APPLICATIONS OF A CLASS OF GENERALIZED BOUNDARY CONDITIONS TO SCATTERING BY A METAL-BACKED DIELECTRIC HALF-PLANE

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Abstract

The problem considered is that of a plane wave incident on a perfectly conducting half-plane with a thin dielectric coating on its upper face. The solution is accomplished by introducing a higher order boundary condition to simulate the effect of the dielectric, thereby allowing the structure to be treated as an infinitesimally thin half-plane. Two types of boundary conditions are developed, one applicable to a low contrast dielectric and the other to a high contrast one. The problem is then solved using a generalized version of the Maliuzhinets method in which certain additional constants are introduced to assure the correct behavior of the field at the edge. Although these constants play no role in the final solution, they are needed to cancel out inadmissible singularities in the solution of the homogeneous difference equation. The final solution is expressed in a uniform manner and some numerical results are presented.

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1. Introduction

A structure of considerable interest in scattering theory is a thin metal-backed dielectric layer, and it is important to develop effective techniques for computing the scattering from plates and other targets formed in this manner. One approach is to simulate the layer using an infinitesimally thin sheet placed at the location of the metal backing, and it is shown that this is possible for a thin homogeneous layer. For materials with low and high dielectric constants, the boundary conditions which must be imposed at the upper surface of the sheet are determined. In both cases these are generalized impedance boundary conditions involving second order field derivatives.

For a semi-infinite metal-backed layer the model is a half-plane subject to the appropriate boundary conditions. Based on these, the solution for an H-polarized plane wave incident in a plane perpendicular to the edge is derived. Since the boundary conditions differ on the two sides, the Wiener-Hopf method produces coupled integral equations which cannot be solved using presently available techniques, and though Maliuzhinets' method is effective, the generalized nature of the conditions requires a modification to the method that is customarily employed. The solution is presented in Sections 3 and 4, and numerical data are included to show the effect of the various terms in the boundary conditions. The analysis is much simpler for E-polarization, and the results for this case are given in an Appendix.

2. Boundary Conditions

The geometry considered is shown in Figure 1, and under the assumption of an electrically thin layer of thickness τ ($k\tau << 1$), we seek a boundary condition which can be applied at the surface $y = 0$ to simulate the effect of the metal-backed layer.

From a Taylor series expansion of the field in the dielectric

$$E_x(0+) = E_x(\tau-) - \tau \frac{\partial}{\partial y} E_x(\tau-)$$

$$= E_x(\tau-) - \tau \frac{\partial}{\partial x} E_y(\tau-) + ik\tau \mu_r Z H_z(\tau-) \quad ,$$

where we have shown only the dependence on y. In these expressions k and Z are the propagation constant and intrinsic impedance respectively of free space, and ε_r and μ_r are the relative permittivity and permeability of the dielectric coating. A time factor $e^{-i\omega t}$ has been assumed and suppressed. From the continuity of the tangential field components at the air-dielectric interface we then obtain

$$E_x(0+) = E_x(\tau+) - \frac{\tau}{\varepsilon_r} \frac{\partial}{\partial x} E_y(\tau+) + ik\tau \mu_r Z H_z(\tau+) \quad ,$$

and when the boundary condition at the perfectly conducting surface $y = 0+$ is imposed, the result is

$$E_x(\tau+) = -ik\tau \mu_r Z H_z(\tau+) + \frac{\tau}{\varepsilon_r} \frac{\partial}{\partial x} E_y(\tau+) \tag{1}$$

which is an equivalent boundary condition at the upper surface of the layer. We can transfer this to the surface y = 0+ by noting that in free space

$$E_x(\tau+) = E_x(0+) + \tau \frac{\partial}{\partial x} E_y(0+) - ik\tau\, ZH_z(0+) \tag{2}$$

Thus, to the leading order in τ, the equivalent boundary condition applied at the surface y = 0+ of a sheet in free space is

$$E_x = pZH_z + \frac{q}{ik}\frac{\partial E_y}{\partial x} \qquad (y = 0+) \tag{3}$$

where

$$p = -ik\tau\,(\mu_r - 1) \quad , \qquad q = ik\tau\left(\frac{1}{\varepsilon_r} - 1\right) \tag{4}$$

Similarly,

$$E_z = -pZH_x + \frac{q}{ik}\frac{\partial E_y}{\partial z} \qquad (y = 0+) \tag{5}$$

with

$$E_x = E_z = 0 \tag{6}$$

on the lower (y = 0-) surface of the sheet.

The boundary conditions (3) and (5) are identical to the ones derived by Weinstein [1] and are valid for a low contrast dielectric. Apart from the modification provided by the derivative terms, they are the conditions for an impedance surface with normalized surface impedance p, and the derivative terms vanish if q = 0 corresponding to a pure magnetic material. We also remark that (3) and (5) can be derived from the transition conditions [2] for an unbacked dielectric layer of thickness 2τ by reflection about the middle.

For a layer of high contrast material such that $|N| >> 1$, where $N = \sqrt{\varepsilon_r \mu_r}$ is the complex refractive index, alternative boundary conditions are required, and well-established ones are [3]

$$E_x = \eta Z H_z , \qquad E_z = -\eta Z H_x \tag{7}$$

with

$$\eta = -i \frac{N}{\varepsilon_r} \tan Nk\tau \tag{8}$$

applied at the upper surface $y = \tau+$ of the layer. As $k\tau$ increases with Im. $N > 0$, $\tan Nk\tau \rightarrow i$ and the surface impedance reduces to that for a lossy half-space occupying $y < \tau$. To transfer the conditions to the surface $y = 0+$, we again expand the field components in Taylor series as shown in (2). When the expansions are inserted into (7) and the terms collected, we obtain

$$E_x = p' Z H_z + \frac{q'}{ik} \frac{\partial E_y}{\partial x} + \frac{r'}{ik} Z \frac{\partial H_y}{\partial z} \tag{9}$$

$$(y = 0+)$$

$$E_z = -p' Z H_x + \frac{q'}{ik} \frac{\partial E_y}{\partial z} - \frac{r'}{ik} Z \frac{\partial H_y}{\partial x} \tag{10}$$

with

$$p' = \frac{\eta + ik\tau}{1 + ik\tau\eta} , \qquad q' = - \frac{ik\tau}{1 + ik\tau\eta} , \qquad r' = \frac{ik\tau\eta}{1 + ik\tau\eta} . \tag{11}$$

The conditions (9) and (10) are similar in form to (3) and (5), but we observe that the transfer to y = 0+ has introduced terms involving H_y as well as E_y.

In the particular case of a plane wave incident in the xy plane the boundary conditions simplify. For H-polarization such that H_z, E_x and E_y are the only non-zero field components, the conditions can be written as

$$\left(\gamma_1 \frac{\partial^2}{\partial x^2} + ik\gamma_2 \frac{\partial}{\partial y} - k^2\right) H_z = 0 \tag{12}$$

with $\gamma_1 = q'/p'$, $\gamma_2 = 1/p'$ in the high contrast case (9), and $\gamma_1 = q/p$, $\gamma_2 = 1/p$ in the low contrast case (3). For E-polarization where E_z, H_x and H_y are the only non-zero components, the boundary conditions are also of the form (12) with H_z replaced by E_z and $\gamma_1 = -r'$, $\gamma_2 = p'$ for (10) and $\gamma_1 = 0$, $\gamma_2 = 1/p$ for (5). It is, therefore, sufficient to consider the boundary condition (12), and we note that this is analogous to the one originally proposed by Karp and Karal [4] as a means of simulating complex planar structures.

