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Quadratic Variation of Functionals of Two-Parameter Wiener Process

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The quadratic variation of functionals of the two-parameter Wiener process of the form $f(W(s, t))$ is investigated, where $W(s, t)$ is the standard two-parameter Wiener process and f is a function on the reals. The existence of the quadratic variation is obtained under the condition that f' is locally absolutely continuous and f'' is locally square integrable.

1. INTRODUCTION

Let $[W(s, t): (s, t) \in R_+^2]$, $R_+^2 = [0, \infty) \times [0, \infty)$, be the standard two-parameter Wiener process defined on a complete probability space $(\Omega, \mathbf{F}, P)$, i.e., a Gaussian stochastic process with $EW(s, t) = 0$ and $EW(s, t)\, W(s', t') = \operatorname{Min}(s, s') \operatorname{Min}(t, t')$. We shall also assume, as we may do without restricting the generality, that $W(s, t; \omega)$ is sample path continuous, i.e., for each ω, $W(\cdot; \omega)$ is a continuous function on R_+^2. Let $\mathbf{F}_{st}$, $(s, t) \in R_+^2$, be the σ-field generated by the random variables $[W(u, v): 0 \leqslant u \leqslant s,\ 0 \leqslant v \leqslant t]$ and augmented by the P-null sets in $\mathbf{F}$.

In order to define the quadratic variation of a two-parameter process we need a notation for rectangles and also the notion of the increment of a process over a rectangle. Suppose (s, t) and (s', t') are in R_+^2. If $s < s'$ and $t < t'$, $((s, t), (s', t')]$

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will denote the rectangle $(s, s'] \times (t, t']$. Similarly one can define $[(s, t), (s', t')]$ and $((s, t), (s', t'))$ in an obvious manner. Now given $A = ((s, t), (s', t')]$ as above, the increment of a two-parameter process $Y(s, t)$, $(s, t) \in R_+^2$, over A is

$$Y(A) = Y(s', t') - Y(s, t') - Y(s', t) + Y(s, t). \tag{1}$$

DEFINITION 1. Let $\Pi = [(s_i, t_j); i = 1, 2,..., m, j = 1, 2,..., n]$ be a partition of the rectangle $T = [(a, b), (c, d)]$ with $(a, b) = (s_1, t_1)$ and $(c, d) = (s_m, t_n)$ and s_i, t_j increasing. Let $D_{ij} = ((s_i, t_j), (s_{i+1}, t_{j+1})]$ and

$$Q^2(Y, \Pi) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} Y^2(D_{ij}). \tag{2}$$

Then $Q^2(Y, \Pi)$ is called the quadratic variation of Y over Π.

Throughout this paper we shall use the following without further explanations:

(i) T will always indicate the rectangle $[(a, b), (c, d)]$ with $a > 0$, $b > 0$.

(ii) Let $[a_{mn}: m = 1, 2,...; n = 1, 2,...]$ be a double sequence of real numbers. Then $\lim_{(m,n)\to\infty} a_{mn} = \tilde{a}$ means that given $\epsilon > 0$, there exists $M > 0$ such that $| a_{mn} - \tilde{a} | < \epsilon$ whenever $m, n \geqslant M$. An obvious modification should be made when a_{mn}'s are not real numbers.

(iii) Let $[\Pi_{mn}: m = 1, 2,...; n = 1, 2,...]$ be a double sequence of partitions of T with

$$\Pi_{mn} = [(s_i^m, t_j^n]: i = 1, 2,..., \kappa(m); j = 1, 2,..., \nu(n)].$$

We shall always assume that $\| \Pi_{mn} \| = \mathrm{Max}_{i,j}[(s_{i+1}^m - s_i^m), (t_{j+1}^n - t_j^n)] \to 0$ as $(m, n) \to \infty$. Furthermore we shall drop the superindices m, n and we shall identify $\kappa(m)$, $\nu(n)$ by m, n, respectively, when no danger of confusion arises.

DEFINITION 2. The quadratic variation of Y over T, $Q_T^2(Y)$, is the limit of $Q^2(Y, \Pi_{mn})$, in some sense, as $(m, n) \to \infty$ if it exists.

The purpose of this paper is to study the quadratic variation of the two-parameter Wiener functionals of the form $Y(s, t) = f(W(s, t))$, where f is a function on the reals. In [1], Cairoli and Walsh obtained a two-parameter version of the Ito formula for C^4 functions. It is easy to see that one can drive the quadratic variation of $f(W(s, t))$, for $f \in C^4$, from their formula. Our result, Theorem 3, says that if f' is locally absolutely continuous and if f'' is locally square integrable then

$$Q_T^2(f(W)) = \iint_T [(f')^2(W(u, v)) + uv(f'')^2(W(u, v))]\, du\, dv$$

in probability.

