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NON-LINEAR VIBRATIONS OF THREE-LAYER BEAMS WITH VISCOELASTIC CORES I. THEORY†

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Approximate equations of motion are developed for large amplitude motions of three-layer axially restrained unsymmetrical beams with viscoelastic cores. The external force consists of a constant plus an oscillatory term. The combination of this form of forcing and the large amplitude motions cause the beam to respond at multiples of the forcing frequency. This can lead to difficulties in the complex modulus approach to viscoelasticity. These are overcome here through use of hereditary integrals and their relationships with complex moduli. Theoretical results on the frequency response of clamped, symmetrical beams are compared with earlier experimental work. On the whole, reasonable agreement is found.

1. INTRODUCTION

Considerable theoretical and experimental work on the linear vibration of sandwich structures with elastic components has been done over the years (see the paper by Habip [1] for a thorough survey of progress made prior to 1965). A major contributor was Yu [2–4]. He developed a general one-dimensional theory for plates, a theory which incorporated the effects of transverse shear deformations and rotatory inertia in both core and face sheets. He also presented a simplified version for two-dimensional motions of sandwich plates with identical face sheets and soft cores. Krajcinovic [5, 6] also presented results on symmetric, three-layer elastic beams using an approach similar to Yu's. However, his approach was unique in that he chose orthogonal displacement functions for the beam as a whole, as a result of which several governing equations were uncoupled. Recently Folie [7] did further work in this area. He developed a theory for the transverse bending of three-layer plates with isotropic outer layers and an orthotropic core. Chan and Cheung [8] numerically solved problems of bending and vibration of multi-layered plates by what is termed the finite strip method. Ahmed [9] used a finite-element method to obtain information on the free vibrations of curved, sandwich beams. Recently, Krishna Murty and Shimpi [10] developed a theory for laminated beams which includes the effects of bending and shear, rotatory and longitudinal inertia, in

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all laminates. Of course, there is now a large body of literature concerning theories for elastic sandwich structures with a large number of components (see the paper by Ben-Amoz [11] for more detail).

Considerable interest has developed in using viscoelastic materials in sandwich configurations, both constrained and unconstrained, as means of reducing vibration levels (see, e.g., the paper by Plunkett [12]). DiTaranto [13] developed linear equations for vibrations of layered beams of finite length, loss factors and natural frequencies being computed by DiTaranto and Blasingame [14]. Jones, Salerno and Savacchio [15] did an analytical and experimental evaluation of damping for such beams, the agreement between theory and experiment being at best fair. Yin, Kelly and Barry [16] presented an experimental study of the effects of constrained layer damping on plates, beams, tubes, and other structural members. Mead and Markus [17, 18] and Mead and DiTaranto [19] gave results on loss factors and resonant frequencies for three-layer sandwich beams. Lu and Douglas [20], in experiments on mechanical impedance, got excellent agreement with results obtained by using the analytical formulation of Mead and Markus [17].

Most of these investigations revealed that core materials made from commonly available viscoelastic materials damped vibration only over a limited frequency range. Grootenhuis [21], Agbasiere and Grootenhuis [22], and Nakra and Grootenhuis [23] looked at sandwiches involving several viscoelastic layers, each one having its peak damping in a different frequency range. Grootenhuis [24] showed that certain unsymmetrical sandwich structures provided more effective damping over a wider frequency range as compared to their symmetric counterparts.

A common feature of the theories involved in the above works was to assume a displacement field for each layer. Governing equations were then arrived at by looking at the equilibrium of a beam element, together with the layer materials constitutive laws and the interface conditions. Recently, Yan and Dowell [25, 26] used the principle of virtual work to arrive at general linear equations for the dynamics of three-layer sandwich plates. Restricting attention to soft cores and ignoring all inertias except the transverse one, they derived dispersion relations which agreed quite well with the full equations as well as with the results given by Mead, Markus and DiTaranto. Yan and Dowell also found good agreement between experiments they performed and their theory for beams.

Work related to the above, that the present authors are aware of, should be cited for completeness: Asnani and Nakra [27], Braunisch [28], Chandrasekharan and Ghosh [29], DiTaranto and McGraw [30], Emerson [31], Jones and Parin [32], O. Markus and S. Markus [33], Nakra [34], Sadasiva Rao [35], and Torvik and Strickland [36].

Even though considerable work has been done on the non-linear dynamics of continuous media (for example, that of Ho, Scott and Eisley [37]), relatively little has been in the area of sandwich structures. Reissner [38], in a study on the static deflection of plates with membrane face sheets, showed that, for core materials with elastic moduli the same order of magnitude as those for the face material, linear theory is adequate provided that the transverse deflections are small compared to the total plate thickness. However, as the core becomes softer and softer, the range of linear behavior decreases and geometric non-linearities become important. The same behavior is anticipated for the dynamic response of sandwich structures with membrane face layers as well as for structures with face layers too thick to be considered membranes. Yu [39, 40] extended his linear analysis of flexural vibrations of elastic sandwich plates with thin face sheets to include geometric non-linearities. Wempner and Baylor [41] also derived equations for the large amplitude motions of elastic sandwich plates with weak cores. Habip [42] used perturbation methods to derive equations for the static deflection of two-layer plates. The first-order terms are Von Karman equations, whereas the second-order terms reflect shear effects. Bert [43] has also contributed to the area.

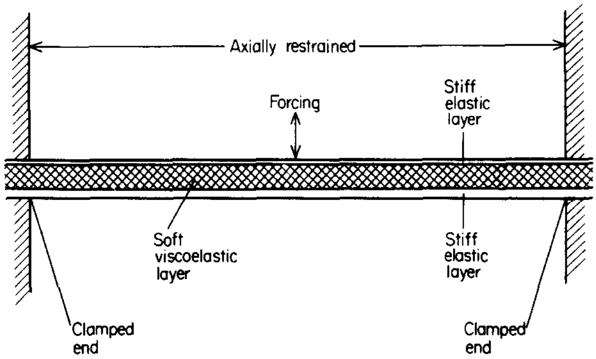

Figure 1. Beam configuration.

In one of the few studies which included experimental results, and the major motivation for the present study, Kovac, Anderson and Scott [44] investigated large amplitude vibrations of harmonically forced symmetric sandwich beams with viscoelastic cores and identical thin face sheets. The ends of the beam were restrained from moving towards each other, and so the transverse vibration induced axial stretching. The face sheets were considered as elastic membranes while the constitutive properties of the core were represented by complex moduli evaluated at the forcing frequency.

