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On Weakly Singular Fredholm Integral Equations with Displacement Kernels

G. R. RICHTER*,†

Department of Mathematics, University of Michigan, 347 West Engineering Building, Ann Arbor, Michigan 48104

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1. INTRODUCTION

In this paper, we consider Fredholm integral equations of the second kind with weakly singular displacement kernels. Generically, our problem is

$$\phi = f + \lambda K\phi$$

where K, an integral transformation of the form $K\psi = \int_0^1 k(|x-y|)\,\psi(y)\,dy$ whose kernel function $k(t)$ is unbounded over $0 \leqslant t \leqslant 1$, is a completely continuous operator on $L^2[0, 1]$. By virtue of the complete continuity of K, we are guaranteed the existence of a unique $L^2[0, 1]$ solution ϕ for each $f \in L^2[0, 1]$, provided the value of λ is such that the homogeneous equation $\phi = \lambda K\phi$ has no nontrivial solutions $\phi \in L^2[0, 1]$.

Solutions of integral equations of this type will in general contain singularities in their derivatives, even for "smooth" inhomogeneous functions f, and our purpose here is to develop a systematic way of characterizing these singularities. This question has particularly important implications for the problem of solving such integral equations numerically, since the success of any numerical procedure depends crucially on one's ability to approximate the solution accurately. In a subsequent paper, the numerical analysis aspects of the problem will be discussed more fully.

In Section 2, we restrict our attention to unbounded displacement kernels of Hilbert–Schmidt type ($k(t) \in L^2[0, 1]$), and develop a means of analyzing the singularities in the solution. In Section 3, we use the results of Section 2 to explicitly characterize these singularities for the important special cases $k(t) = \log t$ and $k(t) = t^\alpha$, $\alpha > -\frac{1}{2}$. Finally, in Section 4, we briefly indicate how the analysis of Section 2 can be extended to accomodate kernels for which $k(t) \in L^1[0, 1]$, e.g., $k(t) = t^\alpha$, $\alpha > -1$.

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† Current address: Department of Computer Science, Rutgers University, New Brunswick, NJ 08903.

2. Hilbert–Schmidt Kernels of Displacement Type

We consider here the integral equation

$$\phi = f + \lambda K\phi \tag{1}$$

where $f \in L^2[0, 1]$ and $K\psi = \int_0^1 k(|x - y|)\,\psi(y)\,dy$, with $k \in L^2[0, 1]$. We first prove the following:

Lemma 1. *For K as defined as above and $\psi \in H^1[0, 1]$ (i.e. $\int_0^1 [\psi'(x)]^2 < \infty$),*

$$(K\psi)' = K\psi' + \psi(0)\,k(x) - \psi(1)\,k(1 - x). \tag{2}$$

Proof. Write

$$\begin{aligned}
K\psi &= \int_0^x k(x - y)\,\psi(y)\,dy + \int_x^1 k(y - x)\,\psi(y)\,dy \\
&= \int_0^x \psi(y) \frac{d}{dy} \left\{ -\int_0^{x-y} k(t)\,dt \right\} dy + \int_x^1 \psi(y) \frac{d}{dy} \left\{ \int_0^{y-x} k(t)\,dt \right\} dy \\
&= \left[\left(-\int_0^{x-y} k(t)\,dt \right) \psi(y) \right]_{y=0}^{y=x} + \left[\left(\int_0^{y-x} k(t)\,dt \right) \psi(y) \right]_{y=x}^{y=1} \\
&\quad + \int_0^x \left(\int_0^{x-y} k(t)\,dt \right) \psi'(y)\,dy - \int_x^1 \left(\int_0^{y-x} k(t)\,dt \right) \psi'(y)\,dy.
\end{aligned}$$

Differentiating, we then obtain

$$(K\psi)' = \psi(0)\,k(x) - \psi(1)\,k(1 - x) + \int_0^1 k(|x - y|)\,\psi(y)\,dy. \qquad \text{Q.E.D.}$$

This result amounts to a commutation formula for the differentiation operator (D) and the integral transform K, viz.

$$DK\psi = KD\psi + \psi(0)\,k(x) - \psi(1)\,k(1 - x).$$

We now impose an additional restriction on K which will enable us to use the commutation formula to characterize the singularities in the solution of $\phi = f + \lambda K\phi$. We will assume that for some positive integer l, the composition K^l maps $L^2[0, 1]$ into $H^1[0, 1]$. (It will be shown in the next section that the integral transforms associated with $\log|x - y|$ and $|x - y|^\alpha$, $\alpha > -\frac{1}{2}$, satisfy this additional hypothesis for $l = 1$ and $l = 2$, respectively).

