_o}{k} \iint_S \underline{J}(\underline{r}')\, g(\underline{r}|\underline{r}')\, dS' \tag{2}$$

where a time factor $e^{-i\omega t}$ has been assumed and suppressed. In eq. (2), Z_o is the intrinsic impedance of free space,

$$g(\underline{r}|\underline{r}') = \frac{e^{ikR}}{4\pi R} \tag{3}$$

is the free space Green function with $R = |\underline{r} - \underline{r}'|$, and

$$\underline{J} = n_\wedge (\underline{H}^+ - \underline{H}^-) \tag{4}$$

is the total current borne by the shell, i.e. the strength of the equivalent electric current sheet. Integration is with respect to the primed coordinates and over one side of the shell (see Figure 1).

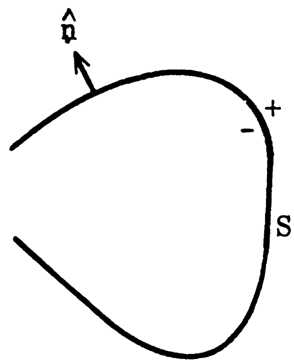

Figure 1: The geometry

In terms of $\underline{\pi}$, the scattered field is $\underline{E}^s(\underline{r}) = \nabla_\wedge \nabla_\wedge \underline{\pi}$, $\underline{H}^s(\underline{r}) = -ikY_o \nabla_\wedge \underline{\pi}$ and hence the total field is

$$\underline{E}(\underline{r}) = \underline{E}^i(\underline{r}) + \frac{iZ_o}{k} \nabla_\wedge \nabla_\wedge \iint_S \underline{J}(\underline{r}') \, g(\underline{r}|\underline{r}') \, dS' \tag{5}$$

$$\underline{H}(\underline{r}) = \underline{H}^i(\underline{r}) + \nabla_\wedge \iint_S \underline{J}(\underline{r}') \, g(\underline{r}|\underline{r}') \, dS' \, . \tag{6}$$

If $\underline{r}$ is not on S, we can interchange the order of differentiation and integration. In particular,

$$\underline{H}(\underline{r}) = \underline{H}^i(\underline{r}) - \iint_S \underline{J}(\underline{r}')_\wedge \nabla g \, dS' \tag{7}$$

since the differentiation is with respect to the unprimed coordinates of the observation point.

To obtain an integral equation for $\underline{J}$, it is natural to consider the tangential components of eq. (7) and then take the limit as $\underline{r}$ tends to a point $\underline{r}_o$ on S. Since (ref. 4)

$$\lim_{\underline{r} \to \underline{r}_o} \hat{n}_\wedge \iint_S \underline{J}(\underline{r}')_\wedge \nabla g \, dS' = \mp \frac{1}{2} \underline{J}(\underline{r}_o) + \iint_S \hat{n}_\wedge \left\{ \underline{J}(\underline{r}')_\wedge \nabla g \right\} dS'$$

with the upper or lower sign accordihg as $\underline{r}_o$ is on the positive or negative side of S,

$$\hat{n}_\wedge \underline{H}^{+,-}(\underline{r}_o) = \hat{n}_\wedge \underline{H}^i(\underline{r}_o) \pm \frac{1}{2}\underline{J}(\underline{r}_o) - \displaystyle{-\!\!\!\!\!\!\iint_S} \hat{n}_\wedge \left\{ \underline{J}(\underline{r}')_\wedge \nabla g \right\} dS' \tag{8}$$

where the slash across the integral sign denotes the Cauchy principal value. Subtraction of the two equations contained in (8) produces only the identity $\underline{J}(\underline{r}_o)=\underline{J}(\underline{r}_o)$, and addition is also fruitless without some prior knowledge of a connection between the tangential components of $\underline{H}$ on the two sides of the current sheet. It is therefore necessary to turn to an electric field integral equation.

For $\underline{r}$ not on S, the differentiation in eq. (5) can be applied to the integrand to give

$$\underline{E}^i(\underline{r}) - \underline{E}(\underline{r}) = -\frac{iZ_o}{k} \iint_S \nabla_\wedge \nabla_\wedge \left\{ \underline{J}(\underline{r}')\, g \right\} dS'$$

$$= \frac{iZ_o}{k} \iint_S \left\{ (\underline{J}(\underline{r}') \cdot \nabla')\nabla g - k^2 \underline{J}(\underline{r}')\, g \right\} dS' . \tag{9}$$

We now form the tangential component and let $\underline{r} \to \underline{r}_o$. Since (ref. 4)

$$\iint_S \underline{J}(\underline{r}')\, g\, dS'$$

is a continuous function of $\underline{r}$ even for $\underline{r}$ on S, and $\hat{n}_\wedge \underline{E}(\underline{r}_o) = 0$ from the boundary condition,

$$ik\, Y_o\, \hat{n}_\wedge \underline{E}^i(\underline{r}_o) = k^2 \iint_S \hat{n}_\wedge \underline{J}(\underline{r}')\, g\, dS' - \lim_{\underline{r} \to \underline{r}_o} \hat{n}_\wedge \iint_S (\underline{J}(\underline{r}') \cdot \nabla')\nabla g\, dS'. \tag{10}$$

This is an integral equation for $\underline{J}$, but a rather unpleasant one, not least because of the higher order non-integrable singularity of the second integral in the limit $\underline{r} = \underline{r}_o$.