In general, agreement between theory and experiment was reasonable. However, the experiment seemed to reveal a superharmonic response when the beam was forced in the vicinity of one-half its linear natural frequency. In this region, the experimental frequency response, as well as the mode shape, deviated considerably from the theoretical predictions. Figure 1 shows the beam configuration and Figure 2 shows results of the investigation.

This paper is the first of two papers summarizing the work directed at explaining the deviation. The prime concern was to determine whether the superharmonic was a structural effect or whether it was due to something in the experiment not accounted for in the theory. Experimentally [44], the beam was forced with an electromagnetic device. When alternating current is passed through an electromagnet, the force felt by a ferromagnetic object is actually a constant bias force plus an alternating force. This bias force effect was not accounted for in the theory. Bennett and Eisley [45], in similar work on homogeneous beams, showed the effect to be negligible, while Meirovitch [46] showed considerable superharmonic could exist

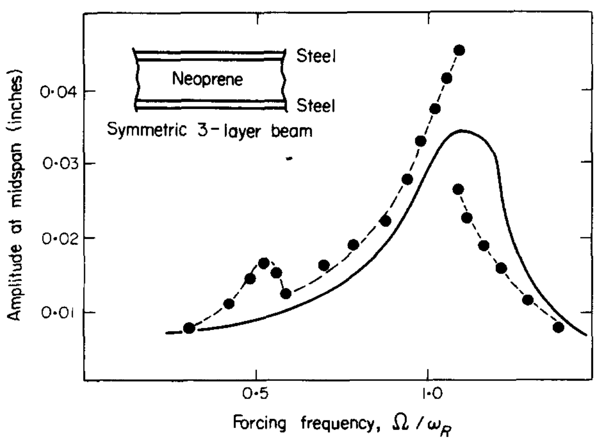

Figure 2. Comparison between theory and experiment from reference [44]. ——, Theory; --●--, experiment; ω_R = linear resonant frequency.

when forcing a cubic hardening system with a constant plus a harmonic force. Of equal importance was the fact that since the superharmonic response did exist, using complex moduli evaluated at the forcing frequency to represent the core material properties was not proper. Thus, in a first attempt to explain the deviation, the response of a three-layer sandwich beam, with a viscoelastic core, to a constant bias force plus a harmonic force was obtained. A summary of that work is presented here. The inclusion of a bias force did not fully account for the superharmonic behavior reported in reference [44] and so it was necessary to conduct a thorough examination of the experimental procedures used in that work and similar investigations. It was found that deficiencies in the experimental study of superharmonic response needed to be corrected. That examination will be presented in a companion paper.

In the present paper the equations governing the non-linear vibrations of a three-layer beam with a viscoelastic core are derived. The equations are valid for beams with dissimilar face layers and are not restricted to beams with membrane face layers. The constitutive properties of the viscoelastic core are treated by using relations between the hereditary integrals and the complex moduli. An approximate solution scheme is presented and numerical results are obtained and compared with the experimental results of reference [44].

2. DEVELOPMENT OF THE EQUATIONS OF MOTION

The equations of motion are derived by using the principle of virtual work. By using a summation convention, the principle of virtual work can be stated as

$$\int_V \left[\rho F_j + \frac{\partial}{\partial x_k}[S_{km}(\delta_{jm} + e_{jm} + \omega_{jm})]\right]\delta u_j \,\mathrm{d}V + \\ + \int_S [T_j - S_{km}(\delta_{jm} + e_{jm} + \omega_{jm})\nu_k]\,\delta u_j \,\mathrm{d}S = 0, \tag{1}$$

where V and S denote the volume and surface of the material in the undeformed state, ρ is the density, F_j the body force per unit mass, S_{km} are Kirchhoff stress components (forces per unit undeformed area), T_j are surface tractions (per unit undeformed area), δ_{jm} stands for the Kronecker delta, δu_j denotes virtual displacements, ν_k are the direction cosines of the unit outward normal to S, and

$$e_{jm} = \tfrac{1}{2}(\partial u_j/\partial x_m + \partial u_m/\partial x_j), \tag{2}$$

$$\omega_{jm} = \tfrac{1}{2}(\partial u_j/\partial x_m - \partial u_m/\partial x_j), \tag{3}$$

where x_j are a set of Cartesian co-ordinates designating a point in the undeformed body. It should be noted at the outset that the theory under development is not aimed at very large deflections. The theory is geared towards deflections of the order of the beam thickness and consequently there will be no distinction between volumes and surfaces before and after deformation. Non-linearity enters because of geometric constraints and will make its presence felt by using Green's non-linear strain measures [47]:

$$\varepsilon_{rs} = e_{rs} + \tfrac{1}{2}(e_{kr} + \omega_{kr})(e_{ks} + \omega_{ks}). \tag{4}$$

Based on experience with homogeneous beam and plate theory, and experience of others, such as Yu and Yan and Dowell, in sandwich constructions, it was felt that a viable theory would be obtained based on the following assumptions: (i) referring to Figure 3, and switching permanently to xyz-notation, one takes all physical quantities to be independent of y; (ii) the normal strains in the thickness direction are negligible; (iii) the normal stresses in the thickness directions are small compared to other stresses; (iv) in equation (4), e_{ij} can be

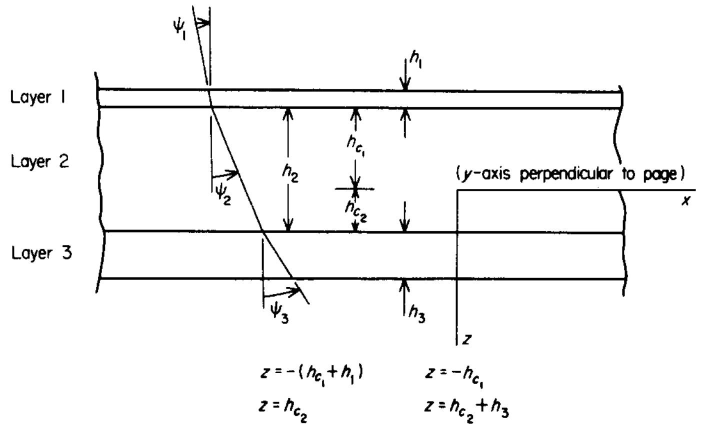

Figure 3. Beam geometry—origin of co-ordinates at mid-span.

