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Causality Structure and the Weierstrass Theorem*

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In a recent paper P. M. Prenter has shown that the Weierstrass theorem can be lifted up to a real separable Hilbert space H. In this paper H is equipped with an identity resolving orthoprojector chain. The Weierstrass type result of Prenter, namely, if f is any continuous function on H, then there exists a finite order approximating polynomic operator on every compact $K \subset H$, is sharpened by the extension: if f is strictly causal (strictly anticausal) then the polynomic approximation can also be strictly causal (strictly anticausal). Other extensions in the same spirit are developed and the results are interpreted in the setting of Volterra operators on L_2.

1. INTRODUCTION

Let X, Y be Hilbert spaces and K a compact subset of X. Let $C(K)$ denote the space of continuous functions on X to Y, restricted to K. The norm on $C(K)$ is uniform

$$\| f - g \| = \max_{x \in K} \| f(x) - g(x) \| .$$

When $X = Y = R$, the real line, the classical Weierstrass theorem states that the polynomials over R are dense in $C(K)$. If $X = R^n$ and $Y = R$ then the Stone–Weierstrass theorem shows that the polynomials in n variables are dense in $C(K)$. Finally if $X = Y = H$, a real separable Hilbert space, then Prenter [1] has shown that the polynomic operators are dense in $C(K)$.

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The above line of development recognizes and utilizes only the topological properties of the functions in question. In most applications, however, the functions also have a causality structure which is equally important. In the present paper we show that the causality structure of a function can, in part, be superimposed on its polynomic approximation without disturbing the Prenter result.

To give some indication of our extension consider $H = L_2(0, T)$ and f a continuous function on H while $K \subset H$ is compact. For arbitrary $\epsilon > 0$ the Prenter result gives a finite polynomic operator g such that

$$\max_{x \in K} \| f(x) - g(x) \| < \epsilon.$$

A minor extension of the Prenter result is that g can be computed by the formula

$$g(x) = \sum_{n=1}^{N} \int_{[0,T]^n} \cdots \int Q_n(\cdot, s_1 \cdots s_n) \prod_1^n x(s_i)\, ds_i ,$$

where each of the N kernels is separable, that is,

$$Q_n(t, s_1, \ldots, s_n) = \sum_{l=1}^{q} \sum_{i,\ldots,k=1}^{m} \Gamma(i, j, \ldots, k, l)\, e_i(s_1)\, e_j(s_2) \cdots e_k(s_n) f_l(t),$$

where $\{e_i\}, \{f_l\} \subset L_2(0, T)$.

A major extension consists of the proof that if f is causal (strictly causal) (example and definition follow) then the above approximation is valid with kernel satisfying

$$Q_n(t, s_1, \ldots, s_n) = 0, \max\{s_i\} > t.$$

With regard to the strict causality condition it suffices for the moment to give an example of such a function in $L_2(0, T)$.

EXAMPLE 1. Let $\hat{h}$ be any Lipschitz continuous function on R and h the map on $L_2(0, T)$ determined by

$$(hx)(t) = \hat{h}(x(t)) \qquad t \in [0, T].$$

For $\{t_j\}$ any sequence satisfying $0 < t_0 < t_1 \cdots < T$ and $\{g_j\} \in l_1$ the map T_1 is computed by

$$(T_1 x)(t) = \sum_j g_j x(t - t_j), \qquad t \in [0, T].$$

The map T_2 is determined for $g \in L_1[0, T]$ by

$$(T_2 x)(t) = \int_0^t g(t - \tau)\, x(\tau)\, d\tau, \qquad t \in [0, T].$$

Then T_1, T_2 and the composition $f = h(T_1 + T_2)$ all satisfy the strict causality requirement described in Section 2.