It could therefore be helpful if we could reduce the order of the singularity even at the expense of introducing surface derivatives of $\underline{J}$, a process equivalent to an integration by parts.

To see whether this is possible, we return to the situation in which $\underline{r}$ is not on S. Since

$$\underline{I} = \iint_S (J(\underline{r}') \cdot \nabla') \nabla g \, dS' = \nabla \iint_S (\hat{n}'_\wedge [\underline{H}]) \cdot \nabla' g \, dS'$$

where $[\underline{H}] = \underline{H}^+(\underline{r}') - \underline{H}^-(\underline{r}')$, we have

$$\underline{I} = \nabla \iint_S \hat{n}' \cdot ([\underline{H}]_\wedge \nabla' g) \, dS'$$

$$= \nabla \iint_S \hat{n}' \cdot \left\{ g \nabla'_\wedge [\underline{H}] - \nabla'_\wedge (g [\underline{H}]) \right\} dS' .$$

But $\hat{n}' \cdot \nabla'_\wedge \underline{H}^{+,-} = -\nabla'_S \cdot \underline{K}^{+,-}$ where $\underline{K}^{+,-} = \hat{n}_\wedge \underline{H}^{+,-}$ is the surface current and ∇'_S is the surface divergence. Hence

$$\hat{n}' \cdot \nabla'_\wedge [\underline{H}] = -\nabla'_S \cdot J$$

and

$$\underline{I} = -\iint_S (\nabla'_S \cdot \underline{J}) \nabla g \, dS' - \nabla \iint_S \hat{n}' \cdot \nabla'_\wedge (g [\underline{H}] \, dS' . \qquad (11)$$

For a closed surface S, the second term on the right hand side vanishes, as can be seen by application of the divergence theorem. For an open surface, however, application of Stokes' theorem gives

$$\nabla \iint_S \hat{n}' \cdot \nabla'_\wedge (g [\underline{H}]) \, dS' = \oint_C \nabla g [\underline{H}] \cdot \hat{t}' \, d\ell'$$

$$= \oint_C \nabla g (\hat{n}'_\wedge \hat{t}') \cdot \underline{J} (\underline{r}') \, d\ell' \qquad (12)$$

where the integration is around the (closed) edge C in the positive direction defined by the unit vector $\hat{t}'$.

We are now left with

$$\underline{E}^i(\underline{r}) - \underline{E}(\underline{r}) = -\frac{iZ_o}{k} \iint_S \left\{ k^2 \underline{J}(\underline{r}') g + (\nabla_s' \cdot \underline{J}) \nabla g \right\} dS'$$

$$- \frac{iZ_o}{k} \oint_C \nabla g \, (\hat{n}' \wedge t') \cdot \underline{J}(\underline{r}') \, d\ell' \tag{13}$$

for $\underline{r}$ not on S, from which an integral equation can be obtained by allowing $\underline{r} \to \underline{r}_o$ and applying the boundary condition. The result is identical to the integral equation given by Poggio and Miller (ref. 5) if and only if the line integral vanishes. This is true for a perfectly conducting surface by virtue of the edge condition, in which case

$$ikY_o \, \hat{n} \wedge \underline{E}^i(\underline{r}_o) = \iint_S \hat{n} \wedge \left\{ k^2 \underline{J}(\underline{r}') g(\underline{r}_o | \underline{r}') + (\nabla_s' \cdot \underline{J}) \nabla g(\underline{r}_o | \underline{r}') \right\} dS' \,. \tag{14}$$

For an imperfectly conducting or resistive surface, however, the line integral in (13) is in general non-zero*, and since its singularity for $\underline{r}$ on S is a non-integrable one, the resulting integral equation is much more difficult to handle.

Either of the equations (10) and (14) is a possible starting point for an analysis of the spherical shell problem. Equation (14) is superficially simpler and has a less singular kernel, but the price that is paid is the occurrence of surface derivatives of the currents in addition to the currents themselves. In contrast, (10) involves only the currents, but because of the highly singular kernel, the limiting operation is more

---

*It is of interest to note that a derivation from the Stratton-Chu equations does not produce a line integral under any circumstances by virtue of the requirement of a closed surface of integration. The validity of the resulting integral equation for a perfectly conducting shell should therefore be regarded as fortuitous.

difficult to perform. Nevertheless, as shown by Liepa et al.(ref. 6), the second derivative singularity which (10) contains can be handled numerically; the equation can also be trivially extended to, say, a resistive shell, but even in the present circumstances it is not self-evident that (14) is a more convenient equation to use. We shall therefore pursue each in turn, starting with (10), and specialize each to the case of an open, perfectly conducting spherical shell of radius a.

