DIFFRACTION BY A PAIR OF IMPEDANCE HALF-PLANES AND AN IMPEDANCE HALF-PLANE
ON A GROUND PLANE

by

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May 1985

This work was performed under Contract No. L5XN-379201-913 with Rockwell International-NAAO.

388967-1-T = RL-2556
ABSTRACT

A rigorous solution is presented for the electromagnetic wave diffraction by (a) A pair of parallel perfectly conducting half planes, (b) A pair of parallel impedance half planes and (c) An impedance half plane in the presence of a ground plane. The above solutions are based on the angular spectrum method which is closely related to the Weiner-Hopf technique. In addition, a GTD solution is given which is valid when the separation distance between the half planes is large. Further, the diffraction of an impedance half plane in the presence of an impedance plane is considered in an approximate manner.
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I. INTRODUCTION

This report describes the solution to the high frequency plane wave diffraction by the four structures illustrated in Fig. 1. These are
(a) Two parallel perfectly conducting half planes.
(b) Two parallel impedance half planes.
(c) An impedance half-plane over a perfectly conducting plane.
(d) An impedance half plane over an impedance plane.

The problem of diffraction by the geometry in Fig. 1(a) was first considered by Heins [1,2] via the traditional Weiner Hopf technique and later by Clemmow [3] via the angular spectrum method. It should be emphasized that the two approaches of analysis are basically equivalent. Usually a solution performed with one method can also be completed with the other. Because of the relative conciseness of the angular spectrum method, this will be employed in our diffraction analysis of the subsequent geometries in Figs. 1(b) through 1(d). Since this analysis is intended for use in the study of diffraction by a thick impedance half plane, the boundary conditions in the inner face of the half planes is not considered. A GTD (geometrical theory of diffraction) solution is also given for all of the configurations in Fig. 1. The GTD solution is generally applicable for $k > \lambda/4$, where $2k$ is the separation distance of the half plane and $\lambda$ is the wavelength.

The solution to the problem of diffraction by the two parallel half planes requires a generalization of the angular spectrum method to include the existence of electric and magnetic currents on the impedance half planes. The generated integral equations in terms of the spectral density functions of these currents are ultimately decoupled
Figure 1. Geometry of the four structures under study (\(\eta_1\) denotes the normalized surface impedance).
on the basis that the magnetic currents vanish when the surface impedance is zero. A GTD solution to this configuration is rather simple provided the diffraction coefficient for the impedance half plane is incorporated in the corresponding GTD solution for the parallel perfectly conducting half planes.

Once the diffracted fields by the geometry in Fig. 1(b) has been obtained, one can then tackle the configurations in Figs. 1(c) and 1(d) by invoking image theory. Clearly, the diffracted field by the geometry in Fig. 1(c) with the perfectly conducting half plane can be obtained rigorously. However, the diffracted field by the geometry in Fig. 1(d) which includes an impedance plane can only be evaluated approximately at this time.

The report contains five chapters. Chapter II presents the formal solution of the diffracted field via the angular spectrum method. Although the $E_z$ solution has already been given by Clemmow [3], this chapter serves the purpose of introducing the method of analysis and in addition presents the solution for the $H_z$ incidence not completed by Clemmow.

Chapter III considers the diffraction by two parallel impedance half planes.

Chapter IV employs the results of Chapter III to formulate solutions for the diffracted field by an impedance half plane parallel to a ground plane or an impedance plane.

Finally, Chapter V discusses the approach to be used in incorporating the above analysis for the diffraction by a step discontinuity or protrusion on a ground or impedance plane.
II. DIFFRACTION BY A PAIR OF PARALLEL PERFECTLY CONDUCTING HALF PLANES

2.1 Introduction

As discussed in the introduction, the electromagnetic problem of a pair of perfectly conducting, half planes has been considered by a number of investigators. First Heins etc [1,2] obtained a Wiener-Hopf solution to the problems of reflection and coupling due to a plane wave incidence. However, a study on the diffraction pattern was not performed. Nevertheless, the difficult task of obtaining the required split functions was accomplished. Clemmow [3] later used the angular spectrum method [4] to obtain a solution for the diffraction by a pair of parallel half planes. His solution again involves the split functions obtained by Heins but is less involved because it avoids the direct computation of the currents on the half planes. Clemmow gives the scattering solution for the soft case (TM) only. However his method is applicable to the hard case (TE) as well. This is indeed accomplished in this chapter.

Lee and Mittra [5] generalized Heins hard case solution (see also Noble [6]) to include the effects of a recessed stub between the parallel half planes. They also appear to be the only authors to date who have published computed results for the scattering by a pair of perfectly conducting half planes and a thick edge (stub has zero recession).

Thus the solution of the diffraction by a pair of perfectly conducting half planes can be found in the literature. However,
these exist in various forms and are in some cases incomplete for our use. It is therefore instructive to rederive the scattering solutions from a pair of perfectly conducting half planes using a consistent method and approach suitable to our needs. In doing this we need to choose a method which can be extended to the scattering solution of the same geometry when it is associated with impedance boundary conditions representing a thin material coating. Clemmow's angular spectrum method was found to give promising results in this direction as it will be examined later.

The angular spectrum method involves the representation of the scattered field by an integral of plane waves. The weighting coefficients corresponding to these plane waves are referred to as the angular (continuous) spectra of the scattered field and are proportional to the surface currents via a Fourier integral transform. A solution to the scattered field then involves the determination of this angular spectrum after invoking the required boundary conditions and certain continuity conditions. Subsequently, the scattered or diffracted field is obtained via an asymptotic evaluation of the integral such as the steepest descent path method.

For the perfectly conducting pair of half planes considered in this section, only electric currents exist on the planar surfaces and thus a single angular spectrum on each half plane is required. However, in the next section, where the pair of impedance half planes is considered, both electric and magnetic angular spectra on each half plane must be included in the solution.
2.2 $H_z$-Incidence

Consider a plane wave (an $e^{j\omega t}$ factor is assumed and suppressed throughout)

$$ H_z^i = e^{jk(x \cos \phi_0 + y \sin \phi_0)} $$

(1)

to be incident on the perfectly conducting geometry in Fig. 1(a).

Employing now the angular spectrum method, the scattered $H$-field can be represented in terms of its angular spectrum as

$$ H_z^{sa} = \pm \int_C \left[ P_1(\cos \alpha) + P_2(\cos \alpha)e^{-j2k\xi \sin \alpha} \right] e^{-jkr \cos(\phi_0 \alpha)} \, d\alpha; \quad y \geq 0 \leq -2\xi $$

(2)

where $C$ is the contour shown in Fig. 2, $P_1(\cos \alpha)$ is associated with the upper half plane and $P_2(\cos \alpha)$ with the lower one. If we define

$$ \lambda = \cos \alpha, $$

(3)

it is clear that in the $\lambda$-plane the $C$ contour traverses from $-\infty$ to $\infty$ and it is indented at $\lambda = -1, -\lambda_0, 1$ as illustrated (note $\lambda_0 = \cos \phi_0$). In the $\alpha$-plane this path corresponds to that for which $\cos \alpha$ remains real. The following outlines a procedure for finding $P_1(\cos \alpha)$ and $P_2(\cos \alpha)$.

According to (2), the scattered field component $E_x^{sa}$ is given by

$$ E_x^{sa} = -Z_0 \int_C \sin \alpha \left[ P_1(\cos \alpha) + P_2(\cos \alpha)e^{+jk\xi \sin \alpha} \right] e^{-jkr \cos(\phi_0 \alpha)} \, d\alpha; \quad y \geq 0 \leq -2\xi $$

(4)
Figure 2. Illustration of the C contour in the $\alpha$ and $\lambda$ complex planes.
where $Z_0$ is the free space impedance. The boundary condition $E_x^s + E_x^i = 0$ on the surfaces $y = 0$, $x > 0$ and $y = -2z$, $x > 0$ require the solution of the following integral equations for $x > 0$,

$$
\int_{-\infty}^{\infty} [P_1(\lambda) + P_2(\lambda)e^{-2jkz\sqrt{1-\lambda^2}}] e^{-j k x \lambda} d\lambda = \sqrt{1 - \lambda^2} e^{-j k x \lambda} e^{-jkx\lambda_0} \quad (5a)
$$

and

$$
\int_{-\infty}^{\infty} [P_1(\lambda)e^{-2jkz\sqrt{1-\lambda^2}} + P_2(\lambda)]e^{-j k x \lambda} d\lambda = \sqrt{1 - \lambda^2} e^{-j2kz\sqrt{1-\lambda^2}} e^{-j k x \lambda} e^{-jkx\lambda_0} \quad (5b)
$$

The above were obtained by noting that

$$
E_x^i = \sin \phi_0 e^{jk(x \cos \phi_0 + y \sin \phi_0)} \quad (6)
$$

and

$$
d\lambda = \sin \alpha \, d\alpha \quad . \quad (7)
$$

In addition, field continuity over the planes $y = 0$, $x < 0$ and $y = -2z$, $x < 0$, requires that $H_z^{sa}$ be continuous across these boundaries. From (2) this condition gives

$$
\int_{-\infty}^{\infty} \left[ \frac{P_1(\lambda)}{\sqrt{1 - \lambda^2}} + \frac{P_2(\lambda)}{\sqrt{1 - \lambda^2}} e^{-j2kz\sqrt{1-\lambda^2}} \right] e^{-j k x \lambda} d\lambda = 0 \quad (8a)
$$

and

$$
\int_{-\infty}^{\infty} \left[ \frac{P_1(\lambda)}{\sqrt{1 - \lambda^2}} e^{-j2kz\sqrt{1-\lambda^2}} + \frac{P_2(\lambda)}{\sqrt{1 - \lambda^2}} \right] e^{-j k x \lambda} d\lambda = 0 \quad . \quad (8b)
$$
for \( x < 0 \). After closing the integration paths by a semi-infinite path in the upper half plane, it is clear that the above are satisfied only if \( P_1(\lambda)/\sqrt{1 - \lambda^2} \) and \( P_2(\lambda)/\sqrt{1 - \lambda^2} \) are free of singularities and zeros (i.e., regular) in the upper \( \lambda \) half plane. This is the region above the contour on the real axis shown in Fig. 2. Denoting such functions as \( U \)-functions and those having the same properties in the lower half plane as \( L \)-functions, we conclude that

\[
P_1(\lambda) = \sqrt{1 + \lambda} U_{1h}(\lambda) \quad (9a)
\]

and

\[
P_2(\lambda) = \sqrt{1 + \lambda} U_{2h}(\lambda) \quad (9b)
\]