2. Mathematical Preliminaries

The properties of multilinear operators have been considered in [1–6] and they shall be very tersely summarized here. If X is a linear space and $X^k = X \times \cdots \times X$ (k fold product) then a k-linear map W_k is a function from X^k to X which is linear in each argument, that is,

$$W_k[x_1, \ldots, \alpha x_i' + \beta x_i'', \ldots, x_k] = \alpha W_k[x_1, \ldots, x_i', \ldots, x_k] + \beta W_k[x_1, \ldots, x_i'', \ldots, x_k]$$

holds for all α, β, x_i', x_i'', and $i = 1, \ldots, k$. Each k-linear function W_k induces a k-power function

$$\hat{W}_k(x) = W_k[x, \ldots, x], \qquad x \in X.$$

A *polynomic function* f is any finite sum

$$f(x) = \sum_{k=0}^{n} \hat{W}_k(x). \tag{1}$$

We say that f is of *order* n if f is computable by this formula where n is the highest multiplicity.

Without loss in generality, we shall work with symmetric multilinear functions, that is, functions which are invariant under permutations of variables. For example W_2 is symmetric if $W_2[x, y] = W_2[y, x]$ for all $x, y \in H$. Our interest primarily is with polynomic functions in which case we shall identify with the symmetric k-linear generator of each k-power term.

The causality structure of interest in this paper is intertwined with the concept of a Hilbert resolution space. This has been dealt with in [7–11] and in conjunction with multipower problems in [2], [3], and [5]. Our review here will summarize the minimal notation necessary to proceed with the article.

Let H denote a Hilbert space and ν a linearly ordered set with minimal and maximal elements a, $b \in \nu$. A family $\mathbb{R} = \{P^t; t \in \nu\}$ of orthoprojectors on H is a resolution of the identity if

(i) $P^a(H) = 0$, $P^b(H) = H$, and $P^k(H) \supseteq P^l(H)$ whenever $k \geqslant l$;

(ii) $\mathbb{R}$ is weakly closed.

In (ii) we insist that if $\{P^i\}$ is a sequence of $\mathbb{R}$ such that $P^i x \to Px$ where P is an orthoprojector then $P \in \mathbb{R}$. In some cases we shall assume that ν is continuous and that $\mathbb{R}$ is *complete* in the sense that for every $k < l$, k, $l \in \nu$ there exists $m \in \nu$ such that $k < m < l$, moreover $\|(P^l - P^k)x\| \to 0$ as $k \to l$ for all $x \in H$. The set H, equipped with $\mathbb{R}$ is called a *Hilbert resolution space*.

A function f on H is said to be *causal* if $P^t x = P^t y \Rightarrow P^t f(x) = P^t f(y)$ for all $t \in \nu$, $x, y \in H$. The statement $P^t f = P^t f P^t$ all $t \in \nu$, is a necessary and sufficient condition for causality. A multilinear map is causal if it is causal linear in each argument. In Section 3 we shall introduce other causality structure properties appropriate to the analysis.

For ease of referral we state here the Prenter–Stone–Weierstrass theorem [1].

THEOREM 1. *If H is a real separable Hilbert space, K a compact set, $f \in C(H)$—the continuous functions on H then there exists a polynomic function p such that*

$$\max_{x \in K} \|f(x) - p(x)\| < \epsilon \qquad \text{for every } \epsilon > 0.$$

3. APPROXIMATIONS OF FINITE RANK

In this section we redevelop some of Prenter's results and sharpen others by constructive methods which facilitate the latter sections. One result is the representation theorem for approximating polynomic operators.

In the following, K is a compact set of the real separable Hilbert space H. Let E_n denote a finite $1/n$ cover of K. Such covers exist for $n = 1, 2, \ldots$ and by taking unions the condition $E_n \subset E_{n+1}$ can easily be met.

Suppose that $\{E_n\}$ is such a nested family of finite covers and that $L_n = \text{span}(E_n)$. Each L_n is finite dimensional, with dimension not exceeding the number of points in its generating E_n. It is also obvious that

$$d(a; L_n) \leqslant d(a; E_n) \leqslant 1/n,$$

where

$$d(a; A) = \inf\{\| a - x \| : x \in A\}.$$

We prove first a modification of a well-known result on compact sets (see Maurin [12] page 151).