dropped compared to ω_{ij} and both can be neglected compared to unity; (v) in each layer, plane sections are taken to remain plane. In a layer, the in-plane displacements are due to extension, shear and bending. The extensional displacement is assumed uniform across the thickness, while the shear and bending are assumed to give rise to a linear variation. In keeping with the assumption of zero normal strain, the transverse displacement is assumed to be the same for each layer. In summary:

$$u_i(x,z,t)=u_i^0(x,t)+z\psi_i(x,t), \qquad i=1,2,3, \tag{5}$$

$$w_i(x,z,t)=w(x,t), \qquad i=1,2,3, \tag{6}$$

where u_i^0 is the extensional in-plane displacement, ψ_i the shear and bending angle (see Figure 3), w the transverse displacement and the index i refers to the layer in question (it will be reserved henceforth for that purpose throughout this work).

Upon using the above-listed assumptions and equations (5) and (6), the principle of virtual work gives, on integrating w.r.t. z,

$$\sum_{i=1}^{3}\int_{-L/2}^{L/2}\left\{\left[-\rho_i(h_i\ddot{u}_i^0+A_{i0}\ddot{\psi}_i)+\frac{\partial}{\partial x}(N_{xxi}+Q_{xzi}\,\omega_{xzi})+(S_{zxi}+S_{zzi}\,\omega_{xzi})|_{z_{i-1}}^{z_i}\right]\delta u_i^0+\right.$$
$$+\left[-\rho_i(A_{i0}\ddot{u}_i^0+I_{i0}\ddot{\psi}_i)+\frac{\partial}{\partial x}(M_{xxi}+M_{xzi}\,\omega_{xzi})-Q_{xzi}+(z(S_{zxi}+S_{zzi}\,\omega_{xzi}))|_{z_{i-1}}^{z_i}\right]\delta\psi_i+$$
$$\left.+\left[-\rho_i h_i\ddot{w}_i+\frac{\partial}{\partial x}(N_{xxi}\,\omega_{zxi}+Q_{xzi})+(S_{zxi}\,\omega_{zxi}+S_{zzi})|_{z_{i-1}}^{z_i}\right]\delta w\right\}\mathrm{d}x+\text{boundary term}=0, \tag{7}$$

where the moment coefficients, h_i, A_{i0} and I_{i0}, and the stress resultants are given by

$$h_i=\int_{z_{i-1}}^{z_i}\mathrm{d}z, \qquad A_{i0}=\int_{z_{i-1}}^{z_i}z\,\mathrm{d}z, \qquad I_{i0}=\int_{z_{i-1}}^{z_i}z^2\,\mathrm{d}z,$$

$$N_{xxi}=\int_{z_{i-1}}^{z_i}S_{xxi}\,\mathrm{d}z, \qquad M_{xxi}=\int_{z_{i-1}}^{z_i}zS_{xxi}\,\mathrm{d}z, \qquad Q_{xzi}=\int_{z_{i-1}}^{z_i}S_{xzi}\,\mathrm{d}z,$$

$$M_{xzi}=\int_{z_{i-1}}^{z_i}zS_{xzi}\,\mathrm{d}z. \tag{8}$$

In the interests of brevity the boundary term in equation (7) has not been written out explicitly since in the ultimate application it will vanish. The term is written out in detail in reference [48].

Upon noting that, at this point, each layer in the sandwich structure is being treated separately and thus δu_i^0, $\delta\psi_i$ and δw are independent, equation (7) yields

$$\sum_{i=1}^{3}\left\{\frac{\partial}{\partial x}(N_{xxi}+Q_{xzi}\,\omega_{xzi})+(S_{zxi}+S_{zzi}\,\omega_{xzi})|_{z_{i-1}}^{z_i}-\rho_i(h_i\,\ddot{u}_i^0+A_{i0}\,\ddot{\psi}_i)\right\}=0, \quad (9)$$

$$\sum_{i=1}^{3}\left\{\frac{\partial}{\partial x}(M_{xxi}+M_{xzi}\,\omega_{xzi})+(z(S_{zxi}+S_{zzi}\,\omega_{xzi}))|_{z_{i-1}}^{z_i}-Q_{xzi}-\rho_i(A_{i0}\,\ddot{u}_i^0+I_{i0}\,\ddot{\psi}_i)\right\}=0, \quad (10)$$

$$\sum_{i=1}^{3}\left\{\frac{\partial}{\partial x}(N_{xxi}\,\omega_{zxi}+Q_{xzi})+(S_{zxi}\,\omega_{zxi}+S_{zzi})|_{z_{i-1}}^{z_i}-\rho_i\,h_i\,\ddot{w}\right\}=0. \quad (11)$$

However, due to interface conditions, equations (9), (10) and (11) are not all independent. Upon specializing to perfect bonds, displacement continuity requires

$$u_1(x,z_1,t)=u_2(x,z_1,t), \quad (12)$$

$$u_2(x,z_2,t)=u_3(x,z_2,t). \quad (13)$$

As S_{zzi} has been taken to be small, equations (12) and (13) require continuity of the shear stress at the interfaces (as can be shown from the boundary term):

$$S_{xz1}(x,z_1,t)=S_{zx2}(x,z_1,t), \quad (14)$$

$$S_{zx2}(s,z_2,t)=S_{zx3}(x,z_2,t). \quad (15)$$

Upon eliminating the interface terms by means of equations (12), (13), (14) and (15), equations (9), (10) and (11) yield

$$\frac{\partial}{\partial x}(N_{xx1}+N_{xx2}+N_{xx3}+Q_{xz1}\,\omega_{xz1}+Q_{xz2}\,\omega_{xz2}+Q_{xz3}\,\omega_{xz3})+(S_{zx3}+S_{zz3}\,\omega_{xz3})_{z_0}-$$
$$-(S_{zx1}+S_{zz1}\,\omega_{xz1})_{z_0}-\rho_1(h_1\,\ddot{u}_1^0+A_{10}\,\ddot{\psi}_1)-\rho_2(h_2\,\ddot{u}_2^0+A_{20}\,\ddot{\psi}_2)$$
$$-\rho_3(h_3\,\ddot{u}_3^0+A_{30}\,\ddot{\psi}_3)=0, \quad (16)$$

