JOURNAL OF ALGEBRA 40, 271–279 (1976)

On a Subgroup of Order $2^{15} \mid GL(5,2) \mid$ in $E_8(\mathbb{C})$, the Dempwolff Group and Aut $(D_8 \circ D_8 \circ D_8)$

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Communicated by Walter Feit

Received December 27, 1974

We analyze a subgroup constructed by Thompson inside $E_8(\mathbb{C})$ and give a new proof that it has a subgroup D which is a nonsplit extension of $GL(5, 2)$ by $\mathbb{F}_2{}^5$. We also obtain a proof that $\mathrm{Auf}(D_8 \circ D_8 \circ D_8)'$ splits over $\mathrm{Inn}(D_8 \circ D_8 \circ D_8)$, which corrects an earlier claim of the author.

1. INTRODUCTION AND STATEMENT OF RESULTS

Recently there has developed some evidence for the existence of four new finite simple groups, the largest of which, called "the monster," involves the other three as sections [7, 10, 19]. One of these was suspected by Thompson to be related to subgroups of groups of type E_8 over various fields. In the course of his investigation, he observed that $E_8(\mathbb{C})$ contains a finite subgroup G such that

$$\begin{aligned}&Q = O_2(G) \text{ is a special group of order } 2^{15}\\ &\text{with } Q' = \Phi(Q) = Z(Q) \text{ elementary of rank 5,}\\ &Q = C_G(Q') \text{ and } G/Q \cong GL(5, 2).\end{aligned}$$

Furthermore, G can be factorized

$$G = QD, \qquad Q \cap D = Q'. \tag{*}$$

Here, D is called the Dempwolff group [5] and is part of a nonsplit short exact sequence

$$1 \to \mathbb{F}_2{}^5 \to D \to GL(5, 2) \to 1. \tag{**}$$

This factorization has been established by P. Smith [16] with the aid of a computer. It is the purpose of this paper to give a different argument that G can be factorized as in (*) (we do not make use of computers). It follows from Section 3 that (**) is nonsplit and from [5] that the isomorphism type of D is uniquely determined.

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For the sake of completeness, we sketch Thompson's construction of G inside $E_8(\mathbb{C})$. Let H be a Cartan subgroup, $H \cong \prod_1^8 \mathbb{C}^\times$, and let $T = \{x \in H \mid x^2 = 1\}$. Then T is elementary abelian of rank 8. We have that $N = N(H)$ satisfies $N/H \cong W$, the Weyl group of type E_8. Recall that $\mathbb{Z}_2 \cong Z(W) \leqslant W'$, $W/Z(W) \cong O^+(8, 2)$, $|W| = 2^{14}3^5 5^2 7$. We may regard T as the "standard module" for $O^+(8, 2)$. In the action of $O^+(8, 2)$ on $T^\#$, there are two orbits, represented by, say, z (a "singular vector") and t (a "nonsingular vector"). Every involution in $E_8(\mathbb{C})$ is conjugate to either t or z (see Sect. 2, Lemma 3). In the 248-dimensional representation on the Lie algebra $\mathscr{L}$, z has trace -8 and t has trace 24. Choose a "maximal totally singular" subspace I of T. Then $|I| = 16$ and $C_{\mathscr{L}}(I)$ is a Cartan subalgebra. Let j be an involution of N which inverts H. Then $H\langle j\rangle = C(I)$. Set $Z = I\langle j\rangle$. Then $C = C(Z)$ lies in N, $C \cap H = T$ and $CH/H\langle j\rangle$ may be identified with $O_2(P)$, where P is the maximal parabolic subgroup of $\Omega^+(8, 2)$ stabilizing I. We have $|O_2(P)| = 2^6$ and $P/O_2(P) \cong GL(4, 2)$. Thus, $|C| = 2^{15}$. Clearly, $N_N(Z)$ induces on Z a group of automorphisms isomorphic to $\mathbb{F}_2{}^4 \cdot GL(4, 2)$. But every involution of Z is conjugate in $E_8(\mathbb{C})$ to z. One next considers a hyperplane $I_0 \neq I$ of Z and sees that $N(I_0) \cap N(Z)$ induces a group of automorphisms on Z isomorphic to $\mathbb{F}_2{}^4 \cdot GL(4, 2)$. It follows that $N(Z)/C \cong GL(5, 2)$. Our group G is $N(Z)$.