LEMMA 1. *If the set $K \subset H$ is compact there exists a sequence of linear finite rank operators $\{S_n\}$ such that*

(a) $S_n \to I$ *pointwise on H*

(b) $S_n \to I$ *uniformly on K.*

Proof. Take S_n to be the orthogonal projection on L_n described above. Then S_n has finite rank. Moreover the containment $L_n \subset L_{n+1}$ and the separability of H implies

$$\| S_n x - x \| \to 0, \qquad x \in H$$

which proves (a) while on K

$$\| S_n x - x \| \leqslant d(x, L_n) \leqslant 1/n, \qquad x \in K$$

holds from which (b) follows.

We turn now in the approximation direction.

LEMMA 2. *If f is a continuous map on H and $K \subset H$ is compact, then there exists a sequence $\{S_n\}$ of finite rank linear operators and a polynomic operator p, such that*

$$\| f(x) - (pS_n)(x) \| < \epsilon, \qquad x \in K$$

which holds for all $\epsilon > 0$ and all $n \geqslant N$.

Proof. A polynomic operator is a finite sum of multipower operators. Recall (see [5, Appendix II]) that for any polynomic operator p there exists a continuous linear operator valued map $T(x, y)$ such that

$$p(x) - p(y) = T(x, y)(x - y), \qquad x, y \in H.$$

In particular,

$$\begin{aligned} \|(pS_n)(x) - p(x)\| &\leqslant \| T(S_n x, x)\| \cdot \| S_n x - x \| \\ &\leqslant M \cdot \| S_n x - x \|, \qquad x \in K, \end{aligned}$$

where M follows from the continuity of T and the compactness of K. Now using the results of Theorem 1 we pick p such that

$$\| f(x) - p(x) \| < \epsilon/2, \qquad x \in K.$$

The lemma then follows from the obvious norm inequalities on the expansion

$$f(x) - (pS_n)(x) = f(x) - p(x) + p(x) - (pS_n)(x), \qquad x \in H.$$

A polynomic operator p is said to be of *finite rank* if there exists a linear projection S of finite rank such that $p = pS$. The last lemma evidently implies that f is approximatable on K by a finite rank polynomic operator. (This result was also noted by Prenter [4].)

LEMMA 3. *Let W be a symmetric, n-linear function which generates a finite rank, n-power function. Then there exists two finite orthonormal sets $\{e_i\}_1^m$, $\{f_l\}_1^q$, such that*

$$W[x, y,..., z] = \sum_{l=1}^{q} \sum_{i,j\cdots k=1}^{m} \langle x, e_i\rangle \langle y, e_j\rangle \cdots \langle z, e_k\rangle \langle f_l , W[e_i , e_j ,..., e_k]\rangle f_l$$

Proof. The n-power case is a transparent extension of the bipower case which we cover here. Since W is of finite rank there exists by Lemma 2 a finite rank projection S such that $W = WS$. Let $\{e_i\}_1^m$ be an orthonormal basis for range (S). Then for $x = \sum \alpha_i e_i$ and $y = \sum \beta_j e_j$

$$W[x, y] = \sum_{i,j=1}^{m} \alpha_i \beta_j W[e_i , e_j].$$

The set $\{W[e_i , e_j]\}$ is finite and hence there exists a finite orthonormal basis $\{f_l\}_1^q$ for span$(\{W[e_i , e_j]\})$. The lemma follows by noting

$$W[e_i , e_j] = \sum_{l=1}^{q} f_l \langle f_l , W[e_i , e_j]\rangle$$

and that

$$\alpha_i = \langle x, e_i\rangle, \qquad \beta_j = \langle y, e_j\rangle.$$

COROLLARY. *If symmetric n linear W on $L_2(\Lambda)$ generates the finite rank n-power function $\hat{W}$, then $\hat{W}$ has a representation*

$$[\hat{W}(x)] (t) = \int_{\Lambda^n} \cdots \int Q(t, s_1 ,..., s_n) \prod_{1}^{n} x(s_i)\, dm(s_i)$$

where functions $\{e_i\}_1^m \{f_l\}_1^q \subset L_2(\Lambda)$ exist and scalars $\{\Gamma(i, j,..., k, l)\}$ exist such that

$$Q(t, s_1 ,..., s_n) = \sum_{l=1}^{q} \sum_{i,j,...,k=1}^{m} \Gamma(i, j,..., k, l)\, e_i(s_1)\, e_j(s_2) \cdots e_k(s_n) f_l(t).$$