SECTION III

INTEGRAL EQUATION (10)

On introducing spherical polar coordinates with origin at the center of the shell, the coordinates of the observation point become (r, θ, ϕ) whilst those of the integration point are (a, θ', ϕ'), implying

$$R = \sqrt{r^2 + a^2 - 2\, ra \cos \gamma} \tag{15}$$

where $$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi' - \phi). \tag{16}$$

Let $$\underline{J}\,(\underline{r}') = J_1(\theta', \phi')\,\hat{\theta}' + J_2\,(\theta', \phi')\,\hat{\phi}' \;. \tag{17}$$

On expressing the unit vectors in terms of the corresponding vectors at the observation point, we have

$$\begin{aligned}\underline{J}\,(\underline{r}') = & \left[J_1\left\{\sin \theta \cos \theta' \cos (\phi'-\phi) - \cos \theta \sin \theta'\right\} - J_2 \sin \theta \sin (\phi'-\phi)\right] \hat{r} \\ & + \left[J_1\left\{\cos \theta \cos \theta' \cos (\phi'-\phi) + \sin\theta \sin\theta'\right\} - J_2 \cos \theta \sin (\phi'-\phi)\right] \hat{\theta} \\ & + \left[J_1 \cos \theta' \sin (\phi'-\phi) + J_2 \cos (\phi'-\phi)\right] \hat{\phi} \;,\end{aligned}$$

and since $\hat{n} = \hat{r}$,

$$\begin{aligned}\hat{n}_\wedge \underline{J}\,(\underline{r}') = & - \left[J_1 \cos \theta' \sin (\phi'-\phi) + J_2 \cos(\phi'-\phi)\right] \hat{\theta} \\ & + \left[J_1\left\{\cos \theta \cos \theta' \cos (\phi'-\phi) + \sin \theta \sin \theta'\right\} - J_2 \cos\theta \sin (\phi'-\phi)\right] \hat{\phi} \,.\end{aligned} \tag{18}$$

Also, from (17),

$$(\underline{J} \cdot \nabla')\, g = \frac{1}{a}\left(J_1 \frac{\partial g}{\partial \theta'} + \frac{J_2}{\sin \theta'} \frac{\partial g}{\partial \phi'}\right) \;.$$

But
$$\frac{\partial R}{\partial \theta'} = \frac{r\,a}{R}\left\{\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\phi'-\phi)\right\}, \tag{19}$$

$$\frac{\partial R}{\partial \phi'} = \frac{r\,a}{R}\sin\theta\sin\theta'\sin(\phi'-\phi) \; . \tag{20}$$

Hence

$$(\underline{J}\cdot\nabla')g = \left[J_1\left\{\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\phi'-\phi)\right\} + J_2\sin\theta\sin(\phi'-\phi)\right]\frac{r}{R}\frac{\partial g}{\partial R}$$

and since

$$\hat{n}_\wedge\nabla = \frac{1}{r}\left(-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

with

$$\frac{\partial R}{\partial \theta} = \frac{r\,a}{R}\left\{\sin\theta\cos\theta' - \cos\theta\sin\theta'\cos(\phi'-\phi)\right\}$$

$$\frac{\partial R}{\partial \phi} = -\frac{r\,a}{R}\sin\theta\sin\theta'\sin(\phi'-\phi) \; ,$$

it follows that

$$\begin{aligned}
\hat{n}_\wedge\nabla\left[(\underline{J}\cdot\nabla')g\right] = &\Bigg(\Big[J_1\cos\theta'\sin(\phi'-\phi) + J_2\cos(\phi'-\phi)\Big]\frac{1}{R}\frac{\partial g}{\partial R} \\
&+ \Big[J_1\left\{\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\phi'-\phi)\right\} + J_2\sin\theta\sin(\phi'-\phi)\Big]\sin\theta'\sin(\phi'-\phi) \\
&\qquad\cdot\frac{r\,a}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial g}{\partial R}\right)\Bigg)\hat{\theta} \\
&+ \Bigg(\Big[-J_1\left\{\sin\theta\sin\theta' + \cos\theta\cos\theta'\cos(\phi'-\phi)\right\} + J_2\cos\theta\sin(\phi'-\phi)\Big]\frac{1}{R}\frac{\partial g}{\partial R} \\
&+ \Big[J_1\left\{\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\phi'-\phi)\right\} + J_2\sin\theta\sin(\phi'-\phi)\Big]\Big\{\sin\theta\cos\theta' \\
&\qquad - \cos\theta\sin\theta'\cos(\phi'-\phi)\Big\}\frac{r\,a}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial g}{\partial R}\right)\Bigg)\hat{\phi} \; .
\end{aligned} \tag{21}$$