\( U_{1h}(\lambda) \) and \( U_{2h}(\lambda) \) are now regular in the upper half plane and such a notation will be employed throughout this report. Note also that the factor \( \sqrt{1 + \lambda} \) is regular in the lower half plane and \( \sqrt{1 - \lambda} \) is regular in the upper half plane. Thus, if we write \( \sqrt{1 - \lambda^2} = \sqrt{1 - \lambda} \sqrt{1 + \lambda} \) and use (9), it is clear that the terms \( P_{1,2}(\lambda)/\sqrt{1 - \lambda^2} \) in (8) are regular in the upper half plane.

In order to find expressions for \( P_{1,2}(\lambda) \) according to (5) and in view of (9), we first proceed to write (5) in a more convenient form. By adding and subtracting (5a) and (5b) we obtain the equivalent integral equations

\[
\int_{-\infty}^{\infty} \sqrt{1 + \lambda} \mathcal{Q}_1(\lambda) \left( 1 + e^{-j2k\epsilon/\sqrt{1 - \lambda^2}} \right) e^{-jkx\lambda} \, d\lambda = \sqrt{1 - \lambda^2}_0 \\
\left( 1 + e^{-j2k\epsilon/\sqrt{1 - \lambda^2}_0} \right) e^{jkx\lambda_0} \quad (10a)
\]
and

\[
\int_{-\infty}^{\infty} \sqrt{1 + \lambda} Q_2(\lambda) \left( 1 - e^{-2jk\sqrt{1-\lambda^2}} \right) e^{-jkx\lambda} d\lambda = \sqrt{1 - \lambda^2_0} \left( 1 - e^{-2jk\sqrt{1-\lambda^2_0}} \right) e^{jkx\lambda_0}
\]  

(10b)

for \( x > 0 \), where

\[
Q_{1+,2}(\lambda) = U_{1h}(\lambda) \pm U_{2h}(\lambda) .
\]  

(11)

By defining the branch of \( \sqrt{1 - \lambda^2} \) to always have a negative imaginary part we then close the path of integration in (10) by a semi-circular contour at infinity of the lower \( \lambda \) half plane. Subsequently, employing Cauchy’s theorem, we find that

\[
\sqrt{1 + \lambda} Q_1(\lambda)(1 + e^{-j2k\sqrt{1-\lambda^2}}) = -\frac{\sqrt{1 - \lambda^2}}{2\pi j} \left( 1 + e^{-j2k\sqrt{1-\lambda^2}} \right) 
\]

\[
\cdot \frac{L_{1h}(\lambda)}{L_{1h}(-\lambda_0)} \frac{1}{\lambda + \lambda_0} 
\]  

(12a)

and

\[
\sqrt{1 + \lambda} Q_2(\lambda)(1 - e^{-j2k\sqrt{1-\lambda^2}}) = -\frac{\sqrt{1 - \lambda^2}}{2\pi j} \left( 1 - e^{-j2k\sqrt{1-\lambda^2}} \right) 
\]

\[
\cdot \frac{L_{2h}(\lambda)}{L_{2h}(-\lambda_0)} \frac{1}{\lambda + \lambda_0} .
\]  

(12b)

As we mentioned earlier the \( L_{1h}(\lambda), L_{2h}(\lambda) \) functions are regular in the lower half plane. The ratio of \( L(\lambda)/L(-\lambda_0) \) is clearly equal to unit at the pole \( \lambda = -\lambda_0 \) so that the residue of this pole gives the result in the right hand side of (10).
A solution to $Q_{1,2}(\lambda)$ requires first the factorization of the expressions $(1 \pm e^{-j2k_0 \sqrt{1-\lambda^2}})$ in terms of $U$ and $L$ functions. This has been accomplished by Heins [1,2] with the $e^{-j\omega t}$ time convention and have been used by Clemmow [3] and Lee and Mittra [5] in certain calculations. We simply quote here that

$$1 + e^{-j2k_0 \sqrt{1-\lambda^2}} = U_1(\lambda)L_1(\lambda)$$  \hspace{1cm} (13a)

and

$$1 - e^{-j2k_0 \sqrt{1-\lambda^2}} = U_2(\lambda)L_2(\lambda)$$  \hspace{1cm} (13b)

The split functions $U_{1,2}(\lambda) = L_{1,2}(-\lambda)$ are regular in the upper half plane and their holomorphic expressions are given in Appendix I according to the $e^{j\omega t}$ time convention employed here. Simplified expressions suitable for small $\lambda$ are also given in Appendix I.

From (12) and (13), and since $Q_{1,2}(\lambda)$ are $U$-functions we find that

$$Q_1(\lambda) = -\frac{1}{2\pi j} \frac{L_1(\lambda_0) \sqrt{1+\lambda_0}}{U_1(\lambda)(\lambda + \lambda_0)}$$  \hspace{1cm} (14a)

and

$$Q_2(\lambda) = -\frac{1}{2\pi j} \frac{L_2(\lambda_0) \sqrt{1+\lambda_0}}{U_2(\lambda)(\lambda + \lambda_0)}$$  \hspace{1cm} (14b)

In addition,

$$L_{1h}(\lambda) = \sqrt{1+\lambda} L_1(\lambda)$$  \hspace{1cm} (15)

and
\[ L_1(-\lambda_0) = \sqrt{1 - \lambda_0} \quad L_2(-\lambda_0) = \sqrt{1 - \lambda_0} U_1(\lambda_0). \quad (16) \]

Finally, from (9), (11), (14) and (3) we find that the angular spectra associated with the upper and lower perfectly conducting half planes are given by

\[ P_{1/2}(\cos \phi) = -\frac{1}{4\pi^2} \frac{2 \cos \phi/2 \cos \phi_0/2}{\cos \phi + \cos \phi_0} \left[ \frac{L_1(\cos \phi_0)}{U_1(\cos \phi)} \pm \frac{L_2(\cos \phi_0)}{U_2(\cos \phi)} \right]. \quad (17) \]

Using (17) in (2) and evaluating the integral via the steepest descent method it is obtained that the scattered field is

\[ H_{z}^{Sa} = \frac{\sqrt{2\pi}}{kr} \left[ P_1(\cos \phi) + P_2(\cos \phi)e^{-j2k\sin \phi}\right]e^{-jkr}\epsilon^{j\pi/4}; \quad (18) \]

or

\[ H_{z}^{Sa} = \frac{1}{\sqrt{2\pi k}} e^{-j\pi/4} \frac{2 \cos \phi/2 \cos \phi_0/2}{\cos \phi + \cos \phi_0} \left[ -L_1(\cos \phi_0) \cos(k\sin \phi) + j \frac{L_2(\cos \phi_0)}{U_2(\cos \phi)} \sin(k\sin \phi) \right] e^{-jkr}\epsilon^{j\pi/4}; \quad (19) \]

Noting that the factor

\[ D_1(\phi, \phi_0) = -\frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{2 \cos \phi/2 \cos \phi_0/2}{\cos \phi + \cos \phi_0} \quad (20) \]
is the hard diffraction coefficient for the edge of a perfectly conducting half plane we may express $H_z^{Sa}$ more compactly as

$$H_z^{Sa} = D_h(\phi, \phi_0) \Delta(\phi, \phi_0; \kappa) \frac{e^{-jk\kappa}}{\sqrt{r}}; \quad (21)$$

where

$$\Delta(\phi, \phi_0; \kappa) = e^{-jk\kappa} \sin \phi \left[ \frac{L_1(\cos \phi_0)}{U_1(\cos \phi)} \cos(k\kappa \sin \phi) + j \frac{L_2(\cos \phi_0)}{U_2(\cos \phi)} \cdot \sin(k\kappa \sin \phi) \right]. \quad (22)$$

It will be clear in the later sections that $\Delta(\phi, \phi_0; \kappa)$ is a geometry dependent factor and corresponds to the interactions between the two half planes. In addition, although not obvious from (22), it can be proven with the use of (13) that (21) satisfies reciprocity as required.

Since for $\kappa = 0$ (see Appendix I or Eq. (13)),

$$L_1(\cos \phi) = U_1(\cos \phi) = \sqrt{2}$$

it follows that

$$\Delta(\phi, \phi_0; 0) = 1$$

and thus (23) reduces to the diffracted field by a perfectly conducting half plane as required. Further, because $D_h(\pi, \pi) = 0$, $H_z^{Sa}$ vanishes when $\phi = \phi_0 = \pi$, i.e., at grazing.
2.3 \( E_Z \)-Incidence

The analysis for finding the scattered field with a plane wave

\[
E^i_Z = e^{jk(x \cos \phi_0 + y \sin \phi_0)}
\]  

(23)

incident onto the geometry of Fig. 1(a) is similar to that presented for the \( H_Z \) (TE) case.