Proof. It is necessary only to note in Lemma 3 that

$$\langle x, e_j\rangle = \int_{\Lambda} x(s)\, e_j(s)\, dm(s)$$

and that $\Gamma(i, j,..., k, l) = \langle f_l , W[e_i , e_j ,..., e_k]\rangle$ are scalars.

We draw these results together in a summarizing theorem.

THEOREM 2. *If f is continuous on H and $K \subset H$ is compact then there exists a finite rank polynomic operator p such that $\|f(x) - p(x)\| < \epsilon$, $x \in K$. Moreover, there exist two finite orthonormal sets $\{e_i\}_1^m$, $\{f\}_1^q$ such that*

$$p(x) = \sum_0^N \hat{W}_n(x), \qquad x \in K$$

where each $\hat{W}_n(x) = W_n[x,..., x]$ is of the form of Lemma 3. If $H = L_2(\Lambda)$ then each $\hat{W}_n$ is an integral operator with a kernel described in the corollary to Lemma 3.

4. THE APPROXIMATION OF STRICTLY CAUSAL FUNCTIONS

In this section H is always real and separable. We shall need also partitions of ν namely *finite* sets $\Omega_N = \{t_j \in \nu; t_{j-1} < t_j$ and $t_0 = a$, $t_N = b\}$. We shall adopt the conventions $P^j = P^{t_j}$ and $\Delta_j = P^j - P^{j-1}$.

DEFINITION 1. A function f on H is said to be *prestrictly causal* if there exists a partition Ω_N such that

$$f = \sum_1^N \Delta_i f P^{i-1}.$$

It is helpful to use the notation PSC to denote the set of all prestrictly causal functions. To avoid confusion $\mathbb{P}$ will denote the *finite order polynomic operators* and as before $C(K)$ denotes the continuous functions restricted to a compact set K.

LEMMA 4. *The set $\mathbb{P} \cap PSC$ is dense in the set $C(K) \cap PSC$.*

Proof. Suppose $f \in C(K) \cap PSC$ and choose $\Delta_i f P^{i-1}$ which suggests the real separable Hilbert space $P^i(H)$ in which $P^i(H) \cap K$ is compact. By Theorem 1 there exists $q^i \in \mathbb{P}$ such that $q^i: P^i(H) \to P^i(H)$ and

$$\sup_{x \in K \cap P^i(H)} \|(\Delta_i f P^{i-1} - q^i)\, x\| < \epsilon/N^{1/2}.$$

Now $q^i = P^{i-1}q^i + \Delta_i q^i$ and from

$$\|(\Delta_i f P^{i-1} - q^i)\, x\|^2 = \|(\Delta_i f P^{i-1} - \Delta_i q^i)\, x\|^2 + \| P^{i-1} q^i x\|^2$$

it is immediate that

$$\sup_{x \in K \cap P^i(H)} \|(\Delta_i f P^{i-1} - \Delta_i q^i)\, x\| < \epsilon/N^{1/2}.$$

Moreover, since $P^i(H) \supset P^{i-1}(H)$ it follows that

$$\sup_{x \in K \cap P^i(H)} \|(\Delta_i f P^{i-1} - \Delta_i q^i) P^{i-1} x \| < \epsilon / N^{1/2}.$$

Now we define $\hat{q}^i = \Delta_i q^i P^{i-1}$, a polynomic operator on H and construct $q = \sum \hat{q}^i$. Noting that $\Delta_j q P^{j-1} = \hat{q}^j$ we see that $q \in \mathbb{P} \cap PSC$. The identity

$$\sup_{x \in K} \left\| \sum (\Delta_i f P^{i-1} - \hat{q}^i) \, x \right\|^2 = \sup_{x \in K} \sum \|(\Delta_i f P^{i-1} - \hat{q}^i) \, x \|^2 \leqslant \sum_1^N \epsilon^2 / N = \epsilon^2$$

thus establishes the asserted density.