3. Solution For H-Polarization

An H-polarized plane wave is incident on a half-plane simulating a semi-infinite metal-backed layer and occupying the portion $x < 0$ of the plane $y = 0$. In terms of the cylindrical polar coordinates ρ, ϕ, z with $x = \rho \cos \phi$, $y = \rho \sin \phi$, the incident field is assumed to be

$$H_z^i = e^{-ik\rho \cos(\phi - \phi_o)} \quad . \tag{13}$$

According to the development in the previous section, on the upper side ($\phi = \pi$) of the half-plane the total (incident plus scattered) field H_z satisfies the generalized impedance condition

$$\left(\gamma_1 \frac{\partial^2}{\partial \rho^2} - \frac{ik\gamma_2}{\rho} \frac{\partial}{\partial \phi} - k^2\right) H_z = 0 \quad , \tag{14}$$

whereas on the perfectly conducting side ($\phi = -\pi$) the boundary condition is

$$\frac{\partial H_z}{\partial \rho} = 0 \quad . \tag{15}$$

Following Maliuzhinets [5] we write

$$H_z(\rho,\phi) = \frac{1}{2\pi i} \int_\gamma e^{-ik\rho \cos\alpha} s(\alpha + \phi) d\alpha \quad , \tag{16}$$

where α is the double loop Sommerfeld path. To satisfy the edge condition it is required that $H_z = O\{(k\rho)^\varepsilon\}$ for small $k\rho$ with $\varepsilon > 0$, and this implies that for large $|\mathrm{Im}.\alpha|$, $s(\alpha) = O\{\exp(-\varepsilon|\mathrm{Im}.\alpha|)\}$. When the boundary conditions are imposed and the differentiations performed, the derivative with respect to ϕ can be eliminated using integration by parts, giving

$$\int_\gamma e^{-ik\rho\cos\alpha} (\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)\, s(\alpha + \pi)\, d\alpha = 0$$

$$\int_\gamma e^{-ik\rho\cos\alpha} \sin\alpha\, s(\alpha - \pi)\, d\alpha = 0$$

where

$$\sin\theta_{1,2} = -\frac{1}{2\gamma_1}\left\{\gamma_2 \pm \sqrt{\gamma_2^2 + 4\gamma_1(1+\gamma_1)}\right\} \tag{17}$$

with Im. $(\cos\theta_{1,2}) > 0$. The necessary and sufficient conditions for these to be satisfied are [6]

$$(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)\, s(\alpha + \pi) = (\sin\alpha - \sin\theta_1)(\sin\alpha - \sin\theta_2)\, s(-\alpha + \pi) + \sin\alpha(A_0 + A_1\cos\alpha)$$

$$s(\alpha - \pi) = -s(-\alpha - \pi) \quad + B_0 + B_1\cos\alpha \tag{18}$$

where A_0, A_1, B_0 and B_1 are arbitrary constants whose presence is required to achieve the desired order in $|\mathrm{Im}.\alpha|$.

To solve (18) let

$$s(\alpha) = g(\alpha)\, t(\alpha) \tag{19}$$

with

$$g(\alpha) = \frac{\Psi(\alpha, \theta_1, \frac{\pi}{2} - i\infty)\ \Psi(\alpha, \theta_2, 0)}{\Psi(\phi_o, \theta_1, \frac{\pi}{2} - i\infty)\ \Psi(\phi_o, \theta_2, 0)} \tag{20}$$

where $\Psi(\alpha, \beta_1, \beta_2)$ is a product of Maliuzhinets [5] half-plane functions ψ_π which are free of poles and zeros in the strip $|\mathrm{Re}.\alpha| \leq \pi$. Since $\psi_\pi(\pm i\infty) = \infty$ (the resulting factor is cancelled by the corresponding one in the denominator of (20)), and

$$\psi_\pi(\alpha - \frac{3\pi}{2})\ \psi_\pi(\alpha - \frac{\pi}{2}) = \left\{\psi_\pi(\frac{\pi}{2})\right\}^2 \cos\frac{1}{4}(\alpha - \pi) \quad ,$$

we have

$$g(\alpha) = \frac{\psi_\pi(\alpha + \frac{3\pi}{2} - \theta_1)\ \psi_\pi(\alpha + \frac{\pi}{2} + \theta_1)\ \psi_\pi(\alpha + \frac{3\pi}{2} - \theta_2)\ \psi_\pi(\alpha + \frac{\pi}{2} + \theta_2) \cos\frac{1}{4}(\alpha - \pi)}{\psi_\pi(\phi_o + \frac{3\pi}{2} - \theta_1)\ \psi_\pi(\phi_o + \frac{\pi}{2} + \theta_1)\ \psi_\pi(\phi_o + \frac{3\pi}{2} - \theta_2)\ \psi_\pi(\phi_o + \frac{\pi}{2} + \theta_2) \cos\frac{1}{4}(\phi_o - \pi)}$$

(21)

Also

$$g(\alpha - 2\pi) = \Gamma\ g(\alpha + 2\pi) \tag{22}$$

with

$$\Gamma = -\frac{(\sin\alpha - \sin\theta_1)(\sin\alpha - \sin\theta_2)}{(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)}$$

being the plane wave reflection coefficient for the coated surface.

In the two equations (18) replace α by $\alpha + \pi$ and $\alpha - \pi$ to obtain

$$(\sin\alpha - \sin\theta_1)(\sin\alpha - \sin\theta_2)\, s(\alpha + 2\pi) + (\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)\, s(\alpha - 2\pi)$$
$$= -\sin\alpha\,(A_0 - A_1\cos\alpha) + (\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)(B_0 - B_1\cos\alpha)\ .$$

From (19) and (22)

$$t(\alpha+2\pi) - t(\alpha-2\pi) = \frac{1}{g(\alpha-2\pi)}\left\{\frac{\sin\alpha}{(\sin\alpha+\sin\theta_1)(\sin\alpha+\sin\theta_2)}(A_0-A_1\cos\alpha) - (B_0-B_1\cos\alpha)\right\}\ , \tag{23}$$

and if the right hand side of (23) is denoted by $h(\alpha)$, a particular solution of the difference equation is

$$t_0(\alpha) = -\sum_{m=0}^{\infty} h(\alpha + 2\pi + 4m\pi)\ .$$

Hence, from (22),

$$t_0(\alpha) = -h(\alpha + 2\pi)\sum_{m=0}^{\infty}\Gamma^m = -\frac{h(\alpha + 2\pi)}{1-\Gamma}\ ,$$

and when the expression for Γ is inserted, we find

$$t_o(\alpha) = \frac{1}{2g(\alpha)p(\alpha)} \left\{ (\sin\alpha+\sin\theta_1)\,(\sin\alpha+\sin\theta_2)\,(B_o-B_1\cos\alpha) - \sin\alpha\,(A_o-A_1\cos\alpha) \right\}$$

(24)

where

$$p(\alpha) = \sin^2\alpha + \sin\theta_1\,\sin\theta_2 \quad . \tag{25}$$

We note that $t_o(\alpha)$ has poles at the zeros of $p(\alpha)$, and if α_p is such that $\sin\alpha_p =$ $i\,(\sin\theta_1\,\sin\theta_2)^{1/2}$, the four poles which lie in the strip $|\mathrm{Re}\,\alpha| \le \pi$ are $\alpha = \pm\,\alpha_p,\ \pm(\pi - \alpha_p)$.

The general expression for $t\,(\alpha)$ is

$$t\,(\alpha) = \sigma(\alpha) + t_o\,(\alpha) \tag{26}$$

where $\sigma(\alpha)$ satisfies

$$\sigma(\alpha \pm \pi) = \sigma(-\alpha \pm \pi) \quad .$$

It is therefore a function of $\sin\alpha/2$. To reproduce the incident field (13), $\sigma(\alpha)$ must have a pole at $\alpha = \phi_o$ with residue unity and, in addition, poles which cancel those of $t_o(\alpha)$. With this in mind we choose

$$\sigma(\alpha) = \left\{ \frac{1/2}{\sin\frac{\alpha}{2} - \sin\frac{\phi_o}{2}} + \frac{f_1(\sin^2\frac{\alpha}{2}) + \sin\frac{\alpha}{2} f_2(\sin^2\frac{\alpha}{2})}{p(\alpha)} \right\} \cos\frac{\phi_o}{2} \tag{27}$$

where f_1 and f_2 are polynomial functions still to be determined, and from (19), (26), (24) and (27) we then have

$$s(\alpha) = g(\alpha) \left\{ \frac{1/2}{\sin\frac{\alpha}{2} - \sin\frac{\phi_o}{2}} + \frac{f_1(\sin^2\frac{\alpha}{2}) + \sin\frac{\alpha}{2} f_2(\sin^2\frac{\alpha}{2})}{p(\alpha)} \right\} \cos\frac{\phi_o}{2}$$
$$+ \frac{1}{2p(\alpha)} \left\{ (\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)(B_o - B_1 \cos\alpha) - \sin\alpha \, (A_o - A_1 \cos\alpha) \right\} \ . \tag{28}$$

Given the functions f_1 and f_2, the constants A_0, A_1, B_0 and B_1 must be chosen to eliminate the poles of $s(\alpha)$ at $\alpha = \pm\, \alpha_p$ and $\pm\, (\pi - \alpha_p)$. Beyond this, the constants play no role in the analysis.

4. Determination of the Field

When the contour α is closed with the aid of two steepest descent paths through $\alpha = \pm\pi$, the poles of the first term in the expression for $s(\alpha + \phi)$ that lie within the strip $|\text{Re.}\ \alpha| < \pi$ are captured, and their residues give rise to the incident and reflected waves. In addition, a surface wave pole may be captured, and the evaluation of these residue contributions is given in Appendix A. The non-residue portion is the diffracted field and this can be written as

$$H_z^d(\rho, \phi) = \frac{1}{2\pi i} \int_{S(\phi)} e^{ik\rho\cos(\alpha-\phi)} \left\{ s(\alpha+\pi) - s(\alpha-\pi) \right\} d\alpha \tag{29}$$

where $S(\phi)$ is a steepest descent path through $\alpha = \phi$.