In terms of its angular spectrum, the scattered \( E \)-field can be represented as

\[
E^s_Z = \int_C \left[ P_1(\cos \alpha) + P_2(\cos \alpha) e^{-j2k\lambda \sin \alpha} \right] e^{-jkr \cos(\phi - \alpha)} d\alpha; \ y > -2\lambda
\]

(24)

and the solution reduces to finding \( P_{1,2}(\cos \phi) \) as before. The boundary condition \( E^s_Z + E^i_Z = 0 \) requires that

\[
\int_{-\infty}^{\infty} \left[ \frac{P_1(\lambda)}{\sqrt{1 - \lambda^2}} + \frac{P_2(\lambda)}{\sqrt{1 - \lambda^2}} e^{-j2k\lambda \sqrt{1 - \lambda^2}} \right] e^{-j\lambda x} d\lambda = -e^{j\lambda x_0}
\]

(25a)

and

\[
\int_{-\infty}^{\infty} \left[ \frac{P_1(\lambda)}{\sqrt{1 - \lambda^2}} e^{-j2k\lambda \sqrt{1 - \lambda^2}} + \frac{P_2(\lambda)}{\sqrt{1 - \lambda^2}} \right] e^{-j\lambda x} d\lambda = -e^{j\lambda x_0} e^{-j2k\lambda \sqrt{1 - \lambda^2}}
\]

(25b)

for \( x > 0 \). In addition, continuity of the tangential \( H \)-field \( (H_x) \) over the region \( x < 0 \) requires that

\[
\int_{-\infty}^{\infty} \left[ P_1(\lambda) + P_2(\lambda) e^{-j2k\lambda \sqrt{1 - \lambda^2}} \right] e^{-j\lambda x} d\lambda = 0
\]

(26a)
and

\[ \int_{-\infty}^{\infty} \left[ P_1(\lambda) e^{-j2k\varepsilon\sqrt{T-\lambda^2}} + P_2(\lambda) \right] e^{-jkx\lambda} \, d\lambda = 0 \quad (26b) \]

for \( x < 0 \). Alternatively, (26) could be formulated by noting that electric currents scatter zero tangential \( H \)-field in the planes which contain them.

Equation (26) clearly implies that \( P_{1,2}(\lambda) \) are \( U \)-functions, i.e., regular in the upper \( \lambda \)-half plane shown in Fig. 2. Therefore,

\[ P_{1,2}(\lambda) = U_{1e}(\lambda) \quad . \quad (27) \]

Similarly to the \( H_z \) case we also find by adding and subtracting (25) and subsequently employing Cauchy's theorem, that

\[ \frac{Q_1(\lambda)}{\sqrt{T-\lambda^2}} \left( 1 + e^{-j2k\varepsilon\sqrt{T-\lambda^2}} \right) = \frac{1}{2\pi j} \left( 1 + e^{-j2k\varepsilon\sqrt{T-\lambda^2}} \right) \frac{L_{1e}(\lambda)}{L_{1e}(-\lambda_0)} \frac{1}{\lambda + \lambda_0} \quad (28a) \]

and

\[ \frac{Q_2(\lambda)}{\sqrt{T-\lambda^2}} \left( 1 - e^{-j2k\varepsilon\sqrt{T-\lambda^2}} \right) = \frac{1}{2\pi j} \left( 1 - e^{-j2k\varepsilon\sqrt{T-\lambda^2}} \right) \frac{L_{2e}(\lambda)}{L_{2e}(-\lambda_0)} \frac{1}{\lambda + \lambda_0} \quad (28b) \]

where

\[ Q_1(\lambda) = P_1(\lambda) + P_2(\lambda) = U_{1e}(\lambda) + U_{2e}(\lambda) \quad (29a) \]

and

\[ Q_2(\lambda) = P_1(\lambda) - P_2(\lambda) = U_{1e}(\lambda) - U_{2e}(\lambda) \quad . \quad (29b) \]

In view of (13) and (29), equations (28) are satisfied by
\[ Q_1(\lambda) = \frac{1}{2\pi j} \frac{L_1(\lambda_0)}{U_1(\lambda)} \frac{\sqrt{1 - \lambda}}{\lambda + \lambda_0} \frac{\sqrt{1 - \lambda_0}}{\lambda + \lambda_0} \]  \tag{30a} \]

and

\[ Q_2(\lambda) = \frac{1}{2\pi j} \frac{L_2(\lambda_0)}{U_2(\lambda)} \frac{\sqrt{1 - \lambda}}{\lambda + \lambda_0} \frac{\sqrt{1 - \lambda_0}}{\lambda + \lambda_0} \]  \tag{30b} \]

The split functions \( U_{1,2}(\lambda) = L_{1,2}(-\lambda) \) are discussed in Appendix I.

From (29) we obtain that

\[ P_{1,2}(\cos \phi) = \frac{1}{4\pi j} \frac{2 \sin \phi/2 \sin \phi_0/2}{\cos \phi + \cos \phi_0} \left[ \frac{L_1(\cos \phi_0)}{U_1(\cos \phi)} \pm \frac{L_2(\cos \phi_0)}{U_2(\cos \phi)} \right] . \]  \tag{31} \]

Substitution in (24) and subsequent evaluation of the integral via the steepest descent method gives

\[ E_{\text{sa}}^z = D_s(\phi, \phi_0) \Delta(\phi, \phi_0; \xi) \frac{e^{-jk r}}{\sqrt{r}} . \]  \tag{32} \]

As expected, \( D_s(\phi, \phi_0) \) is now the soft diffraction coefficient associated with the diffraction by the edge of a perfectly conducting half plane. Specifically

\[ D_s(\phi, \phi_0) = \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{2 \sin \phi/2 \sin \phi_0/2}{\cos \phi + \cos \phi_0} \]  \tag{33} \]

The geometry function \( \Delta(\phi, \phi_0; \xi) \) is the same as in the \( H_z \) incidence and is given in (22).
2.4 A GTD Solution for Large $\varepsilon$

When $\varepsilon > \lambda/4$, a solution for the diffraction by the geometry in Fig. 1(a) in the context of the geometrical theory of diffraction (GTD) [7] may be applicable. This process requires the summation of the contributions from all six ray mechanisms illustrated in Fig. 3. Using the asymptotic expansion of Bowman [8] for large $k\varepsilon$, we find that to the $O(1/k\varepsilon)$,

\[
\begin{align*}
\left\{ \begin{array}{l}
E_{z_i}^a \\
H_{z_i}^a
\end{array} \right\} &= u_{z_i}^1 = D_{s}(\phi, \phi_0) \frac{e^{-jkr}}{h} \\
u_{z_i}^2 &= D_{s}(\phi, \phi_0)e^{-jk\varepsilon}\sin \phi + \sin \phi_0 \frac{e^{-jkr}}{\sqrt{r}}.
\end{align*}
\]

\[\tag{34a}\]

\[
\begin{align*}
u_{z_i}^{12} &= \frac{e^{-j\pi/4}}{\sqrt{4\pi k\varepsilon}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_0} \right) D_{s}(\phi, \phi_0)e^{-jk\varepsilon}\sin \phi_0 \\
&\quad \cdot \left( \sum_{m=0}^{\infty} \frac{e^{-j(2m+1)2k\varepsilon}}{(2m + 1)^{3/2}} \right) e^{-jk\varepsilon}.
\end{align*}
\]

\[\tag{34b}\]

\[
\begin{align*}
u_{z_i}^{21} &= \frac{e^{-j\pi/4}}{\sqrt{4\pi k\varepsilon}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_0} \right) D_{s}(\phi, \phi_0)e^{-jk\varepsilon}\sin \phi \\
&\quad \cdot \left( \sum_{m=0}^{\infty} \frac{e^{-j(2m+1)2k\varepsilon}}{(2m + 1)^{3/2}} \right) e^{-jk\varepsilon}.
\end{align*}
\]

\[\tag{34c}\]
Figure 3. GTD interaction terms associated with two parallel half planes.
\[ u_{z}^{11} = -\frac{e^{-j\pi/4}}{\sqrt{4\pi k_{\lambda}}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_{0}} \right) D_{s}(\phi, \phi_{0}) \left( \sum_{m=1}^{\infty} \frac{e^{-j(2m)2k_{\lambda}}}{(2m)^{3/2}} \right) \frac{e^{-jk_{r}}}{\sqrt{r}} \]  

(34e)

and

\[ u_{z}^{22} = -\frac{e^{-j\pi/4}}{\sqrt{4\pi k_{\lambda}}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_{0}} \right) D_{s}(\phi, \phi_{0}) e^{-j2k_{\lambda}(\sin \phi + \sin \phi_{0})} \]

\[ \cdot \left( \sum_{m=1}^{\infty} \frac{e^{-j(2m)2k_{\lambda}}}{(2m)^{3/2}} \right) \frac{e^{-jk_{r}}}{\sqrt{r}} \]  

(34f)

In the above, \( u_{z} \) denotes components of either \( E_{z}^{sa} \) or \( H_{z}^{sa} \) corresponding to the soft (\( D_{s} \)) or hard (\( D_{h} \)) diffraction coefficients, respectively. Furthermore, the infinite sums in 34(c) through (f) are associated with the multiple interactions between the upper and lower half plane edges as illustrated in Fig. 3(c) through (f). Each term in the \( m \)-series is divided by the factors \((2m + 1)^{3/2}\) or \((2m)^{3/2}\) and as seen from Table I, the first two to three terms are in most cases sufficient depending on the value of \( \lambda \). More precisely, the number of terms needed for good accuracy is less for a larger separation distance between the half planes. In general,

\[ \begin{align*}
\left\{ E_{z}^{sa} \right\} & \sim u_{z}^{1} + u_{z}^{2} + u_{z}^{12} + u_{z}^{21} + u_{z}^{11} + u_{z}^{22} \\
\left\{ H_{z}^{sa} \right\} & \sim
\end{align*} \]  

