NON-LINEAR RESPONSE OF POINT-REACTORS TO STOCHASTIC INPUTS*

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Abstract—Statistical properties of the non-linear response of a point reactor to a white Gaussian reactivity insertion and an external source are investigated through the general Fokker–Planck theory for linear systems with random coefficients. The autocorrelation function and power spectral density of the reactor power are obtained, and the effect of non-linearities on the corner frequencies is discussed. The response to a Gaussian (not necessarily white) reactivity insertion and an arbitrary neutron source is also considered in the absence of delayed neutrons.

1. INTRODUCTION

This paper is concerned with the statistical properties of the power response of a point-reactor to a stochastic reactivity insertion and a stochastic source of external neutrons, in the absence of feedback effects. From a mathematical point of view, the problem is the determination of the statistical description of solutions of a set of linear stochastic differential equations with randomly varying coefficients (parametric noise) and randomly varying inhomogeneous terms (additive noise). Since a complete statistical description of the output is difficult to obtain in general, we shall confine ourselves to the determination of its mean, correlation function and power spectral density, only. The response to parametric noise is quite different than that of additive noise, because the stability of the system is influenced by the variation of the parameters.

The study of differential equations with randomly varying coefficients seems to be rather recent. It is not our intention to survey the literature on this subject in this introduction. The papers by Samuels and Eringen (1959), Ariaratnam and Graefe (1965, I, II and III), Leibowitz (1962), Tikhonov (1959), Astrom (1965), Gray and Caughay (1965) and more recently by Morisson (1972) are the most relevant papers to the present analysis, and contain sufficiently complete list of other references. A more complete survey of literature on the stability of linear systems with stochastic coefficients was presented by Krist (1971).

In the field of reactor system analysis, the first attempt to account for fluctuating parameters was done by Akcasu (1961). He proposed to explain the observed oscillatory trains in EBWR, and the random scrams that occurred at high power levels in terms of the concept of "mean square instability" in a simple mathematical model, i.e. a second order differential equation with a randomly varying damping coefficient. Dutré (1964, 1968) presented an approximate method to determine the autocorrelation function and power spectral density of the reactor power assuming that the reactivity and neutron source variations are white Gaussian noise processes and independent of each other. Williams (1969) attacked the same problem using the Fokker–Planck theory† and discussed the validity of the conventional linearization procedure. Krist and Poncelet (1973) presented a systematic investigation of the effect of randomly varying microscopic parameters on the stability of power reactors again using the Fokker–Planck theory. They concluded among others that the negative cross-correlation of the random parameters tends to stabilize the system in the mean and mean square where as the autocorrelation of the same parameters acts as a destabilizing effect. Most recently, Gotoh applied a diagrammatic technique to study the power correlation function when the reactivity insertion is a Gaussian random process.

In this paper we present the Fokker–Planck theory in a more general form, and determine the autocorrelation function and the power spectral

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† Fokker–Planck theory was also used by Dalfes (1963) to investigate reactor noise due to neutron statistics.

density of the reactor power when the reactivity and source variations are white noise processes but not necessarily independent of each other. We clarify the discrepancy between the results of this paper and those obtained previously. Since the Fokker–Planck equation breaks down when the parametric and the additive noise processes are not white, we formulate the same problem in the absence of delayed neutrons, assuming that the reactivity variations are Gaussian but not necessarily white, and the source noise is arbitrary. As an example we calculate the mean power in the case of narrow-band reactivity noise. We reproduce the results of the Fokker–Planck theory as a special case assuming white Gaussian noise processes.

2. APPLICATION OF THE FOKKER–PLANCK THEORY

The kinetic equations of a point reactor are

$$\frac{dP(t)}{dt} = [K(t) - 1]P(t) + \sum_{i=1}^{I} \lambda_i C_i(t) + S(t) \quad (1a)$$

$$\frac{dC_i(t)}{dt} = a_i P(t) - \lambda_i C_i(t), \quad i = 1, 2, \ldots, I \quad (2a)$$

where $P(t)$ and $C_i(t)$ are the reactor power and delayed neutron precursor densities, respectively, $K(t)$ is the reactivity insertion in dollars, $S(t)$ is the external source of neutrons, λ_i and β_i are the decay constant and relative abundance of delayed neutrons in the ith group, respectively. In these equations the ratio of the mean generation time l to the fraction of delayed neutrons β is chosen as the unit of time so that $l/\beta = 1$. We assume that $K(t)$ and $S(t)$ are random functions of time with the constant mean values K_0 and S_0, respectively. Thus,

$$K(t) = K_0 + k(t) \quad (2a)$$
$$S(t) = S_0 + s(t) \quad (2b)$$

where $\langle k(t) \rangle = 0$ and $\langle s(t) \rangle = 0$. The fluctuations $k(t)$ and $s(t)$ are taken to be stationary random processes with known correlation functions

$$\phi_k(\tau) = \langle k(t)k(t + \tau) \rangle \quad (3a)$$
$$\phi_s(\tau) = \langle s(t)s(t + \tau) \rangle \quad (3b)$$
$$\phi_{ks}(\tau) = \langle k(t)s(t + \tau) \rangle \quad (3c)$$

The power spectral densities, which are the Fourier transforms of these functions will be denoted by $G_k(\omega)$, $G_s(\omega)$ and $G_{ks}(\omega)$, respectively.