Since $p(\alpha)$ is a function of $\sin^2\alpha$,

$$s(\alpha \pm \pi) = g(\alpha \pm \pi) \left\{ \frac{\sin\frac{\phi_o}{2} \pm \cos\frac{\alpha}{2}}{\cos\alpha + \cos\phi_o} + \frac{f_1(\cos^2\frac{\alpha}{2}) \pm \cos\frac{\alpha}{2} f_2(\cos^2\frac{\alpha}{2})}{p(\alpha)} \right\} \cos\frac{\phi_o}{2}$$

$$+ \frac{1}{2p(\alpha)} \left\{ (\sin\alpha - \sin\theta_1)(\sin\alpha - \sin\theta_2)(B_o + B_1 \cos\alpha) + \sin\alpha\,(A_o + A_1 \cos\alpha) \right\} \tag{30}$$

and the last group of terms in (30) does not contribute to $s(\alpha+\pi) - s(\alpha-\pi)$. From (21)

$$\begin{aligned} g(\alpha + \pi) &= G(\alpha, \phi_o)\,(a - x)\,(b - x)\left(\frac{1}{\sqrt{2}} + x\right) \\ g(\alpha - \pi) &= G(\alpha, \phi_o)\,(a - y)\,(b - y)\left(\frac{1}{\sqrt{2}} + y\right) \end{aligned} \tag{31}$$

where

$$G(\alpha, \phi_o) = \left\{ \psi_\pi\left(\frac{\pi}{2}\right) \right\}^8 \left\{ 8\,\Psi\left(\alpha, \theta_1, \frac{\pi}{2} - i\infty\right) \Psi\left(\phi_o, \theta_1, \frac{\pi}{2} - i\infty\right) \Psi\left(\alpha, \theta_2, \frac{\pi}{2} - i\infty\right) \right.$$

$$\left.\Psi\left(\phi_o, \theta_2, \frac{\pi}{2} - i\infty\right) \cos\frac{1}{4}(\alpha - \pi) \cos\frac{1}{4}(\phi_o - \pi)\right\}^{-1}$$

and

$$a = \cos\frac{1}{2}\left(\theta_1 - \frac{\pi}{2}\right), \qquad b = \cos\frac{1}{2}\left(\theta_2 - \frac{\pi}{2}\right)$$

$$x = \cos\frac{1}{2}\left(\alpha - \frac{\pi}{2}\right), \qquad y = \sin\frac{1}{2}\left(\alpha - \frac{\pi}{2}\right). \tag{32}$$

We note that $G(\alpha, \phi_o)$ is symmetric in α and ϕ_o and $O\{\exp(-3/4|\mathrm{Im}.\alpha|)\}$ for large $|\mathrm{Im}.\alpha|$.

From (30) and (31) we now obtain

$$s(\alpha+\pi) - s(\alpha-\pi) = G(\alpha,\phi_o) \cos\frac{\phi_o}{2}\Bigg[(a-x)(b-x)\left(\frac{1}{\sqrt{2}} + x\right)\left\{\frac{\sin\frac{\phi_o}{2} + \cos\frac{\alpha}{2}}{\cos\alpha+\cos\phi_o} + \frac{f_1(\cos^2\frac{\alpha}{2}) + \cos\frac{\alpha}{2} f_2(\cos^2\frac{\alpha}{2})}{p(\alpha)}\right\} - (a-y)(b-y)\left(\frac{1}{\sqrt{2}} + y\right)\left\{\frac{\sin\frac{\phi_o}{2} - \cos\frac{\alpha}{2}}{\cos\alpha+\cos\phi_o} + \frac{f_1(\cos^2\frac{\alpha}{2}) - \cos\frac{\alpha}{2} f_2(\cos^2\frac{\alpha}{2})}{p(\alpha)}\right\}\Bigg],$$

and after much tedious simplification, the coefficient of $G(\alpha, \phi_o) / (\cos\alpha + \cos\phi_o)$ is found to be

$$\sqrt{2}\left(\frac{1}{2} + ab - \frac{a+b}{\sqrt{2}}\right)\cos\frac{\alpha}{2}\cos\frac{\phi_o}{2} + \frac{1}{\sqrt{2}}\left(\frac{1}{2} - \frac{a+b}{\sqrt{2}}\right)\sin\alpha\,\sin\phi_o$$

$$+\frac{1}{\sqrt{2}}\left(1+ab-\frac{a+b}{\sqrt{2}}\right)\left(\cos\frac{\alpha}{2}\sin\phi_o+\cos\frac{\phi_o}{2}\sin\alpha\right)+\frac{1}{\sqrt{2}}\cos\alpha\cos\frac{\alpha}{2}\cos\frac{\phi_o}{2}\left(\sin\frac{\alpha}{2}-\sin\frac{\phi_o}{2}\right) .$$

All of the terms in the above coefficient are of an allowed order in $|\mathrm{Im}.\alpha|$ except the last one, but since

$$\cos\alpha\left(\sin\frac{\alpha}{2}-\sin\frac{\phi_o}{2}\right)=-\left(\cos\alpha\sin\frac{\phi_o}{2}+\cos\phi_o\sin\frac{\alpha}{2}\right)+\sin\frac{\alpha}{2}\left(\cos\alpha+\cos\phi_o\right) ,$$

we can write

$$s(\alpha+\pi)-s(\alpha-\pi)=\frac{1}{\sqrt{2}}\frac{G(\alpha,\phi_o)}{\cos\alpha+\cos\phi_o}\left\{2\left(\frac{1}{2}+ab-\frac{a+b}{\sqrt{2}}\right)\cos\frac{\alpha}{2}\cos\frac{\phi_o}{2}\right.$$
$$+\left(\frac{1}{2}-\frac{a+b}{\sqrt{2}}\right)\sin\alpha\sin\phi_o+\left(1+ab-\frac{a+b}{\sqrt{2}}\right)\left(\cos\frac{\alpha}{2}\sin\phi_o+\cos\frac{\phi_o}{2}\sin\alpha\right)$$
$$\left.-\cos\frac{\alpha}{2}\cos\frac{\phi_o}{2}\left(\cos\alpha\sin\frac{\phi_o}{2}+\cos\phi_o\sin\frac{\alpha}{2}\right)\right\}+\frac{G(\alpha,\phi_o)}{p(\alpha)}\cos\frac{\phi_o}{2}q(\alpha) \qquad (33)$$

where

$$q(\alpha)=(a-x)(b-x)\left(\frac{1}{\sqrt{2}}+x\right)\left\{f_1\left(\cos^2\frac{\alpha}{2}\right)+\cos\frac{\alpha}{2}f_2\left(\cos^2\frac{\alpha}{2}\right)\right\}$$
$$-(a-y)(b-y)\left(\frac{1}{\sqrt{2}}+y\right)\left\{f_1\left(\cos^2\frac{\alpha}{2}\right)-\cos\frac{\alpha}{2}f_2\left(\cos^2\frac{\alpha}{2}\right)\right\}+\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}p(\alpha) .$$

The function $q(\alpha)$ must be zero to satisfy the order conditions, and this requires

$$f_1\left(\cos^2\frac{\alpha}{2}\right)\pm\cos\frac{\alpha}{2}f_2\left(\cos^2\frac{\alpha}{2}\right)=4\left\{(a\pm x)(b\pm x)\left(\frac{1}{\sqrt{2}}\mp x\right)+(a\mp y)(b\mp y)\left(\frac{1}{\sqrt{2}}\pm y\right)\right\}$$

implying

$$f_1(\sin^2\frac{\alpha}{2}) + \sin\frac{\alpha}{2} f_1(\sin^2\frac{\alpha}{2}) = 4\left\{ (a+x)(b+x)(\frac{1}{\sqrt{2}} - x) + (a+y)(b+y)(\frac{1}{\sqrt{2}} - y) \right\} .$$

Thus,

$$\begin{aligned} f_1(\sin^2\frac{\alpha}{2}) &= 4\sqrt{2}\left(\frac{1}{2} + ab - \frac{a+b}{\sqrt{2}}\right) \\ f_2(\sin^2\frac{\alpha}{2}) &= -4\sqrt{2}\left(\frac{3}{2} + ab - \frac{a+b}{\sqrt{2}} - \sin^2\frac{\alpha}{2}\right) , \end{aligned} \tag{34}$$

and the right hand sides are functions of $\sin^2\frac{\alpha}{2}$ as required. The resulting expression for $s(\alpha+\pi) - s(\alpha-\pi)$ is symmetric in α and ϕ_o and $O\left\{\exp\left(-\frac{1}{4}|\mathrm{Im}.\alpha|\right)\right\}$ for large $|\mathrm{Im}.\alpha|$. Hence, for small $k\rho$, $H_z^d = O\left\{(k\rho)^{\frac{1}{4}}\right\}$ in accordance with the desired edge behavior, and the above choice of f_1 and f_2 is the only one that achieves this.