(35)

For relatively large \( \lambda \), say 1.5 \( \lambda \), the scattered field may even be approximated with only the singly diffracted fields from the upper and lower edges, viz for \( \lambda > 1.5 \lambda \).
\[ E_{sz}^{sa} = D_s(\phi, \phi_o) \cos[kz(\sin \phi + \sin \phi_o)] e^{-jkz(\sin \phi + \sin \phi_o)} \frac{e^{-jkr}}{\sqrt{r}}. \]

\[ (36) \]
Table 2.1

Values of $(2m)^{3/2}$ and $(2m+1)^{3/2}$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$(2m)^{3/2}$</th>
<th>$(2m+1)^{3/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>--</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2.83</td>
<td>5.2</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>11.18</td>
</tr>
<tr>
<td>3</td>
<td>14.7</td>
<td>18.5</td>
</tr>
<tr>
<td>4</td>
<td>22.63</td>
<td>27.0</td>
</tr>
</tbody>
</table>
III. DIFFRACTION BY A PAIR OF PARALLEL IMPEDANCE HALF PLANES

3.1 Introduction

The geometry relating to a pair of imperfectly conducting or impedance half planes is depicted in Fig. 1(b). An impedance (opaque) half plane is characterized by the boundary conditions

\[ \bar{\mathbf{E}} - (\hat{n} \cdot \bar{\mathbf{E}})\hat{n} = \eta_1 Z_0 \hat{n} \times \bar{\mathbf{H}} \]  \hspace{1cm} (37)

on its upper and lower surfaces. \( \bar{\mathbf{E}} \) and \( \bar{\mathbf{H}} \) in (37) refer to the total fields and \( \eta_1 = Z_S/Z_0 \) is the normalized impedance on each face of the half plane. \( \eta_1 \) may also correspond to an equivalent surface impedance. For example, in case of a perfectly conducting half plane with a thin material coating on each of its faces, the problem is equivalent to an impedance half plane with

\[ \eta_1 = j \tan k_d t \]  \hspace{1cm} (38)

where \( t \) is the small thickness of the coating and \( k_d \) is the propagation constant associated with the material coating. In terms of the constitutive parameters

\[ \mu_d = \mu_0 (\mu'_r - j\mu''_r) \]  \hspace{1cm} (39a)

\[ \varepsilon_d = \varepsilon_0 (\varepsilon'_r - j\varepsilon''_r) \]  \hspace{1cm} (39b)

we have that

\[ k_d = k_0 (k'_r - jk''_r) \]  \hspace{1cm} (40a)
where

\[ k^t_r = K_{mr} \cos(\phi_d/2) \]  \hspace{1cm} (40b)

\[ k^u_r = K_{mr} \sin(\phi_d/2) \]  \hspace{1cm} (40c)

\[ K_{mr} = \left[ (\mu_r^e \varepsilon_r^e - \varepsilon_r^u \mu_r^u)^2 + (\mu_r^u \varepsilon_r^u + \varepsilon_r^u \mu_r^u)^2 \right]^{1/4} \]  \hspace{1cm} (40d)

and

\[ \phi_d = \tan^{-1} \left[ \frac{\mu_r^u \varepsilon_r^u + \varepsilon_r^u \mu_r^u}{\mu_r^e \varepsilon_r^e - \varepsilon_r^u \mu_r^u} \right]. \]  \hspace{1cm} (40e)

Equation (38) may be even further approximated to give (for very small \( t \)) [9]

\[ \eta = jk_0 t \left( \frac{\varepsilon_d}{\varepsilon_0} - \frac{\varepsilon_0}{\varepsilon_d} \right) \]  \hspace{1cm} (41)

\( \varepsilon_0 \) and \( \mu_0 \) are the free space permittivity and permeability.

The asymptotic solution to the diffraction by a single impedance half plane is given by Senior [10]. In addition, explicit (non-integral) uniform diffraction coefficients for the impedance half plane can be found in [11]. These will be used later in this section. However, the problem of diffraction by a pair of half planes shown in Fig. 1(b) has not been considered previously. In the following we will develop such a solution. The approach of analysis will be based on the angular spectrum method and as such it will be a continuation or generalization of the procedure given in Section II. Once the scattering solutions to the problem in Fig. 1(b) has been developed it will serve as the basis for the diffraction by an impedance half-plane over a ground plane via image theory or, in an approximate manner, over an
impedance plane. Further the angular spectrum method can be used for obtaining the coupling into the parallel plate waveguide composed of a pair of impedance half planes. Subsequently, based on the above results the problem of diffraction or scattering by a coated protrusion can be considered.

3.2 Formulation of the Required Integral Equations

Because of the non-zero impedance on the half planes of Fig. 1(b), both electric and magnetic currents will exist on their surfaces. Therefore, the total scattered field will be given by

\[
\begin{align*}
E^s_z &= E^{es}_z + E^{ms}_z \\
H^s_z &= H^{es}_z + H^{ms}_z
\end{align*}
\]

(42)

depending on the polarization of incidence under consideration. In (42), the components with the superscript es refer to the scattered field associated with the electric currents and those with the superscript ms refer to the scattered field associated with the magnetic currents.

According to the angular spectrum method, the scattered field components due to the configuration depicted in Fig. 1(b) may be respresented as follows (see also Chapter II):

In case of \( E_z \) incidence,

\[
E^{es}_z = \int_C \left[ P_1 e^{i \alpha} + P_2 e^{i \alpha} e^{i \lambda_\alpha \sin \alpha} \right] e^{-jkr \cos(\phi_\alpha)} d\alpha ; \quad y \geq 0
\]

(43a)

and
\[ E_z^{ms} = \pm \int_C [P_{1m}(\cos \alpha) + P_{2m}(\cos \alpha)e^{2jkl \sin \alpha}e^{-jk\cos(\phi \alpha)}] \mathrm{d}a ; \quad y \geq 0 \quad \leq -2l \] (43b)

where C is the contour for which \( \cos \alpha \) remains real and is shown in Fig. 2. The corresponding H-fields are

\[ H_x^{es} = \pm Y_0 \int C \sin \alpha[P_{1e}(\cos \alpha) + P_{2e}(\cos \alpha)e^{2jkl \sin \alpha}e^{-jk\cos(\phi \alpha)}] \mathrm{d}a ; \quad y \geq 0 \quad \leq -2l \] (44a)

and

\[ H_x^{ms} = Y_0 \int C \sin \alpha[P_{1m}(\cos \alpha) + P_{2m}(\cos \alpha)e^{j2kl \sin \alpha}e^{-jk\cos(\phi \alpha)}] \mathrm{d}a ; \quad y \geq 0 \quad \leq -2l \] (44b)

In the above, \( P_{1e}(\cos \alpha) \) and \( P_{1m}(\cos \alpha) \) are the angular spectra associated with the top impedance half plane and similarly \( P_{2e}(\cos \alpha) \) with \( P_{2m}(\cos \alpha) \) correspond to the bottom half plane. Using the parameter \( \lambda = \cos \alpha \), we further find that \( P_{1e}(\lambda) \) correspond to the spectra of the electric currents, \( J_z \), on the respective planes due to the discontinuity of \( H_x^{es} \). Similarly, \( P_{1m}(\alpha)/\sqrt{1-\lambda^2} \) correspond to the spectra of the magnetic currents, \( M_x \), due to the discontinuity of \( E_z^{ms} \) over the \( y = 0, x > 0 \) and \( y = -2l, x < 0 \) planes.

In the case of \( H_z \) incidence, the scattered fields can be represented as
\[ H^{es}_z = \pm \int_C [P_{1e}(\cos \alpha) + P_{2e}(\cos \alpha)e^{j2kz \sin \alpha}]e^{-jkr \cos(\phi + \alpha)}d\alpha; \quad y \geq 0, \quad y \leq -2l \tag{45a} \]

and

\[ H^{ms}_z = \int_C [P_{1m}(\cos \alpha) + P_{2m}(\cos \alpha)e^{j2kz \sin \alpha}]e^{-jkr \cos(\phi - \alpha)}d\alpha; \quad y \geq 0, \quad y \leq -2l \tag{45b} \]

The corresponding E-fields are

\[ E^{es}_x = -Z_0 \int_C \sin \alpha [P_{1e}(\cos \alpha) + P_{2e}(\cos \alpha)e^{j2kz \sin \alpha}]e^{-jkr \cos(\phi + \alpha)}d\alpha; \]

\[ y > 0, \quad y < -2l \tag{46a} \]

and

\[ E^{ms}_x = \pm Z_0 \int_C \sin \alpha [P_{1m}(\cos \alpha) + P_{2m}(\cos \alpha)e^{j2kz \sin \alpha}]e^{-jkr \cos(\phi - \alpha)}d\alpha; \]

\[ y > 0, \quad y < -2l \tag{46b} \]

The determination of the above spectra \( P_{1e}(\lambda) \) and \( P_{1m}(\lambda) \) requires the simultaneous solution of an equal number of integral equations. These can be obtained by imposing the necessary continuity and boundary conditions. For brevity purposes, below we discuss the construction of the required integral equations for the \( E_z \) incidence only. The corresponding equations for the \( H_z \)-incidence can be obtained in a similar manner on the basis of Eqs (45) and (46).