The problem we wish to attack is to determine the correlation and power spectral density matrix of the

outputs $P(t)$, $C_1(t)$, $\ldots$, $C_I(t)$ in terms of the correlation functions (or power spectral densities) of the input noise $k(t)$ and $s(t)$. This problem can be considered as a special case of a more general problem characterized by an $n \times n$ system of stochastic differential equations of the form

$$\frac{dX(t)}{dt} = g[X(t)] + \alpha(t)X(t) + \alpha_0(t) \quad (4)$$

where $X(t)$ is the output (or state) vector, $g(X)$ is a vector whose components are, in general, non-linear functions of vector X, $\alpha(t)$ is an $n \times n$ square matrix whose components $\alpha_{ij}(t)$ are stationary random processes representing the random variations of the coefficients (called the parametric noise), and $\alpha_0(t)$ is a vector whose components $\alpha_{j0}(t)$ are also stationary random processes representing the variations in the external sources (called the additive noise). Without loss of generality we may assume that

$$\langle \alpha_{j\mu}(t) \rangle = 0$$
$$(\mu = 0, 1, 2, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots, n) \quad (5)$$

because we can always include the mean values in $g(X)$. In the case of one group of delayed neutrons, the point kinetic equations (1) are reproduced from the above general form with $X(t) = [P(t), C(t)]$ and

$$g(X) = MX + N \quad (6a)$$

where

$$M = \begin{bmatrix} (K_0 - 1) & \lambda \\ 1 & -\lambda \end{bmatrix}, \quad N = \begin{bmatrix} S_0 \\ 0 \end{bmatrix} \quad (6b)$$

and

$$\alpha(t) = \begin{bmatrix} k(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \alpha_0(t) = \begin{bmatrix} s(t) \\ 0 \end{bmatrix} \quad (6c)$$

If we define an equilibrium state X_0 by

$$MX_0 + N = 0 \quad (7)$$

and consider the incremental output $\delta X(t) = X(t) - X_0$, then the point kinetic equations become

$$\frac{d \, \delta X}{dt} = M \, \delta X + \alpha \, \delta X + \beta_0 \quad (8)$$

where $\beta_0 \equiv \alpha X_0 + \alpha_0$. Since $\langle \alpha_{ij} \rangle = \langle \beta_{j0} \rangle = 0$, (8) is the linear version of (4). The Fokker–Planck theory can be developed using the general form (4) as easily as using (8). It is known that the output $X(t)$ forms a Markov process (Leibowitz, 1962; Ariaratnam and Graefe, 1965) if $\alpha(t)$ and $\alpha_0(t)$ are white noise processes, viz.

$$\langle \alpha_{i\nu}(t)\alpha_{j\nu}(t') \rangle = 2D_{i\mu j\nu}\delta(t - t') \quad (9)$$

where

$$i, j = 1, 2, \ldots, n; \quad \mu, \nu = 0, 1, 2, \ldots, n.$$

If, in addition, they are Gaussian, the first-order probability density $f(\mathbf{X}, t)$ and the conditional probability density $W(\mathbf{X}, t \mid \mathbf{X}_0, t_0)$ for $t > t_0$ of the output Markov process satisfy the Fokker–Planck equation

$$\frac{\partial f(\mathbf{X}, t)}{\partial t} + \frac{\partial}{\partial X_i} (A_i f)$$
$$- \frac{1}{2} \frac{\partial^2}{\partial X_i \, \partial X_j} (B_{ij} f) = 0 \quad (10)$$

where summation convention over repeated indices i and j is implied. The coefficients A_i and B_{ij} are defined by

$$A_i(\mathbf{X}) = \lim_{\delta t \to 0} \frac{\langle \delta X_i \rangle}{\delta t} \quad (11a)$$

and

$$B_{ij}(\mathbf{X}) = \lim_{\delta t \to 0} \frac{\langle \delta X_i \, \delta X_j \rangle}{\delta t}. \quad (11b)$$

Evaluating δX_i from (4), these coefficients are determined (Ariaratnam and Graefe, 1965, III) as

$$A_i(\mathbf{X}) = g_i(\mathbf{X}) + D_{i1lj}X_j + D_{i1l0} \quad (12)$$

and

$$B_{ij}(\mathbf{X}) = 2[D_{iljm}X_lX_m$$
$$+ (D_{ilj0} + D_{i0jl})X_l + D_{i0j0}] \quad (13)$$

again with the summation convention over $l, m = 1, 2, \ldots, n$.