In our way of searching for $Q_T^2(f(W))$, we shall encounter several types of summations (see formulas (3)–(6)); they are worth studying by themselves. Our results concerning the limit behavior of these summations are proved in Section 2 and summarized in Theorems 1 and 2.

Lemmas 1 and 3 are generalization of results given in Wong and Zakai [6]. For the informations used in this paper concerning the sample path properties of the multiparamater Wiener process we refer the reader to Orey and Pruitt [3] and Zimmerman [7]. Finally we may advise the readers to consult Wang [5] for the most recent results on quadratic variation of functionals of one-parameter Wiener processes.

2. Some Preliminary Results

Let Y be adapted to $\mathbf{F}_{st}$. Define

$$B(Y, \Pi_{mn}) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} Y(s_i, t_j)\, W(D_{ij})^2, \tag{3}$$

$$C(Y, \Pi_{mn}) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} Y(s_i, t_j)\, W(H_{ij})^2\, W(K_{ij})^2, \tag{4}$$

$$R(Y, \Pi_{mn}) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} Y(s_i, t_j)\, | D_{ij} |, \tag{5}$$

$$S(Y, \Pi_{mn}) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} Y(s_i, t_j)\, s_i t_j\, | D_{ij} |, \tag{6}$$

where $D_{ij} = ((s_i, t_j), (s_{i+1}, t_{j+1})]$, $H_{ij} = ((0, t_j), (s_i, t_{j+1})]$, $K_{ij} = ((s_i, 0), (s_{i+1}, t_j)]$ and $| D_{ij} |$, $| H_{ij} |$, $| K_{ij} |$ denote the area of the corresponding rectangles.

Lemma 1. *Let $EY^2(s, t) \leqslant M$, for all $(s, t) \in T$ where M is a constant. Then*

$$\lim_{(m,n)\to\infty} E[B(Y, \Pi_{mn}) - R(Y, \Pi_{mn})]^2 = 0. \tag{7}$$

Proof. By (3) and (5),

$$E[B(Y, \Pi_{mn}) - R(Y, \Pi_{mn})]^2$$

$$= E \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} Y^2(s_i, t_j)\, [W(D_{ij})^2 - | D_{ij} |]^2 \tag{8}$$

$$+ E \sum\sum\sum\sum_{(i,j)\neq(p,q)} Y(s_i, t_j)\, Y(s_p, t_q)\, [W(D_{ij})^2 - | D_{ij} |]\, [W(D_{pq})^2 - | D_{pq} |]$$

Clearly, for $(i, j) \neq (p, q)$

$$EY(s_i, t_j)\, Y(s_p, t_q)[W(D_{ij})^2 - |D_{ij}|][W(D_{pq})^2 - |D_{pq}|] = 0,$$

and

$$\begin{aligned} EY^2(s_i, t_j)[W(D_{ij})^2 - |D_{ij}|]^2 &\leqslant ME[W(D_{ij})^2 - |D_{ij}|]^2 \\ &= 2M\,|D_{ij}|^2 \leqslant 2M\,\|\Pi_{mn}\|\,|D_{ij}|. \end{aligned}$$

Hence the left-hand side of (8) is dominated by $2M\,|T|\,\|\Pi_{mn}\|$ which concludes the proof.

LEMMA 2. *Let the process Y be uniformly integrable on T. Then*

$$\lim_{(m,n)\to\infty} E\,|B(Y, \Pi_{mn}) - R(Y, \Pi_{mn})| = 0. \tag{9}$$

Proof. Let

$$Y_N(s, t) = Y(s, t) \qquad \text{if} \quad |Y(s, t)| \leqslant N \text{ and } = 0 \text{ otherwise,}$$

where N is a constant to be determined later. Let $Y^N(s, t) = Y(s, t) - Y_N(s, t)$. Now by uniform integrability of Y, we can choose N big enough such that

$$\begin{aligned} E\,|B(Y^N, \Pi_{mn}) - R(Y^N, \Pi_{mn})| &\leqslant \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} E\,|Y^N(s_i, t_j)|\, E[W(D_{ij})^2 + |D_{ij}|] \\ &\leqslant 2\,|T| \sup_{(s,t)\in T} E\,|Y^N(s, t)| < \tfrac{1}{2}\epsilon. \end{aligned}$$

For this fixed N pick $\delta < \epsilon/4N^2\,|T|$, then by Lemma 1, $\|\Pi_{mn}\| < \delta$ implies that

$$E\,|B(Y_N, \Pi_{mn}) - R(Y_N, \Pi_{mn})| < \tfrac{1}{2}\epsilon,$$

and we are through.

