Three Dimensional Moment Method Simulation of Penetrable Scatterers Consisting of Non-metallic and Circuit Analog Surfaces

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THREE DIMENSIONAL MOMENT METHOD SIMULATION OF PENETRABLE SCATTERERS CONSISTING OF NON-METALLIC AND CIRCUIT ANALOG SHEETS

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Abstract

This report describes the formulation for a mixed integral equation formulation implemented in the code CADRIS. An important aspect of this implementation is the inclusion of Circuit Analog sheet models that is not available in other codes. This capability is combined with modeling capabilities for Resistive sheets, metallic, dielectric and impedance surfaces, and combination of all. The report begins with the introductions of the appropriate integral equations and proceeds to develop their discretization. Several validations are given for PEC and composite scatterers.

1) Surface Integral Equations

Consider the general electromagnetic scattering problem depicted in Fig. 1.1.

![Figure 1.1. Geometry showing the various boundaries](image)

We are interested in a field point \( \mathbf{r} \), located in a closed volume \( V \) or on a regular surface \( S_i (i = 1, 2, \ldots, n) \). Beginning with the vector Green’s theorem,

\[
\int_V \left[ (Q (\nabla' \times \nabla' \times T)) - (T (\nabla' \times \nabla' \times Q)) \right] dV' = \int_{\sum_{i=1}^n S_i} \left( (T \times \nabla' \times Q) - (Q \times \nabla' \times T) \right) ds' = 0
\]

(1)

where, \( Q(r), T(r) \in \mathbb{C}^2; r \in V, \sum_{i=1}^n S_i \). Here the primes refer to the primed(integration) coordinates. To derive the integral equations for the electric and magnetic currents on the surfaces \( S_i, i = 1, 2, \ldots, n \) we set
\( \mathbf{T} = \mathbf{E}(r')(\text{electric field}), \) and \( \mathbf{Q} = \mathbf{I}(r, r') \) with

\[
G(r, r') = \frac{e^{-ik|r-r'|}}{4\pi|r-r'|}, \quad k = \omega\sqrt{\epsilon\mu}
\]  

(2)

Note that when \( r \to r' \) in \( V, G, \nabla G \) and \( \nabla^2 G \) have singularities. To overcome this singularity problem, when integrating \( G \) or its derivatives we exclude an infinitesimal sphere of volume \( V_\delta \to 0 \) and centered at \( r = r' \) and shown in Fig.1.2. We deal with the spherical volume \( V_\delta \) of radius \( \delta \to 0 \) separately by invoking the divergence theorem.

\[
V' = V - V_\delta
\]

\[
S' = \sum_{i=1}^{n} S_i + S_\delta
\]

Figure 1.2. Geometry for singularity problem

Next we introduce into (1) Maxwell equations

\[
\nabla \times \mathbf{E} + j\omega\mu\mathbf{H} = -\mathbf{M}
\]  

(3a)

\[
\nabla \times \mathbf{H} - j\omega\epsilon\mathbf{E} = \mathbf{J}
\]  

(3b)

\[
\nabla \cdot (\mu\mathbf{H}) = \rho_m
\]  

(3c)

\[
\nabla \cdot (\epsilon\mathbf{E}) = \rho
\]  

(3d)

After some straightforward vector manipulations, (1) can be written as;

\[
\int_{V'} [j\omega\mu \mathbf{J} - \mathbf{M} \times \nabla'G - (\rho/\epsilon) \nabla'G]dv' \\
+ \int_{\sum_{i=1}^{n} S_i + S_\delta} j\omega\mu(n' \times \mathbf{H})\Phi - (n' \times \mathbf{E}) \times \nabla'G - (n' \cdot \mathbf{E})\nabla'G]ds' = 0
\]  

(4)

where we have assumed that \( V \) in equation (1) is linear, isotropic and homogeneous.

When \( r' \) is located on one of the \( S_i(i = 1, 2, \ldots) \) surfaces, we proceed to extract the integral singularity noted earlier. Refering to Fig.1.3, we rewrite the surface integrals as,
\[
\int_{S_{i} + S_{o}} \mathbf{n}' \times \mathbf{E} \times \nabla' G = \lim_{\delta \to 0} \left[ \int_{S_{i} - S_{o}} + \int_{S_{o}} + \int_{S_{i}} \right]
\]

(5)

\[
\lim_{\delta \to 0} \left[ \int_{S_{o}} + \int_{S_{i}} \right] = -E(r)[1 - \frac{P}{2\pi}]
\]

(6)

Here, \(P\) is the absolute value of the solid angle subtended by \(S_{i} \delta\) at \(r\) in the limit as \(\delta \to 0\).

\[
P = \begin{cases} 
\pi, & r \in S_i \\
0, & elsewhere 
\end{cases}
\]

(7)

Based on (6), (4) can now be written as

\[
\theta(r)\mathbf{E}(r) = \left[ \int_{V'} [j \omega \mu \mathbf{J} + \mathbf{M} \times \nabla' \Phi - (\rho/\epsilon) \nabla' \Phi] d\mathbf{v}' 
+ \int_{\sum_{i=1}^{n} S_i} j \omega \mu (\mathbf{n}' \times \mathbf{H} - (\mathbf{n}' \times \mathbf{E}) \times \nabla' \Phi - (\mathbf{n}' \times \mathbf{E}) \nabla' \Phi) ds' \right]
\]

(8)

On invoking duality we also obtain the corresponding integral equation for \(\mathbf{H}\) as;

\[
\theta(r)\mathbf{H}(r) = \left[ \int_{V'} [j \omega \mu \mathbf{M} - \mathbf{J} \times \nabla' \Phi - (m/\mu) \nabla' \Phi] d\mathbf{v}' 
+ \int_{\sum_{i=1}^{n} S_i} -j \omega \mu (\mathbf{n}' \times \mathbf{E} \Phi \times (\mathbf{n}' \times \mathbf{H}) - (\mathbf{n}' \times \mathbf{H}) \nabla' \Phi) ds' \right]
\]

(9)

Here \(\int\) denotes the Cauchy Principal Value and \(\theta(r)\) can be given as follows.

\[
\theta(r) = \begin{cases} 
2, & r \in S_i \\
1, & elsewhere 
\end{cases}
\]

(10)

In (8) and (9) since all the sources are contained in the volume \(V'\), this volume integral can be referred to as the ‘source term’. If there are no sources in \(V'\), this integral will be zero. We will assume that the sources are far from the scatterer and represent the source integral by \((\mathbf{E}', \mathbf{H}')\) which later be set a plane wave incident in the scatterer. Equation (8) and (9) can then be written as

\[
\theta(r)\mathbf{E}(r) = \theta \left[ \mathbf{E}'(r) + \int_{\sum_{i=1}^{n} S_i} [-j \omega \mu (\mathbf{n}' \times \mathbf{H} \Phi 
+ (\mathbf{n}' \times \mathbf{E}) \times \nabla' \Phi + (\mathbf{n}' \times \mathbf{E}) \nabla' \Phi) ds'] \right]
\]

(11)
\[ \theta(r)H(r) = \theta \left[ H^i(r) + \int_{S} \left[ j\omega\varepsilon (n' \times E) \Phi + (n' \times H) \times \nabla' \Phi + (n' \cdot H) \nabla' \Phi ds' \right] \right] \]  

(12)

Next we rewrite these in terms of surface current densities \((J, M)\) where

\[ J = n \times H \delta(S) \]  

(13a)

\[ M = E \times n \delta(S). \]  

(13b)

and it follows that

\[ n.E = \frac{-1}{j\omega\varepsilon} \nabla.(n \times H) = \frac{-1}{j\omega\varepsilon} \nabla.J \]  

(14a)

and

\[ n.H = \frac{1}{j\omega\mu} \nabla.(n \times E) = \frac{-1}{j\omega\mu} \nabla.M. \]  

(14b)

Sustituting (13a,b) and (14a,b) into (11) and (12) gives the integral representation

\[ \theta(r)E(r) = E^i(r) - \Lambda J + \Omega M \]  

(15a)

\[ \theta(r)H(r) = H^i(r) - \Omega J - \frac{1}{\eta^2} \Lambda M \]  

(15b)

where \(J\) and \(M\) are unknown surface current densities. Here, \(\Lambda\) and \(\Omega\) are the integro-differential operators given by,

\[ \Lambda \delta f \Gamma(r) = \int_{S} [j\omega\mu \Gamma(r') + \frac{J}{\omega\varepsilon} \nabla(\nabla' \Gamma(r'))] G(r - r')ds' \]  

(16a)

and

\[ \Omega \Gamma(r) = \int_{S} \Gamma(r') \times \nabla G(r - r')ds', \]  

(16b)

and \(\eta = \sqrt{\mu/\varepsilon}\) is the characteristic impedance of the medium.