The solution of the Fokker–Planck equation is available only in some special cases (Wang and Uhlenbeck, 1945; Chandrasekhar, 1943). However, if only the moments of the output are of interest, as is the case in the present paper, it is possible to obtain the relevant moment equations directly from the Fokker–Planck equations, without having to solve it, as

$$\frac{d\langle \mathbf{X}(t) \rangle}{dt} = \langle \mathbf{A}(\mathbf{X}) \rangle, \quad (14)$$

$$\frac{d\langle \mathbf{X}(t)\mathbf{X}^T(t) \rangle}{dt}$$
$$= \langle \mathbf{X}\mathbf{A}^T(\mathbf{X}) \rangle + \langle \mathbf{A}(\mathbf{X})\mathbf{X}^T \rangle + \langle \mathbf{B}(\mathbf{X}) \rangle, \quad (15)$$

$$\frac{\partial \langle \mathbf{X}(t)\mathbf{X}^T(t_0) \rangle}{\partial t} = \langle \mathbf{A}[\mathbf{X}(t)]\mathbf{X}^T(t_0) \rangle, \quad t > t_0. \quad (16)$$

The last equation is obtained from the Fokker–Planck equation for $W(\mathbf{X}, t \mid \mathbf{X}_0, t_0)$ multiplying it by $f(\mathbf{X}_0, t_0)\mathbf{X}\mathbf{X}_0^T$, integrating over $\mathbf{X}$ and $\mathbf{X}_0$ and using

$$\langle \mathbf{X}(t)\mathbf{X}^T(t_0) \rangle$$
$$= \int d\mathbf{X} \int d\mathbf{X}_0 \, \mathbf{X}\mathbf{X}_0^T \, W(\mathbf{X}, t \mid \mathbf{X}_0, t_0) \, f(\mathbf{X}_0, t_0).$$

In the case of one group of delayed neutrons $(a_i = 1)$, we obtain

$$\mathbf{A}(\mathbf{X}) = \mathbf{\Lambda}\mathbf{X} + \mathbf{\Lambda}_0 \quad (17a)$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} (K_0 + \frac{1}{2}G_k - 1) & \lambda \\ 1 & -\lambda \end{bmatrix},$$
$$\mathbf{\Lambda}_0 = \begin{bmatrix} S_0 + \frac{1}{2}G_{ks} \\ 0 \end{bmatrix} \quad (17b)$$

and

$$\mathbf{B}(\mathbf{X}) = \begin{bmatrix} G_kP^2 + 2G_{ks}P + G_s & 0 \\ 0 & 0 \end{bmatrix} \quad (17c)$$

where we have defined

$$\left. \begin{array}{l} \phi_k(\tau) = G_k\delta(\tau) \\ \phi_s(\tau) = G_s\delta(\tau) \\ \phi_{ks}(\tau) = G_{ks}\delta(\tau) \end{array} \right\}. \quad (18)$$

Substituting these into (14), (15 and 16) we obtain

$$\frac{d\langle \mathbf{X} \rangle}{dt} = \mathbf{\Lambda}\langle \mathbf{X} \rangle + \mathbf{\Lambda}_0, \quad (19)$$

$$\frac{d\langle \mathbf{X}\mathbf{X}^T \rangle}{dt} = \langle \mathbf{X}\mathbf{X}^T \rangle\mathbf{\Lambda}^T + \mathbf{\Lambda}\langle \mathbf{X}\mathbf{X}^T \rangle$$
$$+ \langle \mathbf{X} \rangle\mathbf{\Lambda}_0^T + \mathbf{\Lambda}_0\langle \mathbf{X} \rangle^T + \langle \mathbf{B} \rangle, \quad (20)$$

$$\frac{\partial \langle \mathbf{X}(t)\mathbf{X}(t_0)^T \rangle}{\partial t}$$
$$= \mathbf{\Lambda}\langle \mathbf{X}(t)\mathbf{X}^T(t_0) \rangle + \mathbf{\Lambda}_0\langle \mathbf{X}^T \rangle, \quad t > t_0. \quad (21)$$