From (29) and (33) the diffracted field can be written as

$$H_z^d(\rho,\phi) = \frac{1}{4\pi i}\int_{S(\phi)} e^{ik\rho\cos(\alpha-\phi)} H(\alpha,\phi_o)\left(\sec\frac{\alpha+\phi_o}{2} + \sec\frac{\alpha-\phi_o}{2}\right) d\alpha \tag{35}$$

where

$$H(\alpha,\phi_o) = \frac{1}{\sqrt{2}} G(\alpha,\phi_o)\left\{\sin\frac{\alpha}{2}\sin\frac{\phi_o}{2}\left(\sin\frac{\alpha}{2} + \sin\frac{\phi_o}{2}\right) + ab\left(1 + \sin\frac{\alpha}{2} + \sin\frac{\phi_o}{2}\right)\right.$$

$$+\left(\frac{1}{2}-\frac{a+b}{\sqrt{2}}\right)\left(1+2\sin\frac{\alpha}{2}\sin\frac{\phi_o}{2}+\sin\frac{\alpha}{2}+\sin\frac{\phi_o}{2}\right)\Bigg\} \ , \tag{36}$$

and for $k\rho >> 1$ a uniform asymptotic representation of the total field, including optical and surface wave contributions, is given in Appendix A. We note that if $\varepsilon_r = \mu_r = 1$ and $\tau = 0$ or if $\varepsilon_r = 1 + i\infty$, $\mu_r = 1$ and $\tau = 0$,

$$H(\alpha,\phi_o) = \cot\frac{\alpha}{2}\cot\frac{\phi_o}{2}$$

and (35) then reduces to the known expression for the diffracted field of a perfectly conducting half-plane.

5. Numerical Results

Using the uniform expression (A.2) for the diffracted field, scattering patterns were computed for a number of dielectric coatings. The patterns correspond to the low and high contrast boundary conditions, and are compared to those obtained with the standard impedance boundary condition (7).

Figures 2(a) - (c) show the total H_z field patterns for a plane wave incident on a perfectly conducting half-plane whose upper surface is coated with a dielectric layer of thickness $\tau = \lambda/20$ ($k\tau = 0.314$). The field is incident at the angle $\phi_0 = 150$ degrees and the curves correspond to coatings having $\varepsilon_r = 2$, $\mu_r = 1$; $\varepsilon_r = 5 + i0.5$, $\mu_r = 1.5 + i0.1$; and $\varepsilon_r = 7.4 + i1.1$, $\mu_r = 1.4 + i0.67$, the last being a commercially available radar absorber. In all cases the pole at $\pi + \theta_2$, whose location is a primary function of the thickness, is far from the path $S(\phi)$, but the pole at $\pi + \theta_1$ has a strong effect on the diffracted and total fields. The latter pole is associated with the propagation factor of the TE_0 mode in the layer, and for a boundary condition to produce the correct reflected field it is necessary that $-ik\cos\theta_1$ equal the attenuation factor of the TE_0 surface wave mode.

The three curves in each figure were computed using the low contrast, high contrast and standard impedance boundary conditions, and since the curves differ, it is necessary to determine which provides the most accurate picture of the field for a coated half-plane. We do this by examining the reflected field recovered by each solution. As expected, in the case $\varepsilon_r = 2$, $\mu_r = 1$, the solution based on the low contrast boundary conditions accurately reproduces the reflection coefficient of the metal-backed layer and is therefore best. At the other extreme, when $\varepsilon_r = 7.4 + i1.1$, $\mu_r = 1.4 + i0.67$, the solution obtained using the high contrast boundary conditions is accurate to within 6 percent in amplitude and 6 degrees in phase, whereas the low contrast solution is in error by 35 percent in amplitude. The high contrast solution is therefore better, and because of the large refraction index, the solution based on the standard impedance boundary conditions is almost as good. The intermediate case is the coating having $\varepsilon_r = 5 + i0.5$, $\mu_r = 1.5 + i0.1$ in Figure 2(b). Compared with the exact reflected wave, the low contrast solution is high in amplitude by 4.5 percent and low in phase by 8 degrees, and the high contrast solution is low in amplitude by 2 percent and high in phase by 7 degrees. The two solutions are comparable in accuracy, and the differences between the curves can be attributed to the opposite signs of the errors.

The backscatter echowidth patterns for the three half-planes computed using the three boundary conditions are shown in Figures 3(a) - (c), and the above comments are also applicable here.

With boundary conditions such as those presented, a matter of concern is the accuracy as a function of the thickness and material properties of the layer, and the above comparisons merely illustrate the type of accuracy achievable with the boundary conditions described. Nevertheless, the comparisons suggest that some combination of the low and high contrast conditions could prove accurate for all values of ε_r and μ_r, and this will be addressed in a future paper.

Acknowledgements

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Appendix A Uniform Evaluation of the Diffraction Integral

To perform a uniform evaluation of the diffraction integral (35) it is necessary to take into account the geometrical optics poles at $\alpha = \alpha_1^{\pm} = \pm\pi + \phi_o$ and $\alpha = \alpha_2^{\pm} = \pm\pi - \phi_o$, as well as the surface wave pole at $\alpha = \alpha_3 = \pi + \theta_1$, or $\alpha = \alpha_2^{\pm} = \pi + \theta_2$. This last is associated with the function $\psi_\pi(\alpha + \frac{3\pi}{2} - \theta_{1,2})$ appearing in (21), and its presence is apparent when the identity

$$\psi_\pi(\alpha + \frac{3\pi}{2} - \theta_n) = \sin\frac{1}{4}(\alpha - \pi - \theta_n)\ \mathrm{cosec}\frac{1}{4}(\alpha - 2\pi - \theta_n)\ \psi_\pi(\alpha - \frac{\pi}{2} - \theta_n) \qquad \text{(A.1)}$$

is employed in (31).

A straightforward method that assures a uniform evaluation of the integral is the additive procedure discussed in [7]. The method regularizes the integrand by the addition and subtraction of secants. Each secant has an appropriate singularity and a multiplying constant chosen to produce the desired residue. The integrand can then be split into two parts, the first of which is a slowly varying function of α and the second a sum of secants. The latter integral can be evaluated exactly [8], leading to a uniform result. Following this procedure, we obtain

$$H_z^d \sim -\frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi k}}\frac{e^{ik\rho}}{\sqrt{\rho}}\left\{H(\phi,\phi_o)\left[\sec\frac{1}{2}(\phi+\phi_o) + \sec\frac{1}{2}(\phi-\phi_o)\right] - \sum_{n=1}^{4}\frac{D_n}{c_n}\left[1 - F_{k\rho}\ (2k\rho c_n)\right]\right\}$$

(A.2)

with

$$F_{k\rho}(z^2) = \pm 2iz\, e^{-iz^2} \int_{\pm z}^{\infty} e^{iu^2}\, du = \pm 2iz\, F_c(\pm z) \qquad \text{(A.3)}$$

where the upper sign is employed if $-\frac{3\pi}{4} < \arg z < \frac{\pi}{4}$ and the lower sign otherwise. The constants D_n are the residues of the respective poles:

$$D_1 = H(\pm\pi + \phi_o, \phi_o)\, e^{ik\rho\cos(\phi-\phi_o)} = e^{ik\rho\cos(\phi-\phi_o)}$$

$$D_2 = -H(\pm\pi - \phi_o, \phi_o)\, e^{ik\rho\cos(\phi+\phi_o)} \qquad (\phi_o \gtrless 0)$$

where

$$H(\pi - \phi_o, \phi_o) = \frac{(\sin\theta_1 - \sin\phi_o)(\sin\theta_2 - \sin\phi_o)}{(\sin\theta_1 + \sin\phi_o)(\sin\theta_2 + \sin\phi_o)} ,$$

$$H(-\pi - \phi_o, \phi_o) = 1 \;;$$

$$D_3 = -\frac{1}{2} H_+(\pi + \theta_1, \phi_o)\, e^{-ik\rho\cos(\phi-\theta_1)} \left[\operatorname{cosec}\frac{1}{2}(\theta_1 + \phi_o) + \operatorname{cosec}\frac{1}{2}(\theta_1 - \phi_o)\right] ,$$

$$D_4 = -\frac{1}{2} H_+(\pi + \theta_2, \phi_o)\, e^{-ik\rho\cos(\phi-\theta_2)} \left[\operatorname{cosec}\frac{1}{2}(\theta_2 + \phi_o) + \operatorname{cosec}\frac{1}{2}(\theta_2 - \phi_o)\right] .$$