For the \( E_z \)-incidence a set of integral equations can be obtained if we note that \( H^{es}_x \) vanishes in the plane of the electric current sheets and via duality \( E^{ms}_z \) also vanishes in the plane of the magnetic current sheet. These conditions require that
\[ \int_{-\infty}^{\infty} p_{1e}(\lambda) e^{-j \kappa \lambda} d\lambda = 0 \quad \text{for} \quad x < 0, \quad (47a) \]

\[ \int_{-\infty}^{\infty} p_{2e}(\lambda) e^{-j \kappa \lambda} d\lambda = 0 \quad \text{for} \quad x < 0, \quad (47b) \]

\[ \int_{-\infty}^{\infty} \frac{p_{1m}(\lambda)}{\sqrt{1 - \lambda^2}} e^{-j \kappa \lambda} d\lambda = 0 \quad \text{for} \quad x < 0, \quad (47c) \]

\[ \int_{-\infty}^{\infty} \frac{p_{2m}(\lambda)}{\sqrt{1 - \lambda^2}} e^{-j \kappa \lambda} d\lambda = 0 \quad \text{for} \quad x < 0. \quad (47d) \]

Equations (47) are similar to Eqs. (8) and (26) in Chapter II and imply that

\[ p_{1e}(\lambda) = U_{1e}(\lambda) \quad (48a) \]

and

\[ p_{1m}(\lambda) = \sqrt{1 - \lambda^2} U_{1m}(\lambda). \quad (48b) \]

As before, the U-functions refer to regular functions in the upper \( \lambda \)-half plane.

Additional integral equations can be obtained by imposing the impedance boundary condition (see Eq. (37)),

\[ (E_z^S + E_z^i) = -\eta \frac{1}{z_0} (H_x^S + H_x^i) \quad (49) \]

on the outer faces of the top and bottom half planes. The boundary conditions on the inner faces of the half planes are not considered at this time since the ultimate use of this analysis is the study of a thick impedance half plane.
Substitution of Eqs. (42) through (44) into (49) requires on both outer faces of the impedance half planes that

\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) \left\{ [P_{1e}(\lambda) + P_{1m}(\lambda)] + [P_{2e}(\lambda) + P_{2m}(\lambda)] e^{-j2k\sqrt{1 - \lambda^2}} \right\} e^{-jkx\lambda} d\lambda = -\left( 1 - \eta \right) e^{jkx\lambda_0} \]  

(50a)

and

\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) \left\{ [P_{1e}(\lambda) - P_{1m}(\lambda)] e^{-j2k\sqrt{1 - \lambda^2}} + [P_{2e}(\lambda) - P_{2m}(\lambda)] \right\} e^{-jkx\lambda} d\lambda = -\left( 1 - \eta \right) e^{jkx\lambda_0} e^{-j2k\sqrt{1 - \lambda^2}} \]

for \( x > 0 \). Provided the boundary conditions in the inner faces of the half planes are of no interest, equations (50) along with (48) are sufficient for determining \( P_{1e}(\lambda) \) and \( P_{1m}(\lambda) \). Their solution is presented in the next section.

3.3 \( E_z \) Incidence

In this section the solution of equations (50) in view of (48) is outlined. Similarly to the development in Chapter II, once the angular spectra are determined, it is then a trivial task for obtaining the scattered field. We proceed now to solve Eqs. (50) by first putting them in a more suitable form.

Addition and subtraction of Eqs. (50a) and (50b) gives

\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) \left[ Q_{1e}(\lambda)(1 + e^{-j2k\sqrt{1 - \lambda^2}}) + Q_{2m}(\lambda)(1 - 2^{-2j\sqrt{1 - \lambda^2}}) \right] e^{-jkx\lambda} d\lambda = -\left[ (1 + e^{-2j\sqrt{1 - \lambda_0^2}}) - \eta \left( 1 - e^{-2j\sqrt{1 - \lambda_0^2}} \right) \right] e^{jkx\lambda_0} \]

and
\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) \left[ Q_{2e}(\lambda) \left( 1 - e^{-2jkx\sqrt{1 - \lambda^2}} \right) + Q_{1m}(\lambda) \left( 1 + e^{-2jkx\sqrt{1 - \lambda^2}} \right) \right] e^{-jx\lambda} d\lambda = - \left[ \left( 1 - e^{-2jkx\sqrt{1 - \lambda^2}} \right) - \eta_1 \sqrt{1 - \lambda^2} \left( 1 + e^{-2jkx\sqrt{1 - \lambda^2}} \right) \right] e^{jkx\lambda} \]

where \( x > 0 \),

\[ \begin{align*}
Q_{1e}(\lambda) &= P_{1e}(\lambda) + P_{2e}(\lambda) = U_{1e}(\lambda) + U_{2e}(\lambda) , \\
Q_{1m}(\lambda) &= P_{1m}(\lambda) + P_{2m}(\lambda) = \sqrt{1 + \lambda} \left[ U_{1m}(\lambda) + U_{2m}(\lambda) \right] , \\
Q_{2e}(\lambda) &= P_{1e}(\lambda) - P_{2e}(\lambda) = U_{1e}(\lambda) - U_{2e}(\lambda) , \\
\text{and} \\
Q_{2m}(\lambda) &= P_{1m}(\lambda) - P_{2m}(\lambda) = \sqrt{1 + \lambda} \left[ U_{1m}(\lambda) - U_{2m}(\lambda) \right] .
\end{align*} \]

Clearly, according to (52), \( Q_{1e}(\lambda) \) and \( Q_{2e}(\lambda) \) are U-functions. It is also important to note that \( Q_{1e}(\lambda) \) are associated with the electric current spectra while \( Q_{1m}(\lambda) \) are associated with the magnetic current spectra. Thus, \( Q_{1m}(\lambda) \) and \( Q_{2m}(\lambda) \) must vanish when \( \eta_1 = 0 \) (perfectly conducting half planes). Based on this observation and reciprocity considerations, it is concluded that Eqs. (51) are equivalent to the following four equations:
\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) Q_{1e}(\lambda)(1 + e^{-2j\kappa y\sqrt{1 - \lambda^2}})d\lambda = -(1 + e^{-j2\kappa y\sqrt{1 - \lambda^2}})e^{jkx}\lambda_0, \]

\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) Q_{1m}(\lambda)(1 + e^{-j2\kappa y\sqrt{1 - \lambda^2}})d\lambda = \eta_1 \sqrt{1 - \lambda^2} (1 + e^{-j2\kappa y\sqrt{1 - \lambda^2}})e^{jkx}\lambda_0, \]

\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) Q_{2e}(\lambda)(1 - e^{-j2\kappa y\sqrt{1 - \lambda^2}})d\lambda = -(1 - e^{-j2\kappa y\sqrt{1 - \lambda^2}})e^{jkx}\lambda_0, \]

and

\[ \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right) Q_{2m}(\lambda)(1 - e^{-j2\kappa y\sqrt{1 - \lambda^2}})d\lambda = \eta_1 \sqrt{1 - \lambda^2} (1 - e^{-j2\kappa y\sqrt{1 - \lambda^2}})e^{jkx}\lambda_0. \]

(53)

for \( x > 0. \)

We may now close the integration path of (53) by a semi-circle in the lower \( \lambda \) half plane. Subsequently, Cauchy's theorem may be employed to find that (see Chapter II)

\[ p_{1e}(\lambda) + p_{2e}(\lambda) = \frac{1}{2\pi j} \frac{L_1(\lambda_0)}{U_2(\lambda)} \frac{U_1(\lambda)U_3(\lambda_0)}{\lambda + \lambda_0} \]  

(54a)

\[ p_{1e}(\lambda) - p_{2e}(\lambda) = \frac{1}{2\pi j} \frac{L_2(\lambda_0)}{U_2(\lambda)} \frac{U_1(\lambda)U_3(\lambda_0)}{\lambda + \lambda_0} \]  

(54b)

\[ p_{2m}(\lambda) + p_{2m}(\lambda) = -\frac{1}{2\pi j} \frac{L_1(\lambda_0)}{U_1(\lambda)} \frac{\eta_1 \sqrt{1 + \lambda_0} \sqrt{1 + \lambda} U_3(\lambda)U_3(\lambda_0)}{\lambda + \lambda_0} \]  

(54c)

\[ p_{1m}(\lambda) - p_{2m}(\lambda) = -\frac{1}{2\pi j} \frac{L_2(\lambda_0)}{U_2(\lambda)} \frac{\eta_1 \sqrt{1 + \lambda_0} \sqrt{1 + \lambda} U_3(\lambda)U_3(\lambda_0)}{\lambda + \lambda_0} \]  

(54d)
The split function $U_{1,2}(\lambda)$ and $L_{1,2}(\lambda)$ are the same as those in Eqs. (13) and are given in Appendix I. In addition, $U_3(\lambda)$ is another split function which also occurs in the solution of the diffraction by a single impedance half plane. It is found according to

$$
\left(\frac{1}{\sqrt{1 - \lambda^2}} + \eta_1 \right)^{-1} = U_3(\lambda; \eta_1) L_3(\lambda; \eta_1), \quad (55)
$$

where $L_3(-\lambda; \eta_1) = U_3(\lambda; \eta_1)$. Explicit non-integral expressions of $U_3(\lambda)$ for the $e^{i\omega t}$ time convention are given in Appendix II. These were originally obtained by Senior [10] and approximate non-integral expressions can be found in [11].