$$\frac{\partial}{\partial x}(M_{xx1}+M_{xx2}+M_{xx3}+M_{xz1}\,\omega_{xz1}+M_{xz2}\,\omega_{xz2}+M_{xz3}\,\omega_{xz3})+[z(S_{zx3}+S_{zz3}\,\omega_{xz3})]\,z_3-$$
$$-[z(S_{zx1}+S_{zz1}\,\omega_{xz1})]\,z_0-Q_{xz1}-Q_{zx2}-Q_{xz3}-\rho_1(A_{10}\,\ddot{u}_1^0+I_{10}\,\ddot{\psi}_1)$$
$$-\rho_2(A_{20}\,\ddot{u}_2^0+I_{20}\,\ddot{\psi}_2)-\rho_3(A_{30}\,\ddot{u}_3^0+I_{30}\,\ddot{\psi}_3)=0, \quad (17)$$

$$\frac{\partial}{\partial x}(N_{xx1}\,\omega_{xz1}+N_{xx2}\,\omega_{zx2}+N_{xx3}\,\omega_{zx3}+Q_{xz1}+Q_{xz2}+Q_{xz3})+(S_{zx3}\,\omega_{zx3}+S_{zz3})\,z_3-$$
$$-(S_{zx1}\,\omega_{zx1}+S_{zz1})\,z_0-(\rho_1\,h_1+\rho_2\,h_2+\rho_3\,h_3)\,\ddot{w}=0, \quad (18)$$

where the terms $(S_{zx3}+S_{zz3}\,\omega_{xz3})\,z_3$, etc., can be related to the applied tractions T_{x3}, etc.

For the beams that are the subject of the ultimate study, some further simplifications can be made in equations (16), (17) and (18). Attention will be confined to soft cores and S_{xx2} (and consequently M_{xx2}, N_{xx2}) is taken to be negligible. The loadings will be such that transverse motion dominates and rotatory and in-plane inertia will be deleted. Finally, since the work focuses on axially restrained beams, the in-plane stress resultants N_{xx1}, etc., are large, compared to the shear forces. Consequently, terms such as $Q_{xz1}\,\partial\omega_{xz1}/\partial x$ and $\omega_{xz1}\,\partial Q_{xz1}/\partial x$, termed shear curvature and shear buoyancy, by Chu and Herrmann [49], will be deleted in the sequel.

At this stage it is appropriate to consider the constitutive laws for the layer materials. With the materials in the faces taken to be linear, isotropic, elastic solids, Hooke's law gives, for the only non-vanishing stress components,

$$S_{xxi} = E_i \varepsilon_{xxi}, \qquad i = 1, 3, \tag{19}$$

$$S_{xzi} = 2G_i \varepsilon_{xzi}, \qquad i = 1, 3, \tag{20}$$

where E_i and G_i stand for the Young's modulus and shear modulus, respectively. With linear viscoelastic behavior assumed, the constitutive law for the core is given by a hereditary integral

$$S_{xz2} = 2 \int_{-\infty}^{t} Y_2(t-\tau)\, \mathrm{d}\varepsilon_{xz2}(\tau), \tag{21}$$

where Y_2 is the shear relaxation function of the material.

Some further approximations can be meaningfully made at this step. Upon using the strain measures, equations (2), (3) and (4), the assumed displacement field, equations (5) and (6), and the constitutive laws, equations (19), (20) and (21), the interface conditions represent four relations among the seven unknown displacements u_i^0, ψ_i and w. The seven displacements involve kinematic terms such as

$$(\psi_2 + \partial w/\partial x) - (1/G_3) \int_{-\infty}^{t} Y_2(t-\tau)\, \mathrm{d}(\psi_2 + \partial w/\partial x). \tag{22}$$

If the core were elastic, item (22) would be

$$(\psi_2 + \partial w/\partial x) - (G_2/G_3)(\psi_2 + \partial w/\partial x), \tag{23}$$

where G_2 is the shear modulus of the core. For soft cores $G_2/G_3 \ll 1$ and the second term in item (23) can be deleted. By analogy, for the viscoelastic materials considered here, the following kinematic approximation to item (22) is justified:

$$(\psi_2 + \partial w/\partial x) - (1/G_3) \int_{-\infty}^{t} Y_2(t-\tau)\, \mathrm{d}(\psi_2 + \partial w/\partial x) = (\psi_2 + \partial w/\partial x). \tag{24}$$

Upon using this and similar approximations (more detail can be found in reference [48]) the displacement fields become

$$u_1 = u + z_1 \psi + (z_1 - z)\, \partial w/\partial x, \tag{25}$$

$$u_2 = u + z\psi, \tag{26}$$

$$u_3 = u + z_2 \psi + (z_2 - z)\, \partial w/\partial x, \tag{27}$$

where $u \equiv u_2^0$ and $\Psi \equiv \Psi_2$. Also the approximations lead to

$$\psi_1 = -\partial w/\partial x = \psi_3. \tag{28}$$

Upon using equations (24) through (28), the strain components can be calculated from equations (2), (3) and (4). Then the stress resultants can be determined from equations (8). Substituting their values into the differential equations (16), (17) and (18), and choosing the origin of z to be such that

$$E_3 h_3 h_{c2} = E_1 h_1 h_{c1}, \tag{29}$$

one obtains

$$\frac{\partial}{\partial x}\left\{(E_3 h_3 + E_1 h_1)\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - \left[E_3 h_3 h_{c2}\left(1 + \frac{h_3}{2h_{c2}}\right) - E_1 h_1 h_{c1}\left(1 + \frac{h_1}{2h_{c1}}\right)\right]\frac{\partial^2 w}{\partial x^2}\right\} = 0, \tag{30}$$