In the above $H_+(\alpha, \phi_o)$ differs from the expression (30) for $H(\alpha, \phi_o)$ in having $G(\alpha, \phi_o)$ replaced by

$$G_+(\alpha, \phi_o) = \frac{g_+(\alpha + \pi)}{(a-x)\,(b-x)\,(\frac{1}{\sqrt{2}} + x)}$$

where

$$g_+(\alpha) = 2\sqrt{2}\;\frac{\psi_\pi(\alpha - \frac{\pi}{2} - \theta_i)\;\psi_\pi(\alpha + \frac{\pi}{2} + \theta_i)\;\psi_\pi(\alpha + \frac{3\pi}{2} - \theta_j)\;\psi_\pi(\alpha + \frac{\pi}{2} + \theta_j)\,\cos\frac{1}{4}(\alpha - \pi)}{\psi_\pi(\phi_o - \frac{\pi}{2} - \theta_i)\;\psi_\pi(\phi_o + \frac{\pi}{2} + \theta_i)\;\psi_\pi(\phi_o + \frac{3\pi}{2} - \theta_j)\;\psi_\pi(\phi_o + \frac{\pi}{2} + \theta_j)\,\cos\frac{1}{4}(\phi_o - \pi)}$$

with $\theta_i = \theta_1$, $\theta_j = \theta_2$ for D_3 and $\theta_i = \theta_2$, $\theta_j = \theta_1$ for D_4. The quantities c_n in (A.2) are

$$c_1 = \cos\frac{1}{2}\,(\alpha_1^{\pm} - \phi \pm \pi) = \cos\frac{1}{2}\,(\phi - \phi_o)$$

$$c_2 = \cos\frac{1}{2}\,(\alpha_2^{\pm} - \phi \pm \pi) = -\cos\frac{1}{2}\,(\phi + \phi_o)$$

$$c_3 = \cos\frac{1}{2}\,(\alpha_3 - \phi \pm \pi) = \mp\cos\frac{1}{2}\,(\phi - \theta_1)$$

$$c_4 = \cos\frac{1}{2}\,(\alpha_4 - \phi \pm \pi) = \mp\cos\frac{1}{2}\,(\phi - \theta_2)$$

where the upper (lower) sign corresponds to positive (negative) angles of incidence. It should be noted that a uniform evaluation of the surface wave contribution must be performed only when $\phi > 0$ and provided Re. $(\pi + \theta_{1,2} - \phi) < 0$, in which case the surface wave pole is in the vicinity of the saddle point.

To obtain the total field H_z, the residues must be added to the diffracted field (A.2) whenever the corresponding poles are captured in the closure of the contour γ. Alternatively, (A.2) represents the total field when the lower signs are used in the computation of the transition function (A.3).

Appendix B Solution for E-Polarization

If the incident field is E-polarized with

$$E_z^i = e^{-ik\rho \cos(\phi - \phi_o)}$$

the boundary condition on the upper side ($\phi = \pi$) of the half-plane is again (12) with H_z replaced by E_z, and on the lower side ($\phi = -\pi$) the condition is $E_z = 0$. Superficially at least, the problem appears almost identical to that for H-polarization, but in fact the analysis is much simpler.

If $s(\alpha)$ is the spectral function for E_z (see (16)), the first of the equations (18) is unaffected, but the second is replaced by

$$s(\alpha - \pi) = s(-\alpha - \pi) + \sin\alpha \,(B_o + B_1 \cos\alpha) \ .$$

We again write

$$s(\alpha) = g(\alpha)\, t\,(\alpha)$$

where now

$$g(\alpha) = \frac{\Psi(\alpha, \theta_1, \frac{\pi}{2} - i\infty)\, \Psi(\alpha, \theta_2, \frac{\pi}{2} - i\infty)}{\Psi(\phi_o, \theta_1, \frac{\pi}{2} - i\infty)\, \Psi(\phi_o, \theta_2, \frac{\pi}{2} - i\infty)} \ ,$$

and this is $O\left\{\exp\left(\frac{1}{2}|\text{Im}.\alpha|\right)\right\}$ for large $|\text{Im}.\alpha|$. The function $g(\alpha)$ satisfies (22) with Γ reversed in sign, and the difference equation for $t(\alpha)$ is

$$t(\alpha + 2\pi) - t(\alpha - 2\pi) = -\frac{\sin\alpha}{g(\alpha - 2\pi)}\left\{\frac{A_o - A_1\cos\alpha}{(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)} - \left(B_o - B_1\cos\alpha\right)\right\} .$$

A particular solution constructed in the same manner as before is

$$t_o(\alpha) = -\frac{1}{2g(\alpha)}\,\frac{1}{\sin\theta_1 + \sin\theta_2}\left\{(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)(B_o - B_1\cos\alpha) - (A_o - A_1\cos\alpha)\right\} ,$$

and since this is free of poles in the strip $|\text{Re}.\alpha| \le \pi$, it is sufficient to take $A_0 = A_1 = B_0 = B_1 = 0$, implying $t_0(\alpha) = 0$. It follows that f_1 and f_2 must also be zero, and hence

$$s(\alpha) = g(\alpha)\frac{\frac{1}{2}\cos\frac{\phi_o}{2}}{\sin\frac{\alpha}{2} - \sin\frac{\phi_o}{2}} .$$

When the contour is closed an expression for the total field is obtained in the form (29), and because

$$g(a + \pi) = G(\alpha, \phi_o)\,(a\text{-}x)\,(b\text{-}x)$$

$$g(\alpha - \pi) = G(\alpha, \phi_o)\,(a\text{-}y)\,(b\text{-}y)$$

with

$$G(\alpha, \phi_o) = \left\{\psi_\pi\left(\frac{\pi}{2}\right)\right\}^8 \left\{4\Psi(\alpha, \theta_1, \frac{\pi}{2} - i\infty)\, \Psi(\phi_o, \theta_1, \frac{\pi}{2} - i\infty)\, \Psi(\alpha, \theta_2, \frac{\pi}{2} - i\infty)\, \Psi(\phi_o, \theta_2, \frac{\pi}{2} - i\infty)\right\}^{-1} ,$$

we have

$$s(\alpha+\pi) - s(\alpha-\pi) = G(\alpha,\phi_o) \frac{\cos\frac{\phi_o}{2}}{\cos\alpha+\cos\phi_o} \left\{(a-x)\,(b-x)\left(\sin\frac{\phi_o}{2} + \cos\frac{\alpha}{2}\right)\right.$$

$$\left. - (a-y)\,(b-y)\left(\sin\frac{\phi_o}{2} - \cos\frac{\alpha}{2}\right)\right\}$$

$$= \frac{G(\alpha,\phi_o)}{\cos\alpha+\cos\phi_o} \left\{\frac{1}{2}\sin\alpha\,\sin\phi_o + 2ab\cos\frac{\alpha}{2}\cos\frac{\phi_o}{2}\right.$$

$$\left. - \frac{a-b}{\sqrt{2}}\left(\cos\frac{\alpha}{2}\sin\phi_o + \cos\frac{\phi_o}{2}\sin\alpha\right)\right\} .$$

This is symmetric in α and ϕ_o and $O\left\{\exp\left(-\frac{1}{2}\left|\mathrm{Im}.\alpha\right|\right)\right\}$ for large $\left|\mathrm{Im}.\alpha\right|$, implying $E_z = O\left\{(k\rho)^{\frac{1}{2}}\right\}$ for small $k\rho$ in accordance with the required edge behavior. Thus, in spite of the generalized boundary condition imposed on the upper side of the half-plane, Maliuzhinets' method is applicable in its standard form.

With only minor modifications, the analysis in Appendix A is also applicable for this polarization, and, in particular, the diffracted field is again given by (A.2) provided $H(\alpha, \phi_o)$ is replaced by

$$H(\alpha, \phi_o) = G(\alpha, \phi_o) \left\{\sin\frac{\alpha}{2}\sin\frac{\phi_o}{2} + ab - \frac{a-b}{\sqrt{2}}\left(\sin\frac{\alpha}{2} + \sin\frac{\phi_o}{2}\right)\right\} .$$