2) Formulation for Different Type of Boundary Conditions

2.1) PEC Boundary (Metallic Surfaces)

Different types of surface integral formulations have been developed for these kind of surfaces. We give here the very well-known EFIE (Electric Field Integral Equation), MFIE (Magnetic Field Integral Equation) and CFIE (Combined Field Integral Equation) formulations.

a) EFIE Formulation

Consider the PEC surface depicted in Fig.2.1.
Using (15a) and (15b) for the fields outside the PEC surface, we can write,

\[
\theta(r)E(r) = E^i(r) - \Delta J + \Omega M, \quad r \in B_1 \cup \partial B_{12} \tag{17a}
\]

\[
\theta(r)H(r) = H^i(r) - \Omega J - \frac{1}{\eta^2} \Lambda M, \quad r \in B_1 \cup \partial B_{12} \tag{17b}
\]

\[
E_2(r) = H_2(r) = 0, \quad r \in B_2 \tag{17c}
\]

To construct the integral equation, we note that on the metallic surface,

\[
n \times [E_1 - E_2] = n \times E_1 = 0 \tag{18a}
\]

\[
n \times [H_1 - H_2] = n \times H_1 = J \tag{18b}
\]

These imply, \( M = 0 \) and thus (17a) become,

\[
E_1(r) = E^i(r) - \Lambda J(r), \quad r \in B_1 \tag{19a}
\]

and

\[
n \times E^i(r) = n \times \Lambda J(r), \quad r \in \partial B_{12} \tag{19b}
\]

The above are the so called EFIE whose solution gives the unknown current on the surface.

b) MFIE Formulation

From (17b) and (18b) with \( M = 0 \), the magnetic field on the boundary \( B_{12} \) is given by

\[
\frac{1}{2} J(r) = n \times H^i(r) - n \times \Omega J(r), \quad r \in \partial B_{12} \tag{20}
\]

c) CFIE Formulation
The solution of EFIE and MFIE formulation can return non-physical results at internal resonant frequencies. In this case we resort to the CFIE to overcome this problem (ref Peterson-Wilton). CFIE combines the EFIE and MFIE equations in a linear fashion as

\[
\alpha(\text{EFIE}) + (1 - \alpha)\eta(\text{MFIE}) = \text{CFIE} \tag{21a}
\]

or

\[
\alpha(\mathbf{E}_i(\mathbf{r})) + (1 - \alpha)\eta(\mathbf{n} \times \mathbf{H}_i(\mathbf{r})) = (\alpha \Lambda + (1 - \alpha)\eta \mathbf{n} \times \Omega + \frac{1}{2} \mathbf{J}(\mathbf{r})) \tag{21b}
\]

The coefficient \( \alpha \) is arbitrary and possibly complex. Typical \( \alpha \) is set to 1/2 but other choices can be made. Basically the CFIE shifts the resonances of the MFIE and EFIE outside the range of interest.

2.2) Resistive Boundary

Consider the Resistive Boundary surface depicted in (Fig.2.2).

![Figure 2.2. Geometry for application of the resistive boundary condition](image)

For a resistive boundary, the boundary condition on \( \partial \mathcal{B}_{12} \) are

\[
\mathbf{n} \times [\mathbf{E}_1 - \mathbf{E}_2] = 0 \tag{22a}
\]

\[
\mathbf{n} \times [\mathbf{E}_1 + \mathbf{E}_2] = 2\mathcal{R}_e \eta_1 \mathbf{n} \times \mathbf{n} \times [\mathbf{H}_1 - \mathbf{H}_2] \tag{22b}
\]

\[
\mathbf{n} \times \mathbf{E}_1 = -\mathbf{M}_1 \quad ; \quad \mathbf{n} \times \mathbf{H}_1 = \mathbf{J}_1 \tag{22c}
\]

\[
\mathbf{n} \times \mathbf{E}_2 = \mathbf{M}_2 \quad ; \quad \mathbf{n} \times \mathbf{H}_2 = -\mathbf{J}_2. \tag{22d}
\]

where \( \mathcal{R}_e \) refers to the normalized surface resistivity in ohms per square. Fields in the regions \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) can be given as

\[
\theta_1(\mathbf{r})\mathbf{E}_1 = \mathbf{E}_1 - \Lambda_1 \mathbf{J}(\mathbf{r}) + \Omega_1 \mathbf{M}(\mathbf{r}) \tag{23a}
\]

\[
\theta_1(\mathbf{r})\mathbf{H}_1 = \mathbf{H}_1 - \Omega_1 \mathbf{J}_1(\mathbf{r}) + \frac{1}{\eta_1} \Lambda_1 \mathbf{M}_1(\mathbf{r}) \tag{23b}
\]

\[
\theta_2(\mathbf{r})\mathbf{E}_2 = -\Lambda_2 \mathbf{J}_2(\mathbf{r}) + \Omega_2 \mathbf{M}_2(\mathbf{r}) \tag{23c}
\]
\[ \theta_2(\mathbf{r})\mathbf{H}_2 = -\Omega_2\mathbf{J}_2(\mathbf{r}) + \frac{1}{\eta_2^2}\mathbf{M}_2(\mathbf{r}) \]  

(23d)

In (23a–d), indices 1 and 2 are corresponding to the fields and currents in regions \( B_1(\epsilon \rightarrow \epsilon_1, \mu \rightarrow \mu_1) \), and \( B_2(\epsilon \rightarrow \epsilon_2, \mu \rightarrow \mu_2) \), respectively. Using (22a–d) in (23a–d) we obtain

\[
\begin{bmatrix}
\Lambda_1 + R_e \\
\Omega_1 \\
R_e
\end{bmatrix} = 
\begin{bmatrix}
\lambda_1 + \frac{\Omega_1}{\mu_1} \Lambda_1 + \frac{R_e}{\mu_1} \\
\Omega_2 \\
\Lambda_2 + R_e
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_1 \\
\mathbf{M}_1 \\
\mathbf{J}_2
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{E}_1 \\
\mathbf{H}_1 \\
0
\end{bmatrix}
\]  

(24)

If \((\epsilon_1 = \epsilon_2, \mu_1 = \mu_2)\), (24) reduces to,

\[
[\Lambda_1 + 2R_e]\mathbf{J} = \mathbf{E}_1 , \quad \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2.
\]  

(25)

which refers to the case of a resistive sheet boundary in free space.

2.3) Dielectric Boundary

A homogenous penetrable body is depicted in Fig.2.3.

![Image of a dielectric boundary](image)

Figure 2.3. Geometry for constructing integral equations for dielectric boundary

Here we will give four different integral equation formulations for dielectric boundaries. For this situation, the expression for \( \mathbf{E}_{1,2} \) and \( \mathbf{H}_{1,2} \) are given in (23a–d).

A) EFIE formulation:

The problem given in Fig 2.3 can be seperated into two sub-problems each gives the field in one of the regions.
Figure 2.4. Geometry for the application of the EFIE and MFIE for dielectric boundary

a) External Problem  b) Internal Problem

Boundary conditions for the $E$ field for the external and the internal problems are:

\[ n \times E_1 = 0 \tag{26a} \]

and

\[ n \times E_2 = 0 \tag{26b} \]

respectively. From $(23a - d)$ and $(26a, b)$, we obtain the EFIE equations,

\[ n \times E_i = n \times \Lambda_1 J_1 - n \times \Omega_1 M_1 \tag{27a} \]

\[ 0 = n \times \Lambda_2 J_1 - n \times \Omega_2 M_1 \tag{27b} \]

B) MFIE formulation:

Boundary conditions for the Magnetic field for the external and internal problems are;

\[ n \times H_1 = 0 \tag{28a} \]

and

\[ n \times H_2 = 0 \tag{28b} \]

respectively. on the surface of the dielectric body. Using $(28)$ in $(23a - d)$, we arrive to MFIE.

\[ n \times H_i = n \times \Omega_1 J_1 + \frac{1}{\eta_i^2} n \times \Lambda_1 M_1 \tag{29a} \]

\[ 0 = n \times \Omega_2 J_1 + \frac{1}{\eta_2^2} n \times \Lambda_2 M_1 \tag{29b} \]

C) CFIE formulation:

Combining the EFIE and MFIE formulations as outlined in (21), yields

\[ [\alpha E' + (1 - \alpha)n \times H'] = (\alpha \Lambda_1 + (1 - \alpha)n \times \Omega_1)J_1 \]

\[- (\alpha \Omega_1 + (1 - \alpha)\frac{1}{\eta_1^2} n \times \Lambda_1)M_1 \tag{30a} \]

\[ 0 = (\alpha \Lambda_2 + (1 - \alpha)n \times \Omega_2)J_1 \]

\[- (\alpha \Omega_2 + (1 - \alpha)\frac{1}{\eta_2^2} n \times \Lambda_2)M_1 \tag{30b} \]
These are the most general integral equations to be solved for \( J \) and \( M \). They can in general be combined with similar integral equations from other dielectric boundaries for simulating rather complex geometries.

D) PMCHW formulation:

The PMCHW formulation is a special case of the CFIE and is also robust at interior resonant frequencies. The method relies on implying the continuity condition on the surface of the dielectric body. That is,

\[
\mathbf{n} \times [\mathbf{E}_1 - \mathbf{E}_2] = 0
\]  

(31a)

and

\[
\mathbf{n} \times [\mathbf{H}_1 - \mathbf{H}_2] = 0.
\]  

(31b)

Employing these conditions to the fields given in (23a-d), we obtain the PMCHW equations.

\[
\begin{bmatrix}
\Lambda_1 + \Lambda_2 & -\Omega_1 - \Omega_2 \\
\Omega_1 + \Omega_2 & \frac{\mu_1}{\mu_2} \Lambda_1 + \frac{\mu_2}{\mu_1} \Lambda_2
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_1 \\
\mathbf{M}_1
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{E}^i \\
\mathbf{H}^i
\end{bmatrix}
\]  

(32)

2.4) Impedance Boundary

The impedance boundary condition is of the form,

\[
\mathbf{n} \times \mathbf{E}_1 = Z \mathbf{n} \times (\mathbf{n} \times \mathbf{H}_1).
\]  

(33)

Figure 2.5 Geometry for applying the impedance boundary condition

Substituting (33) into (23a, b), we get the surface integral equation.

\[
\begin{bmatrix}
\Lambda_1 + \frac{Z}{\varepsilon_1} & -\Omega_1 \\
\Omega_1 & \frac{\mu_1}{\mu_2} \Lambda_1 + \frac{\mu_2}{\mu_1} \Lambda_2
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_1 \\
\mathbf{M}_1
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{E}^i \\
\mathbf{H}^i
\end{bmatrix}
\]  

(34)

In (34), \( \eta = \sqrt{\frac{\mu_2}{\varepsilon_1}} \) represents the characteristic impedance of the surrounding medium.