LEMMA 3. *Let the process Y be continuous in $L_1(\Omega, P)$ for all $(s, t) \in T$, i.e.,*

$$\lim_{(s',t')\to(s,t)} E\,|Y(s', t') - Y(s, t)| = 0 \qquad \text{whenever} \quad (s, t) \in T.$$

Then

$$\lim_{(m,n)\to\infty} E\,\Big|R(Y, \Pi_{mn}) - \iint_T Y(u, v)\, du\, dv\Big| = 0.$$

Proof. Define $Y_{mn}(s, t) = Y(s_i, t_j)$ for $(s, t) \in D_{ij}$. Then

$$E \mid R(Y, \Pi_{mn}) - \iint_T Y(u, v)\, du\, dv \mid \leqslant \iint_T E \mid Y_{mn}(u, v) - Y(u, v)\mid du\, dv \to 0$$

as $(m, n) \to \infty$. For $Y(s, t)$ is $L_1(\Omega, P)$-continuous and T is compact.

LEMMA 4. *Let $f \in L_1(R)$, then the process $[f(W(s, t)), (s, t) \in T]$ is uniformly integrable and is continuous in $L_1(\Omega, P)$.*

Proof. Since $f \in L_1(R)$ the uniform integrability follows easily by observing that

$$\int_{[|f(W(s,t))|>N]} | f(W(s, t))|\, dP \leqslant (2\pi ab)^{-1/2} \int_{[|f(y)|>N]} |f(y)|\, dy \to 0.$$

as $N \to \infty$. For $f \in C_0^\infty$ the continuity of $f(W(s, t))$ in $L_1(\Omega, P)$ follows immediately by applying the bounded convergence theorem. For the general case, $f \in L_1(R)$, there exist a sequence of functions, $[f_n]_{n=1}^\infty$, in C_0^∞ such that f_n converges to f in $L_1(R)$. Hence

$$\begin{aligned} E\,|f(W(s', t')) - f(W(s, t))| &\leqslant E\,|f(W(s', t')) - f_n(W(s', t'))| \\ &\quad + E\,|f_n(W(s', t')) - f_n(W(s, t))| \\ &\quad + E\,|f_n(W(s, t)) - f(W(s, t))| \\ &\leqslant E\,|f_n(W(s', t')) - f_n(W(s, t))| \\ &\quad + 2(2\pi ab)^{-1/2}\, \|f_n - f\|_{L_1(R)}\,. \end{aligned} \tag{10}$$

Now given $\epsilon > 0$, there exists an n such that $2(2\pi ab)^{-1/2}\,\|f_n - f\| \leqslant \frac{1}{2}\epsilon$. But $f_n \in C_0^\infty$, therefore when (s', t') is sufficiently close to (s, t) the right hand side of (10) is smaller than ϵ. Q.E.D.

Since we are mainly interested in the processes of the form $[f(W(s, t)), (s, t) \in T]$, we summarize our results and state it in the following theorem.

THEOREM 1. *Let $f \in L_1(R)$. Then*

$$\lim_{(m,n)\to\infty} E \left| R(f(W), \Pi_{mn}) - \iint_T f(W(u, v))\, du\, dv \right| = 0 \tag{11}$$

$$\lim_{(m,n)\to\infty} E \left| B(f(W), \Pi_{mn}) - \iint_T f(W(u, v))\, du\, dv \right| = 0. \tag{12}$$

Furthermore, we have convergence in probability in both (11) *and* (12) *if $f \in L_1^{\text{loc}}(R)$.*

Proof. Only the last statement needs justification. Now since

$$P[| W(s, t)| \geqslant N \text{ for some } (s, t) \in T] \leqslant P[\sup_{\substack{0<s<c \\ 0<t<d}} | W(s, t)| \geqslant N]$$
$$\leqslant 16\, P[| W(c, d)| \geqslant N] \to 0 \tag{13}$$

as $N \to \infty$ where the last inequality can be found in either [3, or 7]. This enables us to restrict our attention to a compact set and use the result for $f \in L_1(R)$ to conclude the proof.

THEOREM 2. *Let* $f \in L_1(R)$, *then*

$$\lim_{(m,n)\to\infty} E \left| S(f(W), \Pi_{mn}) - \iint_T uvf(W(u, v))\, du\, dv \right| = 0 \tag{14}$$

$$\lim_{(m,n)\to\infty} E \left| C(f(W), \Pi_{mn}) - \iint_T uvf(W(u, v))\, du\, dv \right| = 0. \tag{15}$$

Furthermore, we have convergence in probability in both (14) *and* (15) *if* $f \in L_1^{\text{loc}}(R)$.