(i) *The behavior of the mean*

It is interesting to write the equations for the mean (19) explicitly

$$\frac{d\langle P \rangle}{dt} = (K_0 + \frac{1}{2}G_k - 1)\langle P \rangle$$
$$+ \lambda\langle C \rangle + S_0 + \frac{1}{2}G_{ks} \quad (22a)$$

$$\frac{d\langle C \rangle}{dt} = \langle P \rangle - \lambda\langle C \rangle \quad (22b)$$

which are different than the conventional equations of reactor kinetics in the absence of fluctuations:

$$\frac{dX(t)}{dt} = MX(t) + N \qquad (23)$$

where M and N were defined in (6b). The stability of (19) is determined by the characteristic values of Λ where as that of the deterministic system (23) depends on the characteristic values of M. The stability of the mean is readily obtained from (22a) as

$$K_0{}^* \equiv K_0 + \tfrac{1}{2}G_k < 0. \qquad (24)$$

If this condition is satisfied the characteristic values of Λ are negative, and $\langle X(t) \rangle$ decays exponentially as

$$\langle X(t) \rangle = e^{\Lambda t}\langle X(0) \rangle + (I - e^{\Lambda t})\langle X \rangle \quad (25a)$$

to its stationary value $\langle X \rangle$ determined by $\Lambda\langle X \rangle + \Lambda_0 = 0$. In the case of one-group of delayed neutrons the latter yields

$$\langle P \rangle = -(S_0 + \tfrac{1}{2}G_{ks})/(K_0 + \tfrac{1}{2}G_k),$$
$$\langle C \rangle = \langle P \rangle/\lambda. \quad (25b)$$

In the absence of fluctuations, i.e. when G_{ks} and G_k are zero, the mean power is $S_0/(-K_0)$. The fluctuations in the reactivity tend to increase this value because $G_k > 0$ and $K_0 < 0$. On the other hand the fluctuations in the external source, when they are correlated with the reactivity fluctuations, can influence the mean power in either direction depending on the sign of the cross-correlation G_{ks}. A positive cross-correlation tends to increase $\langle P \rangle$, and vice versa. In the absence of correlations, the fluctuations in the external source does not affect the mean power.

It is observed in (25) that $\langle P \rangle$ may become negative if $S_0 + (\tfrac{1}{2})G_{ks} < 0$. This non-physical situation, we think, is a consequence of the assumption that $S(t) = S_0 + s(t)$ is Gaussian, which is itself non-physical because $S(t)$ is always positive insofar as it represents a neutron source.

The fact that the equation of the mean is different than that of the conventional reactor kinetics is at variance with the conclusions by Williams (1969), which imply $\Lambda = M$ and $\Lambda_0 = N$ in (19). The origin of this discrepancy lies in the approaches taken in solving differential equations with random coefficients, and in the interpretation of the physical content of such equations, rather than in an error. The "mathematical" approach taken by Williams is applicable to stochastic differential equations essentially of the form

$$\delta X = MX\delta t + \delta\beta X + \delta\beta_0$$

where $\delta\beta$ is an $n \times n$ matrix and $\delta\beta_0$ an $n \times 1$ matrix whose elements are increments of Wiener processes (Ariaratnam and Graefe, 1965 II). The "physical" approach used in this paper (also in Akcasu, 1961, and Krist and Poncelet, 1973) is applicable to stochastic differential equations of the form, in the present context, of

$$\dot{X} = MX + \alpha X + \alpha_0$$

where $\alpha_{ij}(t)$ are white Gaussian processes. The functions $\beta_{ij}(t)$ defined as the Wiener processes are related to $\alpha_{ij}(t)$ by the stochastic integrals

$$\beta_{ij}(t) = \int_0^t du\, \alpha_{ij}(u).$$

Although they may seem to be mathematically equivalent, these two differential equations "model" two different physical systems, as pointed out by Ariaratnam and Graefe (1965 II), and lead to two different Fokker–Planck equations. In the first case the system receives a random impulse in each interval δt whereas in the second case the system receives a continual random disturbance during its time evolution. Due to the pathological nature of the member functions of Wiener processes (for example, they are everywhere continuous but almost nowhere differentiable [Melsa and Sage, 1973]), the mathematical problem in the first case involves the theory of stochastic differential-integral calculus (Doob, 1953). In physical problems however $\beta_{ij}(t)$ do not truly represent Wiener processes but rather $d\beta_{ij}(t)/dt = \alpha_{ij}(t)$ are mathematical approximations to Gaussian processes with very short, but yet finite, correlation time. In this sense, $\beta_{ij}(t)$ are not so pathological in physical problems as to require a different differential–integral calculus (Gray and Caughey, 1965). Leibowitz (1962) pointed out that the difference between the Fokker–Planck equations in these two cases lies in the evaluation of A_i and B_{ij} using (11), which involves the stochastic integral

$$\int_t^{t+\delta t} du\, \alpha(u)X(u).$$

In the theory of stochastic integration the expected value of this integral is "defined" to be zero whereas in the physical approach it is not zero.