Definition 2. A function f is *strictly causal* if it is the limit[1] of a sequence of prestrictly causal operators.

We shall use the notation SC to denote the set of *all strictly causal operators* on H. Lemma 4 sets the stage for the following.

Theorem 3. *The set* $\mathbb{P} \cap SC$ *is dense in* $C(K) \cap SC$.

Proof. If $f \in C(K) \cap SC$ then there exists a sequence $f^n \in C(K) \cap PSC$ such that

$$\sup_{x \in K} \|(f - f^n) \, x \| < \epsilon / 2.$$

For each $f^n \in C(K) \cap PSC$ we have by Lemma 1 a polynomic operator $g^n \in \mathbb{P} \cap PSC$ such that

$$\sup_{x \in K} \|(f^n - g^n) \, x \| < \epsilon / 2.$$

From the identity $f - g^n = (f - f^n) + (f^n - g^n)$ and the obvious norm inequality it follows that

$$\sup_{x \in K} \| f - g^n \| < \epsilon$$

for every $\epsilon > 0$, thus $\mathbb{P} \cap PSC$ is dense in $C(K) \cap SC$.

Now since $C(K)$, SC, and $\mathbb{P}$ are closed under addition and composition, Theorem 3 may be paraphrased in the form that the algebra $\mathbb{P} \cap SC$ is dense in the algebra $C(K) \cap SC$. The detail of this result can be sharpened somewhat.

[1] If f is linear convergence in the usual operator norm, if f is multipower the multipower operator norm suffices (see [2]) in the immediate sequel the uniform limit on all compact subsets is intended.

First note that if $f = g + h$ where g, $h \in C(K) \cap SC$. If p, $q \in \mathbb{P} \cap SC$ are $\epsilon/2$ approximates, respectively, of f, g, then $p + q$ is an ϵ approximate of f. In short, a function can be approximated by approximating its (additive) parts. It is slightly less obvious that approximations are preserved under composition. This is a consequence of the following lemma.

LEMMA 5. *Let $L \in C(K)$, $F \in C(J)$ where K, J are compact and $F(J) \subseteq K$. Let W, $G \in \mathbb{P}$ such that*

$$\sup_{x\in K} \|(L - W)\, x\| \leqslant \epsilon_1\,, \qquad \sup_{x\in J} \|(F - G)\, x\| < \epsilon_2\,.$$

Then there exists a scalar δ such that

$$\sup_{x\in J} \|(L \circ F - W \circ G)\, x\| \leqslant \epsilon_1 + \delta\epsilon_2\,.$$

We shall need here the expansion result for polynomic operators cited in the proof of Lemma 2. If $Q \in \mathbb{P}$ then $x - y$ is a factor of $Q(x) - Q(y)$ in the sense that there exists a bounded linear operator valued function T dependent on x, y (in a polynomic way) such that

$$Q(x) - Q(y) = T(x, y)\,(x - y).$$

Moreover, $\|Q(x) - Q(y)\| \leqslant \|T(x, y)\| \cdot \|x - y\|$ where for any bounded set D there exists a finite scalar $\delta(D)$ such that

$$\sup_{x,y\in D} \|T(x, y)\| \leqslant \delta(D).$$

Turning now to the proof of Lemma 5, let $A = L \circ F$ and $B = W \circ G$. Then

$$\begin{aligned}(A - B)\,(x) &= L(Fx) - W(Gx) = LF(x) - W(Fx) + W(Fx) - W(Gx)\\ &= (L - W)F(x) + T[F(x), G(x)]\,(F(x) - G(x)),\end{aligned}$$

hence

$$\|(A - B)\, x\| \leqslant \|(L - W)F(x)\| + \bar{\delta}\, \|F(x) - G(x)\|$$

Since $F(J) \subseteq K$ we have (here $\bar{\delta} = \delta(J)$)

$$\begin{aligned}\sup_{x\in J} \|(A - B)\, x\| &\leqslant \sup_{y\in K} \|(L - W)\, y\| + \bar{\delta} \sup_{x\in J} \|F(x) - G(x)\|\\ &= \epsilon_1 + \bar{\delta}\epsilon_2\,,\end{aligned}$$

the asserted result.