2.4) CA (Circuit-Analog Boundary)
Consider a thin (penetrable or impenetrable) multilayered sheet (see fig. 2.6). Relations between the tangential components on the two sides of the sheet (consistent with duality and reciprocity) can be written as

\[
\begin{align*}
\mathbf{n} \times [E^+(r) + E^-(r)] &= \hat{R}_e \mathbf{n} \times [H^+(r) - H^-(r)] - \hat{R}_m \mathbf{n} [E^+(r) - E^-(r)] \\
\mathbf{n} \times [H^+(r) + H^-(r)] &= \hat{R}_m \mathbf{n} \times [E^+(r) - E^-(r)] + \hat{R}_e \mathbf{n} [H^+(r) - H^-(r)]
\end{align*}
\] (35a, 35b)

Here \(E^\pm(r)\) and \(H^\pm(r)\) represent the fields on the upper and the lower surfaces of the sheet, respectively; \(R_e\) and \(R_m\) are the electric and the magnetic resistivities and \(R_c\) is a cross coupling term. Rewriting (35a) and (35b) in matrix form, we obtain

\[
\begin{pmatrix} E^- \\ \eta \mathbf{n} \times H^- \end{pmatrix} = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix} \begin{pmatrix} E^+ \\ \eta \mathbf{n} \times H^+ \end{pmatrix}
\] (36)

where \(\chi_{ij}(i, j = 1, 2)\) are given by

\[
\begin{align*}
\chi_{11} &= [1 + (1/2 - R_c)^2/R_e R_m]/[1 - ((1/4 - R_c^2)/R_e R_m)] \\
\chi_{12} &= -1/[R_m[1 - ((1/4 - R_c^2)/R_e R_m)] \\
\chi_{21} &= -1/[R_e[1 - ((1/4 - R_c^2)/R_e R_m)] \\
\chi_{22} &= [1 + (1/2 + R_c)^2/R_e R_m]/[1 - ((1/4 - R_c^2)/R_e R_m)]
\end{align*}
\] (37a, 37b, 37c, 37d)

in which \(R_n(\hat{R}_e = 2\eta R_e, \hat{R}_m = -(2/\eta)R_m, \hat{R}_c = 2R_c)\), \((n = e, m, c)\) stand for the normalized resistivities. Assuming that the thin multilayered sheet can be characterized by its reflection and transmission properties, \(R_n(n = e, m, c)\) can be determined by relating them to the reflection and transmission coefficients of the sheet. For general layered structure, we need two reflection (\(\Gamma^\pm\)) and one transmission coefficient (\(T\)). The corresponding reflection and transmission coefficients from (35a, b) are

\[
\Gamma^\pm(\theta) = (2R_c \cos \theta - 2R_m / \cos \theta \pm 4R_e) / (4(R_e R_m + R_c^2) + 1 + 2R_e \cos \theta + 2R_m / \cos \theta)
\] (38a)
\[ T(\theta) = \frac{(4(R_e^2 + R_m) - 1)}/(4(R_e R_m + R_e^2) + 1 + 2R_e \cos \theta + 2R_m/\cos \theta) \]  

where \( \theta \) stands for the incident angle. At normal incidence \( (\theta = 0) \) the above can be inverted to yield

\[ R_e = \frac{[T^2(0) - (1 + \Gamma^+(0))(1 + \Gamma^-(0))]/[\Gamma^+(0)\Gamma^-(0) - (1 - T^2(0))^2]}/2 \]  

\[ R_m = \frac{[T^2(0) - (1 - \Gamma^+(0))(1 - \Gamma^-(0))]/[\Gamma^+(0)\Gamma^-(0) - (1 - T^2(0))^2]}/2 \]  

\[ R_c = \frac{[\Gamma^-(0) - \Gamma^+(0)]/[\Gamma^+(0)\Gamma^-(0) - (1 - T^2(0))^2]}/2 \]  

Thus, upon having the reflection/transmission coefficients we can extract the corresponding \( R_n(n = e, m, c) \) values.

### 2.4.1) Reduction to simple sheet condition

The transmission line model can be used to relate the above \( R_n(n = e, m, c) \) parameters to the resistivity \( (Z_p) \) and conductivity \( (Z_s) \) values for simple sheets. Equating the reflection and transmission coefficients for the above circuit with those from (4) yields

\[ \text{series} \rightarrow \ R_e \rightarrow \infty, \ R_m = -Z_s/Z_0, \ R_c = 0 \]  

\[ \text{parallel} \rightarrow \ R_e \rightarrow \infty, \ R_m = -Z_s/Z_0, \ R_c = 0 \]

and the \( \chi_{ij}(i, j = 1, 2) \) matrices reduce to

\[ \chi_{ser} = \begin{bmatrix} 1 & Z_s/Z_0 \\ 0 & 1 \end{bmatrix} \]  

and

\[ \chi_{par} = \begin{bmatrix} 1 & 0 \\ -Z_s/Z_p & 1 \end{bmatrix} \].

When (41a,b) is used in (36), we conclude that a single parallel impedance circuit represents a resistive boundary condition, whereas a single series impedance circuit represents a magnetically conductive boundary.

### 2.4.2) Surface Integral Equation

To construct a surface integral equation let us refer to Fig.2.3. In this case interior and the exterior fields can be expressed with \((23a - d)\). The relation in between the tangential field components and the surface currents can also be given with \((22c, d)\). Substituting \((23a - d)\) into (36) with the identification that \( E_1 = E^+, \ E_2 = E^-, \ H_1 = H^+, \ H_2 = H^- \) we obtain the integral equation;

\[ \begin{bmatrix} \lambda_1 - \frac{\chi_{12}}{2\chi_{21}} & -\Omega_1 & -\frac{1}{\eta_2} & 0 \\ \Omega_1 & \frac{1}{\eta_1} \lambda_1 + \frac{\chi_{12}}{2\chi_{21}} & 0 & -\frac{1}{2\eta_2} \chi_{12} \\ \frac{\chi_{12}}{2} (\chi_{12} - \frac{\chi_{12} \chi_{21}}{\chi_{21}}) & 0 & \lambda_2 - \frac{\chi_{12}}{2\chi_{21}} & -\Omega_2 \\ 0 & \frac{1}{2\eta_2} (\chi_{21} - \frac{\chi_{12} \chi_{21}}{\chi_{12}}) & \Omega_2 & \frac{1}{\eta_2} \lambda_2 + \frac{\chi_{12}}{2\chi_{21}} \end{bmatrix} \begin{bmatrix} J_1 \\ M_1 \\ J_2 \\ L_2 \end{bmatrix} = \begin{bmatrix} E^i \\ H^i \\ 0 \\ 0 \end{bmatrix} \]  

\[ (42) \]
3) Composite Structures

Consider the geometry depicted in Fig. 2.8. We are going to give the surface integral equations for different problems implemented in the code CADRIS.

3.1) Problem 1

In our first problem $S_1$ and $S_2$ are dielectric and $S_3$ and $S_4$ are the perfectly conducting surfaces. Electric and the magnetic fields can be written as follows in three different region(see Fig. 2.8.).

\[
\theta(r)E_1(r) = E_1' - \Lambda_1(J_1 + J_4) + \Omega_1 M_1
\]

\[
\theta(r)H_1(r) = H_1' - \Omega_1(J_1 + J_4) + \frac{1}{\eta_1^2} \Lambda_1 M_1
\]

\[
\theta(r)E_2(r) = -\Lambda_2(J_2 - J_3) + \Omega_2(M_2 - M_1)
\]

\[
\theta(r)H_2(r) = -\Omega_2(J_2 - J_3) + \frac{1}{\eta_2^2} \Lambda_2(M_2 - M_1)
\]

\[
\theta(r)E_3(r) = -\Lambda_3(J_3 - J_2) - \Omega_3 M_2
\]

\[
\theta(r)H_3(r) = -\Omega_3(J_3 - J_2) - \frac{1}{\eta_3^2} \Lambda_3 M_2
\]

Boundary conditions on the surfaces $S_1 - S_4$ can be given as

\[
OnS_1: n \times [E_1 - E_2] = 0 \quad ; \quad n \times [H_1 - H_2] = 0
\]

\[
OnS_2: n \times [E_2 - E_3] = 0 \quad ; \quad n \times [H_2 - H_3] = 0
\]

\[
OnS_3: n \times E_3 = 0 \ (EFIE) \quad ; \quad n \times H_3 = J_3 \ (MFIE)
\]

\[
OnS_4: n \times E_1 = 0 \ (EFIE) \quad ; \quad n \times H_1 = J_4 \ (MFIE)
\]

Using (44a - d) in (43a - f), and after some straightforward manipulations we find the surface integral equation as

\[
\mathbf{ZI} = \mathbf{V}
\]

\[
\mathbf{Z} = \begin{bmatrix}
\Lambda_1 + \Lambda_2 & -\Omega_1 - \Omega_2 & -\Lambda_2 & \Omega_2 & 0 & \Lambda_1 \\
\Omega_1 + \Omega_2 & \frac{1}{\eta_1^2} \Lambda_1 + \frac{1}{\eta_2^2} \Lambda_2 & -\Omega_2 & -\frac{1}{\eta_2^2} \Lambda_2 & 0 & \Omega_1 \\
-\Lambda_2 & \Omega_2 & \Lambda_2 + \Lambda_3 & -\Omega_2 - \Omega_3 & -\Lambda_3 & 0 \\
-\Omega_2 & -\frac{1}{\eta_2^2} \Lambda_2 & \Omega_2 + \Omega_3 & \frac{1}{\eta_3^2} \Lambda_2 + \frac{1}{\eta_3^2} \Lambda_3 & -\Omega_3 & 0 \\
0 & 0 & -\Lambda_3 & \Omega_3 & \Lambda_3 & 0 \\
\Lambda_1 & -\Omega_1 & 0 & 0 & 0 & \Lambda_1
\end{bmatrix}
\]
Fig. 2.8 General Problem Geometry
\[
I = \begin{bmatrix}
J_1 \\
M_1 \\
J_2 \\
M_2 \\
J_3 \\
J_4
\end{bmatrix}
\quad V = \begin{bmatrix}
E'(S_1) \\
H'(S_1) \\
0 \\
0 \\
0 \\
E'(S_4)
\end{bmatrix}
\]  \hspace{1cm} (45c)

Problem 2

In the second problem \(S_1\) is a dielectric, \(S_2\) is a resistive and \(S_3\) and \(S_4\) are the perfectly conducting surfaces.