It is concluded by Gray and Caughay (1965) that if $\alpha(t)$ in the stochastic differential equation is "either an approximation to white noise, or a limit of the case where $\alpha(t)$ approaches white noise, then the physical approach," as adapted in this paper, must be used. In this case one is allowed to use the

ordinary differential-integral calculus in solving differential equations with random coefficients (Melsa and Sage, 1973).

(ii) The behavior of the correlation matrix

The discussion of the correlation matrix $\boldsymbol{\phi}(t, t_0) \equiv \langle \mathbf{X}(t)\mathbf{X}^{\mathrm{T}}(t_0) \rangle$ is simplified by writing (30) and (21) directly in terms of the covariance matrix

$$\mathbf{C}(t, t_0) = \boldsymbol{\phi}(t, t_0) - \langle \mathbf{X}(t) \rangle \langle \mathbf{X}^{\mathrm{T}}(t_0) \rangle. \quad (26)$$

The time dependence of the mean $\langle \mathbf{X}(t) \rangle$ is obtained from (19). One finds

$$\frac{d\mathbf{C}(t, t)}{dt} = \mathbf{C}\boldsymbol{\Lambda}^{\mathrm{T}} + \boldsymbol{\Lambda}\mathbf{C} + \langle \mathbf{B} \rangle \quad (27)$$

$$\frac{\partial \mathbf{C}(t, t_0)}{\partial t} = \boldsymbol{\Lambda}\mathbf{C}(t, t_0), \qquad t > t_0. \quad (28)$$

In (27), $\langle \mathbf{B} \rangle$ depends on $\mathbf{C}(t, t_0)$ as we see from (17c). To show this dependence explicitly, we introduce

$$\langle \mathbf{B} \rangle = \boldsymbol{\Gamma}C_{11}(t) + \boldsymbol{\Gamma}_0(t) \quad (29a)$$

where

$$\boldsymbol{\Gamma} = \begin{bmatrix} G_k & 0 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\Gamma}_0 = \begin{bmatrix} G_k\langle P(t) \rangle^2 + 2G_{ks}\langle P(t) \rangle + G_s & 0 \\ 0 & 0 \end{bmatrix}. \quad (29b)$$

The time-dependence of the variance matrix $\mathbf{C}(t, t)$, and the covariance matrix $\mathbf{C}(t, t_0)$ for $t > t_0$ and a fixed t_0 can be investigated through (27) and (28). The time dependence of $\langle P(t) \rangle$ in $\boldsymbol{\Gamma}_0(t)$, which appears in (27) is to be obtained from the equation of the mean.

The stability of $\mathbf{C}(t, t)$ can be investigated by converting (27) into a 3×3 system by introducing a new vector with components $C_{11}(t)$, $C_{12}(t)$ and $C_{22}(t)$, and considering the roots of the characteristic equation of the resulting set of equations:

$$\begin{vmatrix} 2(K_0 + G_k - 1) - s & 2\lambda & 0 \\ 1 & (K_0 + \tfrac{1}{2}G_k - 1 - \lambda) - s & \lambda \\ 0 & 2 & -(2\lambda + s) \end{vmatrix} = 0. \quad (30)$$

One may apply, for example, Hurwitz determinants to obtain the conditions for stability. It is noted that the stability of $\mathbf{C}(t, t)$ is influenced only by the reactivity fluctuations (parametric noise). In the absence of the reactivity fluctuations, viz. $G_k = 0$,

(30) reduces to $K_0 < 0$ which is the stability condition for the deterministic system. In the absence of delayed neutrons ($\lambda = 0$ and $K_0 - 1 \to K_0$), we obtain:

$$K_0 + G_k < 0 \quad (31)$$

for the stability of $\mathbf{C}(t, t)$. In general, (30) imposes an upper bound on G_k beyond which $\mathbf{C}(t, t)$ diverges. Derivation an analytical expression for this upper bound in terms of K_0 and λ seems to be tedius, but straight forward, and will not be attempted here. When G_k is sufficiently small, $\mathbf{C}(t, t)$ approaches a stationary value $\mathbf{C}$ which is determined by

$$\mathbf{C}\boldsymbol{\Lambda}^{\mathrm{T}} + \boldsymbol{\Lambda}\mathbf{C} + \boldsymbol{\Gamma}C_{11} + \boldsymbol{\Gamma}_0 = 0. \quad (32)$$

This equation may be considered as a generalization of the Einstein's relation (Lax, 1960) to parametric noise. The solution of (32) for $\mathbf{C} = \{C_{pp}, C_{pc}, C_{cc}\}$ is found as

$$C_{pp} = \tfrac{1}{2}[G_k\langle P \rangle^2 + 2G_{ks}\langle P \rangle + G_s] \\ \times \left[\frac{K_0{}^*}{K_0{}^* - \lambda} - (K_0{}^* + \tfrac{1}{2}G_k)\right]^{-1}, \quad (33a)$$

$$C_{pc} = C_{cp} = \frac{C_{pp}}{\lambda - K_0{}^*} = \lambda C_{cc}. \quad (33b)$$

Equation (33a) demonstrates the manner in which the reactivity noise and the random fluctuations in the external source contribute to the variance of the reactor power. In particular, the effect of the correlation between these two noise sources is displayed explicitly. The contribution of the reactivity noise is proportional to the square of the mean power, whereas that of the cross-correlation depends on the mean power linearly. The denominator of (33a) is positive for the values of G_k for which a stationary state is possible (recall that $K_0{}^* < 0$ by virtue of (24)).