We now discuss the results of this paper.

THEOREM. *The group $G \leqslant E_8(\mathbb{C})$ of order $2^{15}\,|GL(5, 2)|$ can be factorized $G = O_2(G) \cdot D$, $D \cap O_2(G) = Z(O_2(G)) \cong \mathbb{F}_2{}^5$, where D is the Dempwolff group.*

COROLLARY 1. *Let E be an extra special group of order 2^7, type $+$, i.e., $E \cong D_8 \circ D_8 \circ D_8$. Then* $\operatorname{Inn}(E) \cong \mathbb{F}_2{}^6$, $\operatorname{Out}(E) \cong O^+(6, 2) \cong \Sigma_8$ *and the exact sequence*

$$1 \to \operatorname{Inn}(E) \to \operatorname{Aut}(E)' \to \operatorname{Out}(E)' \to 1$$

is split.

The proof of Corollary 1 may be extracted from the analysis in Section 3. In [9], we erroneously asserted that $\operatorname{Aut}(E)'$ was not split over $\operatorname{Inn}(E)$. Our attention was drawn to this by Yoshida in a letter. He provided generators for a subgroup of $\operatorname{Aut}(E)$ isomorphic to $A_8 \cong \Omega^+(6, 2)$ (these are reproduced in Sect. 4). Furthermore, a recent letter from G. E. Wall informs the author that [2] deals with many of the questions considered in [9], though from a different viewpoint. In particular, the authors correctly assert that $\operatorname{Aut}(E)'$ does split over $\operatorname{Inn}(E)$ and they describe generators for a complement. Recently, G. Bell has announced the result $H^2(\Omega^+(6, 2), \mathbb{F}_2{}^6) = 0$. On the other hand, Corollary 3 below tells us that $H^2(O^+(8, 2), \mathbb{F}_2{}^6) \neq 0$.

COROLLARY 2. *Let* $F \simeq D_8 \circ D_8 \circ \mathbb{Z}_4$ *be of symplectic type and order* 2^6. *Then* $\mathrm{Out}(F) \simeq \mathrm{Sp}(4, 2) \times \mathbb{Z}_2$. *Let* $C_1 = C_{\mathrm{Aut}(F)}(Z(F))$ *and let* C_0 *be the other subgroup of index* 2 *in* $\mathrm{Aut}(F)$ *with* $O_2(C_0) = O_2(C_1)$. *Then*

$$1 \to \mathrm{Inn}(F) \to C_i \to \mathrm{Sp}(4, 2) \to 1$$

splits for $i = 0$ *and does not split for* $i = 1$.

The first assertion follows from Corollary 1. Namely, take F as a maximal subgroup of E. Then C_0 becomes the stabilizer in $\mathrm{Aut}(E)'$ of F and we intersect C_0 with a complement to $\mathrm{Inn}(E)$ in $\mathrm{Aut}(E)'$ to get the splitting for $i = 0$. As for the nonsplitting when $i = 1$, we refer to [9, Corollary 3].

COROLLARY 3. *With* $E \simeq D_8 \circ D_8 \circ D_8$,

$$1 \to \mathrm{Inn}(E) \to \mathrm{Aut}(E) \to \mathrm{Out}(E) \to 1$$

is nonsplit.

If false, the argument of the last paragraph could be used to get a splitting of the sequence in Corollary 2 for $i = 1$.