In this case fields can be as

\[
\theta(r)E_1(r) = E' - \Lambda_1(J_1 + J_4) + \Omega_1M_1 
\]  \hspace{1cm} (46a)

\[
\theta(r)H_1(r) = H' - \Omega_1(J_1 + J_4) + \frac{1}{\eta_1}\Lambda_1M_1 
\]  \hspace{1cm} (46b)

\[
\theta(r)E_2(r) = -\Lambda_2(J_2 - J_1) + \Omega_2(M_2 - M_1) 
\]  \hspace{1cm} (46c)

\[
\theta(r)H_2(r) = -\Omega_2(J_2 - J_1) + \frac{1}{\eta_2^2}\Lambda_2(M_2 - M_1) 
\]  \hspace{1cm} (46d)

\[
\theta(r)E_3(r) = -\Lambda_3(J_3 + J_2) - \Omega_3M_2 
\]  \hspace{1cm} (46e)

\[
\theta(r)H_3(r) = -\Omega_3(J_3 + J_2) - \frac{1}{\eta_3^2}\Lambda_3M_2 
\]  \hspace{1cm} (46f)

Boundary conditions on the surfaces \(S_1 - S_4\) can be given as

\[
OnS_1 : n \times [E_1 - E_2] = 0 \quad ; \quad n \times [H_1 - H_2] = 0 
\]  \hspace{1cm} (47a)

\[
OnS_2 : n \times [E_2 - E_3] = 0 \quad ; \quad n \times [E_2 + E_3] = 2\eta_1 R n \times [H_2 - H_3] 
\]  \hspace{1cm} (47b)

\[
OnS_3 : n \times E_3 = 0(EFIE) \quad ; \quad n \times H_3 = J_3(MFIE) 
\]  \hspace{1cm} (47c)

\[
OnS_4 : n \times E_1 = 0(EFIE) \quad ; \quad n \times H_1 = J_4(MFIE) 
\]  \hspace{1cm} (47d)

Using (47a - d) in (46a - f), and after some straightforward manipulations we find the surface integral equation as
\[
Z = \begin{bmatrix}
\lambda_1 + \lambda_2 & -\Omega_1 - \Omega_2 & -\lambda_2 & \Omega_2 & 0 & 0 & \lambda_1 \\
\Omega_1 + \Omega_2 & \frac{1}{\eta_1} \lambda_1 + \frac{1}{\eta_2} \lambda_2 & -\Omega_2 & -\frac{1}{\eta_2} \lambda_2 & 0 & 0 & \Omega_1 \\
-\lambda_2 & \Omega_2 & \lambda_2 + \frac{\eta R}{2} & -\Omega_2 & -\frac{\eta R}{2} & 0 & 0 \\
-\Omega_2 & -\frac{1}{\eta_2} \lambda_2 & \Omega_2 & \frac{1}{\eta_2} \lambda_2 + \frac{1}{\eta_3} \lambda_3 + \frac{R}{2 \eta_1} & -\Omega_3 & -\Omega_3 & 0 \\
0 & 0 & \frac{\eta R}{2} & \Omega_3 & \lambda_3 + \frac{\eta R}{2} & \lambda_3 & 0 \\
0 & 0 & 0 & \Omega_3 & \lambda_3 & \lambda_3 & 0 \\
\lambda_1 & -\Omega_1 & 0 & 0 & 0 & 0 & \lambda_1
\end{bmatrix}
\]

(48a)

\[
\begin{bmatrix}
J_1 \\
M_1 \\
J_2 \\
M_2 \\
J_{22} \\
J_3 \\
J_4
\end{bmatrix} = \begin{bmatrix}
E'(S_1) \\
H'(S_1) \\
0 \\
0 \\
0 \\
0 \\
E'(S_4)
\end{bmatrix}
\]

(48b)

Problem 3

In the third problem \(S_1\) is a dielectric, \(S_2\) is a CA-boundary and \(S_3\) and \(S_4\) are the perfectly conducting surfaces.

In this case fields can be as

\[
\theta(r) E_1(r) = E' - \Lambda_1 (J_1 + J_4) + \Omega_1 M_1
\]

(49a)

\[
\theta(r) H_1(r) = H' - \Omega_1 (J_1 + J_4) + \frac{1}{\eta_1} \Lambda_1 M_1
\]

(49b)

\[
\theta(r) E_2(r) = -\Lambda_2 (J_2 - J_1) + \Omega_2 (M_2 - M_1)
\]

(49c)

\[
\theta(r) H_2(r) = -\Omega_2 (J_2 - J_1) + \frac{1}{\eta_2} \Lambda_2 (M_2 - M_1)
\]

(49d)

\[
\theta(r) E_3(r) = -\Lambda_3 (J_3 + J_{22}) + \Omega_3 M_{22}
\]

(49e)

\[
\theta(r) H_3(r) = -\Omega_3 (J_3 + J_{22}) - \frac{1}{\eta_3} \Lambda_3 M_{22}
\]

(49f)

boundary conditions on the surfaces \(S_1 - S_4\) can be given as

\[
\text{On} S_1 : \mathbf{n} \times [E_1 - E_2] = 0 \quad ; \quad \mathbf{n} \times [H_1 - H_2] = 0
\]

(50a)
Fig. 2.11 Geometry for Problem-3
\[
OnS_2 : \begin{bmatrix} E^- \\ \eta n \times H^- \end{bmatrix} = \begin{bmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{bmatrix} \begin{bmatrix} E^+ \\ \eta n \times H^+ \end{bmatrix} \quad (50b)
\]

\[
OnS_3 : n \times E_3 = 0 (EFIE) \quad ; \quad n \times H_3 = J_3 (MFIE) \quad (50c)
\]

\[
OnS_4 : n \times E_4 = 0 (EFIE) \quad ; \quad n \times H_4 = J_4 (MFIE) \quad (50d)
\]

Using (50a – d) in (49a – f), and after some straightforward manipulations we find the surface integral equation as

\[
Z = \begin{bmatrix}
\Lambda_1 + \Lambda_2 & -\Omega_1 - \Omega_2 & -\Lambda_2 & 0 & 0 & 0 & \Lambda_1 \\
\Omega_1 + \Omega_2 & \frac{1}{\eta_1} \Lambda_1 + \frac{1}{\eta_2} \Lambda_2 & -\Omega_2 & -\frac{1}{\eta_2} \Lambda_2 & 0 & 0 & 0 & \Omega_1 \\
-\Lambda_2 & \Omega_2 & \Lambda_2 - \frac{\chi_{12} \chi_{22}}{2\chi_{21}} & -\Omega_2 & -\frac{\eta}{\chi_{21}} & 0 & 0 & 0 \\
-\Omega_2 & -\frac{1}{\eta_2} \Lambda_2 & \Omega_1 & \frac{1}{\eta_1} \Lambda_2 + \frac{\chi_{11}}{2\eta_1} & 0 & -\frac{1}{2\eta_{12}} & 0 & 0 \\
0 & 0 & \frac{\eta}{2} (\chi_{12} - \frac{\chi_{11} \chi_{22}}{\chi_{21}}) & 0 & \Lambda_2 - \frac{\chi_{11}}{2\eta_{21}} & -\Omega_2 & \Lambda_3 & 0 \\
0 & 0 & 0 & \frac{1}{\eta_2} (\chi_{21} - \frac{\chi_{11} \chi_{22}}{\chi_{12}}) & \Omega_3 & \frac{1}{\eta_2} \Lambda_3 + \frac{\chi_{22}}{2\eta_{13}} & \Omega_3 & 0 \\
0 & 0 & 0 & 0 & \Lambda_3 & -\Omega_3 & \Lambda_3 & 0 \\
\Lambda_1 & -\Omega_1 & 0 & 0 & 0 & 0 & 0 & \Lambda_1
\end{bmatrix} \quad (51a)
\]

\[
I = \begin{bmatrix} J_1 \\ M_1 \\ J_2 \\ M_2 \\ J_{22} \\ M_{22} \\ J_3 \\ J_4 \end{bmatrix} \quad V = \begin{bmatrix} E^i(S_1) \\ H^i(S_1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E^i(S_4) \end{bmatrix} \quad (51b)
\]

Problem 4

In the fourth problem \(S_1\) and \(S_2\) are dielectric, \(S_3\) is a PEC and \(S_4\) is an impedance surface.