(iii) Power spectral density

In stationary state $\mathbf{C}(t, t_0)$ depends on $(t - t_0)$, and (28) reduces to

$$\frac{d\mathbf{C}(\tau)}{d\tau} = \boldsymbol{\Lambda}\mathbf{C}(\tau), \qquad \tau > 0. \quad (34)$$

The values of $\mathbf{C}(\tau)$ for $\tau < 0$ is obtained from the relation $\mathbf{C}(-\tau) = \mathbf{C}(\tau)^{\mathrm{T}}$. The Laplace transform of (36) yields

$$\overline{\mathbf{C}}(s) = (s - \boldsymbol{\Lambda})^{-1}\mathbf{C} \quad (35)$$

where $\mathbf{C}$ is given by (32). The power spectral density matrix $\mathbf{G}(w)$ is found from $\overline{\mathbf{C}}(s)$ as $\mathbf{G}(w) = \mathbf{C}(iw) + \overline{\mathbf{C}}(-iw)^{\mathrm{T}}$

$$\mathbf{G}(w) = (iw - \boldsymbol{\Lambda})^{-1}\mathbf{C} + \mathbf{C}(-iw - \boldsymbol{\Lambda}^{\mathrm{T}})^{-1} \quad (36)$$

which can also be written as

$$G(w) = (iw - \Lambda)^{-1}\langle B\rangle(-iw - \Lambda^T)^{-1} \quad (37)$$

if one eliminates C in (36) in favor of $\langle B\rangle$ using $\Lambda C + C\Lambda^T + \langle B\rangle = 0$ (Lax, 1966). However, $\langle B\rangle$ still depends on C (cf. 32). The power spectral density of the power fluctuations is $G_{11}(w)$ which we denote by $G_{pp}(w)$. From (37):

$$G_{pp}(w) = |(iw - \Lambda_{11}^{-1}|^2 \langle B_{11}\rangle \quad (38)$$

Using

$$\langle B_{11}\rangle = G_k[C_{pp} + \langle P\rangle^2] + 2G_{ks}\langle P\rangle + G_s$$
$$= 2|K_0^*| [1 + \lambda + |K_0^*|]C_{pp}/(\lambda + |K_0^*|) \quad (39)$$

we obtain the normalized power spectral density as

$$\frac{G_{pp}(\omega)}{C_{pp}}$$
$$= 2|K_0^*|\frac{|K_0^*| + \lambda + 1}{|K_0^*| + \lambda}|(iw - \Lambda)_{11}^{-1}|^2. \quad (40)$$

In these expressions, $(s - \Lambda)_{11}^{-1}$ is the source transfer function of a point reactor with a negative reactivity:

$$(s - \Lambda)_{11}^{-1} = \frac{Z(s)}{1 - K_0^*Z(s)} = \frac{s + \lambda}{(s - w_1)(s - w_2)} \quad (41)$$

where $Z(s)$ is the zero power transfer function, i.e. in the case of one group model $Z(s)^{-1} = s[1 + 1/(s + \lambda)]$, and $-w_1$ and $-w_2$ are the roots of the inhour equation $1 = K_0^*Z(s)$. Thus, the power spectral density becomes

$$\frac{G_{pp}(w)}{C_{pp}} = 2|K_0^*|\frac{|K_0^*| + \lambda + 1}{|K_0^*| + \lambda}$$
$$\times \left|\frac{\lambda + iw}{(w_1 + iw)(w_2 + iw)}\right|^2. \quad (42a)$$

It is concluded that the fluctuations in the external source (additive noise) do not affect the shape of the power spectral density even when it is correlated with the reactivity variations. The effect of the reactivity noise manifests itself in $K_0^* = K_0 + G_k/2$, reducing the magnitude of the negative reactivity from $|K_0|$ to $|K_0 + G_k/2|$. This reduction causes the corner frequencies w_1 and w_2 to shift towards lower frequencies when the magnitude of reactivity noise G_k is increased. The corner frequency $w = \lambda$ associated with the delayed neutrons is not affected. The dependence of the corner frequencies on the magnitude of the input noise is a non-linear effect. These conclusions have been varified by analogue computer studies (Karasulu, 1975).