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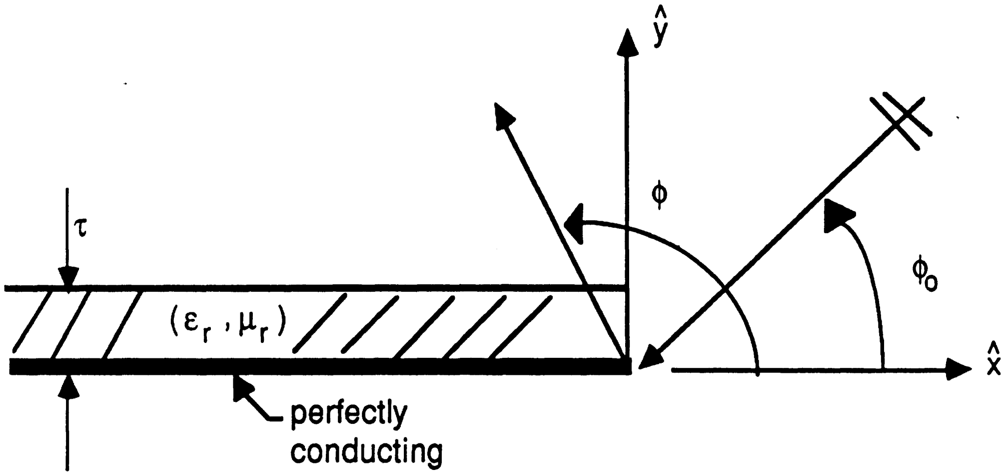
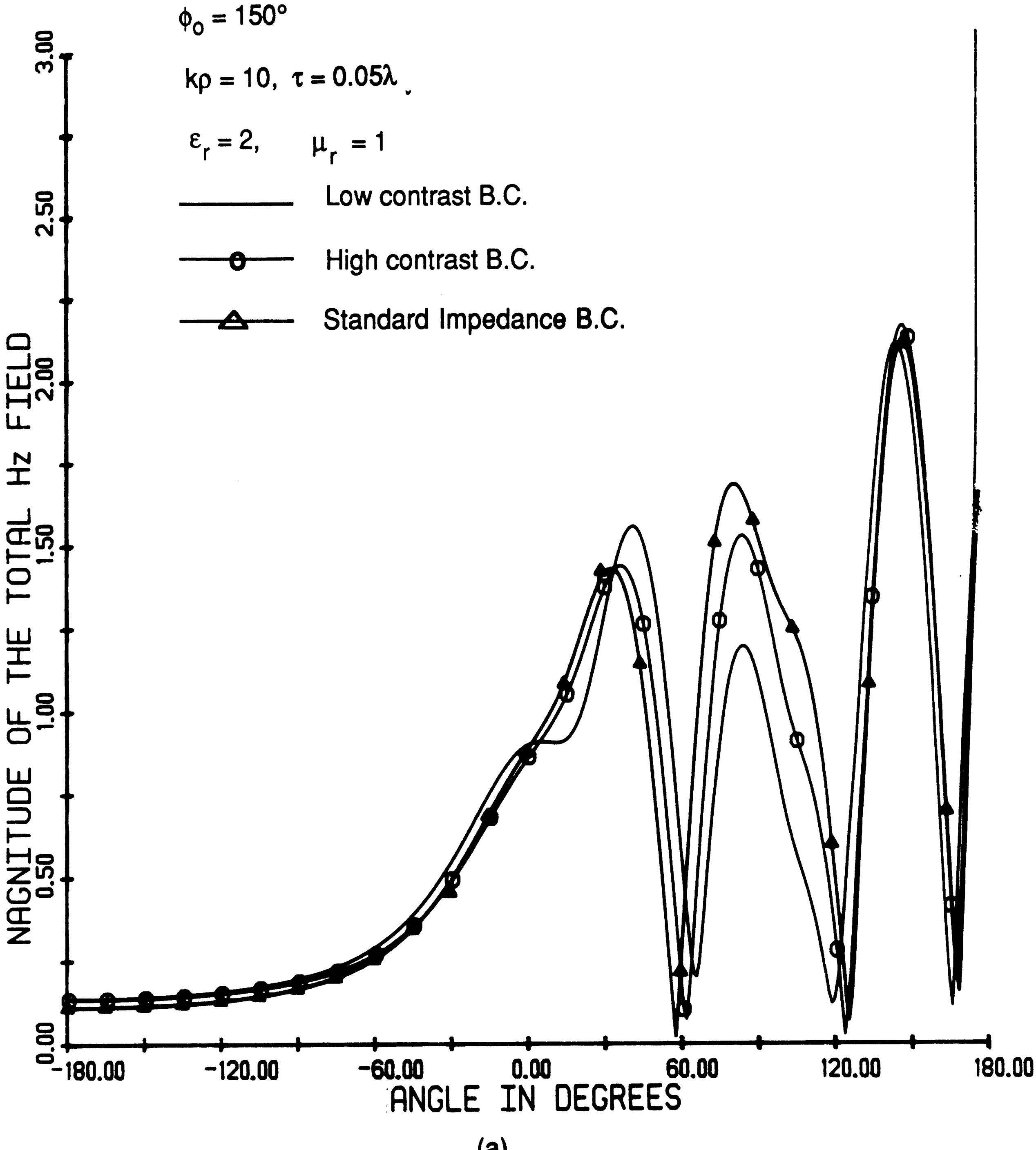
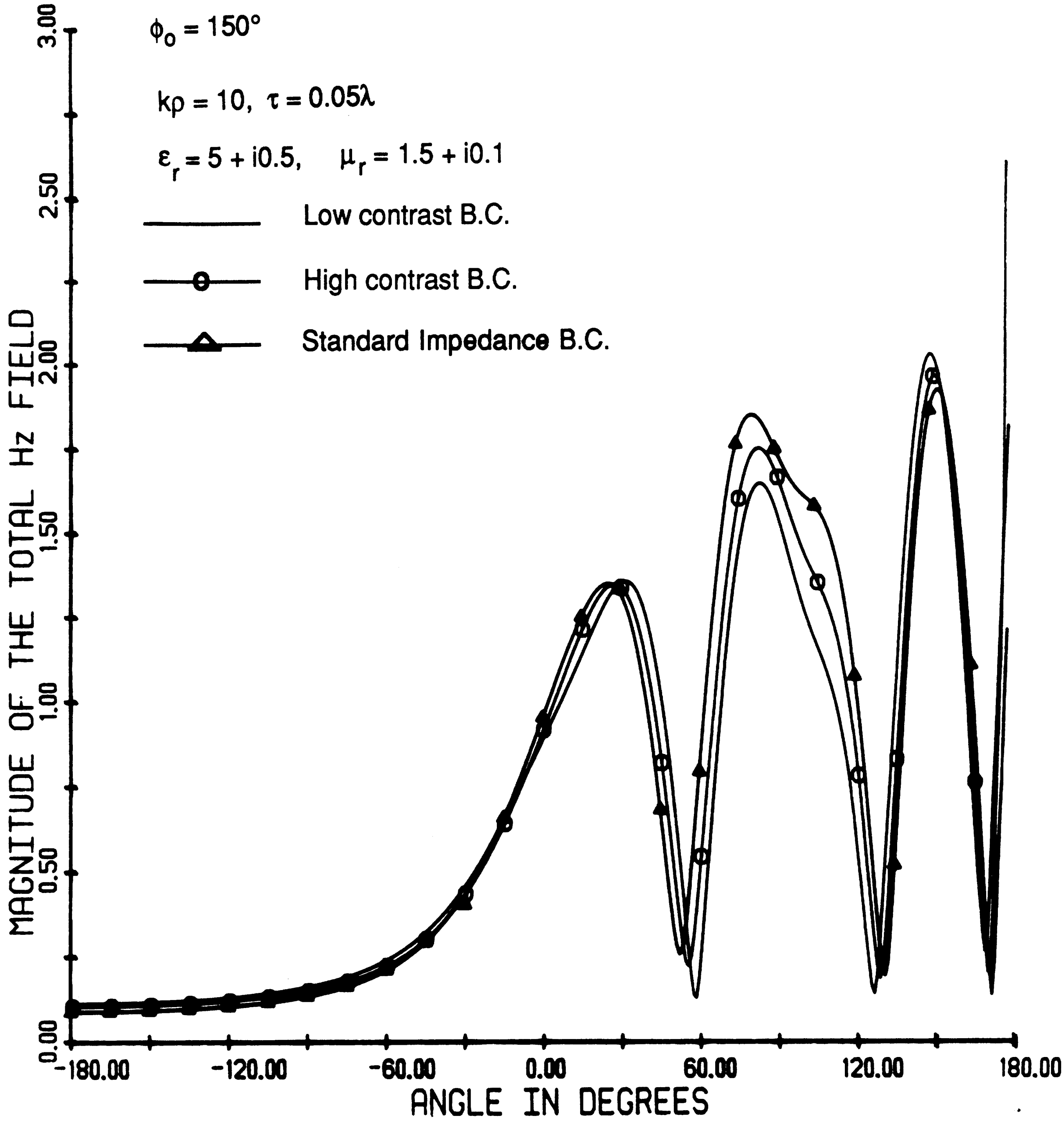