We will now construct functions h_n and g_n such that the nth (distributional) derivative of ϕ satisfies the integral equation

$$D^n\phi = h_n + \lambda K(D^n\phi - g_n) \tag{3a}$$

$$D^n\phi - g_n \in L^2[0, 1]. \tag{3b}$$

The "non-L^2" part of $D^n\phi$ will thus be contained in g_n.

For $n = 0$, we take $g_0 = 0$ and $h_0 = f$. We then proceed inductively to develop recursion formulas for h_{n+1} and g_{n+1}, assuming the validity of (3a) and (3b) for a given integer n. We begin by writing

$$D^n\phi - g_n = h_n - g_n + \lambda K(D^n\phi - g_n), \tag{4}$$

and note the equivalent interated form

$$D^n\phi - g_n = [I + \lambda K + \cdots + (\lambda K)^m](h_n - g_n) + (\lambda K)^{m+1}(D^n\phi - g_n), \tag{5}$$

where m is any nonnegative integer. Taking $m = l$ in (5) and differentiating, we obtain

$$\begin{aligned} D^{n+1}\phi - Dg_n &= D[I + \lambda K + \cdots + (\lambda K)^l](h_n - g_n) \\ &\quad + \lambda DK[\lambda^l K^l(D^n\phi - g_n)]. \end{aligned} \tag{6}$$

Since, by assumption, $D^n\phi - g_n \in L^2[0, 1]$ and K^l maps $L^2[0, 1]$ into $H^1[0, 1]$, we can use the commutation formula to write

$$\begin{aligned} \lambda DK[\lambda^l K^l(D^n\phi - g_n)] &= [\lambda^{l+1}K^l(D^n\phi - g_n)]_0\, k(x) \\ &\quad - [\lambda^{l+1}K^l(D^n\phi - g_n)]_1\, k(1 - x) \\ &\quad + \lambda KD[\lambda^l K^l(D^n\phi - g_n)]. \end{aligned} \tag{7}$$

Letting

$$\begin{aligned} a_n &= [\lambda^{l+1}K^l(D^n\phi - g_n)]_0\,, \\ b_n &= [\lambda^{l+1}K^l(D^n\phi - g_n)]_1\,, \end{aligned} \tag{8}$$

and using (5) with $m = l - 1$, we obtain

$$\begin{aligned} &\lambda DK[\lambda^l K^l(D^n\phi - g_n)] \\ &\quad = a_n k(x) - b_n k(1 - x) \\ &\qquad + \lambda KD[D^n\phi - g_n - [I + \lambda K + \cdots + (\lambda K)^{l-1}](h_n - g_n)]. \end{aligned} \tag{9}$$

Hence, $D^{n+1}\phi = h_{n+1} + \lambda K(D^{n+1}\phi - g_{n+1})$ where

$$\begin{aligned} g_{n+1} &= Dg_n + D[I + \lambda K + \cdots + (\lambda K)^{l-1}](h_n - g_n) \\ h_{n+1} &= Dg_n + D[I + \lambda K + \cdots + (\lambda K)^l](h_n - g_n) + a_n k(x) - b_n k(1 - x) \end{aligned} \tag{10}$$

and

$$D^{n+1}\phi - g_{n+1} \in L^2[0, 1].$$

Moreover, if we define a sequence $\{c_m(x)\}$ by

$$\begin{aligned} c_0 &= f \\ c_{m+1} &= \lambda^l DK^l c_m + a_m k(x) - b_m k(1 - x), \end{aligned} \tag{11}$$

then

$$g_{m+1} = Dg_m + D[I + \lambda K + \cdots + (\lambda K)^{l-1}]\, c_m \qquad \text{for} \quad m = 0, 1,\ldots . \tag{12}$$