When the results of eqs. (20) and (21) are substituted into (10), the electric field integral equation becomes

$$ikY_o \hat{n} \wedge \underline{E}^i(\underline{r}_o) = a^2 \lim_{\underline{r} \to \underline{r}_o} \iint_S \left\{ (f_{11} J_1 + f_{12} J_2)\hat{\phi} - (f_{21} J_1 + f_{22} J_2)\hat{\theta} \right\} \sin\theta' \, d\theta' \, d\phi' \qquad (22)$$

where $\underline{r}_o$ is a point on S and

$$f_{11} = \left\{ \sin\theta \sin\theta' + \cos\theta \cos\theta' \cos(\phi'-\phi) \right\} \left(k^2 g + \frac{1}{R}\frac{\partial g}{\partial R} \right)$$

$$- \left\{ \cos\theta \sin\theta' - \sin\theta \cos\theta' \cos(\phi'-\phi) \right\} \left\{ \sin\theta \cos\theta' - \cos\theta \sin\theta' \cos(\phi'-\phi) \right\} \frac{ra}{R} \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial g}{\partial R} \right) \qquad (23)$$

$$f_{12} = -\sin(\phi'-\phi) \left[\cos\theta \left(k^2 g + \frac{1}{R}\frac{\partial g}{\partial R} \right) + \sin\theta \left\{ \sin\theta \cos\theta' - \cos\theta \sin\theta' \cos(\phi'-\phi) \right\} \frac{ra}{R} \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial g}{\partial R} \right) \right] \qquad (24)$$

$$f_{21} = \sin(\phi'-\phi) \left[\cos\theta' \left(k^2 g + \frac{1}{R}\frac{\partial g}{\partial R} \right) + \sin\theta' \left\{ \cos\theta \sin\theta' - \sin\theta \cos\theta' \cos(\phi'-\phi) \right\} \frac{ra}{R} \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial g}{\partial R} \right) \right] \qquad (25)$$

$$f_{22} = \cos(\phi'-\phi) \left(k^2 g + \frac{1}{R}\frac{\partial g}{\partial R} \right) - \sin\theta \sin\theta' \sin^2(\phi'-\phi) \frac{ra}{R} \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial g}{\partial R} \right) . \qquad (26)$$

If a neighborhood Δ of the point $\underline{r}_o = (a, \theta, \phi)$ is excluded from the integration in (22), R will remain finite over the rest of the surface even in the limit $r = a$, and the limiting operation can be applied to the integrand directly. This is equivalent to putting $r = a$, in which case

$$R = a\sqrt{2(1 - \cos\gamma)} = 2a \left| \sin \frac{\gamma}{2} \right| . \qquad (27)$$

At the point $\theta' = \theta$, $\phi' = \phi$, however, R is infinite when $r = a$, and the resulting singularity is a non-integrable one of the form encountered by Liepa et al (ref. 6). We are then forced to estimate the contribution of the self cell analytically, and the result will actually become infinite as the dimensions of the cell shrink to zero.

Within this cell, $\theta' - \theta$ and $\phi' - \phi$ are small, and hence

$$f_{11} \simeq \left\{k^2 g + \frac{1}{R}\frac{\partial g}{\partial R} + (\theta' - \theta)^2 \frac{ra}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial g}{\partial R}\right)\right\}$$

$$f_{12} \simeq -(\theta' - \phi)\left\{\cos\theta\,(k^2 g + \frac{1}{R}\frac{\partial g}{\partial R}) - (\theta' - \theta)\sin\theta\,\frac{ra}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial g}{\partial R}\right)\right\}$$

$$f_{21} \simeq (\phi' - \phi)\left\{\cos\theta\,(k^2 g + \frac{1}{R}\frac{\partial g}{\partial R}) + (\theta' - \theta)\sin\theta\,\frac{ra}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial g}{\partial R}\right)\right\} \tag{28}$$

$$f_{22} \simeq \left\{k^2 g + \frac{1}{R}\frac{\partial g}{\partial R} + (\phi' - \phi)^2 \sin^2\theta\,\frac{ra}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial g}{\partial R}\right)\right\} .$$