5. The Approximation of Causal Functions

As before K is a compact set in the real separable Hilbert resolution space H. For $f \in C(K)$ we let $f(K)$ denote the image of K under f. Of course continuous images of compact sets are themselves compact. Consider a sequence $\{Q_n\}$ of bounded linear operators on H.

Definition 3. If each Q_n is strictly causal and if $\{Q_n\} \to I$ uniformly on every compact set $K \subset H$, then $\{Q_n\}$ is a *strictly causal Steklov* sequence.

Lemma 6. *If a strictly causal Steklov sequence exists in* $\{H, P^t\}$ *then* $\mathbb{P} \cap SC$ *are dense in* $C(K) \cap C$.

Proof. For $f \in C(K)$ let $\{Q_n\}$ be *strictly* causal Steklov. For arbitrary $\epsilon > 0$ an N exists such that (using the uniform convergence $Q_n \to I$ on $f(K)$)

$$\max_K \|f(x) - Q_n f(x)\| < \epsilon/2, \qquad \text{all } n \geqslant N$$

holds. Since Q_n is linear and strictly causal and f is causal, the composition $Q_n f$ is strictly causal (see [14, Proposition 4.5]).

Now by Lemma 4, there exists a strictly causal polynomial q such that

$$\max_{x \in K} \|(Q_n f)(x) - q(x)\| < \epsilon/2.$$

Hence

$$\max_{x \in K} \|f(x) - q(x)\| \leqslant \max_{x \in K} \|f(x) - (Q_n f)(x)\| + \max_{x \in K} \|(Q_n f)(x) - q(x)\| = \epsilon,$$

which completes the proof.

We turn now to the existence question of strictly causal Steklov functions.

Example 2. Here $H = L_2(0, 1)$ and we recognize in the functions

$$(P^t x)(\beta) = \begin{cases} x(\beta), & \beta \leqslant t \\ 0, & \beta > t \end{cases}$$

a resolution of the identity. In this setting consider the functions Q_n defined by

$$(Q_n x)(t) = n \int_{t-1/n}^{t} x(s)\, ds, \qquad n = 1, 2, \ldots. \tag{2}$$

It is clear that each Q_n is bounded, linear, and causal, moreover in view of Example 1, each Q_n is strictly causal. These functions are in fact obvious variations on the classical Steklov functions.

Using existing proofs (see [13, Section 1.10]) with only minor changes it follows that

$$\| Q_n \| \leqslant 1, \qquad \{Q_n\} \to I,$$

where the convergence is uniform on every compact K. Thus Eq. (2) provides a strictly causal Steklov sequence.

Extending the definition of Q_n to finite products of $L_2(0, T)$ in the natural way we arrive at the following.

THEOREM 4. *If $\{H, P^t\}$ is a finite product of real separable L_2 equipped with the truncation projections then $\mathbb{P} \cap PSC$, $\mathbb{P} \cap SC$, and $\mathbb{P} \cap C$ are dense in $C(K) \cap C$ on every compact $K \subset H$.*

The theorem follows from Lemma 6 and the knowledge that PSC is dense in SC while $\mathbb{P} \cap SC \subset \mathbb{P} \cap C$.

It is helpful to recognize at this point a distinction between L_2 and l_2 spaces.