Using (12) repeatedly with $g_0 = 0$, we get

$$g_n = \sum_{m=0}^{n-1} D^{n-m}[I + \lambda K + \cdots + (\lambda K)^{l-1}]\, c_m . \tag{13}$$

Finally, we have $D^n\phi = g_n +$ an $L^2[0, 1]$ function, and upon integration —with the integral operator denoted by S—we obtain

$$\phi(x) = \psi_n(x) + H^n[0, 1] \text{ function } \left(H^n[0, 1] = \left\{f(x) \,\middle|\, \int_0^1 [f^{(n)}]^2 < \infty\right\}\right) \tag{14}$$

where $\psi_n(x) = \sum_{m=0}^{n-1} S^m[I + \lambda K + \cdots + (\lambda K)^{l-1}]\, c_m(x)$.

It is, in general, impossible to ascertain a priori the values of the scalars a_n and b_n which appear in the recursion formulas for the functions $c_m(x)$, since they involve derivatives of the solution. However, it is possible to ferret out the singular (non-$H^n[0, 1]$) terms in $\psi_n(x)$ and thus characterize the solution ϕ as a linear combination of certain known singular functions plus an $H^n[0, 1]$ function. For numerical purposes, this is all that is needed to construct accurate approximations to the solution.

We collect these results in the following.

THEOREM 1. *Let* $K\psi = \int_0^1 k(|\, x - y\,|)\, \psi(y)\, dy$, *where* $k \in L^2[0, 1]$, *and let* $K^l\colon L^2[0, 1] \to H^1[0, 1]$. *Then with the sequence of functions* $\{c_m(x)\}$ *defined as in* (11), *the solution of the integral equation* $\phi = f + \lambda K\phi$ $(f \in L^2[0, 1])$ *can be written*

$$\phi(x) = \psi_n(x) + \textit{an } H^n[0, 1] \textit{ function},$$

with $\psi_n(x)$ *defined by* (14).

We have at this point obtained a prescription which can be used to characterize the singular terms which may arise in the solution of $\phi = f + \lambda K\phi$. Of course, there need not be any singularities at all in the solution of such a problem—take ϕ to be an analytic function and set $f = \phi - \lambda K\phi$, for instance. This is an anomalous situation, however, in which the values of the scalars a_n and b_n in (8) are such that ψ_n in (14) becomes an $H^n[0, 1]$ function. In a heuristic sense, one may anticipate singularities in the solution of an inhomogeneous integral equation when the eigenfunctions of its kernel exhibit singularities. This is guaranteed to be the case when the conditions of the following theorem are met.

THEOREM 2. *Let $k(x)$ be an $L^2[0, 1]$ function such that no nontrivial linear combination of $k(x)$ and $k(1 - x)$ yields an $H^1[0, 1]$ function. Also, let K: $H^1[0, 1] \to H^1[0, 1]$ and $K^l: L^2[0, 1] \to H^1[0, 1]$ for some positive integer l, where $K\psi = \int_0^1 k(|x - y|)\,\psi(y)\,dy$. Then for each eigenvalue of K, the corresponding eigenspace is a subspace of $H^1[0, 1]$ but not of $H^2[0, 1]$.*

Proof. We first show that all eigenfunctions are in $H^1[0, 1]$. From the L^2 theory of Fredholm integral equations [3], we know that the eigenfunctions —solutions of $\phi = \lambda K\phi$—are in $L^2[0, 1]$. We can also write $\phi = \lambda^l K^l \phi$, from which it follows by hypothesis that $\phi \in H^1[0, 1]$.

Now suppose that for a given eigenvalue λ of K, all associated eigenfunctions are in $H^2[0, 1]$. Let ϕ be such an eigenfunction, i.e., $\phi = \lambda K\phi$ with $\|\phi\| \neq 0$. Using Lemma 1, we obtain

$$D\phi = \lambda\phi(0)\,k(x) - \lambda\phi(1)\,k(1 - x) + \lambda KD\phi.$$

Since $KD\phi \in H^1[0, 1]$ and no nontrivial linear combination of $k(x)$ and $k(1 - x)$ can yield an $H^1[0, 1]$ function, we must have $\phi(0) = \phi(1) = 0$, in which case $D\phi$ satisfies the same homogeneous equation. A repetition of this argument leads to the conclusion that for any eigenfunction ϕ associated with λ, all derivatives of ϕ satisfy the same homogeneous equation and also vanish at $x = 0$ and $x = 1$. But λ has finite multiplicity, p say, so there exist scalars $\alpha_1, ..., \alpha_p$ such that