Also

$$R \simeq \sqrt{\epsilon^2 + ra\left\{(\theta' - \theta)^2 + \sin^2\theta\,(\phi' - \phi)^2\right\}} \tag{29}$$

where, for brevity, we have written $r - a = \epsilon$. It will be assumed that ϵ is a small non-zero quantity, but it is important that no approximation with respect to r, e.g. replacing ra by a^2, be made at this stage. Furthermore,

$$k^2 g + \frac{1}{R}\frac{\partial g}{\partial R} = -\frac{1}{4\pi R^3}\left\{1 - ikR - (kR)^2\right\} e^{ikR}$$

$$= -\frac{1}{4\pi R^3}\left\{1 + O(\overline{kR}^2)\right\} \tag{30}$$

and

$$\frac{ra}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial g}{\partial R}\right) = \frac{3\,ra}{4\pi R^5}\left\{1 - ikR - \frac{1}{3}(kR)^2\right\} e^{ikR}$$

$$= \frac{3\,ra}{4\pi R^5}\left\{1 + O(\overline{kR}^2)\right\} . \tag{31}$$

To illustrate the evaluation of the self-cell contribution, consider

$$I = \iint_{\Delta} \frac{1}{R^5} (\theta'-\theta)^2 \, d\theta' \, d\phi' = \frac{1}{r^2 a^2 \sin\theta} \iint_{\Delta} \frac{x^2 \, dx \, dy}{(\epsilon^2 + x^2 + y^2)^{5/2}}$$

where we have written $x = \sqrt{ra}\,(\theta'-\theta)$, $y = \sqrt{ra}\,\sin\theta\,(\phi'-\phi)$. If we now approximate Δ by a circular disk of radius ℓ , the substitution $x = \rho\cos\alpha$, $y = \rho\sin\alpha$ reduces the integral to

$$I = \frac{1}{r^2 a^2 \sin\theta} \int_0^{2\pi} \int_0^{\ell} \frac{\rho^3 \cos^2\alpha \, d\rho \, d\alpha}{(\epsilon^2 + \rho^2)^{5/2}} = \frac{\pi}{r^2 a^2 \sin\theta} \int_0^{\ell} \frac{\rho^3 \, d\rho}{(\epsilon^2 + \rho^2)^{5/2}}$$

which can be evaluated to give

$$\iint_{\Delta} \frac{1}{R^5} (\theta'-\theta)^2 \, d\theta' \, d\phi' = \frac{\pi}{3 r^2 a^2 \sin\theta} \left\{ \frac{2}{|\epsilon|} - \frac{3}{(\ell^2 + \epsilon^2)^{1/2}} + \frac{\epsilon^2}{(\ell^2 + \epsilon^2)^{3/2}} \right\} ,$$

where $\pi\ell^2$ is the area of the cell. Similarly,

$$\iint_{\Delta} \frac{1}{R^5} (\phi'-\phi)^2 \, d\theta' \, d\phi' = \frac{\pi}{3 r^2 a^2 \sin^3\theta} \left\{ \frac{2}{|\epsilon|} - \frac{3}{(\ell^2 + \epsilon^2)^{1/2}} + \frac{\epsilon^2}{(\ell^2 + \epsilon^2)^{3/2}} \right\}$$

whereas

$$\iint_{\Delta} \frac{1}{R^5} (\theta'-\theta)(\phi'-\phi) \, d\theta' \, d\phi' = 0 \; ;$$

and

$$\iint_{\Delta} \frac{1}{R^3} \, d\theta' \, d\phi' = \frac{2\pi}{ra \sin\theta} \left(\frac{1}{|\epsilon|} - \frac{1}{(\ell^2 + \epsilon^2)^{1/2}} \right)$$

whereas

$$\iint_{\Delta} \frac{1}{R^3} (\phi' - \phi) \, d\theta' \, d\phi' = 0 \, .$$

Using these results in conjunction with eqs. (28), (30) and (31), the self cell contribution to the integral in (22) is found to be

$$\iint_{\Delta}\left\{(f_{11}J_1+f_{12}J_2)\hat{\phi}-(f_{21}J_1+f_{22}J_2)\hat{\theta}\right\}\sin\theta'\,d\theta'\,d\phi'$$

$$=\frac{1}{4ra}(J_1\hat{\phi}-J_2\hat{\theta})\left\{-\frac{2}{|\epsilon|}+\frac{2}{(\ell^2+\epsilon^2)^{1/2}}+\frac{2}{|\epsilon|}-\frac{3}{(\ell^2+\epsilon^2)^{1/2}}+\frac{\epsilon^2}{(\ell^2+\epsilon^2)^{3/2}}\right\}$$

$$=-\frac{1}{4ra}\frac{\ell^2}{(\ell^2+\epsilon^2)^{3/2}}(J_1\hat{\phi}-J_2\hat{\theta})\ . \tag{32}$$

For $\ell \neq 0$ this tends to

$$-\frac{1}{4a^2\ell}(J_1\hat{\phi}-J_2\hat{\theta})$$

as $\epsilon \to 0$.