COROLLARY 4. *Let* D *be the Dempwolff group and* A *a hyperplane of* $O_2(D)$. *Set* $M = N_D(A)$. *Then* M *contains a normal subgroup* $W \simeq \mathbb{Z}_4^4$, $W = C_M(W)$, *such that* $M/W \simeq \hat{A}_8$, *the covering group of* A_8. *Furthermore,* M *does not split over* W *but does contain a perfect subgroup* M_0 *such that* $O_2(D) = A \times Z(M_0)$ *and* $M_0/Z(M_0)$ *is a nonsplit extension of* $GL(4, 2)$ *by* $\mathbb{F}_2^4$.

Again, this is not a new result, just a new proof.

Finally, we mention that the existence of D shows $H^2(GL(5, 2), \mathbb{F}_2^5) \neq 0$. This nonvanishing was an unsettled case in Dempwolff's work [5, 6]. One now has the complete result: $\dim_{\mathbb{F}_2} H^2(GL(n, 2), \mathbb{F}_2^n) = 1$ precisely when $n = 3, 4, 5$ and is 0 otherwise (see [5, 6] for more details).

We thank John Thompson for explaining the construction of G and for pointing out an error in our original proof.

2. PRELIMINARY RESULTS

We present some lemmas which are needed to prove the main theorem. Our notation is standard and follows [8]. In addition, we use $\mathbb{Z}_n^s$, $\mathbb{F}_2^s$ and 2_ϵ^{1+2s} to denote, respectively, the direct product of s copies of $\mathbb{Z}_n$, an s-dimensional $\mathbb{F}_2$-vector space and an extra special group of order 2^{2s+1}, type $\epsilon = +, -$ (see [9] for a discussion).

LEMMA 1 (D. G. Higman [11] or Alperin, Gorenstein [1]). *For V any faithful 4-dimensional $\mathbb{F}_2GL(4, 2)$-module, $H^1(GL(4, 2), V) = 0$. Consequently, any exact sequence of $\mathbb{F}_2$-modules $0 \to V \to W \to \mathbb{F}_2 \to 0$, $0 \to \mathbb{F}_2 \to W \to V \to 0$ is split.*

The next lemma is slightly stronger than what we need later.

LEMMA 2. *Let V be an n-dimensional vector space over $\mathbb{F}_2$ and let $G = GL(V) = GL(n, 2)$. Then $\Lambda^2(V)$ is an irreducible G-module and, for $n \geqslant 3$, $V \otimes V$ is a uniserial G-module with composition factors isomorphic to $\Lambda^2(V)$, V, $\Lambda^2(V)$, in that order.*

Proof. We show that $\Lambda^2(V)$ is irreducible by induction on n. For $n \leqslant 3$, this is easily verified, so we assume $n \geqslant 4$. Take a basis $v_1, \ldots, v_n$ for V. Then $\{v_i \wedge v_j \mid 1 \leqslant i < j \leqslant n\}$ is a basis for $\Lambda^2(V)$. Let $W = \langle v_1, \ldots, v_{n-1}\rangle$ and let $H = \{g \in G \mid W = W^g,\ v_n = v_n{}^g\} \cong GL(n-1, 2)$. Set $A = \langle v_i \wedge v_n \mid 1 \leqslant i \leqslant n-1\rangle$, $B = \langle v_i \wedge v_j \mid 1 \leqslant i < j \leqslant n-1\rangle$. Then $\Lambda^2(V) = A \oplus B$ is an H-decomposition. By induction, B is H-irreducible and A is H-irreducible since A is the standard module for H. Since $n \geqslant 4$, $\dim A = n \neq \binom{n}{2} = \dim B$, whence $A \ncong B$. Thus, any proper G-submodule of $\Lambda^2(V)$ lies in A or B. On the other hand, $g \in G$ defined by $g\colon v_i \mapsto v_i$, $1 \leqslant i \leqslant n-2$, $g\colon v_{n-1} \mapsto v_n$, $g\colon v_n \mapsto v_{n-1}$ satisfies $A^g \neq A$, $B^g \neq B$. Therefore, $\Lambda^2(V)$ is irreducible.