From (54) we finally obtain that

$$
P_{1e}(\cos \phi) = \frac{1}{4\pi j} \frac{U_3(\cos \phi; \eta_1) U_3(\cos \phi_0; \eta_1)}{\cos \phi + \cos \phi_0} \left[ \frac{L_3(\cos \phi_0)}{U_1(\cos \phi)} \pm \frac{L_2(\cos \phi_0)}{U_2(\cos \phi)} \right] \quad (56a)
$$

and

$$
P_{1m}(\sin \phi) = \frac{1}{4\pi j} \frac{2\eta_1 \cos \phi/2 \cos \phi_0/2}{\cos \phi + \cos \phi_0} \frac{U_3(\cos \phi; \eta_1) U_3(\cos \phi_0; \eta_1)}{\cos \phi + \cos \phi_0} \left[ \frac{L_3(\cos \phi_0)}{U_1(\cos \phi)} \pm \frac{L_2(\cos \phi_0)}{U_2(\cos \phi)} \right]. \quad (56b)
$$

Substituting (56) into (43) and (42), it is found that an asymptotic expression for the scattered field is

$$
E_{z}^{sb} \sim D_S(\phi, \phi_0; \eta_1) \Delta(\phi, \phi_0; \xi) \frac{e^{-jkr}}{\sqrt{r}}. \quad (57)
$$
\( \Delta(\phi, \phi_0; \lambda) \) is again given by (22) and \( D^I_s(\phi, \phi_0; \eta_1) \) is the diffraction coefficients for the edge of an impedance half plane given by

\[
D^I_s(\phi, \phi_0; \eta_1) = \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{1 - 2\eta_1 \cos \phi/2 \cos \phi_0/2}{\cos \phi + \cos \phi_0} U_3(\cos \phi; \eta_1)U_3(\cos \phi_0; \eta_1)
\]

(58)

When \( \eta_1 \to 0 \) we find from the expressions in Appendix II that

\[
U_3(\cos \phi; \eta_1)|_{\eta_1 \to 0} = \sqrt{2} \sin \phi/2
\]

(59)

and thus Eqs. (57) and (58) reduces to Eqs. (32) and (33), respectively. This was to be expected.

3.4 \( H_z \) Incidence

The scattered field associated with the \( H_z \) incidence on the pair of impedance half planes in Fig. 1(b) requires the use of Eqs. (45) and (46) and to follow the procedure outlined for the \( E_z \) incidence. As seen, this is a rather tedious process. However, this is not necessary since we may invoke the principle of duality and obtain that

\[
H^s_{z} - D^I_h(\phi, \phi_0; \eta_1)\Delta(\phi, \phi_0; \lambda) \frac{e^{-jk\lambda}}{\sqrt{r}}
\]

(60)

where

\[
D^I_h(\phi, \phi_0; \eta_1) = D^I_s(\phi, \phi_0; 1/\eta_1)
\]

(61)

Again, when \( \eta_1 \to 0 \), the diffraction coefficient \( D^I_h \) should reduce to that in (20). Indeed, we note that

\[
U_3(\cos \phi; 1/\eta_1)|_{\eta_1 \to 0} = \sqrt{\eta_1}
\]

(62)
and from (58) and (61) it is clear that the solution in (60) reduces to (19) or (21).

3.5 A GTD Solution for Large \( \lambda \)

Similarly to the perfectly conducting geometry in Fig. 1(a) we may also obtain a GTD solution to the geometry in Fig. 1(b). Equation (35) is still valid, however, all of the diffraction components in (34) must be modified to include the dependence on the surface impedance, \( \eta \). Assuming that the inner walls of the half planes are perfectly conducting, it is relatively easy to see that these take the form,

\[
u_z^1 = \frac{D_s(\phi, \phi_0; \eta)}{h} e^{-jkr} \sqrt{r} ,
\]

\[
u_z^2 = \frac{D_s(\phi, \phi_0; \eta)}{h} e^{-j2k\lambda} \left( \sin \phi + \sin \phi_0 \right) e^{-jk2 \lambda} \sqrt{r} ,
\]

\[
u_z^{12} = \frac{e^{-j\pi/4}}{\sqrt{4\pi k\lambda}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_0} \right) \frac{D_s(\phi, \phi_0; \eta)}{h} e^{-j2k\lambda} \sin \phi_0
\]

\[
\sum_{m=0}^{\infty} \left( \Gamma_0 \right)^2 m e^{-j(2m+1)2k\lambda} \left( \frac{2m + 1}{(2m + 1)^{3/2}} \right) \sqrt{r} ,
\]

\[
u_z^{21} = \frac{e^{-j\pi/4}}{\sqrt{4\pi k\lambda}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_0} \right) \frac{D_s(\phi, \phi_0; \eta)}{h} e^{-jk\lambda} \sin \phi
\]

\[
\sum_{m=0}^{\infty} \left( \Gamma_0 \right)^2 m e^{-j(2m+1)2k\lambda} \left( \frac{2m + 1}{(2m + 1)^{3/2}} \right) \sqrt{r} ,
\]
\[ u_{1z}^{11} = \frac{e^{-j\pi/4}}{\sqrt{4\pi k \xi}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_0} \right) D_{S_{h}}^{I}(\phi, \phi_0; \eta_1) \]

\[ \cdot \left[ \sum_{m=1}^{\infty} \left( \Gamma_0 \right)^m \frac{e^{-j(2m)2k \xi}}{(2m)^{3/2}} \right] \frac{e^{-jkr}}{\sqrt{r}} , \] 

(63e)

and

\[ u_{2z}^{22} = \frac{e^{-j\pi/4}}{\sqrt{4\pi k \xi}} \left( \frac{1}{\cos \phi} + \frac{1}{\cos \phi_0} \right) D_{S_{h}}^{I}(\phi, \phi_0; \eta_1) e^{-j2k \xi (\sin \phi + \sin \phi_0)} \]

\[ \cdot \left[ \sum_{m=1}^{\infty} \left( \Gamma_0 \right)^m \frac{e^{-j(2m)2k \xi}}{(2m)^{3/2}} \right] \frac{e^{-jkr}}{\sqrt{r}} \] 

(63f)

where

\[ \Gamma_0 = \pm 1 . \] 

(64)

Clearly, this factor stems from the multiple reflections between the half planes. As mentioned in Chapter II, only two to three interactions need to be considered for sufficiently accurate results. Again, for \( \xi > 1.5 \lambda \) the scattered field can be safely approximated by

\[ \begin{align*}
E_{s_{b_z}} & \sim D_{S_{h}}^{I}(\phi, \phi_0; \eta_1) \cos[k \xi (\sin \phi + \sin \phi_0)] e^{-jk \xi (\sin \phi + \sin \phi_0)} \frac{e^{-jkr}}{\sqrt{r}} . \\
H_{s_{b_z}} & \end{align*} \] 

(65)
IV. DIFFRACTION BY AN IMPEDANCE HALF PLANE IN THE PRESENCE OF A GROUND OR IMPEDANCE PLANE

4.1 Diffraction by an Impedance Half Plane in the Presence of a Ground Plane

The solution of the diffracted field by the geometry in Fig. 1(c) can be obtained from the respective solution by the geometry in Fig. 1(b) via image theory. In particular, as shown in Fig. 4, the problem of diffraction by the geometry in Fig. 1(c) is equivalent to the sum of two diffracted fields from a pair of parallel impedance half planes. One of the diffracted fields is due to the direct wave and the other is due to an image wave. If the incident wave is given by

$$u^i_z = \begin{Bmatrix} E^i_z \\ H^i_z \end{Bmatrix} = e^{jk(x \cos \phi_o + y \sin \phi_o)} ,$$ (66)

the image wave satisfying the necessary boundary conditions on the ground plane is

$$u^I_z = \begin{Bmatrix} E^I_z \\ H^I_z \end{Bmatrix} = \frac{-1}{\pi} e^{jk(x \cos \phi_I + y \sin \phi_I)} e^{-jk \epsilon \sin \phi_o}$$

$$= e^{-jk \epsilon \sin \phi_o} e^{-jk \epsilon \sin \phi_o}$$ (67)

where

$$\phi_I = 2\pi - \phi_o .$$ (68)

The diffracted field due to a plane wave incident on the geometry of Fig. 1(c) is now expressed by
Figure 4. Illustration of the application of image theory to the problem of diffraction by the geometry in Fig. 1(c).
\[ E^s_{Z}(\phi, \phi_0) = E^s_{Z}(\phi, \phi_0) - E^s_{Z}(\phi, 2\pi - \phi_0) e^{-2jk\ell \sin \phi_0} \]  
(69a)

for the \( E_Z \)-incidence and by

\[ H^s_{Z}(\phi, \phi_0) = H^s_{Z}(\phi, \phi_0) + H^s_{Z}(\phi, 2\pi - \phi_0) e^{-2jk\ell \sin \phi_0} \]  
(69b)

in case of the \( H_Z \)-incidence. \( E^s_{Z}(\phi, \phi_0) \) and \( H^s_{Z}(\phi, \phi_0) \) are the scattered fields associated with the geometry in Fig. 1(b) and are given by Eqs. (57) and (60), respectively. In addition, \( E^s_{Z}(\phi, 2\pi - \phi_0) \) and \( H^s_{Z}(\phi, 2\pi - \phi_0) \) are given by the same expressions provided the incidence angle is set to \( 2\pi - \phi_0 \) instead of the usual \( \phi_0 \). We further note that the GTD expression for \( E^s_{Z} \) and \( H^s_{Z} \) is Section 5 of Chapter III may be used in (69) whenever applicable.

### 4.2 Diffraction by an Impedance Half Plane in the Presence of an Impedance Plane

The geometry corresponding to this problem is shown in Fig. 1(d). This is already the same geometry as that in Fig. 1(c) when \( \eta_2 = 0 \). The diffracted field by the structure in Fig. 1(c) was obtained in the previous section after invoking image theory and in connection with the solution for the diffraction by the geometry in Fig. 1(b). A similar formulation can then be used for obtaining the diffracted field by the structure in Fig. 1(d). However, in this case a more detailed analysis is required in order to determine the extent of the application of image theory when the ground plane is replaced by an impedance plane.