In this case fields can be written as

\[
\theta(r) E_i(r) = E^i - \Lambda_1 (J_1 + J_4) + \Omega_1 (M_1 + M_4) \quad (52a)
\]
\[ \theta(r)\mathbf{H}_1(r) = \mathbf{H}^i - \Omega_1(J_1 + J_4) - \frac{1}{\eta_1^2} \Lambda_1(M_1 + M_4) \]  
\[ \theta(r)\mathbf{E}_2(r) = -\Lambda_2(J_2 - J_1) + \Omega_2(M_2 - M_1) \]  
\[ \theta(r)\mathbf{H}_2(r) = -\Omega_2(J_2 - J_1) + \frac{1}{\eta_2^2} \Lambda_2(M_2 - M_1) \]  
\[ \theta(r)\mathbf{E}_3(r) = -\Lambda_3(J_3 - J_2) + \Omega_3 M_2 \]  
\[ \theta(r)\mathbf{H}_3(r) = -\Omega_3(J_3 - J_2) - \frac{1}{\eta_3^2} \Lambda_3 M_2 \]  

boundary conditions on the surfaces \( S_1 - S_4 \) can be given as

\[ \text{On} S_1 : \mathbf{n} \times [\mathbf{E}_1 - \mathbf{E}_2] = 0 \quad ; \quad \mathbf{n} \times [\mathbf{H}_1 - \mathbf{H}_2] = 0 \]  
\[ \text{On} S_2 : \mathbf{n} \times [\mathbf{E}_2 - \mathbf{E}_3] = 0 \quad ; \quad \mathbf{n} \times [\mathbf{H}_2 - \mathbf{H}_3] = 0 \]  
\[ \text{On} S_3 : \mathbf{n} \times \mathbf{E}_3 = 0 \text{ (EFIE)} \quad ; \quad \mathbf{n} \times \mathbf{H}_3 = J_3 \text{ (MFIE)} \]  
\[ \text{On} S_4 : \mathbf{n} \times \mathbf{E}_1 = Z \mathbf{n} \times \mathbf{n} \times \mathbf{H}_1 \]  

Using (54a - d) in (53a - f), and after some straightforward manipulations we find the surface integral equation as

\[
Z = \begin{bmatrix}
\Lambda_1 + \Lambda_2 & -\Omega_1 - \Omega_2 & -\Lambda_2 & \Omega_2 & 0 & \Lambda_1 & -\Omega_1 \\
\Omega_1 + \Omega_2 & \frac{1}{\eta_1^2} \Lambda_1 + \frac{1}{\eta_2^2} \Lambda_2 & -\Omega_2 & -\frac{1}{\eta_2^2} \Lambda_2 & 0 & \Omega_1 & \frac{1}{\eta_1^2} \Lambda_1 \\
-\Lambda_2 & \Omega_2 & \Lambda_2 + \Lambda_3 & -\Omega_2 - \Omega_3 & -\Lambda_3 & 0 & 0 \\
-\Omega_2 & -\frac{1}{\eta_2^2} \Lambda_2 & \Omega_2 + \Omega_3 & \frac{1}{\eta_2^2} \Lambda_1 + \frac{1}{\eta_3^2} \Lambda_3 & -\Omega_3 & 0 & 0 \\
0 & 0 & -\Lambda_3 & \Omega_3 & \Lambda_3 & 0 & 0 \\
\Lambda_1 & -\Omega_1 & 0 & 0 & 0 & \Lambda_1 + \frac{Z}{2} & -\Omega_1 \\
\Omega_1 & \frac{1}{\eta_1^2} \Lambda_1 & 0 & 0 & 0 & \Omega_1 & \frac{1}{\eta_1^2} \Lambda_1 + \frac{1}{\eta_2^2} \Lambda_2 
\end{bmatrix}
\]  

\[ \mathbf{I} = \begin{bmatrix}
\mathbf{J}_1 \\
\mathbf{M}_1 \\
\mathbf{J}_2 \\
\mathbf{M}_2 \\
\mathbf{J}_3 \\
\mathbf{M}_3 \\
\mathbf{J}_4 \\
\mathbf{M}_4 
\end{bmatrix} \quad \mathbf{V} = \begin{bmatrix}
\mathbf{E}^i(S_1) \\
\mathbf{H}^i(S_1) \\
0 \\
0 \\
0 \\
\mathbf{E}^i(S_4) \\
\mathbf{H}^i(S_4) 
\end{bmatrix} \]  

\[ \text{Problem 5} \]
In the fifth problem $S_1$ is a dielectric, $S_2$ is a resistive, $S_3$ is a PEC and $S_4$ is an impedance surface.

In this case fields can be as

$$\theta(\mathbf{r})\mathbf{E}_1(\mathbf{r}) = \mathbf{E'} - \Lambda_1(\mathbf{J}_1 + \mathbf{J}_4) + \Omega_1(\mathbf{M}_1 + \mathbf{M}_4) \quad (56a)$$

$$\theta(\mathbf{r})\mathbf{H}_1(\mathbf{r}) = \mathbf{H'} - \Omega_1(\mathbf{J}_1 + \mathbf{J}_4) + -\frac{1}{\eta_1^2}\Lambda_1(\mathbf{M}_1 + \mathbf{M}_4) \quad (56b)$$

$$\theta(\mathbf{r})\mathbf{E}_2(\mathbf{r}) = -\Lambda_2(\mathbf{J}_2 - \mathbf{J}_1) + \Omega_2(\mathbf{M}_2 - \mathbf{M}_1) \quad (56c)$$

$$\theta(\mathbf{r})\mathbf{H}_2(\mathbf{r}) = -\Omega_2(\mathbf{J}_2 - \mathbf{J}_1) + -\frac{1}{\eta_2^2}\Lambda_2(\mathbf{M}_2 - \mathbf{M}_1) \quad (56d)$$

$$\theta(\mathbf{r})\mathbf{E}_3(\mathbf{r}) = -\Lambda_3(\mathbf{J}_3 + \mathbf{J}_{22}) + -\Omega_3\mathbf{M}_2 \quad (56e)$$

$$\theta(\mathbf{r})\mathbf{H}_3(\mathbf{r}) = -\Omega_3(\mathbf{J}_3 + \mathbf{J}_{22}) + -\frac{1}{\eta_3^2}\Lambda_3\mathbf{M}_2 \quad (56f)$$

boundary conditions on the surfaces $S_1 - S_4$ can be given as

$$OnS_1 : \mathbf{n} \times [\mathbf{E}_1 - \mathbf{E}_2] = 0 \quad ; \quad \mathbf{n} \times [\mathbf{H}_1 - \mathbf{H}_2] = 0 \quad (57a)$$

$$OnS_2 : \mathbf{n} \times [\mathbf{E}_2 - \mathbf{E}_3] = 0 \quad ; \quad \mathbf{n} \times [\mathbf{E}_2 + \mathbf{E}_3] = 2\eta_1 R\mathbf{n} \times [\mathbf{H}_2 - \mathbf{H}_3] \quad (57b)$$

$$OnS_3 : \mathbf{n} \times \mathbf{E}_3 = 0 ; \mathbf{E}FIE \quad ; \quad \mathbf{n} \times \mathbf{H}_3 = \mathbf{J}_3 ; MFI$$

$$OnS_4 : \mathbf{n} \times \mathbf{E}_1 = Z\mathbf{n} \times \mathbf{H}_1 \quad (57d)$$

Using $(57a - d)$ in $(56a - f)$, and after some straightforward manipulations we find the surface integral equation as

$$Z = \begin{bmatrix}
\Lambda_1 + \Lambda_2 & -\Omega_1 - \Omega_2 & -\Lambda_2 & \Omega_2 & 0 & 0 & \Lambda_1 & -\Omega_1 \\
\Omega_1 + \Omega_2 & \frac{1}{\eta_1^2}\Lambda_1 + \frac{1}{\eta_2^2}\Lambda_2 & -\Lambda_2 & -\frac{1}{\eta_2^2}\Lambda_2 & 0 & 0 & \Omega_1 & \frac{1}{\eta_1^2}\Lambda_1 \\
-\Lambda_2 & \Omega_2 & \Lambda_2 + \frac{Z R}{2} & -\Omega_2 & -\frac{Z R}{2} & 0 & 0 & 0 \\
-\Omega_2 & -\frac{1}{\eta_3^2}\Lambda_2 & \Omega_2 & \frac{1}{\eta_2^2}\Lambda_2 + \frac{1}{\eta_3^2}\Lambda_3 + \frac{R}{2\eta_1} & -\Omega_3 & -\Omega_3 & 0 & 0 \\
0 & 0 & \frac{Z R}{2} & \Omega_3 & \Lambda_3 + \frac{Z R}{2} & \Lambda_3 & 0 & 0 \\
0 & 0 & 0 & \Omega_3 & \Lambda_3 & \Lambda_3 & 0 & 0 \\
\Lambda_1 & -\Omega_1 & 0 & 0 & 0 & 0 & \Lambda_1 + \frac{Z}{2} & -\Omega_1 \\
\Omega_1 & \frac{1}{\eta_1^2}\Lambda_1 & 0 & 0 & 0 & 0 & \Omega_1 & \frac{1}{\eta_1^2}\Lambda_1 + \frac{1}{2Z} \end{bmatrix} \quad (58a)$$
\[ I = \begin{bmatrix} J_1 \\ M_1 \\ J_2 \\ M_2 \\ J_{22} \\ J_3 \\ J_4 \\ M_4 \end{bmatrix}, \quad V = \begin{bmatrix} E'(S_1) \\ H'(S_1) \\ 0 \\ 0 \\ 0 \\ 0 \\ E'(S_4) \\ H'(S_4) \end{bmatrix} \] (58b)

Problem 6

In the sixth problem \( S_1 \) is a dielectric, \( S_2 \) is a CABC \( S_3 \) is a PEC and \( S_4 \) is an Impedance surface.