Proof. Since $f \in L_1(R)$, by Lemma 4. $f(W(s, t))$ is continuous in $L_1(\Omega, P)$ and this enables us to adopt the argument given in Lemma 3 to show that (14) holds true. To show (15), first, we assume $f \in C^\infty$ and then at the end we shall indicate how one could remove this assumption.

$$E[C(f(W), \Pi_{mn}) - S(f(W), \Pi_{mn})]^2$$

$$= E\left[\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(W(s_i, t_j))\, [W(H_{ij})^2\, W(K_{ij})^2 - s_i t_j \mid D_{ij} |]\right]^2$$

$$= \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} E[f^2(W(s_i, t_j))\, [W(H_{ij})^2\, W(K_{ij})^2 - s_i t_j \mid D_{ij} |]^2] \tag{16}$$

$$+ \sum\sum_{(i,j)\neq(p,q)}\sum\sum E[f(W(s_i, t_j))\, f(W(s_p, t_q))\, [W(H_{ij})^2\, W(K_{ij})^2 - s_i t_j \mid D_{ij} |]$$

$$\times\, [W(H_{pq})^2\, W(K_{pq})^2 - s_p t_q \mid D_{pq} |]]$$

$$= I + II.$$

Since $f \in C_0^\infty$, there exists a constant M such that $|f(x)| \leqslant M$ for all $x \in R$. Also note that $|H_{ij}||K_{ij}| = s_i t_j |D_{ij}|$. Thus we have

$$I = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} 8E[f^2(W(s_i, t_j))] (s_i t_j | D_{ij} |)^2$$

$$\leqslant 8M^2 c^2 d^2 \| \Pi_{mn} \|^2 \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} | D_{ij} | = C \| \Pi_{mn} \|^2 \to 0,$$

as $(m, n) \to \infty$. Here C is a constant which does not depend on m and n. From now on we shall use such a C to represent a "universal constant" in the sense that it does not depend on m and n but it may vary from place to place.

To prove that II also goes to zero, it suffices to discuss the summation in II in the following three cases: (i) $p > i$ and $q > j$, (ii) $p > i$ and $q = j$, and (iii) $p > i$ and $q < j$. For other cases follow by symmetry.

Case (i). This portion of II is zero because $[W(H_{pq})^2 W(K_{pq})^2 - s_p t_q | D_{pq} |]$ is independent of all other factors within the big bracket and it has mean zero.

Case (ii). In this case a fairly straightforward computation yields that

$$| E[\cdots]| = 2 | K_{pj} | | H_{ij} |^2 E[f(W(s_i, t_j)) f(W(s_p, t_j)) W(K_{ij})^2]$$
$$\leqslant C | K_{pj} | | H_{ij} |^2 | K_{ij} | \leqslant C | D_{ij} | (s_{p+1} - s_p) \| \Pi_{mn} \|.$$

Hence

$$\left| \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{p=i+1}^{m-1} E[\cdots] \right| \leqslant C \| \Pi_{mn} \| \to 0, \qquad \text{as} \quad (m, n) \to \infty.$$

Case (iii). In this case we also let the reader to verify that:

$$E[\cdots] = | H_{ij} | | K_{pq} | [2 | D_{iq} |^2 E[f(W(s_i, t_j)) f(W(s_p, t_q))]$$
$$+ 4 | D_{iq} | E[f(W(s_p, t_q)) f(W(s_i, t_j)) W(K_{iq}) W(H_{iq})]$$
$$+ E[f(W(s_i, t_j)) f(W(s_p, t_q))(W(H_{iq})^2 - | H_{iq} |)(W(K_{iq})^2 - | K_{iq} |)]]$$

Now since $| K_{iq} | | H_{iq} | = s_i t_q | D_{iq} | \leqslant cd | D_{iq} |$ and

$$| E[f(W(s_i, t_j)) f(W(s_p, t_q)) W(H_{iq}) W(K_{iq})]|$$
$$\leqslant M^2 E[| W(H_{iq}) W(K_{iq})|] \leqslant M^2(| H_{iq} | | K_{iq} |)^{1/2} \leqslant M^2(cd)^{1/2} | D_{iq} |^{1/2}.$$