$$\sum_{j=1}^{p} \alpha_j D^j \phi = 0$$

The general solution of this constant coefficient ordinary differential equation consists of (exponential and/or polynomial) entire functions—hence ϕ is an entire function. But $\phi(x)$ has a zero of infinite multiplicity at $x = 0$ (and at $x = 1$ also), so $\phi(x)$ must be identically zero, in which case it cannot be an eigenfunction. From this contradiction, we conclude that for no eigenvalue λ is the corresponding eigenspace a subspace of $H^2[0, 1]$. Q.E.D.

3. Application of the Results of Section 2 to Some Specific Kernels

Here we will use the results of the previous section to characterize the singularities which appear in the solutions of $\phi(x) = f(x) + \lambda \int_0^1 k(|x - y|)\,\phi(y)\,dy$, where $k(t) = \log t$ and $k(t) = t^\alpha$ for $\alpha > -\frac{1}{2}$. We begin with the logarithmic kernel.

$$K\psi = \int_0^1 \log|x - y|\,\psi(y)\,dy. \tag{A}$$

LEMMA 2. $K: L^2[0, 1] \to H^1[0, 1]$.

Proof. Let $\psi \in L^2[0, 1]$ and define an extension $\bar{\psi} \in L^2(-\infty, \infty)$ of ψ as follows:

$$\begin{aligned} \bar{\psi}(x) &= \psi(x), \qquad x \in [0, 1] \\ &= 0, \qquad x \notin [0, 1]. \end{aligned}$$

Now let $\beta(x) = D\{\int_{-\infty}^{\infty} \log | x - y | \bar{\psi}(y)\, dy\}$. As shown in [2], $\beta(x)$ is the Hilbert transform of $\bar{\psi}$,

$$\beta(x) = \text{P.V.} \left\{ \int_{-\infty}^{\infty} \frac{\bar{\psi}(y)}{x - y}\, dy \right\},$$

and the Hilbert transform maps $L^2(-\infty, \infty)$ into itself. Hence $\beta \in L^2(-\infty, \infty)$, and since $DK\psi$ is the restriction of β to $[0, 1]$, we obtain $DK\psi \in L^2[0, 1]$ and $K\psi \in H^1[0, 1]$.

Using Theorem 1 with $l = 1$ we can therefore write the solution of $\phi = f + \lambda K\phi$ ($f \in L^2[0, 1]$) in the form

$$\phi = \sum_{m=0}^{n-1} S^m c_m + H^n[0, 1] \text{ function} \tag{15}$$

with $c_0 = f$,

$$c_{m+1} = \lambda DKc_m + a_m \log x - b_m \log(1 - x) \qquad m = 0, 1, 2, \ldots .$$

For the case in which $f \in H^n[0, 1]$, we have the following.

LEMMA 3. *Let* $f \in H^n[0, 1]$. *Then*

$$\begin{aligned} c_m(x) = {} & [\log x, \ldots, (\log x)^m][1, x, \ldots, x^{m-1}] \\ & + [\log(1 - x), \ldots, (\log(1 - x))^m][1, 1 - x, \ldots, (1 - x)^{m-1}] \\ & + H^n[0, 1] \textit{ function}. \end{aligned}$$

(For purposes of clarity, the notation $[\phi_1(x), \ldots, \phi_m(x)][\psi_1(x), \ldots, \psi_n(x)]$ is used here and subsequently to signify a linear combination of the form

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \phi_i(x)\, \psi_j(x)).$$

To establish the validity of this statement, we need the following two results:

(1) $$\begin{aligned} f \in H^n \Rightarrow DKf = {} & [\log x][1, x, \ldots, x^{n-1}] \\ & + [\log(1 - x)][1, 1 - x, \ldots, (1 - x)^{n-1}] \\ & + H^n[0, 1] \text{ function} \end{aligned}$$

$$(2)\quad \begin{aligned} DK\{(\log x)^r x^s\} &= [\log x,\dots,(\log x)^{r+1}][x^s] \\ &\quad + [\log(1-x)][(1-x)^r,\dots,(1-x)^{n-1}] \\ &\quad + H^n[0,1] \text{ function} \\ DK\{(\log(1-x))^r(1-x)^s\} &= [\log(1-x),\dots,(\log(1-x))^{r+1}][(1-x)^s] \\ &\quad + [\log x][x^r,\dots,x^{n-1}] \\ &\quad + H^n[0,1] \text{ function.} \end{aligned}$$

(1) can be obtained by writing

$$D^nKf = KD^nf + \sum_{k=0}^{n-1} D^{n-1-k}[(D^kf)_0 \log x - (D^kf)_1 \log(1-x)]$$

through repeated use of the commutation formula, and then integrating $n-1$ times. (2) can be verified by writing $K\{(\log x)^r x^s\} = I_1(x) + I_2(x)$ where $I_1(x) = \int_0^x \log(x-y)(\log y)^r y^s \, dy$, $I_2(x) = \int_x^1 \log(y-x)(\log y)^r y^s \, dy$. The variable change $y = xt$ in the first integral easily yields $I_1(x) = [x^{s+1}][\log x,\dots,(\log x)^{r+1}]$. Writing $\log(y-x)$ in the second integral as $\log y + \log(1 - x/y)$ and expanding $\log(1-x/y)$ in a Taylor series about $x/y = 0$ yields $I_2(x) = [x^{s+1}][\log x,\dots,(\log x)^{r+1}]$ + an $H^\infty[0, 1-\delta]$ function for $\delta \in (0,1)$. Finally, expansion of $y^s(\log y)^r$ in a Taylor series about $y = 1$ yields $I_2(x) = [\log(1-x)][(1-x)^{r+1},\dots,(1-x)^{n-1}]$ + an $H^\infty[\delta, 1]$ function for $\delta \in (0,1)$. The first part of (2) then follows upon differentiating $I_1(x)$ and $I_2(x)$; the second part is obtained through symmetry.

Now for $m = 0$, the assertion of the lemma is obviously true since the bracketed terms are void, while for $m = 1$, its validity follows immediately from (1). Its validity for arbitrary m follows easily by induction using (1) and (2). Q.E.D.

Upon integrating $c_m(x)$ m times and applying (15), we obtain the following.

THEOREM 2. *Let $K\psi = \int_0^1 \log|x-y|\,\psi(y)\,dy$. Then for $f \in H^n[0,1]$, the solution of the integral equation $\phi = f + \lambda K\phi$ can be written in the form*

$$\phi(x) = \left\{ \begin{array}{l} [x\log x][1,\dots,x^{n-2}] \\ \quad + [(x\log x)^2][1,\dots,x^{n-3}] \\ \quad + \\ \quad \vdots \\ \quad + [(x\log x)^{n-1}][1] \end{array} \right\} + \left\{ \begin{array}{l} [(1-x)\log(1-x)][1,\dots,(1-x)^{n-2}] \\ \quad + [((1-x)\log(1-x))^2][1,\dots,(1-x)^{n-3}] \\ \quad + \\ \quad \vdots \\ \quad + [((1-x)\log(1-x))^{n-1}][1] \end{array} \right\} + H^n[0,1] \text{ function}$$

$$K = \int_0^1 | x - y |^\alpha \psi(y)\, dy, \qquad \alpha \in (-1/2, 0). \tag{B}$$

We first prove:

LEMMA 4. *For* $s < \frac{1}{2}$, $K\colon H^s \to H^{s+1+\alpha}$.