Figure 5 depicts the 12 actual diffraction mechanisms that exist in relation to the geometry in Fig. 1(d) and their equivalent when
the geometry is replaced with that in Fig. 1(b) which consists of two parallel impedance half planes. Figure 5(a) illustrates the scattering mechanisms associated with the direct field or equivalent along the direction of incidence and Fig. 5(b) those simulated by an equivalent image field. The factor appearing by the equivalent incident rays must multiply the scattered field produced by that mechanism in order to fully simulate the actual one. These factors are basically the plane wave reflection coefficients for an impedance plane as a function of the angle of incidence. More precisely,

\[
\Gamma(\phi) = \begin{cases} 
\Gamma_s(\phi) = \frac{n_1 \sin \phi - 1}{n_1 \sin \phi + 1} \\
\Gamma_h(\phi) = \frac{n_1 - \sin \phi}{n_1 + \sin \phi}
\end{cases}
\] (70)

which reduces to \(\Gamma_o\) in Eq. (64) when \(\phi = \pi/2\). Note that \(\Gamma_s(\phi)\) refers to the \(E_z\) incidence while \(\Gamma_h(\phi)\) refers to the \(H_z\)-incidence.

Based on our previous GTD analyses (see Fig. 3) given in Section 2.4 and 3.5, it is now a relatively simple task to develop a similar GTD solution for the diffracted fields associated with the mechanisms in Fig. 5. As before (see Eq. (35)), the total diffracted field for large \(\kappa\) is given by

\[
u_{sd}^{ud} = E_{zs}^{sd} H_{zs}^{sd} \begin{cases} 
- u_z^{1} + u_z^{2} + u_z^{12} + u_z^{21} + u_z^{11} + u_z^{22} \\
+ (u_{zI}^{1} + u_{zI}^{2} + u_{zI}^{12} + u_{zI}^{21} + u_{zI}^{11} + u_{zI}^{22})e^{-2j\kappa \sin \phi_0}
\end{cases}
\] (71)
Figure 5. Actual mechanisms for the structure in Figure 1(d) and their equivalent.
where the components with the subscript \( I \) refer to the image mechanisms.

From Fig. 5(a) we observe that \( u^2_z \) is the same as in (63a). In addition

\[
\begin{align*}
u^2_z &= \tau_s(\phi_0)\tau_s(\phi) u_z^{2b}, \\
u^{12}_z &= \tau_s(\phi_0)\tau_0 u_z^{12b}, \\
u^{21}_z &= \tau_s(\phi)\tau_0 u_z^{21b}, \\
u^{11}_z &= (\tau_0)^2 u_z^{11b}
\end{align*}
\] (72a) (72b) (72c) (72d)

and

\[
\begin{align*}
u^{22}_z &= \tau_s(\phi_0)\tau_s(\phi)(\tau_0)^2 u_z^{22b}, 
\end{align*}
\] (72e)

where the superscript \( b \) indicates components corresponding to the structure in Fig. 1(b) and are given in (63).

The components associated with the image mechanisms in Fig. 5(b) can be expressed in a similar manner as follows:

\[
\begin{align*}
u^2_{zI} &= \tau_s(\phi_0)u_z^{1b}(\phi,\phi_I), \\
u^2_{zI} &= \tau_s(\phi)u_z^{2b}(\phi,\phi_I), \\
u^{12}_z &= \tau_0 u_z^{12b}(\phi,\phi_I), \\
u^{21}_z &= \tau_s(\phi)\tau_s(\phi_0)\tau_0 u_z^{21b}(\phi,\phi_I),
\end{align*}
\] (73a) (73b) (73c) (73d)
\[ u_{z1}^{11} = \frac{\Gamma_s(\phi_0)(\Gamma_0)^2}{h} u_{z1}^{1b}(\phi,\phi_1) \]  
\[ u_{z1}^{22} = \frac{\Gamma_s(\phi)(\Gamma_0)^2}{h} u_{z1}^{2b}(\phi,\phi_1) \]  
(73e)  
(73f)

and \( u_{z1}^{jb}(\phi,\phi_1) \) are again found from Eqs. (63) provided \( \phi_0 \) is replaced by \( \phi_1 = 2\pi - \phi_0 \).

The above GTD solution is only valid for large \( \lambda \). However, when \( \lambda \) is small the field associated with the multiple interactions cannot be considered optical and we must then develop a solution in terms of the expressions given in Eqs. (57) and (60) as done in the previous section. Unfortunately, an exact solution is not possible since the ray mechanisms in Figs. 5(a) and 5(b) (see also Eqs. (72) and (73)) are not all multiplied by the same factor. Thus, an approximate solution will suffice at this point. It is important, though, that the contribution of the first order terms, \( u_z^1, u_z^2, u_{z1}^1 \), and \( u_{z1}^2 \) be calculated correctly since these dominate the total diffracted field. Fortunately, this can be accomplished by rewriting the diffracted field corresponding to the structure in Fig. 1(b) in a more convenient form.

In particular, Eqs. (57) and (60) can be written as follows:

\[
\begin{bmatrix}
E_{z}^{sb} \\
H_{z}^{sb}
\end{bmatrix}
= \begin{bmatrix}
E_{z1}^{sb} + E_{z2}^{sb} \\
H_{z1}^{sb} + H_{z2}^{sb}
\end{bmatrix}. \tag{74}
\]

The components with subscript 1 correspond to the total diffracted field by the top impedance half plane and those with subscript 2 to the diffracted field by the bottom impedance half plane. Note that the
total diffracted field, for example, by the top half plane includes the components $u_{Z}^{1}$, $u_{Z}^{2}$, and $u_{Z}^{11}$. From Eqs. (56) and (43), this diffracted field is given by

$$
\begin{align*}
E_{Z}^{sb} & = D_{S}^{I}(\phi, \phi_{0}; \eta_{1}) \Delta_{+}(\phi, \phi_{0}; \xi) \frac{e^{-jkr}}{\sqrt{r}} \\
H_{Z}^{sb} & = \left\{ D_{S}^{I}(\phi, \phi_{0}; \eta_{1}) + D_{S}^{I}(\phi, \phi_{0}; \eta_{1}) \left[ \Delta_{+}(\phi, \phi_{0}; \xi) - 1 \right] \right\} \frac{e^{-jkr}}{r} \quad (75)
\end{align*}
$$

and thus the contribution of $u_{Z}^{1}$ is now explicit. Similarly, for the diffracted field by the bottom half plane we have

$$
\begin{align*}
E_{Z}^{sb} & = \left\{ D_{S}^{I}(\phi, \phi_{0}; \eta_{1}) e^{-j2k\xi \sin \phi_{0}} + D_{S}^{I}(\phi, \phi_{0}; \eta_{1}) \Delta_{-}(\phi, \phi_{0}; \xi) \\
& \quad - e^{-j2k\xi \sin \phi_{0}} \right\} e^{-j2k\xi \sin \phi_{0}} \frac{e^{-jkr}}{\sqrt{r}} \quad (76)
\end{align*}
$$

The functions $\Delta_{\pm}(\phi, \phi_{0}; \xi)$ are defined by

$$
\Delta_{\pm}(\phi, \phi_{0}; \xi) = \frac{1}{2} \left\{ \frac{L_{1}(\cos \phi_{0})}{U_{1}(\cos \phi)} \pm \frac{L_{2}(\cos \phi_{0})}{U_{2}(\cos \phi)} \right\} \quad (77)
$$

and $D_{S}^{I}(\phi, \phi_{0}; \eta_{1})$ is given in (58).

We may now construct an approximate solution for the diffracted field by the structure in Fig. 1(d). Looking at Fig. 5, we have that for small $\xi$ (less than one wavelength)

$$
\begin{align*}
u_{Z}^{sd} & = \left\{ E_{Z}^{sd}, H_{Z}^{sd} \right\} = u_{Z}^{top} + u_{Z}^{bot} + \left( u_{Z}^{I1} + u_{Z}^{bot} \right) e^{-2j\xi \sin \phi_{0}} \quad (78)
\end{align*}
$$
where

\[
\begin{align}
\mathbf{u}_{\text{top}}^z & = \left\{ D_{s}^{I} (\phi, \phi; \eta_1) + \Gamma(\phi, \phi) \Gamma_0 D_{s}^{I} (\phi, \phi; \eta_1) \left[ \Delta_+ (\phi, \phi; \xi) - 1 \right] \right\} \frac{e^{-jkr}}{\sqrt{r}}, \\
\mathbf{u}_{\text{top}}^{z \text{i}} & = \left\{ \Gamma_s (\phi, \phi) D_{s}^{I} (\phi, \phi; \eta_1) + \Gamma_0 D_{s}^{I} (\phi, \phi; \eta_1) \left[ \Delta_+ (\phi, \phi; \xi) - 1 \right] \right\} \frac{e^{-jkr}}{\sqrt{r}}, \\
\mathbf{u}_{\text{bot}}^z & = \left\{ \Gamma_s (\phi, \phi) D_{s}^{I} (\phi, \phi; \eta_1) e^{-j2k_\xi \sin \phi} + \Gamma_s (\phi, \phi) \Gamma_0 D_{s}^{I} (\phi, \phi; \eta_1) \\
& \quad \left[ \Delta_- (\phi, \phi; \xi) - e^{-j2k_\xi \sin \phi} \right] \right\} \frac{e^{-jkr}}{\sqrt{r}}, \\
\mathbf{u}_{\text{bot}}^{z \text{i}} & = \left\{ \Gamma_s (\phi, \phi) D_{s}^{I} (\phi, \phi; \eta_1) e^{j2k_\xi \sin \phi} + \Gamma_s (\phi, \phi) \Gamma_0 D_{s}^{I} (\phi, \phi; \eta_1) \\
& \quad \left[ \Delta_- (\phi, \phi; \xi) - e^{j2k_\xi \sin \phi} \right] \right\} \frac{e^{jkr}}{\sqrt{r}}.
\end{align}
\]  

(79a) 
(79b) 
(79c) 
(79d)

The above take into consideration that backscattering is of primary interest here. We further note that they are rather conservative approximations and thus they would most likely give a higher result than the correct one. This is true provided $|\Gamma_0| < 1$ as usual. In fact, the smaller the value of $|\Gamma_0|$ or $|\Gamma_s|$, the better the approximation in (79) become. For this case only the dominant terms of (79) need to be considered. These are the first terms from each of the expressions (79a) through (79d).
V. CONCLUSIONS AND FUTURE STUDIES

In this study we presented rigorous solutions to the problems of diffraction for the first three configurations in Fig. 5. The diffraction by the last geometry in Fig. 5 was considered only in an approximate manner. Further, the configurations in Fig. 1(c) and 1(d) were based on the rigorous solution to the diffraction by two parallel impedance half planes. The last was obtained via a generalization of the angular spectrum method to account for the concurrent existence of both electric and magnetic currents.