(COUNTER) EXAMPLE 3. Consider real l_2 with

$$\nu = \{0, 1, 2,\ldots\}, \qquad x = (x_1, x_2, \ldots, x_n, \ldots) \in l_2$$

and the functions $\{P^j : j = 1,\ldots\}$ defined by

$$(P^j x)_k = \begin{cases} x_k, & k \leqslant j \\ 0, & k > j. \end{cases}$$

It is easily verified that $\{l_2, P^j\}$ is a Hilbert resolution space. Suppose now that Ω_N is any mesh which includes the points 0, 1. Let Q be linear and pre-strictly causal, that is

$$Q = \sum_{i=1}^{N} \Delta_i Q P^{i-1}.$$

However $\Delta_1 = P^1$ and $P^{i-1} = 0$ for $i = 1$, and hence Qx is a sequence of the form $(0, QP^1x, QP^2x,\ldots)$. Obviously

$$x - Qx = (x_1, x_2 - QP^1x, x_3 - QP^2x,\ldots),$$

hence

$$\| x - Qx \| \geqslant | x_1 |, \qquad x = (x_1, x_2, \ldots).$$

If $x_1 = 0$ then $P^1x = 0$ and the problem passes to the second component namely

$$\| x - Qx \| \geqslant | x_2 |, \qquad x = (0, x_2, \ldots).$$

Since the single point set $K = \{(1, 0, 0,\ldots)\}$ is compact it follows easily that there are *no* strictly causal Steklov functions on l_2.

Examples 2 and 3 point out that Theorem 4 cannot be lifted up to abstract $\{H, P^t\}$ without additional structure assumptions on the underlying order set ν. At this writing we are not prepared to venture in this direction.

6. Closure

In our development we have focused attention on the concept of causality and in particular strict causality. The literature on causality structure has several additional concepts of equal importance. In closing we take note of two.

Strict and prestrict anticausality are the duals of strict and prestrict causality, respectively, in that P^t is replaced by $I - P^t$ in all defining equations. If one does this systematically, in Sections 2, 3, and 4 the proofs still hold for the dualized versions of the lemmas and theorems contained therein. In short if AC, SAC, and $PSAC$ denote anticausal, strict anticausal, and prestrict anticausal sets then the subalgebra $PSAC \cap \mathbb{P}$ is dense in the subalgebra $PSAC \cap C(K)$ on K, while the subalgebra $SAC \cap \mathbb{P}$ is dense in the subalgebra $SAC \cap C(K)$ on K, and finally Theorem 4 dualizes.

An operator on $\{H, P^t\}$ is said to be memoryless if it is both causal and anticausal. It is known (see [10]) that in some cases (including linear Hilbert–Schmidt operators) an operator can be decomposed into a unique sum of a strictly anticausal, a strictly causal, and a memoryless part. Suppose a theorem to the effect that $M \cap \mathbb{P}$ was dense in $M \cap C(K)$ on K was available where M denotes the set of all bounded memoryless functions. Suppose also that $f \in C(K)$ is of the form $f = f_{sc} + f_{sa} + f_m$ where $f_{sc} \in SC \cap C(K)$, $f_{sa} \in SAC \cap C(K)$, and $f_m \in M \cap C(K)$, respectively. One could then construct a polynomic approximation p to f of the form $p = p_c + p_a + p_m$ where $p_{sc} \in SC \cap \mathbb{P}$, $p_a \in SAC \cap \mathbb{P}$, and $p_m \in M \cap \mathbb{P}$. In this way polynomic approximations which preserve the subalgebras M, C, SC, AC, and SAC would result.

Movement in this direction, however, is blocked by a result which is proved in [15] namely: In $L_2(0, 1)$ there exists compact sets K such that *no* memoryless multipower operators exist on K of order $n \geqslant 2$. For contrast in l_2 memoryless multipower operators of all orders exist define on all of l_2 in abundance.

In a related direction note that if $f \in C \cap C(K)$ in the l_2 setting has the form $f = f_{sc} + f_m$ then Theorem 3 provides a $p_{sc} \in SC \cap \mathbb{P}$ and (it can be shown [15]) that a $p_m \in M \cap \mathbb{P}$ exist such that $p = p_{sc} + p_m$ is a polynomic fit to f. In short, the loss of the strictly causal Steklov functions is replaced by the existence of memoryless multipower functions. One might then conjecture that the abstraction of Theorem 4 will reflect this tradeoff.

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