In this case fields can be as

\[ \theta(r)E_1(r) = E^i - \Lambda_1(J_1 + J_4) + \Omega_1(M_1 + M_4) \] (59a)

\[ \theta(r)H_1(r) = H^i - \Omega_1(J_1 + J_4) + \frac{1}{\eta_1^2} \Lambda_1(M_1 + M_4) \] (59b)

\[ \theta(r)E_2(r) = -\Lambda_2(J_2 - J_1) + \Omega_2(M_2 - M_1) \] (59c)

\[ \theta(r)H_2(r) = -\Omega_2(J_2 - J_1) + \frac{1}{\eta_2^2} \Lambda_2(M_2 - M_1) \] (59d)

\[ \theta(r)E_3(r) = -\Lambda_3(J_3 + J_{22}) + \Omega_3M_{22} \] (59e)

\[ \theta(r)H_3(r) = -\Omega_3(J_3 + J_{22}) - \frac{1}{\eta_3^2} \Lambda_3M_{22} \] (59f)

boundary conditions on the surfaces \( S_1 - S_4 \) can be given as

\[ \text{On}S_1 : n \times [E_1 - E_2] = 0 ; \quad n \times [H_1 - H_2] = 0 \] (60a)

\[ \begin{bmatrix} E^- \\ \eta n \times H^- \end{bmatrix} = \begin{bmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{bmatrix} \begin{bmatrix} E^+ \\ \eta n \times H^+ \end{bmatrix} \] (60b)

\[ \text{On}S_3 : n \times E_3 = 0 (EFIE) ; \quad n \times H_3 = J_3 (MFIE) \] (60c)

\[ \text{On}S_4 : n \times E_1 = Zn \times n \times H_1 \] (60d)

Using (60a - d) in (59a - f), and after some straight forward manipulations we find the surface integral equation as
\[ Z = \begin{bmatrix} \Lambda_1 + \Lambda_2 & -\Omega_1 - \Omega_2 & -\Lambda_2 & \Omega_2 & 0 & 0 & 0 & \Lambda_1 & -\Omega_1 \\ \Omega_1 + \Omega_2 & \frac{1}{n_1} \Lambda_1 + \frac{1}{n_2} \Lambda_2 & -\Omega_2 & -\frac{1}{n_2} \Lambda_2 & 0 & 0 & 0 & \Omega_1 & \frac{1}{n_1} \Lambda_1 \\ -\Lambda_2 & \Omega_2 & \Lambda_2 - \frac{x_{12} x_{22}}{2x_{11}} & -\Omega_2 & -\frac{y}{2x_{21}} & 0 & 0 & 0 & 0 \\ -\Omega_2 & -\frac{1}{n_2} \Lambda_2 & \Omega_1 & \frac{1}{n_1} \Lambda_2 + \frac{x_{11}}{2x_{12}} & 0 & -\frac{1}{2x_{12}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_2 - \frac{2x_{11}}{2x_{21}} & -\Omega_2 & \Lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2x_{21}} (x_{21} - \frac{x_{11} x_{22}}{x_{12}}) & \Omega_3 & \frac{1}{n_2} \Lambda_3 + \frac{x_{12}}{2x_{12}} & \Omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_3 & -\Omega_3 & \Lambda_3 & 0 & 0 \\ \Omega_1 & -\Omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_1 + \frac{z}{2} & -\Omega_1 \end{bmatrix} \]  

\[ J_1 = \begin{bmatrix} J_1 \\ M_1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} J_2 \\ M_2 \end{bmatrix}, \quad J_3 = \begin{bmatrix} J_3 \end{bmatrix}, \quad J_4 = \begin{bmatrix} J_4 \\ M_4 \end{bmatrix} \]

\[ I = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \]

\[ E^i(S_1) = \begin{bmatrix} E^i(S_1) \\ H^i(S_1) \end{bmatrix}, \quad E^i(S_4) = \begin{bmatrix} E^i(S_4) \end{bmatrix}, \quad H^i(S_4) = \begin{bmatrix} H^i(S_4) \end{bmatrix} \]

**Problem 7**

In the second problem \( S_1 \) is a dielectric, \( S_3 \) and \( S_4 \) are PEC surfaces.

In this case fields can be as

\[ \theta(r)E_1(r) = E^i - \Lambda_1 (J_1 + J_4) + \Omega_1 M_1 \]  

\[ \theta(r)H_1(r) = H^i - \Omega_1 (J_1 + J_4) + \frac{1}{n_1} \Lambda_1 M_1 \]  

\[ \theta(r)E_2(r) = -\Lambda_2 (J_3 - J_1) - \Omega_2 M_4 \]  

\[ \theta(r)H_2(r) = -\Omega_2 (J_3 - J_1) + \frac{1}{n_2} \Lambda_2 M_1 \]
Fig. 2.14 Geometry for Problem-7

\[ J_4 \]

\[ J_3 \]

\[ J_1, M_1 \]

PEC

Dielectric
boundary conditions on the surfaces \( S_1 \) – \( S_4 \) can be given as

\[
OnS_1 : \mathbf{n} \times [\mathbf{E}_1 - \mathbf{E}_2] = 0 \quad ; \quad \mathbf{n} \times [\mathbf{H}_1 - \mathbf{H}_2] = 0
\]  
(63a)

\[
OnS_3 : \mathbf{n} \times \mathbf{E}_3 = 0(\text{EFIE}) \quad ; \quad \mathbf{n} \times \mathbf{H}_3 = \mathbf{J}_3(\text{MFIE})
\]  
(63b)

\[
OnS_4 : \mathbf{n} \times \mathbf{E}_4 = 0(\text{EFIE}) \quad ; \quad \mathbf{n} \times \mathbf{H}_1 = \mathbf{J}_4(\text{MFIE})
\]  
(63c)

Using (63a – c) in (62a – d), and after some straightforward manipulations we find the surface integral equation as

\[
Z = \begin{bmatrix}
\Lambda_1 + \Lambda_2 & -\Omega_1 - \Omega_2 & -\Lambda_2 & \Lambda_1 \\
\Omega_1 + \Omega_2 & \frac{1}{\eta_1} \Lambda_1 + \frac{1}{\eta_2} \Lambda_2 & -\Omega_2 & \Omega_1 \\
-\Lambda_2 & \Omega_2 & \Lambda_2 & 0 \\
\Lambda_1 & -\Omega_1 & 0 & \Lambda_1 \\
\end{bmatrix}
\]  
(64a)

\[
\mathbf{I} = \begin{bmatrix}
\mathbf{J}_1 \\
\mathbf{M}_1 \\
\mathbf{J}_3 \\
\mathbf{J}_4 \\
\end{bmatrix} \quad \mathbf{V} = \begin{bmatrix}
\mathbf{E}'(S_1) \\
\mathbf{H}'(S_1) \\
0 \\
\mathbf{E}'(S_4) \\
\end{bmatrix}
\]  
(64b)

**Problem 8**

In the eight problem \( S_1 \) is a dielectric, \( S_3 \) is a PEC and \( S_4 \) is an impedance surface.

In this case fields can be as

\[
\theta(\mathbf{r})\mathbf{E}_1(\mathbf{r}) = \mathbf{E}' - \Lambda_1(\mathbf{J}_1 + \mathbf{J}_4) + \Omega_1(\mathbf{M}_1 + \mathbf{M}_4)
\]  
(65a)

\[
\theta(\mathbf{r})\mathbf{H}_1(\mathbf{r}) = \mathbf{H}' - \Omega_1(\mathbf{J}_1 + \mathbf{J}_4) - \frac{1}{\eta_1} \Lambda_1 \mathbf{M}_1
\]  
(65b)

\[
\theta(\mathbf{r})\mathbf{E}_2(\mathbf{r}) = -\Lambda_2(\mathbf{J}_3 - \mathbf{J}_1) - \Omega_2 \mathbf{M}_1
\]  
(65c)

\[
\theta(\mathbf{r})\mathbf{H}_2(\mathbf{r}) = -\Omega_2(\mathbf{J}_3 - \mathbf{J}_1) + \frac{1}{\eta_2} \Lambda_2 \mathbf{M}_1
\]  
(65d)

boundary conditions on the surfaces \( S_1 \) – \( S_4 \) can be given as

\[
OnS_1 : \mathbf{n} \times [\mathbf{E}_1 - \mathbf{E}_2] = 0 \quad ; \quad \mathbf{n} \times [\mathbf{H}_1 - \mathbf{H}_2] = 0
\]  
(66a)

\[
OnS_3 : \mathbf{n} \times \mathbf{E}_3 = 0(\text{EFIE}) \quad ; \quad \mathbf{n} \times \mathbf{H}_3 = \mathbf{J}_3(\text{MFIE})
\]  
(66b)

\[
OnS_4 : \mathbf{n} \times \mathbf{E}_4 = Z\mathbf{n} \times \mathbf{n} \times \mathbf{H}_1
\]  
(66c)
Fig. 2.14 Geometry for Problem-8

Dielectric

PEC

Impedance

$J_1, M_1$

$J_3$

$J_4, M_4$
Using \((66a - c)\) in \((65a - d)\), and after some straightforward manipulations we find the surface integral equation as

\[
Z = \begin{bmatrix}
\lambda_1 + \lambda_2 & -\Omega_1 - \Omega_2 & -\lambda_2 & \lambda_1 & -\Omega_1 \\
\Omega_1 + \Omega_2 & \frac{1}{\eta_1} \lambda_1 + \frac{1}{\eta_2} \lambda_2 & -\Omega_2 & \Omega_1 & \frac{1}{\eta_1} \lambda_1 \\
-\lambda_2 & \Omega_2 & \lambda_2 & 0 & 0 \\
\lambda_1 & -\Omega_1 & 0 & \lambda_1 + \frac{\eta_2}{\eta_1} & -\Omega_1 \\
\Omega_1 & \frac{1}{\eta_1} \lambda_1 & 0 & \Omega_1 & \frac{1}{\eta_1} \lambda_1 + \frac{1}{\eta_2}
\end{bmatrix}
\]

\[(67a)\]

\[
I = \begin{bmatrix}
J_1 \\
M_1 \\
J_3 \\
J_4 \\
M_4
\end{bmatrix}
\quad V = \begin{bmatrix}
E^i(S_1) \\
H^i(S_1) \\
0 \\
E^i(S_4) \\
H^i(S_4)
\end{bmatrix}
\]

\[(67b)\]

**Problem 9**

In the eight problem \(S_2\) is a resistive, \(S_3\) and \(S_4\) are PEC surfaces.