$$\frac{\partial}{\partial x}\left\{\left[E_3 h_3 h_{c2}\left(1 + \frac{h_3}{2h_{c2}}\right) - E_1 h_1 h_{c1}\left(1 + \frac{h_1}{2h_{c1}}\right)\right]\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w^2}{\partial x}\right)\right] + \right.$$
$$+\left[E_3 h_3 h_{c2}^2\left(1 + \frac{h_3}{2h_{c2}}\right) + E_1 h_1 h_{c1}^2\left(1 + \frac{h_1}{2h_{c1}}\right)\right]\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) -$$
$$\left. -\left[E_3 h_3 h_{c2}^2\left(1 + \frac{h_3}{h_{c2}} + \frac{h_3^2}{3h_{c2}^2}\right) + E_1 h_1 h_{c1}^2\left(1 + \frac{h_1}{h_{c1}} + \frac{h_1^2}{3h_{c1}^2}\right)\right]\frac{\partial^2 w}{\partial x^2}\right\} -$$
$$-h_2\left(1 + \frac{h_1}{h_2} + \frac{h_3}{h_1}\right)\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}\left(\psi + \frac{\partial w}{\partial x}\right) = 0, \tag{31}$$

$$\frac{\partial}{\partial x}\left\{(E_3 h_3 + E_1 h_1)\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - \left[E_3 h_3 h_{c2}\left(1 + \frac{h_3}{2h_{c2}}\right) - \right.\right.$$
$$\left.\left. - E_1 h_1 h_{c1}\left(1 + \frac{h_1}{2h_{c1}}\right)\right]\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x}\right\} + h_2\left(1 + \frac{h_1}{h_2} + \frac{h_3}{h_2}\right)\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}\left(\psi + \frac{\partial w}{\partial x}\right) +$$
$$+ q(x,t) = (\rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3)\,\ddot{w}, \tag{32}$$

where q is the transverse load per unit length.

Integrating equation (30) twice with respect to x and considering a fixed end beam with axial restraints, i.e.,

$$u\left(\frac{L}{2}, t\right) = 0 = u\left(-\frac{L}{2}, t\right) = \frac{\partial w}{\partial x}\left(\frac{L}{2}, t\right) = \frac{\partial w}{\partial x}\left(-\frac{L}{2}, t\right), \tag{33}$$

one obtains, after some manipulation

$$\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = [(E_3 h_3^2 - E_1 h_1^2)/2(E_1 h_1 + E_3 h_3)]\frac{\partial^2 w}{\partial x^2} + \frac{1}{2L}\int_{-L/2}^{L/2}\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x. \tag{34}$$

A rather interesting feature can be seen in equation (34). For unsymmetric beams it is seen that the in-plane displacement is coupled to the out-of-plane displacement through first-order terms. However, for symmetric beams, i.e., $E_1 = E_3$, $h_1 = h_3$, then the first term on the right side of equation (34) drops and coupling is through a second-order effect.

Upon using equation (34), the differential equations (31) and (32) become

$$\frac{\partial}{\partial \chi}\left[K_{11}\frac{\partial^2 \bar{w}}{\partial \chi^2} + K_{12}\left(\frac{\partial \bar{\psi}}{\partial \chi} + \frac{\partial^2 \bar{w}}{\partial \chi^2}\right) - K_{13}\frac{\partial^2 \bar{w}}{\partial \chi^2}\right] - h_2\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}\left(\bar{\psi} + \frac{\partial \bar{w}}{\partial \chi}\right) = 0, \tag{35}$$

$$K_{15}\left\{\int_{-1/2}^{1/2}\left(\frac{\partial \bar{w}}{\partial \chi}\right)^2 \mathrm{d}\chi\right\}\frac{\partial \bar{w}}{\partial \chi} + \bar{h}_2\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}\left(\frac{\partial \bar{\psi}}{\partial \chi} + \frac{\partial^2 \bar{w}}{\partial \chi^2}\right) + \bar{q}(\chi, t) - m\ddot{\bar{w}} = 0, \tag{36}$$

where

$$h_T = h_1 + h_2 + h_3, \qquad \bar{w} = w/h_T, \qquad \bar{\psi} = \psi/h_T, \qquad \chi = x/L,$$

$$K_{11} = \frac{h_T^4}{L^4}\left[E_3\left(\frac{h_3}{h_T}\right)^2 - E_1\left(\frac{h_1}{h_T}\right)^2\right]^2\bigg/\left(4E_1\frac{h_1}{h_T} + 4E_3\frac{h_3}{h_T}\right),$$

$$K_{12} = \left[E_3\frac{h_3}{h_T}\frac{h_{c2}^2}{h_T^2}\left(1 + \frac{h_3}{2h_{c2}}\right) + E_1\frac{h_1}{h_T}\frac{h_{c1}^2}{h_T^2}\left(1 + \frac{h_1}{2h_{c1}}\right)\right]\left(\frac{h_T}{L}\right)^4,$$

$$K_{13} = \left[E_3\frac{h_3}{h_T}\frac{h_{c2}^2}{h_T^2}\left(1 + \frac{h_3}{h_{c2}} + \frac{h_3^2}{3h_{c2}^2}\right) + E_1\frac{h_1}{h_T}\frac{h_{c1}^2}{h_T^2}\left(1 + \frac{h_1}{h_{c1}} + \frac{h_1^2}{3h_{c1}^2}\right)\right]\left(\frac{h_T}{L}\right)^4,$$

$$K_{15} = \left[E_1\frac{h_1}{2h_T} + E_3\frac{h_3}{2h_T}\right]\left(\frac{h_T}{L}\right)^4,$$

$$K_{16} = \left[E_1\frac{h_1^2}{h_T^2} - E_3\frac{h_3^2}{2h_T^2}\right]\left(\frac{h_T}{L}\right)^4,$$

$$m = (\rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3)h_T, \quad \bar{h}_2 = (h_T/L)^2, \quad q(\chi L, t) = \bar{q}(\chi, t).$$

Before closing this section, some remarks should be made on the possibility of making further simplifications by taking the face sheets to be very thin. The condition for locating the co-ordinate axis is equation (29). It, together with the relation $h_2 = h_{c1} + h_{c2}$, leads to

$$h_{c1} = [E_3 h_3/(E_1 h_1 + E_3 h_3)]\,h_2, \tag{37}$$

$$h_{c2} = [E_1 h_1/(E_1 h_1 + E_3 h_3)]\,h_2. \tag{38}$$

By thin face sheets is meant

$$h_1/h_{c1} \ll 1, \qquad h_3/h_{c2} \ll 1. \tag{39}$$

Upon calculating these ratios by using equations (37) and (38), it is seen that it is difficult to meet both the inequalities in equation (19) if E_1 and E_3 differ by even a factor of two or three. Hence, no thin face sheet assumptions are used here.