The integral equation (22) can now be written as

$$i\,Y_o\,\hat{n}\wedge\underline{E}^i(\theta,\phi)=-\frac{1}{4k\ell}\left\{J_1(\theta,\phi)\hat{\phi}-J_2(\theta,\phi)\hat{\theta}\right\}$$
$$+\frac{a^2}{k}\iint_{S-\Delta}\left\{(f_{11}J_1+f_{12}J_2)\hat{\phi}-(f_{21}J_1+f_{22}J_2)\hat{\theta}\right\}\sin\theta'\,d\theta'\,d\phi' \tag{33}$$

leading to the following two scalar but coupled integral equations for the current components J_1 and J_2 :

$$iY_oE^i_\theta(\theta,\phi)=-\frac{1}{4k\ell}J_1(\theta,\phi)+\frac{a^2}{k}\iint_{S-\Delta}\left\{f_{11}J_1(\theta',\phi')+f_{12}J_2(\theta',\phi')\right\}\sin\theta'd\theta'\,d\phi' \tag{34}$$

$$iY_oE^i_\phi(\theta,\phi)=-\frac{1}{4k\ell}J_2(\theta,\phi)+\frac{a^2}{k}\iint_{S-\Delta}\left\{f_{21}J_1(\theta',\phi')+f_{22}J_2(\theta',\phi')\right\}\sin\theta'\,d\theta'd\phi' \tag{35}$$

Numerical techniques for the solution of integral equations with this type of self-cell contribution have been discussed by Knott and Senior (ref. 7) and Liepa et al (ref. 6), and apart from the fact that the integrals are now surface integrals, the equations have the same character as the ones which they have treated.

Nevertheless, the kernels k_{11}, etc are rather formidable (see eqs. (23) through (26)), and because of the derivatives which they contain, it is natural to think of eliminating at least some of them by integration by parts. This can indeed be done using eqs. (19) and (20), but in view of the disk-like approximation to the self cell which (34) and (35) imply, it is easier to carry out the integration prior to taking the limit $\underline{r} - \underline{r}_0$. Not surprisingly, the end result involves surface derivatives of the currents and is precisely that which could have been obtained by starting with the integral equation (14).

SECTION IV

INTEGRAL EQUATION (14)

If we again introduce the spherical polar coordinates (r, θ, ϕ) and use eq. (18), the integral equation becomes

$$ikY_o \hat{n} \wedge \underline{E}^i(\underline{r}_o) = \lim_{\underline{r} \to \underline{r}_o} \iint_S \Bigg\{ \Bigg(- \Big[J_1 \cos\theta' \sin(\phi' - \phi) + J_2 \cos(\phi' - \phi) \Big] \hat{\theta}$$

$$+ \Big[J_1 \Big\{ \cos\theta \cos\theta' \cos(\phi - \phi) + \sin\theta \sin\theta' \Big\} - J_2 \cos\theta \sin(\phi' - \phi) \Big] \hat{\phi} \Bigg) k^2 g$$

$$+ \frac{1}{a \sin\theta'} \left[\frac{\partial}{\partial\theta'} (J_1 \sin\theta') + \frac{\partial}{\partial\phi'} J_2 \right] \left(- \frac{1}{r \sin\theta} \frac{\partial g}{\partial\phi} \hat{\theta} + \frac{1}{r} \frac{\partial g}{\partial\theta} \hat{\phi} \right) \Bigg\} a^2 \sin\theta' \, d\theta' \, d\phi' \quad . (36$$

Since the integral is continuous as $\underline{r} \to \underline{r}_o$, the limit can be applied to the integrand directly, as was done in writing (14). Moreover, $\frac{\partial g}{\partial \phi} = - \frac{\partial g}{\partial \phi'}$, which allows us to eliminate this derivative using integration by parts. The result then is the pair of coupled integral equations

$$ikY_o E^i_\theta(\theta, \phi) = \iint_S \Bigg\{ \Big[J_1 \Big\{ \cos\theta \cos\theta' \cos(\phi' - \phi) + \sin\theta \sin\theta' \Big\} - J_2 \cos\theta \sin(\phi' - \phi) \Big] k^2 a^2 \sin\theta' g$$

$$+ \left[\frac{\partial}{\partial\theta'} (J_1 \sin\theta') + \frac{\partial}{\partial\phi'} J_2 \right] \frac{\partial g}{\partial\theta} \Bigg\} d\theta' \, d\phi' \quad , \tag{37}$$

$$ikY_o E^i_\phi(\theta, \phi) = \iint_S \Bigg\{ \Big[J_1 \cos\theta' \sin(\phi' - \phi) + J_2 \cos(\phi' - \phi) \Big] k^2 a^2 \sin\theta'$$

$$+ \left[\frac{\partial^2}{\partial\theta' \partial\phi'} (J_1 \sin\theta') + \frac{\partial^2 J_2}{\partial\phi'^2} \right] \Bigg\} g \, d\theta' \, d\phi' \quad . \tag{38}$$

These are identical to the ones obtained on integrating (34) and (35) by parts and there seems little doubt that they are preferable to (34) and (35) for numerical solution. We shall therefore concentrate on them.