In this case fields can be as

\[
\theta(r)E_1(r) = E^i - \lambda_1(J_2 + J_4) + \Omega_1 M_2
\]

\[(68a)\]

\[
\theta(r)H_1(r) = H^i - \Omega_1(J_2 + J_4) + \frac{1}{\eta_1} \lambda_1 M_2
\]

\[(68b)\]

\[
\theta(r)E_2(r) = -\lambda_2(J_3 + J_{22}) - \Omega_2 M_2
\]

\[(68c)\]

\[
\theta(r)H_2(r) = -\Omega_2(J_3 + J_{22}) + \frac{1}{\eta_2} \Omega_2 M_2
\]

\[(68d)\]

Boundary conditions on the surfaces \(S_1 - S_4\) can be given as

\[
OnS_2 : \mathbf{n} \times [E_2 - E_3] = 0 \quad ; \quad \mathbf{n} \times [E_2 + E_3] = 2\eta_1 \mathbf{Rn} \times \mathbf{n} \times [H_2 - H_3]
\]

\[(69a)\]

\[
OnS_3 : \mathbf{n} \times E_3 = 0 (EFFE) \quad ; \quad \mathbf{n} \times H_3 = J_3 (MFIE)
\]

\[(69b)\]

\[
OnS_4 : \mathbf{n} \times E_1 = 0 (EFFE) \quad ; \quad \mathbf{n} \times H_1 = J_4 (MFIE)
\]

\[(69c)\]

Using \((69a - c)\) in \((68a - d)\), and after some straightforward manipulations we find the surface integral equation as
Fig. 2.14 Geometry for Problem-9
\[ Z = \begin{bmatrix} \Lambda_1 + \frac{n_1 R}{2} & -\Omega_1 & -\frac{n_1 R}{2} & 0 & \Lambda_1 \\ \Omega_1 & -\frac{1}{\eta_1} \Lambda_1 + \frac{R}{2n_1} & -\Omega_2 & -\Omega_2 & \Omega_1 \\ \frac{n_1 R}{2} & \Omega_2 & \Lambda_2 + \frac{n_1 R}{2} & \Lambda_2 & 0 \\ 0 & \Omega_2 & \Lambda_2 & \Lambda_2 & 0 \\ \Lambda_1 & -\Omega_1 & 0 & 0 & \Lambda_1 \end{bmatrix} \quad (70a) \]

\[ I = \begin{bmatrix} J_2 \\ M_2 \\ J_{22} \\ J_3 \\ J_4 \end{bmatrix}, \quad V = \begin{bmatrix} E^i(S_1) \\ H^i(S_1) \\ 0 \\ 0 \\ E^i(S_4) \end{bmatrix} \quad (70b) \]

**Problem 10**

In the eight problem \( S_2 \) is a resistive, \( S_3 \) is a PEC and \( S_4 \) is an impedance surface.

In this case fields can be as

\[ \theta(r)E_1(r) = E^i - \Lambda_1(J_2 + J_4) + \Omega_1(M_2 + M_4) \quad (71a) \]

\[ \theta(r)H_1(r) = H^i - \Omega_1(J_2 + J_4) + \frac{1}{\eta_1^2} \Lambda_1(M_2 + M_4) \quad (71b) \]

\[ \theta(r)E_2(r) = -\Lambda_2(J_3 + J_{22}) - \Omega_2 M_2 \quad (71c) \]

\[ \theta(r)H_2(r) = -\Omega_2(J_3 + J_{22}) + \frac{1}{\eta_2^2} \Omega_2 M_2 \quad (71d) \]

boundary conditions on the surfaces \( S_1 - S_4 \) can be given as

\[ \text{On} S_2 : n \times [E_2 - E_3] = 0 \quad ; \quad n \times [E_2 + E_3] = 2\eta_1 R n \times [H_2 - H_3] \quad (73a) \]

\[ \text{On} S_3 : n \times E_3 = 0(E F I E) \quad ; \quad n \times H_3 = J_3(M F I E) \quad (73b) \]

\[ \text{On} S_4 : n \times E_1 = Z n \times n \times H_1 \quad (73c) \]

Using (73a – c) in (72a – d), and after some straight forward manipulations we find the surface integral equation as
\[
\begin{bmatrix}
\Lambda_1 + \eta_1 R \\
\Omega_1 \\
\frac{\eta_1 R}{2} \\
0 \\
\eta_1 \Lambda_1
\end{bmatrix}
\begin{bmatrix}
-\Omega_1 \\
-\Omega_2 \\
\Lambda_2 + \frac{\eta_1 R}{2} \\
0 \\
\frac{1}{\eta_1} \Lambda_1
\end{bmatrix}
\begin{bmatrix}
0 \\
\Lambda_1 \\
0 \\
0 \\
\frac{1}{\eta_1} \Lambda_1 + \frac{R}{2}
\end{bmatrix}
\begin{bmatrix}
E'(S_1) \\
H'(S_1) \\
0 \\
0 \\
E'(S_4) \\
H'(S_4)
\end{bmatrix}
\]

\[Z = \begin{bmatrix}
J_2 \\
M_2 \\
J_{22} \\
J_3 \\
J_4 \\
M_4
\end{bmatrix}
\begin{bmatrix}
I
\end{bmatrix}
\]

(74a)

Problem 11

In the eight problem \( S_2 \) is a CA boundary, \( S_3 \) and \( S_4 \) are PEC surfaces.

In this case fields can be as

\[
\theta(r)E_1(r) = E' - \Lambda_1(J_2 + J_4) + \Omega_1 M_2
\]

(75a)

\[
\theta(r)H_1(r) = H' - \Omega_1(J_2 + J_4) - \frac{1}{\eta_1} \Lambda_1 M_2
\]

(75b)

\[
\theta(r)E_2(r) = -\Lambda_2(J_3 + J_{22}) - \Omega_2 M_2
\]

(75c)

\[
\theta(r)H_2(r) = -\Omega_2(J_3 + J_{22}) - \frac{1}{\eta_2} \Omega_2 M_{22}
\]

(75d)

boundary conditions on the surfaces \( S_1 \) - \( S_4 \) can be given as

\[
OnS_2 : \begin{bmatrix}
E^- \\
\eta n \times H^-
\end{bmatrix} = \begin{bmatrix}
\chi_{11} & \chi_{12} \\
\chi_{21} & \chi_{22}
\end{bmatrix} \begin{bmatrix}
E^+ \\
\eta n \times H^+
\end{bmatrix}
\]

(76a)

\[
OnS_3 : \mathbf{n} \times \mathbf{E}_3 = 0(\text{EFIE}) \quad ; \quad \mathbf{n} \times \mathbf{H}_3 = \mathbf{J}_3(\text{MFIE})
\]

(76b)

\[
OnS_4 : \mathbf{n} \times \mathbf{E}_1 = 0(\text{EFIE}) \quad ; \quad \mathbf{n} \times \mathbf{H}_1 = \mathbf{J}_4(\text{MFIE})
\]

(76c)

Using (76a - c) in (75a - d), and after some straight forward manipulations we find the surface integral equation as
Fig. 2.16 Geometry for Problem 11

CA boundary

PEC

$J_1$, $J_3$, $J_2$, $M_2$, $M_{22}$, $J_{22}$
\[
Z = \begin{bmatrix}
\Lambda_1 - \frac{\chi_{12}}{2\chi_{21}} & -\Omega_1 & -\frac{\eta}{2\chi_{21}} & 0 & 0 & \Lambda_1 \\
\Omega_1 & \frac{1}{\eta^2} \Lambda_1 + \frac{\chi_{12}}{2\eta \chi_{12}} & 0 & -\frac{1}{2\eta \chi_{12}} & 0 & \Omega_1 \\
0 & \frac{1}{2\eta} (\chi_{21} - \frac{\chi_{12}}{\chi_{12}}) & \Lambda_2 - \frac{\eta \chi_{21}}{2\chi_{21}} & -\Omega_2 & \Lambda_2 & 0 \\
0 & 0 & \frac{1}{\eta^2} \Lambda_2 + \frac{\chi_{21}}{2\eta \chi_{12}} & \Omega_2 & 0 & 0 \\
\Lambda_1 & -\Omega_1 & 0 & 0 & 0 & \Lambda_1 \\
\end{bmatrix}
\] (77a)

\[
I = \begin{bmatrix}
J_2 \\
M_2 \\
J_{22} \\
M_{22} \\
J_3 \\
J_4 \\
\end{bmatrix}, \quad V = \begin{bmatrix}
E^i(S_1) \\
H^i(S_1) \\
\end{bmatrix} \quad (77b)

Problem 12

In the eight problem $S_2$ is a CA boundary, $S_3$ is a PEC and $S_4$ is an Impedance surface.