Set $M = V \otimes V$, $M_0 = 0$, $M_1 = \langle x \otimes y + y \otimes x \mid x, y \in V\rangle$, $M_2 = \langle x \otimes x \mid x \in V\rangle$, $M_3 = M$. One can easily see that M_{i+1}/M_i, $i = 0, 1, 2$, are isomorphic to, respectively, $\Lambda^2(V)$, V, $\Lambda^2(V)$, and so are irreducible. Hence, we must show that the composition series $M_0 < M_1 < M_2 < M_3$ is the only composition series. It suffices to show that M_{i+2}/M_i is indecomposable for G, $i = 0, 1$. But each of these module extensions $0 \to M_{i+1}/M_i \to M_{i+2}/M_i \to M_{i+2}/M_{i+1} \to 0$ is nonsplit under the action of $\langle g\rangle$, where $g \in G$ induces a transvection on V (an easy exercise). This completes the proof.

LEMMA 3 (Jacobson [14]). *An automorphism of finite order on a complex Lie algebra of type E_8 centralizes a Cartan subalgebra, hence is conjugate in $E_8(\mathbb{C})$ to an element of a Cartan subgroup.*

LEMMA 4. *Let $E \cong 2_+^{1+2n}$. Say $E \lhd L$ and $C_L(E) = Z(E)$. Assume that L has normal elementary abelian subgroups B_1 and B_2 so that $E = B_1B_2$, B_1/E' is a faithful module for L/E and B_2/E' is its dual module. Then L contains a subgroup L_1 such that $L = EL_1$ and $E \cap L_1 = E'$.*

Proof. Let $\{i, j\} = \{1, 2\}$. In the action of B_i on B_j, B_i/E' permutes regularly the hyperplanes of B_j not containing E'. Let D_k be a hyperplane of B_k with $E' \cap D_k = 1$, $k = 1, 2$. Then $L_1 = N_L(D_1) \cap N_L(D_2)$ does the job.

LEMMA 5. $\mathrm{Aut}(\mathbb{Z}_4^{2n})$, $n \geqslant 2$, *does not contain a subgroup isomorphic to* A_4 *in which an element of order* 3 *acts fixed point freely on* $\mathbb{Z}_4{}^n$ *and which is faithful on the Frattini factor of* $\mathbb{Z}_4{}^n$. *In particular*, $1 \to \mathbb{Z}_2^{4n^2} \to \mathrm{Aut}(\mathbb{Z}_4^{2n}) \to GL(2n, 2) \to 1$ *is nonsplit for* $n \geqslant 2$.

Proof. We may suppose the existence of $S \leqslant \mathrm{Aut}(W)$, $W \simeq \mathbb{Z}_2^{4n}$, to violate our conclusion. Let $V = O_2(S)$ and let $h \in S$, $|h| = 3$. Then, h operates fixed point freely on WV, whence WV has class at most 2 [15]. On the other hand, if $x \in V^{\#}$, x is an involution, whence x inverts any $[w, x]$, $w \in W$. Since $[W, x]$ has exponent 4, WV cannot have class 2, contradiction.

The interested reader should see G. Higman [12], who shows that $SL(2, 2^n)$, $n \geqslant 2$, cannot act faithfully on a 2-group in such a way that elements of order 3 are fixed point free, unless the 2-group is elementary abelian. Since $GL(2n, 2)$ contains a copy of $SL(2, 2^n)$, Lemma 5 follows from this result.

3. PROOF OF THE MAIN THEOREM

Let G and Q be as in Section 1. Recall that Q' and Q/Q' are irreducible modules for $G/Q \simeq GL(5, 2)$, Q' and Q/Q' are elementary abelian of ranks 5 and 10, respectively. We wish to prove that

$$1 \to Q/Q' \to G/Q' \to G/Q \to 1 \tag{1}$$

is the split extension.

