Note

On Tactical Configurations with No Four-Cycles

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Communicated by the Managing Editors

Received March 25, 1975

An improved lower bound is given for the band sizes of tactical configurations of rank exceeding two having no 4-cycles. This bound is applied to find an optimal configuration with certain specified parameters. A formula is given for the maximum number of cycle types one must examine to demonstrate that a rank r configuration has no g-cycle. This result, which has appeared in our earlier work as the number of types of closed walks, answers a question of Longyear.

Longyear [4] pointed out that certain questions about block designs and Latin squares can be formulated in terms of tactical configurations. By definition, a tactical configuration T of rank r is an r-partite graph with sets of independent points A_1, A_2, ..., A_r satisfying the following regularity condition. For each i ≠ j, all points in A_i have equally many neighbors in A_j. This common value is called the i, j-degree and is denoted by d_{ij}. Of course, d_{ij} does not necessarily equal d_{ji}. The sets A_i are called bands and for convenience, we write a_i = |A_i|. Counting the i, j lines in two different ways provides the useful identity

\[ a_i d_{ij} = a_j d_{ji}. \tag{1} \]

Longyear [3] examined the problem of determining the smallest tactical configurations with prescribed degrees d_{ij} and girth g. In particular, for girth 6 one has for all i ≠ j the bound

\[ a_i \geq 1 + d_{i6}(d_{ii} - 1). \tag{2} \]
When the rank $r$ is 2, this lower bound is often attained, and the Bruck–Chowla–Ryser theorem (see [5, Theorem 32, p. 115]) provides conditions on $d_{12}$ and $d_{21}$ for which this bound is not attained. However, little is known for rank $r > 2$, although inequality (2) has been used as a lower bound [3]. We now present a bound which is better for ranks exceeding 2, and which reduces to (2) when the rank is 2.

**Theorem 1.** A tactical configuration $T$ which has no 4-cycle must satisfy for all $i$:

$$a_i \geq 1 + \sum_{j \neq i} d_{ij}(d_{ji} - 1).$$

(3)

Proof. For every $j \neq i$, each point in $A_j$ covers $\binom{d_{ij}}{2}$ pairs of points in $A_i$. Furthermore, the same pair in $A_i$ cannot be covered twice by points in $A_j$, for that would constitute a 4-cycle. Consequently, we have the inequality

$$\binom{a_i}{2} \geq \sum_{j \neq i} a_j \binom{d_{ij}}{2} = \sum_{j \neq i} a_j d_{ij}(d_{ji} - 1)/2.$$  

(4)

To this we apply identity (1) to obtain

$$\binom{a_i}{2} \geq \sum_{j \neq i} a_j d_{ij}(d_{ji} - 1)/2.$$  

which reduces to inequality (3).

Let us define the condensed graph $G(T)$ of the tactical configuration $T$ to have $r$ points $v_1, v_2, \ldots, v_r$ with $v_i$ adjacent to $v_j$ if and only if $d_{ij} \neq 0$. We expect the bound (3) to be very tight (and often exact) for tactical configurations of girth 6 provided $G(T)$ also has girth exceeding 5. We are less optimistic when $G(T)$ has smaller girth, for then there are additional possibilities for short cycles in $T$.

For example, Longyear [3] mentions that for $d_{12} = d_{21} = d_{22} = 3$, and $d_{13} = d_{31} = 0$, inequality (2) yields 7 as a lower bound for each band size, whereas 21 was the smallest size actually obtained by construction. Our new bound yields $a_2 \geq 13$ and the identity (1) ensures that $a_1 = a_2 = a_3$. This minimum is easily attained by construction: We label the points of the three bands $u_i, v_i$, and $w_i$, respectively, with $0 \leq i \leq 12$. Reducing subscripts mod 13 as needed, we define $v_i$ to be adjacent to $u_i, u_{i-1}, u_{i-4}, w_i, w_{i-2}$, and $w_{i-7}$. By inspection, no pair of points in $A_2$ is covered twice, and so $T$ has no 4-cycle. Since $T$ is obviously a bigraph, its girth must be at least 6, as desired.

We observe that in this example the condensed graph is the star $K_{1,2}$. Whenever $G(T)$ is a star, the existence of a tactical configuration with
minimum band sizes given by (3) and (1) may be viewed as a packing question. For convenience, suppose $A_i$ is the band corresponding to the star's center. Then the question is: Can one simultaneously pack, for every $i > 1$, $a_i$ line-disjoint copies of $K_{a_i}$ into $K_{a_i}$? In the preceding example, 26 copies of $K_9$ were packed into $K_{19}$ without a single line to spare!

Finally, Longyear [3] asked how many distinct cycle types one must examine to show that a rank $r$ configuration $T$ has no $g$-cycle. Now every $g$-cycle in $T$ induces a closed $g$-walk in $G(T)$ in a natural manner. Clearly, there will be as many as $2g$ such walks possible if we consider each possible starting point and both possible directions for traversing the cycle. It is appropriate to view these walks as equivalent. Longyear's question may then be restated as asking how many inequivalent $g$-walks there are in $G(T)$. But this question has already been answered in our previous paper [2]. Obviously, the maximum possible number of cycle types occurs when we allow $G(T) = K_r$.

**Theorem 2.** The number of inequivalent closed $g$-walks in $K_r$ is given by

$$
\frac{1}{2g} \sum_{d | g} \phi \left( \frac{g}{d} \right) \left[ (r - 1)^d + (r - 1)(-1)^d \right] + e(g) \frac{r(r - 1)^{g/2}}{4},
$$

where

$$
e(g) = \begin{cases} 
1 & \text{if } g \text{ is even}, \\
0 & \text{if } g \text{ is odd}.
\end{cases}
$$

If $G(T) \equiv K_r$, there will be fewer walks, and the exact number was given in [2]. It resembles expression (5) and depends upon the spectrum of $G(T)$.

In particular, to show that $T$ has girth exceeding 5, we must evaluate (5) for $g = 3, 4, \text{ and } 5$. This yields 1 cycle type for rank 2, $1 + 6 + 3 = 10$ cycle types for rank 3, and $4 + 21 + 24 = 49$ cycle types for rank 4. These sums for ranks 3 and 4 were incorrectly reported as 13 and 67 in [3].

In conclusion, we note that the number of cycle types increases so rapidly that a detailed examination of cases quickly becomes unmanageable. However, if we restrict the condensed graph to be a bigraph, the number of cycle types is drastically reduced. In particular, odd cycles are forbidden, and so, to show that $T$ has girth exceeding 5, we need only examine cycle types of length 4 in $K_{m,n}$. This number can be evaluated from a formula resembling (5) which happens to simplify to give $(\binom{m+1}{2})(\binom{n+1}{2})$. Thus, for ranks 3 and 4, the number of cycle types has been reduced from 10 to 3 and from 49 to 9.
REFERENCES