SECTION V

PLANE WAVE AT SYMMETRICAL INCIDENCE

For a plane wave incident at any angle, the incidence electric and magnetic fields can be expanded in Fourier series in the azimuthal angle ϕ using a cylindrical mode expansion. Each such mode then excites currents J_1 and J_2 having the corresponding ϕ dependence, and this allows us not only to eliminate the ϕ' derivatives from eqs. (37) and (38) but also to convert the surface integrals to one-dimensional (line) integrals.

A special case is that in which the incident field is a plane wave at normal incidence on the circular aperture, and this is the one that we pursue. Let

$$\underline{E}^i = \hat{x}\, e^{-ikz} \quad , \quad \underline{H}^i = -\hat{y}\, Y_0\, e^{-ikz} \quad . \tag{39}$$

Then on the shell

$$E^i_\theta(\theta, \phi) = \cos\theta \cos\phi \; e^{-ika\cos\theta}$$

$$E^i_\phi(\theta, \phi) = -\sin\phi \; e^{-ika\cos\theta}$$

and, from symmetry,

$$J_1(\theta, \phi) = J_1(\theta)\cos\phi \; , \qquad J_2(\theta, \phi) = J_2(\theta)\sin\phi \quad . \tag{40}$$

Writing $\psi' = \phi' - \phi$, we have

$$\cos\phi' = \cos\psi' \cos\phi - \sin\psi' \sin\phi$$

$$\sin\phi' = \sin\psi' \cos\phi + \cos\psi' \sin\phi$$

and since g is a periodic function of ψ' with period 2π, terms which are odd functions of ψ' integrate to zero. When these terms are eliminated, eqs. (37) and (38) become

$$ikY_o \cos\theta\, e^{-ika\cos\theta} = \iint_S \left\{(ka)^2 \sin\theta' \left[J_1(\theta')(\cos\theta\cos\theta'\cos\psi' + \sin\theta\sin\theta')\cos\psi'\right.\right.$$

$$\left.\left. - J_2(\theta')\cos\theta\sin^2\psi'\right] g + \left[\frac{\partial}{\partial\theta'}(J_1\sin\theta') + J_2(\theta')\right]\cos\psi'\frac{\partial g}{\partial\theta}\right\} d\theta'\, d\psi' ,$$

$$ikY_o e^{-ika\cos\theta} = \iint_S \left\{(ka)^2 \sin\theta' \left[J_1(\theta')\cos\theta'\sin^2\psi' - J_2(\theta')\cos^2\psi'\right] g\right.$$

$$\left. + \left[\frac{\partial}{\partial\theta'}(J_1\sin\theta') + J_2(\theta')\right]\cos\psi'\, g\right\} d\theta'\, d\psi' ,$$

which can be written as

$$ikY_o \cos\theta\, e^{-ika\cos\theta} = \int_{\theta_o}^{\pi} \left\{(ka)^2 \sin\theta' \left[J_1(\theta')\left\{\cos\theta\cos\theta' K_2(\theta,\theta')\right.\right.\right.$$

$$\left.\left. + \sin\theta\sin\theta' K_1(\theta,\theta')\right\} - J_2(\theta')\cos\theta\left\{K_o(\theta,\theta') - K_2(\theta,\theta')\right\}\right]$$

$$\left. + \left[\frac{\partial}{\partial\theta'}(J_1\sin\theta') + J_2(\theta')\right]\frac{\partial}{\partial\theta}K_1(\theta,\theta')\right\} d\theta' \qquad (41)$$

$$ikY_o e^{-ika\cos\theta} = \int_{\theta_o}^{\pi} \left\{(ka)^2 \sin\theta' \left[J_1(\theta')\cos\theta'\left\{K_o(\theta,\theta') - K_2(\theta,\theta')\right\}\right.\right.$$

$$\left.\left. - J_2(\theta')K_2(\theta,\theta')\right] + \left[\frac{\partial}{\partial\theta'}(J_1\sin\theta') + J_2(\theta')\right] K_1(\theta,\theta')\right\} d\theta' \qquad (42)$$

where θ_o is the half angle of the aperture and

$$K_m(\theta,\theta') = \int_0^{2\pi} g\cos^m\psi'\, d\psi' , \qquad m = 0,\ 1 \text{ or } 2 . \qquad (43)$$

The eqs. (41) and (42) are coupled integral equations similar to those obtained by Sancer and Varvatsis (ref. 8) for a solid body of revolution illuminated by a plane wave at axial incidence, and though the present problem is more complicated because of the infinitesimally thin shell and the consequent necessity of using the E field integral equation, it is somewhat disappointing that the spherical geometry has not produced a greater simplification.