Thus,

$$\begin{aligned}|E[\cdots]| &\leqslant 4M^2(cd)^{1/2}\,|D_{iq}|^{3/2}\,|H_{ij}|\,|K_{pq}| + 2M^2\,|D_{iq}|^2\,|H_{ij}|\,|K_{pq}| \\ &\quad + |H_{ij}|\,|K_{pq}|\,|E[f(W(s_i, t_j))\,f(W(s_p, t_q))(W(H_{iq})^2 - |H_{iq}|) \\ &\quad \times (W(K_{iq})^2 - |K_{iq}|)]| \\ &= I' + II' + III'.\end{aligned}$$

Taking the corresponding sum of I', we get

$$\begin{aligned}\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\sum_{p=i}^{m-1}\sum_{q=1}^{j-1} I' &\leqslant 4M^2(cd)^{1/2}\,\|\Pi_{mn}\|\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\sum_{p=1}^{m-1}\sum_{q=1}^{n-1}|H_{ij}|\,|K_{pq}|\,|D_{iq}| \\ &\leqslant C\,\|\Pi_{mn}\| \to 0, \qquad \text{as} \quad (m, n) \to \infty.\end{aligned}$$

By a similar argument the relevant sum of II' also tends to zero as $(m, n) \to \infty$. Now let us use Δ and Θ for the rectangles $((0, 0), (s_i, t_q)]$ and $((s_{i+1}, 0), (s_p, t_q)]$, respectively. Since f is uniformly continuous ($f \in C_0^\infty$), given $\epsilon > 0$ one can find $\delta(\epsilon)$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta(\epsilon)$. Then

$$\begin{aligned}&|E[f(W(s_i, t_j))\,f(W(s_p, t_q))(W(H_{iq})^2 - |H_{iq}|)(W(K_{iq})^2 - |K_{iq}|)]| \\ &\quad = |E[f(W(s_i, t_j))[f(W(\Delta \cup \Theta) + W(K_{iq})) - f(W(\Delta \cup \Theta))] \\ &\qquad \times (W(H_{iq})^2 - |H_{iq}|)(W(K_{iq})^2 - |K_{iq}|)]| \\ &\quad = |E[\cdots;\,|W(K_{iq})| < \delta] + E[\cdots;\,|W(K_{iq})| \geqslant \delta]| \\ &\quad \leqslant \epsilon M E(|\,W(H_{iq})^2 - |H_{iq}|\,|)\, E(|\,W(K_{iq})^2 - |K_{iq}|\,|) \\ &\qquad + 2M^2 E(|\,W(H_{iq})^2 - |H_{iq}|\,|)\, E[|\,W(K_{iq})^2 - |K_{iq}|\,|;\,|W(K_{iq})| \geqslant \delta] \\ &\quad \leqslant 4\epsilon M\,|H_{iq}|\,|K_{iq}| + 4M^2\,|H_{iq}|[E[W(K_{iq})^2;\,|W(K_{iq})| \geqslant \delta] \\ &\qquad + |K_{iq}|\,P[|W(K_{iq})| \geqslant \delta]] \\ &\quad \leqslant 4\epsilon M\,|H_{iq}|\,|K_{iq}| + 4(3^{1/2})\,M^2[\,H_{iq}|\,|K_{iq}|\,P^{1/2}[|W(K_{iq})| \geqslant \delta] \\ &\qquad + 4M^2|H_{iq}|\,|K_{iq}|\,P[|W(K_{iq})| \geqslant \delta] \\ &\quad \leqslant 4\epsilon M|H_{iq}|\,|K_{iq}| + 12M^2|H_{iq}|\,|K_{iq}|\,P^{1/2}[|W(K_{iq})| \geqslant \delta], \qquad (17)\end{aligned}$$

where $W(\Delta \cup \Theta) =^{\text{def}} W(\Delta) + W(\Theta)$. Since $W(K_{iq})^2 - |K_{iq}|$ is independent of $f(W(s_i, t_j))\,f(W(\Delta \cup \Theta))(W(H_{iq})^2 - |H_{iq}|)$ with mean zero, the first equality holds and the upper estimate for $E[W(K_{iq})^2;\,|W(K_{iq})| \geqslant \delta]$ is obtained by using Schwarz inequality. Now it is clear that the very right hand side of (17) can be made smaller than $C\epsilon\,|H_{iq}|\,|K_{iq}|$ for sufficiently large m and n. Therefore,

$$\begin{aligned}\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\sum_{p=i}^{m-1}\sum_{q=1}^{j-1} III' &\leqslant C\epsilon\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\sum_{p=1}^{m-1}\sum_{q=1}^{n-1}|H_{ij}|\,|K_{pq}|\,|H_{iq}|\,|K_{iq}| \\ &\leqslant C\epsilon.\end{aligned}$$