Proof. Let $\psi \in H^s$ where $s < \frac{1}{2}$. Then define an extension $\bar{\psi} \in H^s(-\infty, \infty)$ as follows:

$$\begin{aligned} \bar{\psi}(x) &= \psi(x), \qquad & x \in [0, 1] \\ &= 0, & x \notin [0, 1]. \end{aligned}$$

Let $\beta(x) = D^{s+1+\alpha}\{\int_{-\infty}^{\infty} | x - y |^\alpha \bar{\psi}(y)\, dy\}$. With the Fourier transform defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\xi x} f(x)\, dx,$$

we obtain

$$\begin{aligned} \hat{\beta}(\xi) &= (i\xi)^{1+s+\alpha} \frac{k}{\xi^{1+\alpha}} \hat{\bar{\psi}}(\xi) \\ &= i^{1+s+\alpha} k \xi^s \bar{\psi}(\xi), \end{aligned}$$

where $k = (2/\pi)^{1/2}\, \Gamma(\alpha + 1) \cos((\alpha + 1)\pi/2)$. Since $\bar{\psi} \in H^s(-\infty, \infty)$, $\int_{-\infty}^{\infty} | \hat{\bar{\psi}}(\xi)|^2 \, | \xi |^{2s}\, d\xi < \infty$. Hence $\hat{\beta}(\xi) \in L^2(-\infty, \infty)$ and $\beta(x) \in L^2(-\infty, \infty)$. Now $D^{s+1+\alpha}K\psi$ is the restriction of $\beta(x)$ to $[0, 1]$; therefore, $D^{s+1+\alpha}K\psi \in L^2[0, 1]$ and $K\psi \in H^{s+1+\alpha}[0, 1]$.

COROLLARY. $K^2\colon L^2[0, 1] \to H^1[0, 1]$.

Proof.

$$K\colon L^2[0, 1] \to H^{1+\alpha}[0, 1] \subset H^{(1/2)-\delta}[0, 1], \qquad \delta > 0.$$

$$K\colon H^{(1/2)-\delta}[0, 1] \to H^{(3/2)+\alpha-\delta}[0, 1]$$

and for

$$0 < \delta < \alpha + \tfrac{1}{2}, \qquad H^{(3/2)+\alpha-\delta} \subset H^1[0, 1].$$

Using Theorem 1 with $l = 2$, we may thus write the solution of $\phi = f + \lambda K\phi$ ($f \in L^2[0, 1]$) in the form

$$\phi(x) = \sum_{m=0}^{n-1} S^m(I + \lambda K)\, c_m + H^n[0, 1] \text{ function.} \tag{16}$$

If we again consider the special case where $f \in H^n[0, 1]$, the following result can be established.

THEOREM 3. *Let $K\psi = \int_0^1 |x - y|^\alpha \psi(y)\,dy$ where α is an irrational number between $-\frac{1}{2}$ and 0. Then for $f \in H^n[0, 1]$, the solution of the integral equation $\phi = f + \lambda K\phi$ can be written in the form*

$$\phi(x) = \begin{Bmatrix} x^{1+\alpha}[1, x,\ldots, x^{n-2}] \\ \quad + x^{2(1+\alpha)}[1, x,\ldots, x^{n-1}] \\ \quad + \\ \quad \vdots \\ \quad + x^{(2n-1)(1+\alpha)}[1] \\ \quad + x^{2n(1+\alpha)}[1] \end{Bmatrix}$$

$$+ \begin{Bmatrix} (1-x)^{1+\alpha}\,[1, 1-x,\ldots, (1-x)^{n-2}] \\ \quad + (1-x)^{2(1+\alpha)}\,[1, 1-x,\ldots, (1-x)^{n-1}] \\ \quad + \\ \quad \vdots \\ \quad + (1-x)^{(2n-1)(1+\alpha)}[1] \\ \quad + (1-x)^{2n(1+\alpha)}\,[1] \end{Bmatrix}$$

$$+ H^n[0, 1] \text{ function}$$

This may be proved by first showing that

$$\begin{aligned} c_m(x) &= [1, x^{1+2\alpha},\ldots, x^{(m-1)(1+2\alpha)}][x^\alpha, x^{1+2\alpha}] \\ &\quad + [1, (1-x)^{1+2\alpha},\ldots, (1-x)^{(m-1)(1+2\alpha)}][(1-x), (1-x)^{1+2\alpha}] \\ &\quad + H^n[0, 1] \text{ function} \end{aligned}$$

and then applying (16). The details are entirely analagous to those used in the proofs of Lemma 4 and Theorem 2, and will be omitted for the sake of brevity. For irrational α the exponents $j(1 + \alpha)$ in the statement of the theorem become integers at regular intervals, and the foregoing result requires modification. The correct modification is to include $(\log x)^p$ as a factor the pth time an integer exponent occurs, while leaving unaltered the terms which do not involve integer exponents.