3. FREQUENCY RESPONSE

In this section, attention will be directed towards obtaining information on the response of the sandwich beam to a force consisting of a constant and a harmonic term. Of necessity, approximate means will be used. The spatial dependence of the integro-partial differential equations (35) and (36) will be eliminated by employing Galerkin's method. Then, the time aspect of the problem will be handled by means of the method of harmonic balance.

The transverse load, $\bar{q}$, is now assumed to be of the form

$$\bar{q}(\chi, t) = a(\chi)(F_0 + F_1 \sin \Omega t), \tag{40}$$

where

$$a(\chi) = \begin{cases} 0, & -1/2 < \chi < -1/40 \\ 20, & -1/40 < \chi < 1/40 \\ 0, & 1/40 < \chi < 1/2, \end{cases} \tag{41}$$

F_0, F_1 are constants, and Ω denotes the driving frequency. The form of a given by equation (41) is meant to simulate the spatial dependence of the electromagnetic force on the beam. On the assumption that steady-state conditions exist, the response is taken as

$$\bar{w}(X, t) = W(X)\,T_1(t), \tag{42}$$

$$\bar{\psi}(X, t) = \Psi(X)\,T_2(t). \tag{43}$$

In the Kovac, Anderson and Scott work [44], as well as in the experimental work of the present authors, the beams were clamped at both ends. Also, they were forced at their center, so that $\bar{w}$ and $\bar{\psi}$ are even and odd functions of χ, respectively. With these items borne in mind, the following Galerkin functions, which are approximations to the first linear mode shape, are used:

$$W(X) = 16(X^2 - 1/4)^2, \tag{44}$$

$$\Psi(X) = 12\sqrt{3}X(X^2 - 1/4). \tag{45}$$

It should be noted that with this choice, the boundary term mentioned in connection with equation (7) is zero.

Upon using equations (40) through (45), Galerkin's method applied to equations (35) and (36) yields

$$(\bar{K}_{11} + \bar{\bar{K}}_{12} - \bar{K}_{13})T_1(t) + \bar{K}_{12}T_2(t) - H_2\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}T_2(\tau) - \bar{H}_2\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}T_1(\tau) = 0, \tag{46}$$

$$(\bar{K}_{15} - M)T_1(t) + H_3\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}T_2(\tau) + H_3\int_{-\infty}^{t} Y_2(t-\tau)\,\mathrm{d}T_1(\tau) + \bar{F}_0 + \bar{F}_1\sin\Omega t = 0, \tag{47}$$

where

$$\bar{K}_{11} = -64{\cdot}5K_{11}, \qquad \bar{K}_{12} = -21{\cdot}7K_{12},$$
$$\bar{\bar{K}}_{12} = -66{\cdot}5K_{12}, \qquad \bar{K}_{13} = -66{\cdot}5K_{13},$$
$$H_2 = 0{\cdot}514h_2, \qquad \bar{H}_2 = 1{\cdot}58h_2,$$
$$\bar{K}_{15} = -23{\cdot}8K_{15}, \qquad H_3 = 1{\cdot}58h_2,$$
$$\bar{H}_3 = -4{\cdot}88h_2, \qquad M = 0{\cdot}406\,m,$$

$$\bar{F}_0 = F_0\int_{-1/2}^{1/2} a(X)\,W(X)\,\mathrm{d}X, \qquad \bar{F}_1 = F_1\int_{-1/2}^{1/2} a(X)\,W(X)\,\mathrm{d}X.$$

The method of harmonic balance will now be employed to obtain frequency response information from equations (46) and (47). With generality in mind, the following time-dependence of the response is assumed:

$$T_1(t) = W_0 + W_1\sin\Omega t + W_2\cos\Omega t + W_3\sin 2\Omega t + W_4\cos 2\Omega t + W_5\sin 3\Omega t + W_6\cos 3\Omega t, \tag{48}$$

$$T_2(t) = V_0 + V_1\sin\Omega t + V_2\cos\Omega t + V_2\cos\Omega t + V_3\sin 2\Omega t + V_4\cos 2\Omega t + V_5\sin 3\Omega t + V_6\cos 3\Omega t. \tag{49}$$

Substituting equations (48) and (49) into equations (46) and (47), gives, on using the method of harmonic balance, with harmonics of the third order being retained,

$$(\bar{K}_{11}+\bar{K}_{12}-\bar{K}_{13})\,[W_0+W_1\sin\Omega t+W_2\cos\Omega t+W_3\sin 2\Omega t+W_4\cos 2\Omega t+$$
$$+W_5\sin 3\Omega t+W_6\cos 3\Omega t]+\bar{K}_{12}[V_0+V_1\sin\Omega t+V_2\cos\Omega t+V_3\sin 2\Omega t+$$
$$+V_4\cos 2\Omega t+V_5\sin 3\Omega t+V_6\cos 3\Omega t]-H_2\int_{-\infty}^{t}Y_2(t-\tau)\,\mathrm{d}(V_0+V_1\sin\Omega\tau+$$
$$+V_2\cos\Omega\tau+V_3\sin 2\Omega\tau+V_4\cos 2\Omega\tau+V_5\sin 3\Omega\tau+V_6\cos 3\Omega\tau)-$$
$$-\bar{H}_2\int_{-\infty}^{t}Y_2(t-\tau)\,\mathrm{d}(W_0+W_1\sin\Omega\tau+W_2\cos\Omega\tau+W_3\sin 2\Omega\tau+W_4\cos 2\Omega\tau+$$
$$+W_5\sin 3\Omega\tau+W_6\cos 3\Omega\tau)=0, \qquad (50)$$

and

$$\bar{K}_{15}\{B_0+B_1\sin(\Omega t)+B_2\cos(\Omega t)+B_3\sin(2\Omega t)+B_4\cos(2\Omega t)+B_5\sin(3\Omega t)+$$
$$+B_6\cos(3\Omega t)+H_3\int_{-\infty}^{t}Y_2(t-\tau)\,\mathrm{d}\{V_0+V_1\sin(\Omega\tau)+V_2\cos(\Omega\tau)+V_3\sin(2\Omega\tau)+$$
$$+V_4\cos(2\Omega\tau)+V_5\sin(3\Omega\tau)+V_6\cos(3\Omega\tau)\}+H_3\int_{-\infty}^{t}Y_2(t-\tau)\,\mathrm{d}\{W_0+W_1\sin(\Omega\tau)+$$
$$+W_2\cos(\Omega\tau)+W_3\sin(2\Omega\tau)+W_4\cos(2\Omega\tau)+W_5\sin(3\Omega\tau)+W_6\cos(3\Omega\tau)\}+$$
$$+M\Omega^2(W_1\sin(\Omega t)+W_2\cos(\Omega t)+4W_3\sin(2\Omega t)+4W_4\cos(2\Omega t)+$$
$$+9W_5\sin(3\Omega t)+9W_6\cos(3\Omega t)\}+\bar{F}_0+\bar{F}_1\sin(\Omega t)=0, \qquad (51)$$

where the B_i's are polynomials in the W_i's resulting from the non-linear term in the differential equation and are defined in the Appendix.