Figure 1. Geometry of the dielectrically coated half-plane

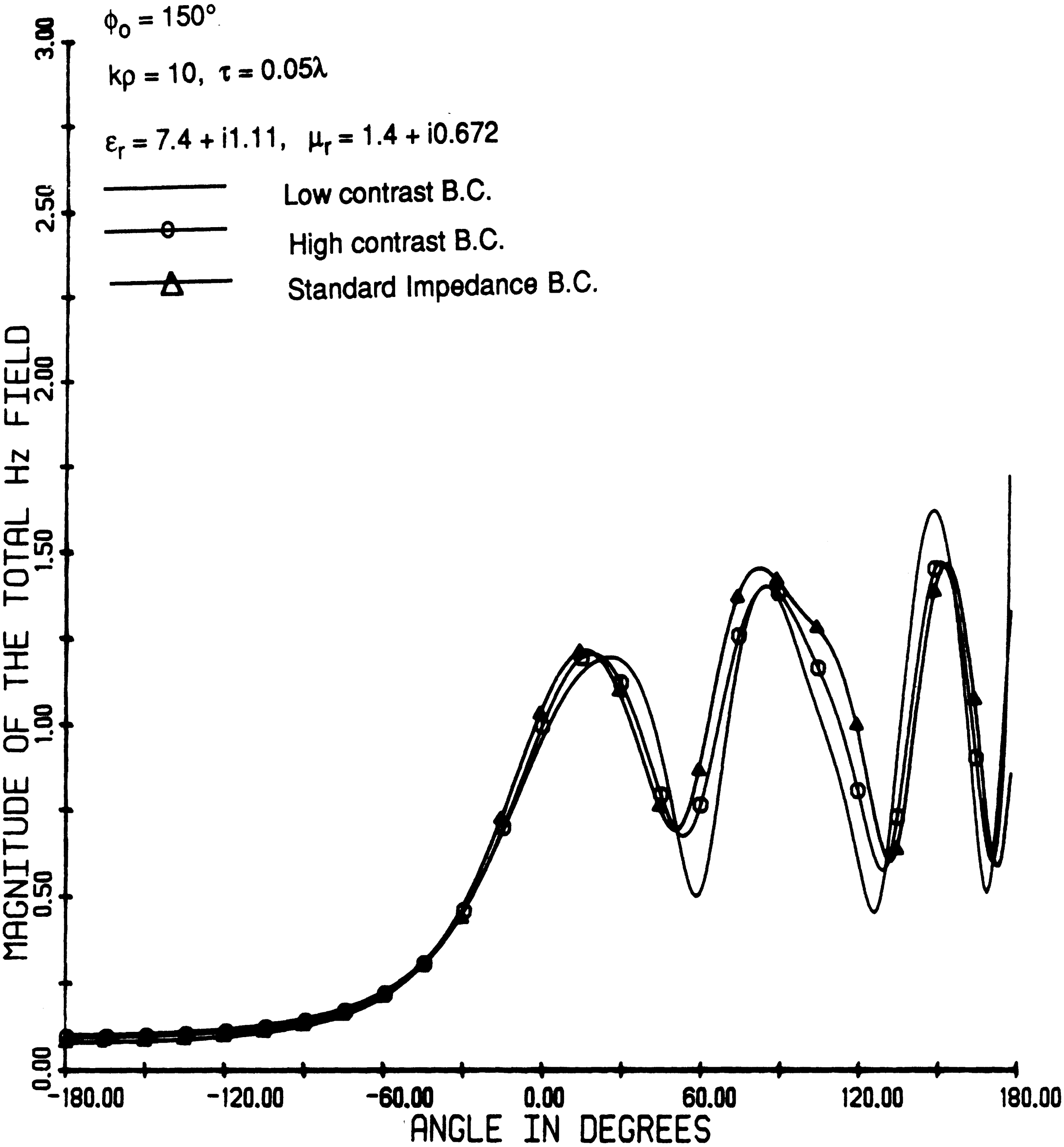


(a)

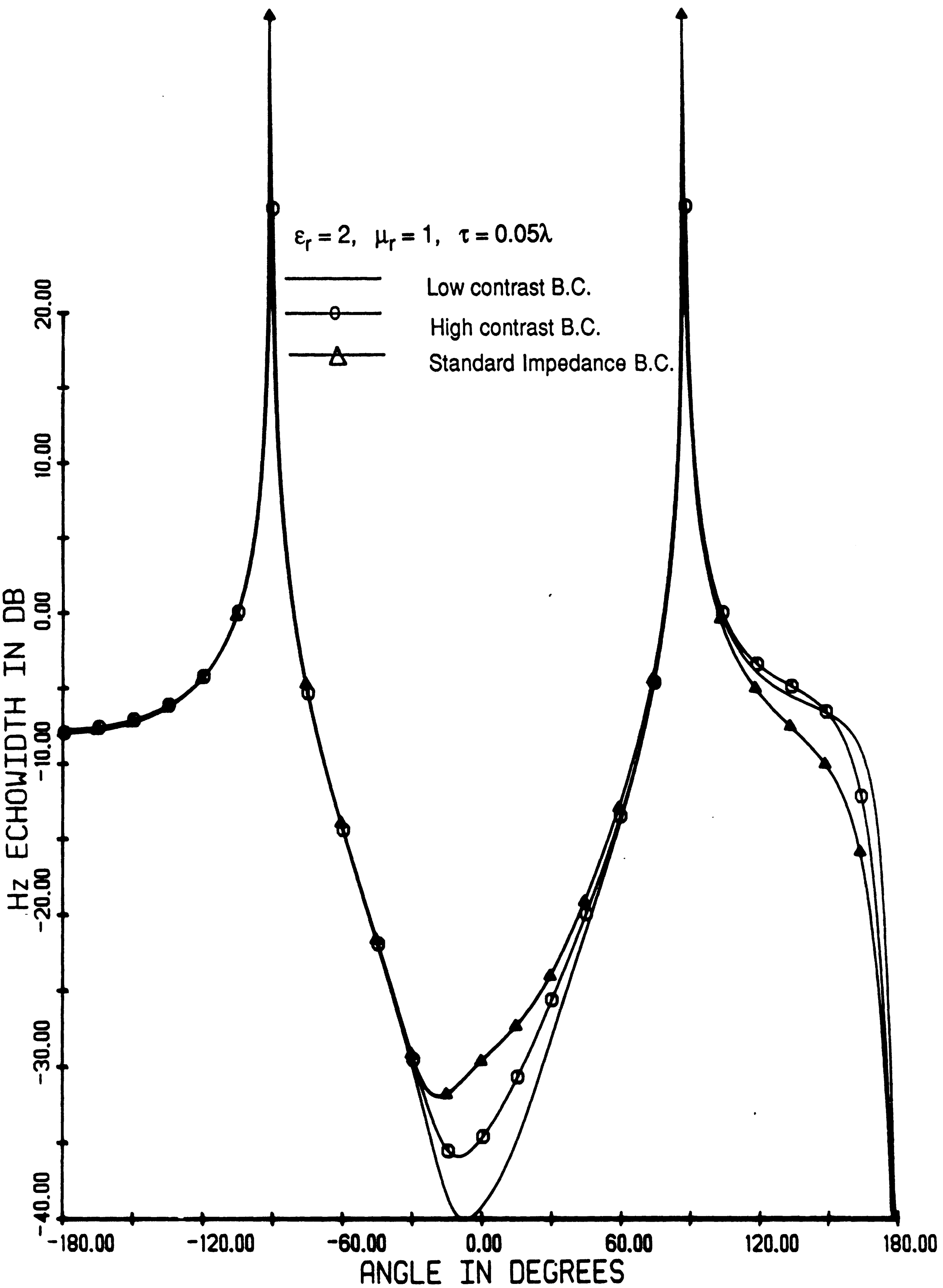
Fig. 2. Total H_z patterns for a plane wave incident on a perfectly conducting half-plane coated with a $\lambda/20$ thick dielectric layer; comparison of solutions based on the low contrast, high contrast and standard impedance boundary conditions. (a) $\varepsilon_r = 2$, $\mu_r = 1$ (b) $\varepsilon_r = 5 + i0.5$, $\mu_r = 1.5 + i0.1$ (c) $\varepsilon_r = 7.4 + i1.11$, $\mu_r = 1.4 + i0.672$.