Let I be a hyperplane of Q'. Set $N = N_G(I)$. Then $N > Q$ and N/Q is a split extension $\mathbb{F}_2{}^4 \cdot GL(4, 2)$. Take $Y < N$, $Y > Q$ so that $Y \cap O_2(N) = Q$ and $Y \cdot O_2(N) = N$. Then $Y/Q \simeq GL(4, 2)$. It suffices, by Gaschütz' theorem [13, I.17.4] to prove that

$$1 \to Q/Q' \to N/Q' \to N/Q \to 1 \tag{2}$$

is the split extension. From the discussion in Section 1 on the construction of G, we get that (in the notation of Sect. 1)

$$\begin{array}{l} \text{the subgroup } U = T\langle j\rangle \text{ contains } Z = Q' = I\langle j\rangle; \\ \text{as a module for } N,\ Q/Z \text{ has composition series} \\ 1 < U/Z < Q/Z; \text{ furthermore, } U/Z \simeq \mathbb{Z}_2{}^4, \\ Q/U \simeq \mathbb{Z}_2{}^6 \text{ and } Q/I \simeq \mathbb{Z}_2{}^4 \times 2_+^{1+6}. \end{array} \tag{3}$$

Now, consider the action of Y on $Z(Q/I)$, elementary of rank 5. Since $Z(Q/I)$ covers U, $Z(Q/I)/(Q'/I)$ may be regarded as the standard module for $Y/Q \simeq GL(4, 2)$. By Lemma 1, $Z(Q/I)$ is completely reducible for Y. In fact, we may even assume that T/I complements Q'/I by taking Y to lie in $N_G(T)$. Now consider Y/T. Since $T \ngeq Q' = Z$, Q/T is nonabelian. Since Y operates irreducibly on Q/U, Q/T is extraspecial. As in Section 1, Y/U may be identified with the parabolic subgroup P of $\Omega^+(8, 2)$. Since P splits over $O_2(P)$, there is $K < H$ so that $KQ = Y$, $K \cap Q = U$ and K is decomposable on T (Corollary 1 now follows).

We wish to produce a subgroup N_0 of $O_2(N)$ such that K normalizes N_0, N_0 covers $O_2(N)/Q$ and $N_0 \cap Q = U$. Namely, K normalizes I and T, hence also $H\langle j\rangle$ (the centralizer in $E_8(\mathbb{C})$ of T). Using the fact that K normalizes Z, this means that K normalizes $X = \{x \in H\langle j\rangle \mid [x, Z] \leqslant I\}$. Since j inverts H and $C_X(j) = T\langle j\rangle = U$, it is not hard to see that X/U is elementary of order 16, so that $N_0 = X$ works.

We now look at images modulo I. Let $^-$ denote the quotient map $N \to \bar{N} = N/I$. Since the K-module $\Lambda^2(I)$ does not have a quotient isomorphic to I, N_0/Z is abelian (see Lemma 2). Since the K-chief factors of N_0/Z are isomorphic to I and I^* ($\not\simeq I$), N_0/Z is elementary abelian. Now, $U = N_{N_0}(T)$, whence $\bar{N}_0$ is extra special of order 2^9, type $+$. Since $\mathrm{Out}(\bar{N}_0) \simeq O^+(8, 2)$ and since N/N_0 acts faithfully on the Frattini factor of $\bar{N}_0$ (as the parabolic subgroup of $\Omega^+(8, 2)$ stabilizing the maximal isotropic subspace $\bar{U}$), we observe that there is a complement to $O_2(N/N_0) = QN_0/N_0$ in N/N_0, say $L/N_0 \simeq GL(4, 2)$, which normalizes a complement, say $\bar{N}_1$, in $\bar{N}_0$ to the maximal abelian subgroup $\bar{U}$. Let N_1 be the preimage of $\bar{N}_1$ in N. Then, N_1 has order 2^9, $N_1 > Z$ and $[Z, N_1] = I$. Taking L to be as above, we use Lemma 4 to produce a subgroup L_1 of L such that $L = N_0L_1$ and $N_0 \cap L_1 = Z$. The group N_1L_1 satisfies $N = QN_1L_1$ and $Q \cap N_1L_1 = Z$, i.e., the sequence (2) splits.