The integrals in equations (50) and (51) are treated by realizing that for a harmonic shearing strain, $\gamma(t)=\gamma_0 e^{j\Omega t}$, the shear stress, $S(t)$, can be represented by the complex modulus representation, resulting in

$$S(t)=[G_2'(\Omega)+jG_2''(\Omega)]\,\gamma(t)=\int_{-\infty}^{t}Y_2(t-\tau)\,\mathrm{d}\gamma(\tau).$$

From this relation it can be shown that the relaxation function Y_2 is related to the complex modulus through the Fourier integrals

$$G_2'(\Omega)=\Omega\int_{-\infty}^{t}Y_2(\xi)\sin\Omega\xi\,\mathrm{d}\xi, \qquad (52)$$

$$G_2''(\Omega)=\Omega\int_{-\infty}^{t}Y_2(\xi)\cos\Omega\xi\,\mathrm{d}\xi. \qquad (53)$$

By means of equations (52) and (53), the various integrals in equations (50) and (51) can be evaluated. The results, on defining $\bar{K}_T=\bar{K}_{11}+\bar{K}_{12}-\bar{K}_{13}$, are 14 coupled, non-linear algebraic equations:

$$\bar{K}_T W_0+\bar{K}_{12}V_0-H_2G_2'(0)\,V_0-\bar{H}_2G_2'(0)\,W_0=0, \qquad (54)$$

$$\bar{K}_T W_1+\bar{K}_{12}V_1-H_2G_2'(\Omega)\,V_1-H_2G_2''(\Omega)\,V_2-\bar{H}_2G_2'(\Omega)\,W_1+\bar{H}_2G_2''(\Omega)\,W_2=0, \qquad (55)$$

$$\bar{K}_T W_2 + \bar{K}_{12} V_2 - H_2 G_2''(\Omega) V_1 - H_2 G_2'(\Omega) V_2 - \bar{H}_2 G_2''(\Omega) W_1 - \bar{H}_2 G_2'(\Omega) W_2 = 0, \quad (56)$$

$$\bar{K}_T W_3 + \bar{K}_{12} V_3 - H_2 G_2'(2\Omega) V_3 + H_2 G_2''(2\Omega) V_4 - \bar{H}_2 G_2'(2\Omega) W_3 + \bar{H}_2 G_2''(2\Omega) W_4 = 0, \quad (57)$$

$$\bar{K}_T W_4 + \bar{K}_{12} V_4 - H_2 G_2''(\Omega) V_3 - H_2 G_2'(2\Omega) V_4 - \bar{H}_2 G_2''(2\Omega) W_3 - \bar{H}_2 G_2'(2\Omega) W_4 = 0, \quad (58)$$

$$\bar{K}_T W_5 + \bar{K}_{12} V_5 - H_2 G_2'(3\Omega) V_5 + H_2 G_2''(3\Omega) V_6 - - \bar{H}_2 G_2'(3\Omega) W_5 + \bar{H}_2 G_2''(3\Omega) W_6 = 0, \quad (59)$$

$$\bar{K}_T W_6 + \bar{K}_{12} V_6 - H_2 G_2''(3\Omega) V_5 - H_2 G_2'(3\Omega) V_6 - - \bar{H}_2 G_2''(3\Omega) W_5 - \bar{H}_2 G_2'(3\Omega) W_6 = 0, \quad (60)$$

$$\bar{K}_{15} B_0 + H_3 G_2'(0) V_0 + \bar{H}_3 G_2'(0) W_0 + \bar{F}_0 = 0, \quad (61)$$

$$\bar{K}_{15} B_1 + H_3 G_2'(\Omega) V_1 - H_3 G_2''(\Omega) V_2 + \bar{H}_3 G_2'(\Omega) W_1 - - \bar{H}_3 G_2''(\Omega) W_2 + M\Omega^2 W_1 + \bar{F}_1 = 0, \quad (62)$$

$$\bar{K}_{15} B_2 + H_3 G_2''(\Omega) V_1 + H_3 G_2'(\Omega) V_2 + \bar{H}_3 G_2''(\Omega) W_1 + + \bar{H}_3 G_2'(\Omega) W_2 + M\Omega^2 W_2 = 0, \quad (63)$$

$$\bar{K}_{15} B_3 + H_3 G_2'(2\Omega) V_3 - H_3 G_2''(2\Omega) V_4 + \bar{H}_3 G_2'(2\Omega) W_3 - - \bar{H}_3 G_2''(2\Omega) + 4M\Omega^2 W_3 = 0, \quad (64)$$

$$\bar{K}_{15} B_4 + H_3 G_2''(2\Omega) V_3 + H_3 G_2'(2\Omega) V_4 + \bar{H}_3 G_2''(2\Omega) W_3 + + \bar{H}_3 G_2'(2\Omega) W_4 + 4M\Omega^2 W_4 = 0, \quad (65)$$

$$\bar{K}_{15} B_5 + H_3 G_2'(3\Omega) V_5 + H_3 G_2''(3\Omega) V_6 - \bar{H}_3 G_2'(3\Omega) W_5 - - \bar{H}_3 G_2''(3\Omega) W_6 + 9M\Omega^2 W_5 = 0, \quad (66)$$

$$\bar{K}_{15} B_6 + H_3 G_2''(3\Omega) V_5 + H_3 G_2'(3\Omega) V_6 + \bar{H}_3 G_2''(3\Omega) W_5 + \bar{H}_3 G_2'(3\Omega) W_6 + 9M\Omega^2 W_6 = 0. \quad (67)$$

4. NUMERICAL RESULTS

Though formidable in appearance, the algebraic equations (54) through (67) can be handled numerically. The method used here was one developed by Brown and Conte [50, 51]. It is a gradient method with the gradient computed numerically rather than having to enter the partial derivatives explicitly. The W_i and V_i are functions of frequency and frequency-response curves for r.m.s. transverse displacement at mid-span were calculated from equation (42). Results were obtained for the symmetric beams reported on by Kovac, Anderson and Scott [44], and for unsymmetric beams not previously investigated. Complex modulus data was supplied by the B. F. Goodrich Co., in connection with the work in reference [44].