Thus we can conclude that $| C(f(W), \Pi_{mn}) - S(f(W), \Pi_{mn})|$ converges to zero in $L_2(\Omega, P)$, hence in $L_1(\Omega, P)$, for $f \in C_0^\infty$. For the general case, $f \in L_1(R)$, we know that there exist $[f_n: n = 1, 2, \ldots]$ in C_0^∞ with $\|f_n - f\|_{L_1(R)} \to 0$ as $n \to \infty$. Then it is easy to see that given $\epsilon > 0$, one can find an N such that

$$E[| C(f_N(W), \Pi_{mn}) - C(f(W), \Pi_{mn})|] < \epsilon,$$

and

$$E[| S(f_N(W), \Pi_{mn}) - S(f(W), \Pi_{mn})|] < \epsilon,$$

where N does not depend on m and n. This is sufficient to carry out the proof. Finally to get the convergence in probability for $f \in L_1^{\text{loc}}(R)$, one can follow the proof given in Theorem 1.

3. Main Result

Before stating our main result in the following theorem we need the notion of the maximal function. We shall only give the definition here and we refer the reader to Stein [4] for the properties of this function which will be used in this work.

Definition 3. Let f be a real valued function on R. Then

$$[Mf](x) = \sup_{r>0} (1/r) \int_{|y-x|<r} |f(y)| \, dy$$

is called the maximal function of f.

Theorem 3. *Let f be a real valued function on the reals such that f' exists and is locally absolutely continuous and let $f'' \in L_2(R)$. Then*

$$\lim_{(m,n)\to\infty} E \,\Big| Q_T^2(f(W), \Pi_{mn}) - \iint_T [(f')^2 (W(u, v)) + uv(f'')^2 (W(u, v))] \, du \, dv \Big| = 0. \tag{18}$$

Furthermore, we have convergence in probability if $f'' \in L_2^{\text{loc}}(R)$.

Proof. We claim that

$$\lim E \left[\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [(fW)(D_{ij}) - f'(W(s_i, t_j)) \, W(D_{ij}) - f''(W(s_i, t_j)) \, W(H_{ij}) \, W(K_{ij})]^2 \right] = 0, \tag{19}$$

where $(fW)(D_{ij})$ is defined in formula (1). Once we prove this, then

$$E\left|Q_T^2(f(W),\Pi_{mn}) - \iint_T [(f')^2 (W(u,v)) + uv(f'')^2 (W(u,v))]\, du\, dv\right|$$

$$\leqslant E\left|\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} [(fW)(D_{ij})]^2 - (f')^2 (W(s_i, t_j))\, W(D_{ij})^2 - (f'')^2 (W(s_i, t_j))\, W(H_{ij})^2\, W(K_{ij})^2\right|$$

$$+ E\left|\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \left[(f')^2 (W(s_i, t_j))\, W(D_{ij})^2 - \iint_T (f')^2 (W(u,v))\, du\, dv\right]\right|$$

$$+ E\left|\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \left[(f'')^2 (W(s_i, t_j))\, W(H_{ij})^2\, W(K_{ij})^2 - \iint_T uv(f'')^2 (W(u,v))\, du\, dv\right]\right|$$

$$= I_{mn} + II_{mn} + III_{mn}.$$

By Theorem 2,

$$\lim_{(m,n)\to\infty} III_{mn} = 0. \tag{20}$$

Furthermore, since for all $x \in R$

$$|f'(x)| = \left|f'(0) + \int_0^x f''(t)\, dt\right| \leqslant |f'(0)| + (|x|)^{1/2}\, \|f''\|_{L_2(R)}. \tag{21}$$

Now an easy argument shows that there exist constants M_1 and M_2 such that for all $(s, t) \in T$,

$$E[(f')^2(W(s,t))] \leqslant M_1, \qquad E[(f')^4(W(s,t))] \leqslant M_2. \tag{22}$$

It follows that $(f')^2(W(s, t))$, $(s, t) \in T$, is uniformly integrable. Hence by Lemmas 1 and 2,

$$\lim_{(m,n)\to\infty} II_{mn} = 0. \tag{23}$$

To see how I_{mn} also goes to zero, note that

$$I_{mn} \leqslant E\left[\left|\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} [[(fW)(D_{ij})]^2 - [f'(W(s_i, t_j))\, W(D_{ij}) + f''(W(s_i, t_j))\, W(H_{ij})\, W(K_{ij})]^2]\right|\right]$$

$$+ 2E\left[\left|\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} (f'f'')(W(s_i, t_j))\, W(D_{ij})\, W(H_{ij})\, W(K_{ij})\right|\right]$$

$$= E\,|A_{mn}| + E\,|B_{mn}|, \qquad \text{(say)}.$$

By (21) and our assumption that $f'' \in L_2(R)$ and an easy computation, $E[(f'f'')^2(W(s, t))]$, $(s, t) \in T$, is bounded by a constant and we obtain

$$E(B_{mn})^2 = E\left[\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (f'f'')^2 (W(s_i, t_j)) [W(D_{ij}) W(H_{ij}) W(K_{ij})]^2\right]$$
$$\leqslant C \| \Pi_{mn} \| \to 0, \qquad \text{as} \quad (m, n) \to \infty.$$