Since we now have (1) a split extension, take $D < G$ so that $G = QD$, $Q \cap D = Z$. It remains to show that

$$1 \to Z \to D \to GL(5, 2) \to 1 \tag{4}$$

is not the split extension to finish the proof of the Main Theorem. The argument can be extended to prove Corollary 4. Namely, we also prove that

if $M = N_D(A)$, where A is some hyperplane of Z,
M contains a normal subgroup $W = C_M(W) \simeq \mathbb{Z}_4^4$
such that $M/W \simeq \hat{A}_8$, the covering group of $A_8 \simeq GL(4, 2)$,
and M does not split over W. (5)

Our argument relies very heavily on the ideas in [18]. Let $\mathscr{A}$ denote the set of hyperplanes of Z. By previous remarks, $\mathscr{L}_A = C_{\mathscr{L}}(A)$ is a Cartan subalgebra of $\mathscr{L}$. Since $\mathscr{L} = \langle \mathscr{L}_A \mid A \in \mathscr{A}\rangle$, $|\mathscr{A}| = 31$ and $\dim \mathscr{L}_A = 8$, $\mathscr{L}$ is actually a direct sum of the $\mathscr{L}_A$.

Consider how M acts on $\mathscr{L}_A$. Let $\varphi \neq 1$ be a constituent and let W be the kernel of the action. Now, $Z \nleqslant W$, but if $WZ < O_2(M)$, then $A = \ker(\varphi)$ and M/A has a noncentral normal abelian subgroup, whence φ is induced [4, 50.7]. Since $M/O_2(M) \simeq GL(4, 2)$ and every noncentral chief factor of M in $O_2(M)$ is the standard module, this means $\varphi(1) \geqslant 15$, against $\dim \mathscr{L}_A = 8$. We now have that $WZ = O_2(M)$ and it remains to show that $M/W \simeq \hat{A}_8$ and W is homocyclic of exponent 4.

Assume that W is not homocyclic of exponent 4. Since the standard module for $GL(4, 2)$ is not a quotient of its exterior square (Lemma 2), W is abelian, hence elementary abelian. Choose an involution $i \in W \backslash Z$. Then i acts as a transvection on Z and centralizes precisely one hyperplane of Z. Let i' be any conjugate of i lying in $M \backslash Z$. Since i' induces a transvection on A, $i'Z$ lies in the center of a Sylow 2-group of M/Z. For any such i', we claim that its trace on $\mathscr{L}_A$ is a fixed number, say s. If $M/W \simeq \hat{A}_8$, this is clear, because $\hat{A}_8$ contains one class of involutions outside its center. If $M/Z \simeq A_8 \times \mathbb{Z}_2$, the character table of A_8 shows that the 8-dimensional representation on $\mathscr{L}_A$ has trace the standard degree 8 permutation character, whence i', being in the center of a Sylow 2-subgroup of M/W, has trace 0. The claim then implies that on all of $\mathscr{L}$, i' has trace $8 + 30s$, which must equal -8 or 24. As no integer s has this property, we have our contradiction, and so W is homocyclic.

It now follows from Lemma 5 that $M/W \simeq \hat{A}_8$.

Finally, we must show that M does not split over W. It follows from the homocyclicity of W that

$$\text{two involutions of } D\backslash Z \text{ which centralize the same hyperplane do not commute.} \tag{6}$$

Assuming M does split over W, we take a complement $M_1 \simeq \hat{A}_8$. By Lemma 1, there is $u \in Z$ so that $Z = A \oplus \langle u \rangle$, as M_1-modules. The covering group of $GL(4, 2)$ has the property that transvections lift to involutions and furthermore that the subgroup centralizing a hyperplane of $\mathbb{F}_2^4$ lifts to an elementary abelian group. Since M_1 is completely reducible on Z, this clearly conflicts with (6). The nonsplitting, plus the fact that M/Z does split over W/Z, implies Corollary 4.