Results show that superharmonic response is present, but it does not have a significant effect on the frequency response curve. Figure 4 shows results for the beam of Figure 2. Included in Figure 4 is the theoretical ratio of the magnitude of the 2Ω component of response to the magnitude of the Ω component as a function of forcing frequency. As indicated, the ratio is largest when forcing the beam near one-half its resonant frequency. Curves similar to those in Figure 4 were found for the unsymmetrical beams to be reported on later. Results

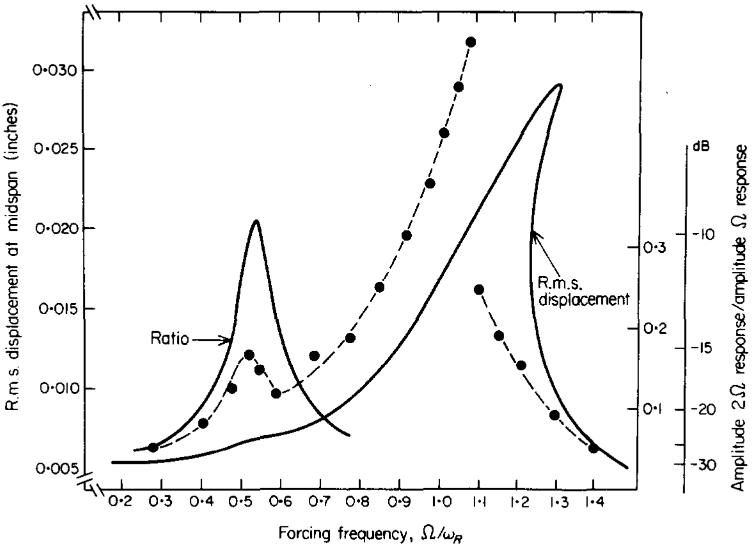

Figure 4. Superharmonic response of symmetric three-layer beam. ——, Theory; --●--, experiment (reference [44]); ω_R = linear resonant frequency.

show that for both symmetric and unsymmetric beams, the superharmonic response is much more pronounced if the bias force is larger than the harmonic force. However, it seems clear that some other effect was responsible for the large increase in response amplitude originally observed when forcing the beam at one-half its resonant frequency. This will be discussed at some length in the paper reporting the experimental results.

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APPENDIX

$$B_0 = C_0 W_0 + (1/2)(C_1 W_1 + C_2 W_2 + C_3 W_3 + C_4 W_4 + C_5 W_5 + C_6 W_6),$$

$$B_1 = C_1 W_0 + C_0 W_1 + (1/2)(-C_4 W_1 + C_3 W_2 + C_2 W_3 - C_6 W_3 - C_1 W_4 + C_5 W_4 + C_4 W_5 - C_8 W_5 - C_3 W_6 + C_7 W_6),$$

$$B_2 = C_2 W_0 + C_0 W_2 + (1/2)(C_3 W_1 + C_4 W_2 + C_1 W_3 + C_5 W_3 + C_2 W_4 + C_6 W_4 + C_3 W_5 + C_7 W_5 + C_4 W_6 + C_8 W_6),$$

$$B_3 = C_3 W_0 + C_0 W_3 + (1/2)(C_2 W_1 - C_6 W_1 + C_1 W_2 + C_5 W_2 - C_8 W_3 + C_7 W_4 + C_2 W_5 - C_{10} W_5 - C_1 W_6 + C_9 W_6),$$

$$B_4 = C_4 W_0 + C_0 W_4 + (1/2)(-C_1 W_1 + C_5 W_1 + C_2 W_2 + C_6 W_2 + C_7 W_3 + C_8 W_4 + C_1 W_5 + C_9 W_5 + C_2 W_6 + C_{10} W_6),$$

$$B_5 = C_5 W_0 + C_0 W_5 + (1/2)(C_4 W_1 - C_8 W_1 + C_3 W_2 + C_7 W_2 + C_2 W_3 - C_{10} W_3 + C_1 W_4 + C_9 W_4 + C_{11} W_6 - C_{12} W_5),$$

$$B_6 = C_6 W_0 + C_0 W_6 + (1/2)(-C_3 W_1 + C_7 W_1 + C_4 W_2 + C_8 W_2 - C_1 W_3 + C_9 W_3 + C_2 W_4 + C_{10} W_4 + C_{11} W_5 + C_{12} W_6),$$

$$C_0 = W_0^2 + (1/2)(W_1^2 + W_2^2 + W_3^2 + W_4^2 + W_5^2 + W_6^2),$$

$$C_1 = 2W_0 W_1 - W_1 W_4 + W_2 W_3 - W_3 W_6 + W_4 W_5,$$

$$C_2 = 2W_0 W_2 + W_1 W_3 + W_2 W_4 + W_3 W_5 + W_4 W_6,$$

$$C_3 = 2W_0 W_3 + W_1 W_2 - W_1 W_6 + W_2 W_5,$$

$$C_4 = (1/2)(W_2^2 - W_1^2) + 2W_0 W_4 + W_1 W_5 + W_2 W_6,$$

$$C_5 = 2W_0 W_5 + W_1 W_4 + W_2 W_3,$$

$$C_6 = 2W_0 W_6 - W_1 W_3 + W_2 W_4,$$

$$C_7 = W_1 W_6 + W_2 W_5 + W_3 W_4,$$

$$C_8 = (1/2)(W_4^2 - W_3^2) + W_2 W_6 - W_1 W_5,$$

$$C_9 = W_3 W_6 + W_4 W_5,$$

$$C_{10} = W_4 W_6 - W_3 W_5,$$

$$C_{11} = W_5 W_6,$$

$$C_{12} = \tfrac{1}{2}(W_6^2 - W_5^2).$$