The Fokker–Planck theory is not applicable when the reactivity and source variations are not white noise processes. In such cases, one constructs an approximate solution of the point kinetic equations for an arbitrary reactivity insertion and external source (exact solutions are in general not available) and then investigates the statistical properties of the output when the inputs are random functions of time. The conventional linearized kinetics is the simplest method of approximation which removes the parametric noise completely. A more accurate method of approximation is the logarithmic linearization procedure (Akcasu, et al., 1971). We postpone the application of this technique to noise analysis to a later paper. In the absence of delayed neutrons, however, the point kinetic equations can be solved exactly for arbitrary reactivity input and external source. Hence, an exact noise theory can be developed in this special case as shown below.

3. POINT-REACTOR MODEL WITHOUT DELAYED NEUTRONS

The kinetic equation for such a reactor is*

$$\frac{dP(t)}{dt} = K(t)P(t) + S(t) \quad (43)$$

which is readily solved as†

$$P(t) = \int_0^\infty du\, S(t - u) \exp\left[\int_0^u dv\, K(t - v)\right] \quad (44a)$$

$$= \frac{\partial}{\partial\alpha}\int_0^\infty du \exp\left[\alpha S(t - u) + \int_0^u dv\, K(t - v)\right]\Bigg|_{\alpha=0}. \quad (44b)$$

If $S(t)$ and $K(t)$ are correlated and both Gaussian, the second form proves to be more convenient [Dutré, 1968]. If they are independent, the first form is adequate, and one does not have to restrict $S(t)$ to Gaussian processes for an exact analysis. To illustrate this point, we calculate the mean power $\langle P(t)\rangle$ first assuming $S(t)$ and $K(t)$ to be Gaussian. Using the following property of a Gaussian process

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* Here again we use the ordinary differential–integral calculus to solve (43) because $k(t)$ always has a finite correlation time insofar as it represents reactivity variations in a reactor (see the discussions at the end of Section 2).

† In (44) the initial value of $P(t)$ does not appear because we have used

$$\lim_{t_0\to -\infty} P(t_0) \exp\left(\int_{t_0}^t K(u)\, du\right) = 0$$

which is justified in stationary systems.

$\xi(t)$ (e.g. Tikhonov, 1959)

$$\langle \exp [\xi] \rangle = \exp \{\langle \xi \rangle + \tfrac{1}{2} Var[\xi]\} \qquad (45)$$

we obtain $\langle P(t) \rangle$ from (44b) after a few steps as

$$\langle P(t) \rangle = \int_0^\infty du \left[S_0 + \int_0^u dv\, \phi_{ks}(v) \right]$$
$$\times \exp\left\{ \left[K_0 + \frac{1}{2} \int_{-u}^{+u} dv \left(1 - \frac{|v|}{u} \right) \phi_{ks}(v) \right] u \right\}. \qquad (46)$$

In this derivation, $s(t)$ and $k(t)$ are not restricted to white noise processes. In this sense, (46) is an extention of (25b) in which $\phi_k(\tau) = G_k \delta(\tau)$ and $\phi_{ks}(\tau) = G_{ks} \delta(\tau)$ were assumed. Under the same conditions, (46) reduces to (25b) provided $K_0{}^* = K_0 + G_k/2 < 0$, which is the condition for the stability of the mean.

If $S(t)$ and $K(t)$ are independent of each other, and only $K(t)$ is Gaussian, we use (44a) and reproduce (46) directly with $\phi_{ks} = 0$, by simply taking the ensemble average of the integrand. Since $S(t)$ and $K(t)$ are uncorrelated, $\langle S(t - u) \rangle$ and $\langle \exp [\int_0^u dv K(t - v)] \rangle$ are evaluated separately. In what follows we shall use (44c) and assume $S(t)$ and $K(t)$ to be always independent of each other.

We can express $\langle P(t) \rangle$ in terms of the power spectral density $G_k(w)$ of the reactivity noise as

$$\langle P(t) \rangle = S_0 \int_0^\infty du$$
$$\times \exp\left\{ u K_0 + \frac{1}{2} \int_{-\infty}^{+\infty} df\, G_k(w) \left[\frac{\sin wu/2}{w/2} \right]^2 \right\} \qquad (47)$$

which is more convenient if $G_k(w)$, rather than $\phi_k(\tau)$, is specified (Williams, 1971). For example, suppose the reactivity insertion is a narrow-band noise centered about a frequency w_0 so that

$$G_k(w) = \pi \sigma_k{}^2 [\delta(w + w_0) + \delta(w - w_0)] \qquad (48)$$

where $\sigma_k{}^2$ is the variance of the process. Substitution of (48) into (47) yields

$$\langle P(t) \rangle = S_0 \int_0^\infty du$$
$$\times \exp\left\{ u K_0 + \frac{1}{2} \sigma_k{}^2 \left[\frac{\sin w_0 u/2}{w_0/2} \right]^2 \right\}. \qquad (49)$$

Note that $\langle P(t) \rangle$ does not involve the power spectral density of the source process, demonstrating the difference between the roles played by the parametric and additive noise processes.