The proofs of the Theorem and Corollaries 1 and 4 are now completed.

4. An Explicit Splitting of $\mathrm{Aut}(2_+^{1+6})'$ over $\mathrm{Inn}(2_+^{1+6})$

We now reproduce Yoshida's embedding of $\Omega^+(6, 2)$ in $\mathrm{Aut}(E)$, where $E \simeq 2_+^{1+6}$. We take $E = \langle v_i, z \mid 1 \leqslant i \leqslant 6 \rangle$, where these generators are involutions, z is central and $[v_i, v_j] = z$ precisely when $i + j = 7$ and

$[v_i, v_j] = 1$ otherwise. Define automorphisms x_r ($r = 0, 1, 2, 01, 02, 012$), w_s ($s = 0, 1, 2$) of E as follows: for $y = x_r$ or w_s, the (y, v_i)-entry of the table below indicates the image of v_i under y.

	v_1	v_2	v_3	v_4	v_5	v_6
x_0	v_1v_2z	v_2	v_3z	v_4	v_5v_6	v_6
x_1	v_1z	v_2v_3	v_3	v_4v_5	v_5	v_6
x_2	v_1	v_2v_4z	v_3v_5z	v_4	v_5	v_6z
x_{01}	v_1v_3z	v_2z	v_3	v_4v_6	v_5	v_6
x_{02}	v_1v_4z	v_2	v_3v_6z	v_4	v_5z	v_6
x_{012}	v_1v_5z	v_2v_6z	v_3	v_4z	v_5	v_6
w_0	v_2	v_1	v_3z	v_4	v_6	v_5
w_1	v_1z	v_3	v_2	v_5	v_4	v_6
w_2	v_1	v_4	v_5	v_2	v_3	v_6z

One checks directly that each x_r and w_s is indeed an automorphism and furthermore has period 2. In addition,

$$[x_r, x_{r'}] = 1, \quad \text{except for} \quad [x_0, x_1] = x_{01},$$
$$[x_0, x_2] = x_{02}, \qquad [x_1, x_{02}] = x_{012} \qquad \text{and} \qquad [x_2, x_{01}] = x_{012},$$

so that the x_r generate a 2-group of order 2^6, isomorphic to a Sylow 2-group of $\Omega^+(6, 2)$. We also have

$$(w_0w_1)^3 = (w_0w_2)^3 = (w_1w_2)^2 = 1,$$
$$(w_0x_0)^3 = (w_1x_1)^3 = (w_2x_2)^3 = 1,$$

and, under conjugation,

$$\begin{array}{lll} w_0\colon x_1 \leftrightarrow x_{01} & w_1\colon x_0 \leftrightarrow x_{01} & w_2\colon x_0 \leftrightarrow x_{02} \\ \quad x_2 \leftrightarrow x_{02} & \quad x_2 \text{ is fixed} & \quad x_1 \text{ is fixed} \\ \quad x_{012} \text{ is fixed} & \quad x_{02} \leftrightarrow x_{012} & \quad x_{01} \leftrightarrow x_{012} \end{array}$$

Now set $B = \langle x_r \mid r = 0, 1, 2, 01, 02, 012\rangle$, $N = \langle w_s \mid s = 0, 1, 2\rangle$. Then N is a Weyl group of type $A_3 = D_3$, i.e., $N \simeq \Sigma_4$. The above information implies that the set BNB is a group (this can be proven by formal arguments usually employed in the study of B, N pairs; see [3] or [17]). Since BNB is a union of double cosets BwB, for $w \in N$, we can compute that $|BNB| = |\Omega^+(6, 2)|$. This means that BNB complements $\mathrm{Inn}(E)$ in $\mathrm{Aut}(E)'$, as claimed.

Yoshida remarks that one can see the splitting by looking at the subgroup of the Weyl group of E_8 which maps onto the parabolic subgroup of $\Omega^+(8, 2)$ which stabilizes a singular vector.

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