On the other hand the identity $a^2 - b^2 = (a - b)^2 + 2b(a - b)$ gives us

$$E | A_{mn} | \leqslant E\left[\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [(fW)(D_{ij}) - f'(W(s_i, t_j)) W(D_{ij}) - f''(W(s_i, t_j)) W(H_{ij}) W(K_{ij})]^2\right]$$
$$+ 2E\left[\left| \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [f'(W(s_i, t_j)) W(D_{ij}) + f''(W(s_i, t_j)) W(H_{ij}) W(K_{ij})] [(fW)(D_{ij}) - f'(W(s_i, t_j)) W(D_{ij}) - f''(W(s_i, t_j)) W(H_{ij}) W(K_{ij})]\right|\right]$$
$$\leqslant E \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [\cdots]^2 + 2\left[E \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [f'(W(s_i, t_j)) W(D_{ij}) + f''(W(s_i, t_j)) W(H_{ij}) W(K_{ij})]^2\right]^{1/2} E\left[\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [\cdots]^2\right]^{1/2} \tag{24}$$

where $E \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [\cdots]$ is the expectation given in (19). Hence, clearly $E | A_{mn} | \to 0$ as $(m, n) \to \infty$. Consequently I_{mn} goes to zero and (18) holds true.

It remains only to establish (19). Observe that,

$$\begin{aligned}(fW)(D_{ij}) &= [f(W(s_i, t_j) + W(H_{ij}) + W(K_{ij}) + W(D_{ij})) \\ &\quad - f(W(s_i, t_j) + W(H_{ij}) + W(K_{ij}))] \\ &\quad + [f(W(s_i, t_j) + W(H_{ij}) + W(K_{ij})) \\ &\quad - f(W(s_i, t_j) + W(H_{ij})) - f(W(s_i, t_j) + W(K_{ij})) + f(W(s_i, t_j))] \\ &= \zeta_{ij} + \xi_{ij}. \end{aligned} \tag{25}$$

Now for fixed $\omega \in \Omega$, using mean value theorem we obtain

$$\begin{aligned}&[\zeta_{ij} - f'(W(s_i, t_j)) W(D_{ij})]^2 \\ &\quad = W(D_{ij})^2 [f'(W(s_i, t_j) + W(H_{ij}) + W(K_{ij}) + \theta_{ij} W(D_{ij})) - f'(W(s_i, t_j))] \\ &\quad \leqslant C W(D_{ij})^2 [W(H_{ij} \cup K_{ij})^2 + W(D_{ij})^2] [Mf'']^2 (W(s_i, t_j)),\end{aligned}$$

where $|\theta_{ij}(\omega)| \leqslant 1$. Hence

$$E \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [\zeta_{ij} - f'(W(s_i, t_j))\, W(D_{ij})]^2$$

$$\leqslant C \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [3\,|D_{ij}|^2 + (s_i(t_{j+1} - t_j) + t_j(s_{i+1} - s_i))\,|D_{ij}|]\, \|[Mf'']\|_{L_2(R)}^2$$

$$\leqslant C \|f''\|_{L_2(R)}^2\, |T|\, \|\Pi_{mn}\| \to 0, \qquad \text{as} \quad (m, n) \to \infty. \tag{26}$$

where the last inequality follows from the fact that $\|[Mf]\|_{L^2(R)} \leqslant C \|f\|_{L^2(R)}$. To handle ξ_{ij}, note that

$$E[\xi_{ij} - f''(W(s_i, t_j))\, W(H_{ij})\, W(K_{ij})]^2$$

$$= E\left[\int_0^{W(H_{ij})} \int_0^{W(K_{ij})} [f''(W(s_i, t_j) + u + v) - f''(W(s_i, t_j))]\, du\, dv\right]^2$$

$$\leqslant E\left[|W(H_{ij})|\,|W(K_{ij})| \int_0^{|W(H_{ij})|} \int_0^{|W(K_{ij})|} [f''(W(s_i, t_j) + u + v) - f''(W(s_i, t_j))]^2\, du\, dv\right]$$