The above results can be readily used and extended to the analysis of a small step discontinuity or a protrusion associated with impedance boundary conditions as shown in Fig. 6(a). This will involve the study of the scattering by a parallel plate waveguide with a recessed stub as shown in Fig. 6(b). For \( d \rightarrow 0 \), the configuration in Fig. 6(b) reduces to that in 6(a). In case of \( E_z \) incidence and \( \varepsilon < \lambda/4 \), the dominant diffraction emanates from the edges only and thus the effect of the stub can be totally neglected. Consequently, the analysis in this report is sufficient for the step discontinuities provided \( \varepsilon < \lambda/4 \).

For the \( H_z \) incidence, the diffraction by the waveguide edge is negligible when \( \phi \) and \( \phi_0 \) are near 180 degrees, but there exists a strong coupling into waveguide modes. Thus, the scattering due to the presence of the stub will be dominant in that region. Again, assuming that \( \varepsilon < \lambda \), only the TEM mode will be non-decaying. Subsequently, once the coupling coefficient, \( C_o(\phi) \), the reflection coefficient, \( R_o \),
Figure 6.  
(a) Geometry of a protrusion  
(b) Geometry to be analyzed  
(c) - (f) Illustration of coupling reflection and launching mechanisms
and the launching coefficient, \( L_0(\phi) \), are found (see Fig. 6), the scattered field can be expressed as [5]

\[
H_z^s = P(\phi, \phi_0) \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} e^{-jkr} \frac{1}{\sqrt{r}}
\]

where

\[
P(\phi, \phi_0) = S(\phi, \phi_0) + \frac{L_0(\phi)C_0(\phi_0) \Gamma_0}{1 - R_0 \Gamma_0}.
\]

In the above \( S(\phi, \phi_0) \) is the diffraction pattern of the waveguide edges and can be extracted from Eqs. (21) or (60). Further \( \Gamma_0 \) is the plane wave reflection coefficient given in (64).
APPENDIX I. DEFINITION OF THE SPLIT FUNCTIONS $U_1(\lambda)$ AND $U_2(\lambda)$

We wish to factorize the functions $(1 \pm e^{-j2k\lambda \sqrt{1-\lambda^2}})$ such that (see also Eqs. (13))

$$1 + e^{-j2k\lambda \sqrt{1-\lambda^2}} = U_1(\lambda)L_1(\lambda)$$  \hspace{1cm} (A.1)

and

$$1 - e^{-j2k\lambda \sqrt{1-\lambda^2}} = U_2(\lambda)L_2(\lambda).$$  \hspace{1cm} (A.2)

The $U$-functions are regular in the upper $\lambda$-half plane shown in Fig. A.2 and the $L$-functions have similar properties in the lower $\lambda$-half plane.

Assuming an $e^{j\omega t}$ time dependence and following the procedure in [6] we find that $U_1(\lambda)$ is given by

$$U_1(\lambda) = L_1(-\lambda) = \sqrt{2} \exp[-\chi_1(\lambda) - \tau(\lambda)] \prod_{n=1,3,5}^{\infty} \left( - \frac{2j\lambda}{n\pi} \right) (k + j\gamma_n) e^{j2k\lambda \sqrt{1-\lambda^2}}$$

\hspace{1cm} (A.3)

where

$$\chi_1(\lambda) = -\frac{jk\lambda^2}{\pi} \left[ 1 - C + \ln \left( \frac{\pi}{2k\lambda} \right) \right] - \frac{k\lambda^2}{2}$$ \hspace{1cm} (A.4)

$$\tau(\lambda) = \frac{\pi}{\pi} \begin{cases} \pi - \cos^{-1}(\lambda) & \lambda > 0 \\ \cos^{-1}(-\lambda) & \lambda < 0 \end{cases}$$ \hspace{1cm} (A.5)

$$\gamma = \begin{cases} k\sqrt{1-\lambda^2} - \tau, & \text{or} \\ jk\sqrt{1-\lambda^2}, & \end{cases}$$ \hspace{1cm} (A.6)
\[ \gamma_n = \begin{cases} \frac{k}{j} \left( \frac{n\pi}{2k^2} \right)^2 - 1 \text{, or} \\ \frac{j}{k} \frac{1 - \left( \frac{n\pi}{2k\xi} \right)^2}{1 - \left( \frac{n\pi}{2k\xi} \right)^2} \end{cases}, \quad (A.7) \]

and

\[ C = 0.5772 \quad (A.8) \]

Note that \( X_2(\lambda) \) was chosen such that \( U_{1\lambda}(\lambda) \) is asymptotic to a constant as \( |\lambda| \to \infty \).

Similarly we find that

\[ U_{2\lambda}(\lambda) = L_{2\lambda}(-\lambda) = e^{i\pi/4} \sqrt{2k\xi} \sqrt{1 - \lambda} \exp[-X_2(\lambda) - T(\lambda)] \lim_{n=2,4,6} \infty \]

\[ \left( -\frac{2j\lambda}{n\pi} \right) (k\lambda + j\gamma_n) e^{j2k\xi\lambda/n\pi}, \quad (A.9) \]

where

\[ X_2(\lambda) = -\frac{j}{\pi} \frac{k\xi\lambda}{1 - C + 1n \left( \frac{2\pi}{k\xi} \right) - \frac{k\xi\lambda}{2}} \quad (A.10) \]

and all other functions and parameters were defined in (A.5) through (A.8).

The above expressions for \( U_{1,2\lambda}(\lambda) \) appear cumbersome because they consist of infinite homomorphic products. However, for our intended calculations where \( \xi < \lambda/4 \), we find that the achieved amplitude accuracy is within five percent by retaining only the first three terms of the product series. Further, the accuracy is improved for smaller \( \lambda \). Note that \( 2\xi \) is the separation distance between the two half planes shown in Fig. 1(a).
APPENDIX II. DEFINITION OF THE SPLIT FUNCTION $U_3(\lambda)$

The function

$$F(\lambda;\eta) = \left(\frac{1}{\sqrt{1 - \lambda^2}} + \eta\right)^{-1} \quad (B.1)$$

occurs in the solution of the problem of diffraction by an impedance half plane or by a set of parallel half planes as discussed in Chapter III. To obtain the final form of the solution to the above problems, it is first required that the function in (B.1) be factorized in the form (see also Eq. (55))

$$\left(\frac{1}{\sqrt{1 - \lambda^2}} + \eta\right)^{-1} = U_3(\lambda;\eta)L_3(\lambda;\eta) \quad (B.2)$$

As usual $U_3(\lambda;\eta)$ is regular in the upper $\lambda$-half plane shown in Fig. A.2 and $L_3(\lambda;\eta)$ has the same property in the lower $\lambda$-half plane.

Expressions for $U_3(\lambda;\eta) = L_3(-\lambda;\eta)$ are given by Senior [10] in an integral form and approximate explicit ones were later found by Volakis and Senior [12]. We simply state below the results in [10] and [11]. Assuming an $e^{j\omega t}$ time dependence we have that

$$U_3(\lambda = \cos \phi;\eta) = \frac{2[2 \cos x(1 - \cos \phi)]^{1/2}}{(\sqrt{2} \sin \frac{\phi - \chi}{2} + 1)(\sqrt{2} \sin \frac{\phi + \chi}{2} + 1) \cdot \left\{\psi_\pi(\pi-\phi+\chi)\psi_\pi(\pi-\phi-\chi) + \psi_\pi(\pi/2)\psi_\pi(\pi/2)\right\}^2}, \quad (B.3)$$

where
\[ \cos \chi = 1/n \]  
(B.4)

and \(\psi_n(\alpha)\) is the Maliuzhinets [12] function given by

\[
\psi_n(\alpha) = \exp \left\{ -\frac{1}{8\pi} \int_0^\alpha \frac{\pi \sin u - 2\sqrt{2} \pi \sin u/2 + 2u}{\cos u} \, du \right\}
\]

\[
\approx \begin{cases} 
1 - 0.0139 \, \lambda^2, & 0 \leq \sigma \leq \pi/2, \quad \tau \leq 4.6 \\
1.05302 \left[ \cos \frac{1}{4}(\alpha - j\gamma_0) \right]^{1/2} \exp \frac{j\alpha}{2} e^{j\alpha}, & 0 \leq \sigma \leq \pi/2, \quad \tau > 4.6
\end{cases}
\]

(B.5)

with \(\alpha = \sigma + j\tau\) and \(\gamma_0 = 0.69315\). For other values of \(\sigma\), the identity

\[
\psi_n(\alpha) = 0.93242 \frac{\cos \left( \frac{\alpha}{4} - \frac{\pi}{8} \right)}{\psi_n(\alpha - \pi)}
\]

may be employed as many times as required before the approximation in (B.5) can be used. It is shown in[12] that this approximation gives a maximum amplitude error of 0.27 percent and a corresponding phase error of 2.4 percent.
References