The mathematical details of this section are, to be sure, somewhat oppressive; however, there is no known alternative way to obtain these results. Fortunately, for numerical purposes, one typically needs only the singular terms in the first few derivatives of the solution, in which case some of the more arduous calculations can be circumvented.

As an illustration of the use of these results, consider the problem of approximating, via Galerkin's method, the eigenfunctions of the logarithmic kernel

$$\phi(x) = \lambda \int_0^1 \log | x - y | \phi(y) \, dy.$$

One immediately encounters the problem of choosing a finite dimensional subspace of $L^2[0, 1]$ in which the eigenfunctions can be approximated accurately. It follows from Theorem 2 that all eigenfunctions can be expressed as a linear combination of $x \log x$, $(1 - x) \log(1 - x)$ and an $H^2[0, 1]$ function. Moreover, any H^2 function can be approximated to within $O(h^2)$ by a piecewise linear polynomial over a uniform mesh of width h[1]. Hence if the approximating subspace is chosen to consist of piecewise linear basis functions, and includes, in addition, the two non-H^2 functions $x \log x$ and $(1 - x) \log(1 - x)$, the existence of an $O(h^2)$ approximation to the solution is guaranteed. Without the inclusion of the two singular functions in the basis, the full $O(h^2)$ approximating capability of the piecewise linear polynomials will not be realized.

5. Extension to L^1 Kernels of Displacement Type

We now very briefly consider integral equations $\phi = f + \lambda K\phi$ where

$$K\psi = \int_0^1 k(| x - y |) \psi(y) \, dy$$

with $k \in L^1[0, 1]$. A procedure similar to that already developed for $L^2[0, 1]$ kernels can be used to characterize the behavior of the solution, provided we make the assumption that for some positive integer l, K^l: $H^{-1/2}[0, 1] \to H^{1/2+\delta}[0, 1]$ for some $\delta > 0$. The following amended versions of Lemma 1 and Theorem 1 are easily established:

Lemma 1′. *Let $\psi \in H^s$ for some $s > \frac{1}{2}$ (ψ is thus continuous) and let $k \in L^1[0, 1]$. Then*

$$DK\psi = KD\psi + \psi(0) k(x) - \psi(1) k(1 - x).$$

Theorem 1′. *Let $K\psi = \int_0^1 k(| x - y |) \psi(y) \, dy$, where $k \in L^1[0, 1]$, and let K^l: $H^{-1/2}[0, 1]$ into $H^{1/2+\delta}[0, 1]$ for some $\delta > 0$. Then with the sequence of functions $\{c_m(x)\}$ as defined in* (12), *the solution of the integral equation $\phi = f + \lambda K\phi$ ($f \in L^2[0, 1]$) can be written*

$$\phi(x) = \psi_n(x) + \textit{an } H^{n-(1/2)}[0, 1] \textit{ function},$$

with $\psi_n(x)$ defined by (14). *If these results are applied to $\phi = f + \lambda K\phi$ where $k(t) = t^\alpha(\alpha > -1)$ and $f \in H^{n-(1/2)}$, it can be shown that ϕ has the form:*

$$\phi(x) = \left\{ \begin{array}{l} x^{1+\alpha}[1,\ldots, x^n] \\ \quad + x^{2(1+\alpha)}[1,\ldots, x^n] \\ \quad + \\ \quad \vdots \\ \quad + x^{nl(1+\alpha)}[1,\ldots, x^n] \end{array} \right\}$$

$$+ \left\{ \begin{array}{l} +(1-x)^{1+\alpha}\,[1,\ldots,(1-x)^n] \\ \quad + (1-x)^{2(1+\alpha)}\,[1,\ldots,(1-x)^n] \\ \quad + \\ \quad \vdots \\ \quad + (1-x)^{nl(1+\alpha)}\,[1,\ldots,(1-x)^n] \end{array} \right\}$$

$$+ H^{n-\frac{1}{2}}[0, 1] \text{ function.}$$

For rational α, those of the above functions which have integer exponents must be modified by logarithmic factors in the manner previously described.

References

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