$$\leqslant 2E\left[|W(H_{ij})|\,|W(K_{ij})| \int_0^{|W(H_{ij})|} \int_0^{|W(K_{ij})|} [(f'')^2\,(W(s_i, t_j) + u + v) + (f'')^2\,(W(s_i, t_j))]\, du\, dv\right]$$

$$= 2E\left[|W(H_{ij})|\,|W(K_{ij})| \int_0^{|W(H_{ij})|} \int_0^{|W(K_{ij})|} E[(f'')^2\,(W(s_i, t_j) + u + v) + (f'')^2\,(W(s_i, t_j))]\, du\, dv\right]$$

$$\leqslant C \|f''\|_{L_2(R)}^2\, E[W(H_{ij})^2\, W(K_{ij})^2] \leqslant C\,|H_{ij}|\,|K_{ij}| \leqslant C\,|D_{ij}|. \tag{27}$$

By an easy argument, now, we can assume f to be of compact support. Since $f''(x)$ exists almost everywhere, by Hobson [2], p. 370, we know

$$\lim_{(r,s)\to 0} [f(x + r + s) - f(x + s) - f(x + r) + f(x)]/rs = f''(x) \qquad \text{a.e.}, \tag{28}$$

where $(r, s) \to 0$ means $|r| + |s| \to 0$. Applying Egorov's theorem, we have uniform convergence in (28) on a set G with $m(G^c)$ as small as we please. Here

$m(G^c)$ is the Lebesgue measure of the complement of the set G with respect to the support of the function f. Now given $\epsilon > 0$, we can choose G and δ such that

$$\| f'' I_{G^C} \|_{L_2(R)} < \epsilon, \tag{29}$$

$$| f(x + r + s) - f(x + s) - f(x + r) + f(x) - f''(x) rs | < \epsilon | rs |, \tag{30}$$

$x \in G$, $| r | + | s | < \delta$ and I_{G^c} is the indicator function of G^c. Furthermore, using Schwarz inequality, we can find γ such that $| H_{ij} | + | K_{ij} | < \gamma$ implies

$$E[W(H_{ij})^2 I_{[\delta,\infty)} (| W(H_{ij})|] < \epsilon | H_{ij} |$$

and

$$E[W(K_{ij})^2 I_{[\delta,\infty)} (| W(K_{ij})|)] < \epsilon | K_{ij} |.$$

$$\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} E[\xi_{ij} - f''(W(s_i, t_j)) W(H_{ij}) W(K_{ij})]^2$$

$$\leqslant \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [E[[\cdots]^2; W(s_i, t_j) \in G, | W(H_{ij})| + | W(K_{ij})| < \delta]$$

$$+ E[[\cdots]^2; W(s_i, t_j) \notin G] + E[[\cdots]^2; | W(H_{ij})| + | W(K_{ij})| \geqslant \delta]]. \tag{31}$$

By the same argument as the one given in (27), the sum corresponding to the second term on the right is no greater than,

$$\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} C \| f'' I_{G^c} \|_{L_2(R)} E[W(H_{ij})^2 W(K_{ij})^2] \leqslant C \| f'' I_{G^c} \|_{L_2(R)} \leqslant C\epsilon,$$

by (29). Similarly, the third term on the right is no greater than

$$\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} C \| f'' \|_{L_2(R)} E[W(H_{ij})^2 I_{[\delta,\infty)}(| W(H_{ij})|) W(K_{ij})^2 I_{[\delta,\infty)}(| W(K_{ij})|)] \leqslant C\epsilon^2,$$

when $\| \Pi_{mn} \| < \text{Min}(\gamma/(c - a), \gamma/(d - b))$, by (31). Now by (30) it is obvious that the first term on the right is also bounded by $C\epsilon$. Consequently for sufficiently large m and n,

$$\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} [\xi_{ij} - f''(W(s_i, t_j)) W(H_{ij}) W(K_{ij})]^2 \leqslant C\epsilon. \tag{32}$$

Now (32) and (26) imply the truth of (19) as we claimed.

For the case $f'' \in L_2^{\text{loc}}(R)$, we refer the reader to the proof of Theorem 1.

Remark 1. Theorem 3 does not hold true in that generality, for it is not hard to see that Lemma 4 fails. However, one can impose more conditions either on the function f or on the partitions Π_{mn} in order to achieve the same result.

Remark 2. A generalization of Theorem 3 also holds true, by a standard limiting argument, for the case when T is a region in the positive plane that could be "exhausted" by rectangles.

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