Part of the difficulty lies with the functions $K_m(\theta, \theta')$, and it would certainly be nice if these integrals could be evaluated analytically and conveniently. Unfortunately, the only obvious method of evaluation other than numerical introduces expansions in spherical modes and produces the divergent series which we have been at pains to avoid. Since (ref. 9)

$$g(\underline{r}|\underline{r}') = \frac{ik}{4\pi} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\gamma) \begin{cases} j_n(ka)\, h_n^{(1)}(kr) , & r > a \\ j_n(kr)\, h_n^{(1)}(ka) , & r < a \end{cases}$$

where $P_n(\cos\gamma)$ is the Legendre polynomial, and

$$P_n(\cos\gamma) = P_n(\cos\theta)\, P_n(\cos\theta') + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)\, P_n^m(\cos\theta') \cos m\psi'$$

it follows that

$$\int_0^{2\pi} g(\underline{r}|\underline{r}')\, d\psi' = \frac{ik}{2} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta)\, P_n(\cos\theta') \begin{cases} j_n(ka)\, h_n^{(1)}(kr), & r > a \\ j_n(kr)\, h_n^{(1)}(ka), & r < a \end{cases} .$$

This diverges when $r = a$ regardless of θ and θ', and differentiation with respect to θ or θ' only makes things worse.

In actual fact, the integral expressions (43) are infinite only when $\theta' = \theta$, reflecting the singular nature of the integral equations, and though the singularities of (41) and (42) are integrable, it is necessary to estimate the self cell contributions analytically in any application of the method of moments. The dominant singularity is that provided by $\partial K_1/\partial\theta$ in eq. (41), but since

$$\cos\psi' \frac{\partial R}{\partial\theta} = -\frac{\partial R}{\partial\theta'} + \cot\theta \sin\psi' \frac{\partial R}{\partial\psi'} ,$$

it follows that

$$\frac{\partial}{\partial\theta} K_1(\theta,\theta') = -\frac{\partial}{\partial\theta'} K_o(\theta,\theta') - \cot\theta\, K_1(\theta,\theta') .$$

Hence,

$$\int_{\theta_o}^{\pi} A(\theta')\frac{\partial}{\partial\theta} K_1(\theta,\theta')\,d\theta' = -\int_{\theta_o}^{\pi} A(\theta')\frac{\partial}{\partial\theta'}K_o(\theta,\theta')d\theta' - \cot\theta \int_{\theta_o}^{\pi} A(\theta')K_1(\theta,\theta')\,d\theta' \tag{44}$$

where

$$A(\theta') = \frac{\partial}{\partial\theta'}(J_1 \sin\theta') + J_2(\theta') .$$

and if the self cell extends from $\theta - \delta_1$ to $\theta + \delta_1$, the self cell contribution to the first term on the right hand side of (44) is simply

$$A(\theta)\left\{K_o(\theta,\theta-\delta_1) - K_o(\theta,\theta+\delta_1)\right\}.$$

For the second integral, the usual disk approximation to the self cell yields

$$\int_{\theta-\delta_1}^{\theta+\delta_1} A(\theta')K_m(\theta,\theta')\,d\theta' = \frac{\ell}{2a^2} \tag{45}$$

where

$$\ell = a\sqrt{\frac{\sin\theta\,\delta_1\delta_2}{\pi}}$$

is the equivalent radius of the cell.

No attempt has yet been made to program eqs. (41) and (42) for numerical solution.

REFERENCES

1. T.B.A. Senior and G.A. Desjardins, Field Penetration into a Spherical Cavity, Interaction Note No. 142, Air Force Weapons Laboratory, August 1973. See also IEEE Trans. EMC-16, 205-208, 1974.

2. S. Chang and T.B.A. Senior, Scattering by a Spherical Shell with a Circular Aperture, 1363-5-T, The University of Michigan Radiation Laboratory, Ann Arbor, Michigan, 1969. See Interaction Note No. 141.

3. T.B.A. Senior, Field Penetration into a Cylindrical Cavity, Interaction Note No. 221, Air Force Weapons Laboratory, January 1975.

4. J. van Bladel, Electromagnetic Fields, McGraw-Hill Book Co., New York, 1964, p. 354.

5. A.J. Poggio and E.K. Miller, "Integral Equation Solutions of Three-Dimensional Scattering Problems", in Computer Techniques for Electromagnetics (Ed. R. Mittra), Pergamon Press, New York, 1973, p. 167.

6. V.V. Liepa, E.F. Knott and T.B.A. Senior, Scattering from Two-Dimensional Bodies with Absorber Sheets, 011764-2-T, The University of Michigan Radiation Laboratory, Ann Arbor, Michigan, 1974.

7. E.F. Knott and T.B.A. Senior, Non-Specular Radar Cross Section Study, 011764-1-T, The University of Michigan Radiation Laboratory, Ann Arbor, Michigan, 1974.

8. M.I. Sancer and A.D. Varvatsis, Calculation of the Induced Surface Current Density on a Perfectly Conducting Body of Revolution, Interaction Note No. 101, 1972.

9. J.A. Stratton, Electromagnetic Theory, McGraw-Hill Book Co., New York, 1941, pp. 408, 414.