$\phi_o = 150°$
$k\rho = 10,\ \tau = 0.05\lambda$
$\varepsilon_r = 5 + i0.5,\quad \mu_r = 1.5 + i0.1$
Low contrast B.C.
High contrast B.C.
Standard Impedance B.C.
MAGNITUDE OF THE TOTAL Hz FIELD
ANGLE IN DEGREES
0.00
0.50
1.00
1.50
2.00
2.50
3.00
-180.00
-120.00
-60.00
0.00
60.00
120.00
180.00

(b)

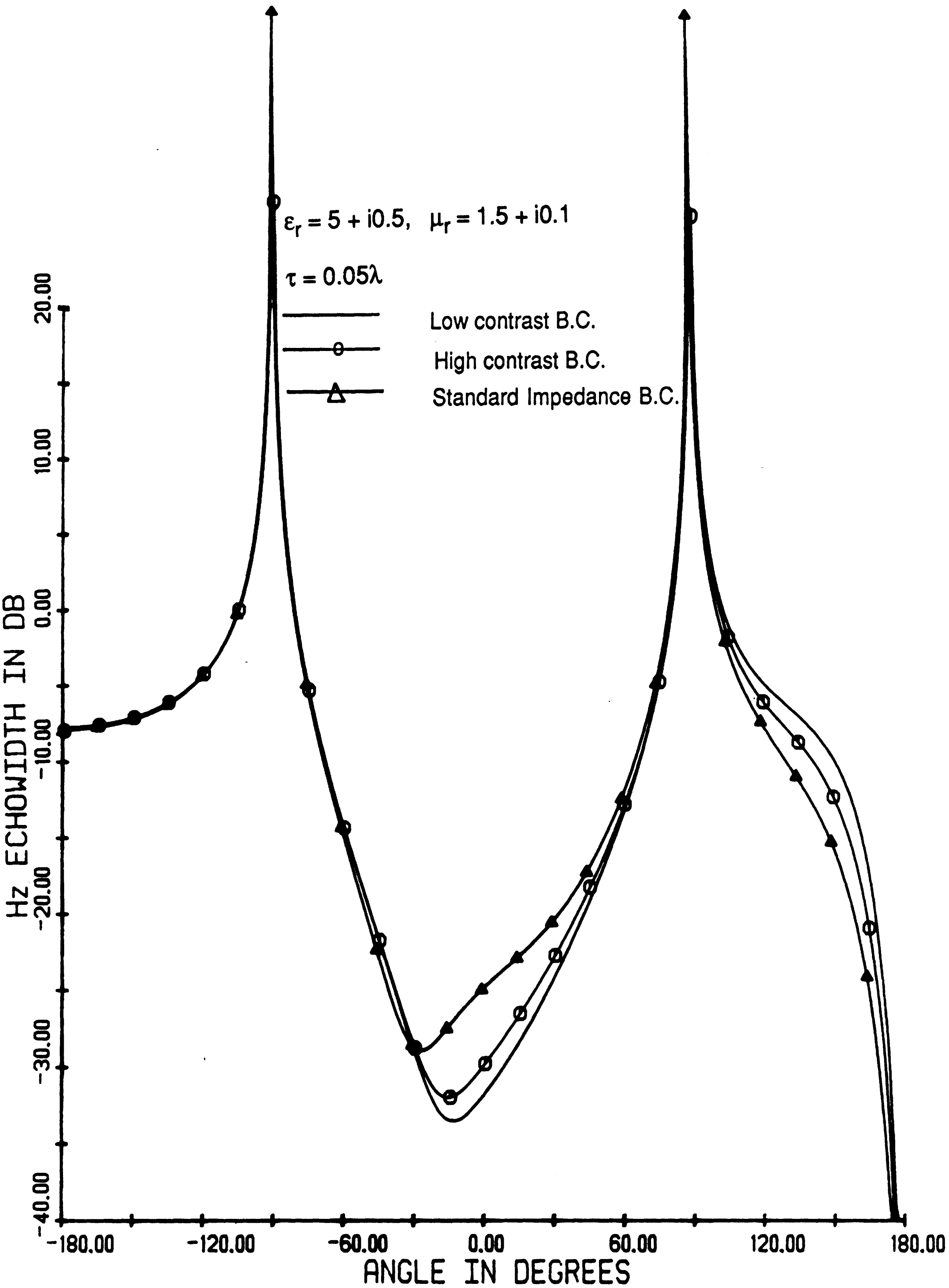

$\phi_o = 150°$
$k\rho = 10,\ \tau = 0.05\lambda$
$\varepsilon_r = 7.4 + i1.11,\ \mu_r = 1.4 + i0.672$
Low contrast B.C.
High contrast B.C.
Standard Impedance B.C.
MAGNITUDE OF THE TOTAL Hz FIELD
0.00
0.50
1.00
1.50
2.00
2.50
3.00
-180.00
-120.00
-60.00
0.00
60.00
120.00
180.00
ANGLE IN DEGREES

(c)



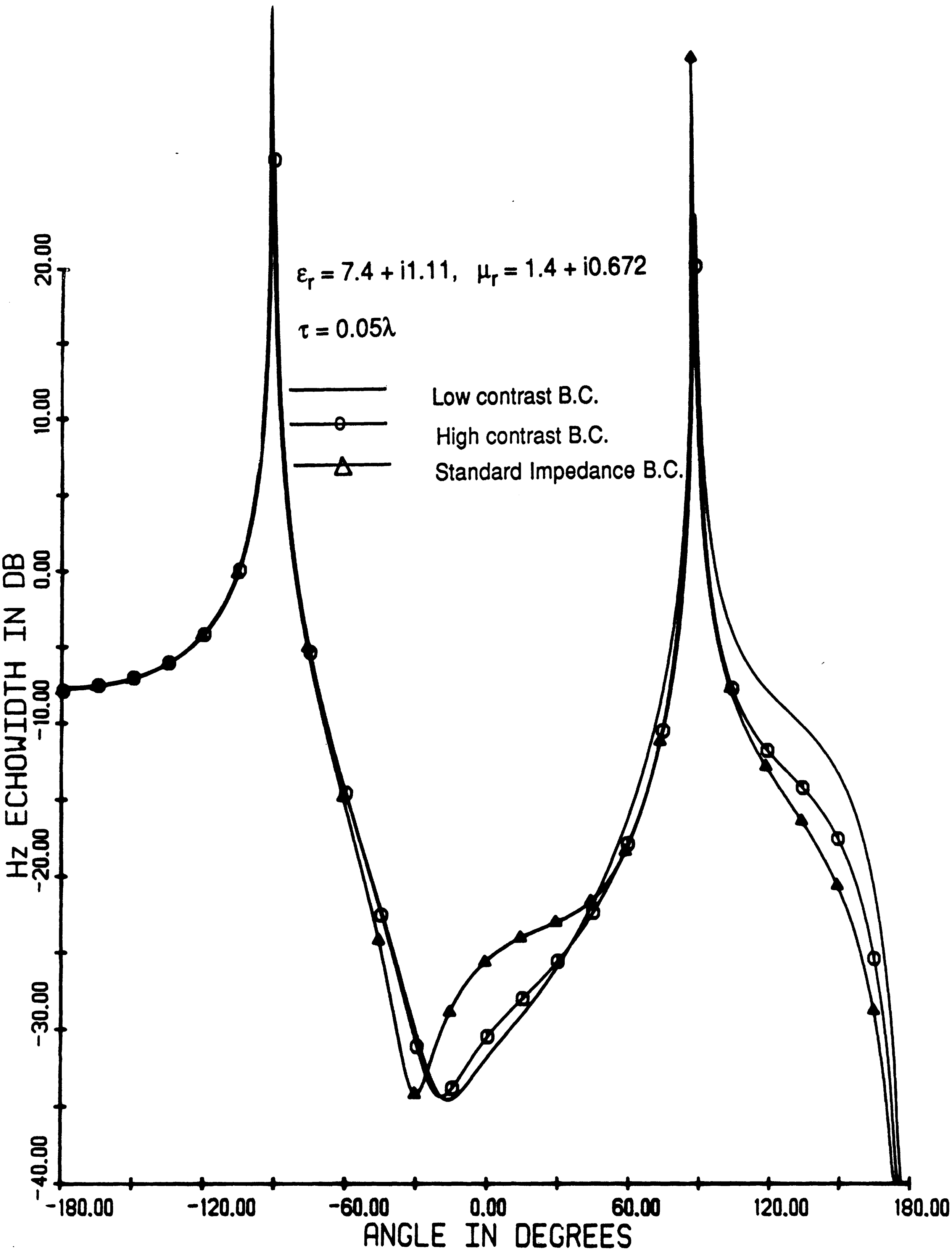
(a)

Fig. 3. Backscatter H_z echowidth for a plane wave incident on a perfectly conducting half-plane coated with a $\lambda/20$ thick dielectric layer; comparison of solutions based on the low contrast, high contrast and standard impedance boundary conditions. (a) $\varepsilon_r = 2$, $\mu_r = 1$ (b) $\varepsilon_r = 5 + i0.5$, $\mu_r = 1.5 + i0.1$ (c) $\varepsilon_r = 7.4 + i1.11$, $\mu_r = 1.4 + i0.672$.



$\varepsilon_r = 5 + i0.5$, $\mu_r = 1.5 + i0.1$
$\tau = 0.05\lambda$
Low contrast B.C.
High contrast B.C.
Standard Impedance B.C.
Hz ECHOWIDTH IN DB
20.00
10.00
0.00
-10.00
-20.00
-30.00
-40.00
-180.00
-120.00
-60.00
0.00
60.00
120.00
180.00
ANGLE IN DEGREES

(b)



$\varepsilon_r = 7.4 + i1.11, \ \mu_r = 1.4 + i0.672$
$\tau = 0.05\lambda$
Low contrast B.C.
High contrast B.C.
Standard Impedance B.C.
Hz ECHOWIDTH IN DB
20.00
10.00
0.00
-10.00
-20.00
-30.00
-40.00
-180.00
-120.00
-60.00
0.00
60.00
120.00
180.00
ANGLE IN DEGREES

(c)