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Note

On Tactical Configurations with No Four-Cycles

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An improved lower bound is given for the band sizes of tactical configurations of rank exceeding two having no 4-cycles. This bound is applied to find an optimal configuration with certain specified parameters. A formula is given for the maximum number of cycle types one must examine to demonstrate that a rank r configuration has no g-cycle. This result, which has appeared in our earlier work as the number of types of closed walks, answers a question of Longyear.

Longyear [4] pointed out that certain questions about block designs and Latin squares can be formulated in terms of tactical configurations. By definition, a *tactical configuration* T of rank r is an r-partite graph with sets of independent points $A_1, A_2, \ldots, A_r$ satisfying the following regularity condition. For each $i \neq j$, all points in A_i have equally many neighbors in A_j. This common value is called the *i, j-degree* and is denoted by d_{ij}. Of course, d_{ij} does not necessarily equal d_{ji}. The sets A_i are called *bands* and for convenience, we write $a_i = |A_i|$. Counting the i, j lines in two different ways provides the useful identity

$$a_i d_{ij} = a_j d_{ji}. \tag{1}$$

Longyear [3] examined the problem of determining the smallest tactical configurations with prescribed degrees d_{ij} and girth g. In particular, for girth 6 one has for all $i \neq j$ the bound

$$a_i \geqslant 1 + d_{ij}(d_{ji} - 1). \tag{2}$$

When the rank r is 2, this lower bound is often attained, and the Bruck–Chowla–Ryser theorem (see [5, Theorem 32, p. 115]) provides conditions on d_{12} and d_{21} for which this bound is not attained. However, little is known for rank $r > 2$, although inequality (2) has been used as a lower bound [3]. We now present a bound which is better for ranks exceeding 2, and which reduces to (2) when the rank is 2.

THEOREM 1. *A tactical configuration T which has no 4-cycle must satisfy for all i:*

$$a_i \geqslant 1 + \sum_{j \neq i} d_{ij}(d_{ji} - 1). \tag{3}$$

Proof. For every $j \neq i$, each point in A_j covers $\binom{d_{ji}}{2}$ pairs of points in A_i. Furthermore, the same pair in A_i cannot be covered twice by points in A_j, for that would constitute a 4-cycle. Consequently, we have the inequality

$$\binom{a_i}{2} \geqslant \sum_{j \neq i} a_j \binom{d_{ji}}{2} = \sum_{j \neq i} a_j d_{ji}(d_{ji} - 1)/2. \tag{4}$$

To this we apply identity (1) to obtain

$$\binom{a_i}{2} \geqslant \sum_{j \neq i} a_i d_{ij}(d_{ji} - 1)/2.$$

which reduces to inequality (3).

Let us define the condensed graph $G(T)$ of the tactical configuration T to have r points $v_1, v_2, \ldots, v_r$ with v_i adjacent to v_j if and only if $d_{ij} \neq 0$. We expect the bound (3) to be very tight (and often exact) for tactical configurations of girth 6 provided $G(T)$ also has girth exceeding 5. We are less optimistic when $G(T)$ has smaller girth, for then there are additional possibilities for short cycles in T.

For example, Longyear [3] mentions that for $d_{12} = d_{21} = d_{23} = d_{32} = 3$, and $d_{13} = d_{31} = 0$, inequality (2) yields 7 as a lower bound for each band size, whereas 21 was the smallest size actually obtained by construction. Our new bound yields $a_2 \geqslant 13$ and the identity (1) ensures that $a_1 = a_2 = a_3$. This minimum is easily attained by construction: We label the points of the three bands u_i, v_i, and w_i, respectively, with $0 \leqslant i \leqslant 12$. Reducing subscripts mod 13 as needed, we define v_i to be adjacent to u_i, u_{i-1}, u_{i-4}, w_i, w_{i-2}, and w_{i-7}. By inspection, no pair of points in A_2 is covered twice, and so T has no 4-cycle. Since T is obviously a bigraph, its girth must be at least 6, as desired.

We observe that in this example the condensed graph is the star $K_{1,2}$. Whenever $G(T)$ is a star, the existence of a tactical configuration with minimum band sizes given by (3) and (1) may be viewed as a packing question. For convenience, suppose A_1 is the band corresponding to the star's center. Then the question is: Can one simultaneously pack, for every $i > 1$, a_i line-disjoint copies of $K_{d_{i1}}$ into K_{a_1}? In the preceding example, 26 copies of K_3 were packed into K_{13} without a single line to spare!

Finally, Longyear [3] asked how many distinct cycle types one must examine to show that a rank r configuration T has no g-cycle. Now every g-cycle in T induces a closed g-walk in $G(T)$ in a natural manner. Clearly, there will be as many as $2g$ such walks possible if we consider each possible starting point and both possible directions for traversing the cycle. It is appropriate to view these walks as *equivalent*. Longyear's question may then be restated as asking how many *inequivalent* g-walks there are in $G(T)$. But this question has already been answered in our previous paper [2]. Obviously, the maximum possible number of cycle types occurs when we allow $G(T) = K_r$.

THEOREM 2. *The number of inequivalent closed g-walks in K_r is given by*

$$\frac{1}{2g} \sum_{d \mid g} \phi\left(\frac{g}{d}\right) [(r-1)^d + (r-1)(-1)^d] + e(g) \frac{r(r-1)^{g/2}}{4}, \tag{5}$$

where

$$e(g) = 1 \quad \text{if } g \text{ is even,}$$
$$ = 0 \quad \text{if } g \text{ is odd.}$$

If $G(T) \neq K_r$, there will be fewer walks, and the exact number was given in [2]. It resembles expression (5) and depends upon the spectrum of $G(T)$.

In particular, to show that T has girth exceeding 5, we must evaluate (5) for $g = 3, 4$, and 5. This yields 1 cycle type for rank 2, $1 + 6 + 3 = 10$ cycle types for rank 3, and $4 + 21 + 24 = 49$ cycle types for rank 4. These sums for ranks 3 and 4 were incorrectly reported as 13 and 67 in [3].

In conclusion, we note that the number of cycle types increases so rapidly that a detailed examination of cases quickly becomes unmanageable. However, if we restrict the condensed graph to be a bigraph, the number of cycle types is drastically reduced. In particular, odd cycles are forbidden, and so, to show that T has girth exceeding 5, we need only examine cycle types of length 4 in $K_{m,n}$. This number can be evaluated from a formula resembling (5) which happens to simplify to give $\binom{n+1}{2}\binom{m+1}{2}$. Thus, for ranks 3 and 4, the number of cycle types has been reduced from 10 to 3 and from 49 to 9.

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