In this case fields can be as

\[
\theta(r)E_1(r) = E^i - \Lambda_1 (J_2 + J_4) + \Omega_1 (M_2 + M_4) \quad (78a)
\]

\[
\theta(r)H_1(r) = H^i - \Omega_1 (J_2 + J_4) + \frac{1}{\eta^2} \Lambda_1 (M_2 + M_4) \quad (78b)
\]

\[
\theta(r)E_2(r) = -\Lambda_2 (J_3 + J_{22}) - \Omega_2 M_{22} \quad (78c)
\]

\[
\theta(r)H_2(r) = -\Omega_2 (J_3 + J_{22}) - \frac{1}{\eta^2} \Omega_2 M_{22} \quad (78d)
\]

boundary conditions on the surfaces $S_1 - S_4$ can be given as

\[
OnS_2 : \begin{bmatrix}
E^- \\
\eta n \times H^- \\
\end{bmatrix} = \begin{bmatrix}
\chi_{11} & \chi_{12} \\
\chi_{21} & \chi_{22} \\
\end{bmatrix} \begin{bmatrix}
E^+ \\
\eta n \times H^+ \\
\end{bmatrix} \quad (79a)
\]

\[
OnS_3 : n \times E_3 = 0 (EFIE) \quad ; \quad n \times H_3 = J_3 (MFIE) \quad (79b)
\]

\[
OnS_4 : n \times E_4 = Z n \times n \times H_1 \quad (79c)
\]

Using (79a - c) in (78a - d). and after some straight forward manipulations we find the surface integral equation as

24
Fig. 2.17 Geometry for Problem-12
\[ Z = \begin{bmatrix}
\Lambda_1 - \frac{n \chi_{22}}{2 \chi_{21}} & -\Omega_1 & -\frac{n}{2 \chi_{21}} & 0 & 0 & \Lambda_1 & -\Omega_1 \\
\Omega_1 & \frac{1}{\eta_1} \Lambda_1 + \frac{\chi_{12}}{2 \eta_1 \chi_{12}} & 0 & -\frac{1}{2 \eta_1 \chi_{12}} & 0 & \Omega_1 & -\frac{1}{\eta_1} \Lambda_1 \\
\frac{n}{2} (\chi_{12} - \frac{\chi_{12} \chi_{22}}{\chi_{21}}) & 0 & \Lambda_2 - \frac{n \chi_{22}}{2 \chi_{21}} & -\Omega_2 & \Lambda_2 & 0 & 0 \\
0 & \frac{1}{2 \eta} (\chi_{21} - \frac{\chi_{12} \chi_{22}}{\chi_{12}}) & \Omega_2 & \frac{1}{\eta_2} \Lambda_2 + \frac{\chi_{22}}{2 \eta_2 \chi_{12}} & \Omega_2 & 0 & 0 \\
0 & 0 & \Lambda_2 & -\Omega_2 & \Lambda_2 & 0 & 0 \\
\Lambda_1 & -\Omega_1 & 0 & 0 & 0 & \Lambda_1 + \frac{n}{2} & -\Omega_1 \\
\Omega_1 & \frac{1}{\eta} \Lambda_1 & 0 & 0 & 0 & \Omega_1 & \frac{1}{\eta_1} \Lambda_1 + \frac{n}{2} 
\end{bmatrix} \tag{80a}
\]

\[ I = \begin{bmatrix}
J_2 \\
M_2 \\
J_{22} \\
M_{22} \\
J_3 \\
J_4 \\
M_4 
\end{bmatrix} \quad V = \begin{bmatrix}
E^i(S_1) \\
H^i(S_1) \\
0 \\
0 \\
E^i(S_4) \\
H^i(S_4) 
\end{bmatrix} \tag{80b}
\]

**Problem 13**

In the thirteenth problem $S_3$ is a PEC and $S_4$ is an Impedance surface.

In this case fields can be as

\[ \theta(r) E_1(r) = E^i - \Lambda_1 (J_3 + J_4) + \Omega_1 M_4 \tag{81a} \]

\[ \theta(r) H_1(r) = H^i - \Omega_1 (J_3 + J_4) + \frac{1}{\eta_1} \Lambda_1 M_4 \tag{81b} \]

boundary conditions on the surfaces $S_1 - S_4$ can be given as

\[ OnS_3 : \mathbf{n} \times E_3 = 0(\mathrm{EFIE}) \quad ; \quad \mathbf{n} \times H_3 = J_3(\mathrm{MFIE}) \tag{82a} \]

\[ OnS_4 : \mathbf{n} \times E_1 = Z \mathbf{n} \times \mathbf{n} \times H_1 \tag{82b} \]

Using (82a, b) in (81a, b), and after some straightforward manipulations we find the surface integral equation as
\[
Z = \begin{bmatrix}
\Lambda_1 & \Lambda_1 & -\Omega_1 \\
\Lambda_1 & \Lambda_1 + \frac{\zeta}{2} & -\Omega_1 \\
\Omega_1 & \Omega_1 & \frac{1}{\eta_i^2} \Lambda_1 + \frac{1}{2} \Omega_1
\end{bmatrix}
\]  \hspace{1cm} (83a)

\[
I = \begin{bmatrix}
J_3 \\
J_4 \\
M_4
\end{bmatrix} \quad V = \begin{bmatrix}
E^i(S_3) \\
E^i(S_4) \\
H^i(S_4)
\end{bmatrix}
\]  \hspace{1cm} (83b)

**Problem 14**

In the thirteenth problem $S_3$ is an impedance and $S_4$ is a PEC surface.

In this case fields can be as

\[
\theta(r)E_1(r) = E^i - \Lambda_1 (J_3 + J_4) + \Omega_1 M_3
\]  \hspace{1cm} (84a)

\[
\theta(r)H_1(r) = H^i - \Omega_1 (J_3 + J_4) + \frac{1}{\eta_i^2} \Lambda_1 M_3
\]  \hspace{1cm} (84b)

boundary conditions on the surfaces $S_1 - S_4$ can be given as

\[
OnS_3 : \mathbf{n} \times \mathbf{E}_1 = \mathbf{Z} \mathbf{n} \times \mathbf{n} \times \mathbf{H}_1
\]  \hspace{1cm} (85a)

\[
OnS_4 : \mathbf{n} \times \mathbf{E}_1 = 0(\text{EFIE}) ; \quad \mathbf{n} \times \mathbf{H}_1 = \mathbf{J}_4(\text{MFIE})
\]  \hspace{1cm} (85b)

Using (85a, b) in (84a, b), and after some straightforward manipulations we find the surface integral equation as

\[
Z = \begin{bmatrix}
\Lambda_1 + \frac{\zeta}{2} & -\Omega_1 & \Lambda_1 \\
\Omega_1 & \frac{1}{\eta_i^2} \Lambda_1 + \frac{1}{2} \Omega_1 & \Lambda_1 \\
\Lambda_1 & -\Omega_1 & \Lambda_1
\end{bmatrix}
\]  \hspace{1cm} (86a)

\[
I = \begin{bmatrix}
J_3 \\
M_3 \\
J_4
\end{bmatrix} \quad V = \begin{bmatrix}
E^i(S_3) \\
H^i(S_3) \\
E^i(S_4)
\end{bmatrix}
\]  \hspace{1cm} (86b)
TEST STRUCTURES (SINGLE SURFACES)
Fig. 3.1 Monostatic Radar Cross Section from a Box and Induced Electric Currents
Fig. 3.2 Monostatic Radar Cross Section from a Cube and Induced Electric Currents

TEST STRUCTURES-II (PEC Cube-450 Unknowns)
TEST STRUCTURES-III (PEC Sphere- 540 Unknowns)

Fig. 3.3 Monostatic Radar Cross Section from a Sphere and Induced Electric Currents
Fig. 3.4 Monostatic Radar Cross Section from a plate and Induced Electric Currents
TEST STRUCTURES-V (NASA Almond-2GHz-2130 Unknowns)

Fig. 3.5 Monostatic Radar Cross Section from 14” Nasa Almond and Induced Electric Currents
TEST STRUCTURES-V(Dielectric Sphere(εᵣ=1.75-j0.3)-360 Unknowns)

Fig.3.6 Bistatic Radar Cross Section from a dielectric Sphere and Induced Electric and Magnetic Currents
TEST STRUCTURES-V (Resistive Sheet-1λX1λ-320 Unknowns)

Fig. 3.7 Monostatic Radar Cross Section from a Resistive Plate and Induced Electric Currents
TEST STRUCTURES-V (Resistive Sheet-1240 Unknowns)

Fig. 3.8 Monostatic Radar Cross Section from a Resistive Plate and Induced Electric Currents
Fig. 3.9 Induced Electric Currents

a) Co-polar \(Z_p = Z_0/4\)
b) Cross-polar \(Z_p = Z_0/4\)
TEST STRUCTURES (COMPOSITE SURFACES)
Example (Validation Geometry-II)

Dielectric ($\varepsilon_r = 2.6$)
S3: PEC
S4: PEC

Fig. 3.10 Bistatic Radar Cross Section from a metal backed dielectric cylinder
Actual Blade \((2 \lambda \times 0.5 \lambda)\) 6350 Unknowns

Fig. 3.11 Bistatic Radar Cross Section from a blade (2 dielectric and 2 PEC surfaces)
Actual Blade (2λ×0.5λ) 7049 Unknowns

Dielectric ($\varepsilon_r=(2.,0.),\mu_r=(1.,0.)$)
R-Card ( $R=(187.5,-187.5)$)

PEC

Fig.3.12 Bistatic Radar Cross Section from a blade ( R-coated case )
Actual Blade (2λX0.5λ) 7748 Unknowns

![Graphs showing bistatic radar cross section from a blade, CA-boundary](image)

Fig.3.13 Bistatic Radar Cross Section from a blade (CA-boundary)