The autocorrelation function of the power response is obtained from (44a):

$$\langle P(t) P(t + \tau) \rangle$$
$$= \int_0^\infty du \int_0^\infty dv\, [S_0{}^2 + \phi_s(u - v + \tau)]$$
$$\times \langle \exp [Q(t, u) + Q(t + \tau, v)] \rangle \qquad (50a)$$

where

$$Q(t, u) \equiv \int_0^u dv\, K(t - v) \qquad (50b)$$

The following relations are needed to proceed further:

$$\langle \exp [Q(t, u) + Q(t + \tau, v)] \rangle$$
$$= \exp \{\langle Q(t, u) \rangle + \langle Q(t + \tau, v) \rangle$$
$$+ \tfrac{1}{2}[Var [Q(t, u)] + Var [Q(t + \tau, v)] +$$
$$+ 2 \operatorname{Cov} [Q(t, u), Q(t + \tau, v)]\}, \qquad (51)$$

$$\langle Q(t, u) \rangle = u K_0, \qquad (52)$$

$$Var [Q(t, u)] = \int_0^u dx \int_0^u dy\, \phi_k(x - y)$$
$$= \int_{-u}^{+u} dx [u - |x|] \phi_k(x), \qquad (53)$$

$$\operatorname{Cov} [Q(t, u), Q(t + \tau, v)]$$
$$= \int_0^u dx \int_0^v dy\, \phi_k(x - y + \tau). \qquad (54)$$

With these results, we can calculate $\langle P(t) P(t + \tau) \rangle \equiv \phi_{pp}(\tau)$ for any arbitrary random source, and a Gaussian reactivity insertion with an arbitrary autocorrelation function. For example we substitute $\phi_k(\tau) = \sigma_k{}^2 \cos w_0 \tau$ and $\phi_s = 0$ in the case of a narrow-band reactivity noise and a constant source. We shall present the results in the case of white noise, i.e. $\phi_s(\tau) = G_s \delta(\tau)$ and $\phi_k(\tau) = G_k \delta(\tau)$ to compare with the results obtained in Section 2. In this case, (53) and (54) become

$$Var [Q(t, u)] = u G_k \qquad (55a)$$

$$\operatorname{Cov} [Q(t, u), Q(t + \tau, v)] = G_k \begin{cases} \begin{rcases} [u H(v - \tau - u) + (v - \tau) H(u - v + \tau)], & v > \tau \\ 0, & v < \tau \end{rcases} \tau > 0 \\ \begin{rcases} [v H(u - |\tau| - v) + (u - |\tau|) H(v + |\tau| - u)], & u > |\tau| \\ 0, & u < \tau \end{rcases} \tau < 0 \end{cases} \qquad (55b)$$

2

Substituting (55) into (50a), performing the double integration over u and v, and substracting $\langle P \rangle^2$ one obtains after somewhat lengthy calculation

$$C_{pp}(\tau) = \frac{1}{2[|K_0^*| + G_k/2]}$$

$$\times \left[\left(\frac{S_0}{K_0^*} \right)^2 G_k + G_s \right] e^{-|K_0^*|\tau}. \quad (56)$$

In the derivation of this result one has to assume $K_0 + G_k < 0$ which is the condition for stability of the mean (cf. 31). The variance $C_{pp}(0)$ determined from (56) is identical to (33a), which was obtained through the Fokker–Planck equation, when the delayed neutrons are ignored, and G_{ks} is set to zero in the latter. It is interesting to notice that (33a) was obtained with the assumption that $S(t)$ is Gaussian, whereas (56) is free from this restriction. The power spectral density $G_{pp}(w)$ obtained from (56) is also identical to (42a) obtained in the Fokker–Planck theory in the absence of delayed neutrons.

4. CONCLUSIONS

It is shown that the autocorrelation function and spectral density of the power response of a point reactor can be determined exactly through the Fokker–Planck theory when the source and reactivity noise are white Gaussian processes, which may be correlated in general. The effect of the non-linearity on the shape of the power spectral density is manifested a as shift towards lower frequencies in the corner frequencies when the magnitude of the reactivity is increased.

An exact noise analysis when the reactivity noise is a non-white Gaussian process and independent of the source noise which is allowed to be an arbitrary random process, is still possible in the absence of delayed neutrons. We have demonstrated this by calculating explicitly the mean power for a narrow-band reactivity noise.

In the presence of delayed neutrons, there is no exact solution when the reactivity noise is non-white. An approximate noise analysis based on logarithmic linearization is currently being developed.

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