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A Nonsimple Conjoint Measurement Model

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The nonsimple conjoint measurement model examined in this paper maps each (a_1, a_2, a_3) in $A_1 \times A_2 \times A_3$ into $\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3)$, where each of ω_1, ω_2, ϕ_2, and ϕ_3 is a real-valued function, so as to preserve a binary relation $\succcurlyeq$ on $A_1 \times A_2 \times A_3$ by $\geqslant$ in the numerical system.

The principle structure of the present model is similar to the structure of the simple models. But in the nonsimple model, we define two different identity elements of A_1 for its different multiplicative effects on the other two components A_2 and A_3, whereas in the simple model, one identity element for each component is defined.

INTRODUCTION

The development of simple conjoint measurement models (Krantz, Luce, Suppes, and Tversky, 1971) makes real-valued scales for psychological measurement possible under certain conditions and offers a measurement-free technique in testing hypotheses about such composition rules. Some models hypothesized in psychology are not simple composition rules so the development of axiom systems for nonsimple models will offer measurement-free techniques for testing such theories.

An axiom system for a familiar but nonsimple model will be presented. Hopefully, the principle of these axiom systems can be extended to construct more psychological models.

The nonsimple conjoint measurement model examined in this paper maps each (a_1, a_2, a_3) in $A_1 \times A_2 \times A_3$ into $\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3)$, where each of ω_1, ω_2, ϕ_2, and ϕ_3 is a real-valued function, so as to preserve a binary relation $\succcurlyeq$ on $A_1 \times A_2 \times A_3$ by $\geqslant$ in the numerical system.

The principle structure of the present model is similar to the structure of the simple models. But in the nonsimple model, we define two different identity elements of A_1 for its different multiplicative effects on the other two components A_2 and A_3, whereas in the simple model, one identity element for each component is defined. The principle

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of using multiple identity elements for one component can be generalized to construct the axiom systems for the more complicated models, e.g., $\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_2)\,\phi_3(a_3) + \omega_3(a_1)\,\phi_4(a_4)$.

Definitions

Definition 1. $\geqslant$ is a binary relation on $A_1 \times A_2 \times A_3$. Two elements (a_1, a_2, a_3) and (b_1, b_2, b_3) of $A_1 \times A_2 \times A_3$ are equivalent, $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$ if and only if they satisfy the following two conditions:

$$(a_1, a_2, a_3) \geqslant (b_1, b_2, b_3)$$

and

$$(b_1, b_2, b_3) \geqslant (a_1, a_2, a_3).$$

We first define the cancellation condition known as the Thomson condition.

Definition 2. A relation $\sim$ on $\{a_1\} \times A_2 \times A_3$, where $a_1 \in A_1$, satisfies the Thomsen condition provided that for every $a_2, b_2, k_2 \in A_2$ and $a_3, b_3, k_3 \in A_3$, if

$$(a_1, a_2, k_3) \sim (a_1, k_2, b_3)$$

and

$$(a_1, k_2, a_3) \sim (a_1, b_2, k_3),$$

then

$$(a_1, a_2, a_3) \sim (a_1, b_2, b_3).$$

The Thomsen condition is a necessary condition for the simple additive model. It is clear that the nonsimple model examined is an addtive model on $\{a_1\} \times A_2 \times A_3$.

Definition 3. (1) A_2^0 and A_3^0 are the subsets of A_2 and A_3, respectively, such that if $a_2^0, b_2^0 \in A_2^0$ and $a_3^0, b_3^0 \in A_3^0$, then

$$(a_1, a_2^0, a_3) \sim (a_1, b_2^0, a_3) \quad \text{for all} \quad a_1 \in A_1 \quad \text{and} \quad a_3 \in A_3;$$

$$(a_1, a_2, a_3^0) \sim (a_1, a_2, b_3^0) \quad \text{for all} \quad a_1 \in A_1 \quad \text{and} \quad a_2 \in A_2;$$

$$(a_1, a_2^0, a_3^0) \sim (b_1, a_2^0, a_3^0) \quad \text{for all} \quad a_1, b_1 \in A_1.$$

(2) A_1^0 and A_1^{00} are the subsets of A_1, such that if $a_1^0 \in A_1^0$ and $a_1^{00} \in A_1^{00}$, then

$$(a_1^0, a_2, a_3^0) \sim (a_1^0, b_2, a_3^0) \quad \text{for all} \quad a_2, b_2 \in A_2 \quad \text{and} \quad a_3^0 \in A_3^0;$$

$$(a_1^{00}, a_2^0, a_3) \sim (a_1^{00}, a_2^0, b_3) \quad \text{for all} \quad a_3, b_3 \in A_3 \quad \text{and} \quad a_2^0 \in A_2^0.$$

A_2^0 and A_3^0 are the sets of zero elements of A_2 and A_3, respectively. A_1^0 and A^{00} are two sets of zero elements of A_1 and have degenerate effects on A_2 and A_3, respectively.

DEFINITION 4. (1) $\{a_1\} \times A_2$ is independent of $\{a_1\} \times A_3$ if and only if for $a_2, b_2 \in A_2$, $(a_1, a_2, x_3) \succsim (a_1, b_2, x_3)$ for some $x_3 \in A_3$ implies that $(a_1, a_2, y_3) \succsim (a_1, b_2, y_3)$ for every $y_3 \in A_3$.

$\{a_1\} \times A_2$ and $\{a_1\} \times A_3$ are mutually independent if and only if $\{a_1\} \times A_2$ is independent of $\{a_1\} \times A_3$ and $\{a_1\} \times A_3$ is independent of $\{a_1\} \times A_2$.

DEFINITION 5. A_1 is sign dependent on $A_2 \times A_3^0$ if and only if A_2 can be partitioned into three sets, A_2^+, A_2^0, and A_2^- such that for any $a_1, b_1 \in A_1$ and $a_3^0 \in A_3^0$ the following conditions hold.

(1) $(a_1, x_2, a_3^0) \succsim (b_1, x_2, a_3^0)$ for some $x_2 \in A_2^+$, then
$(a_1, y_2, a_3^0) \succsim (b_1, y_2, a_3^0)$ for every $y_2 \in A_2^+$, and
$(b_1, y_2, a_3^0) \succsim (a_1, y_2, a_3^0)$ for every $y_2 \in A_2^-$.

(2) $(a_1, x_2, a_3^0) \succsim (b_1, x_2, a_3^0)$ for some $x_2 \in A_2^-$, then
$(a_1, y_2, a_3^0) \succsim (b_1, x_2, a_3^0)$ for every $y_2 \in A_2^-$, and
$(b_1, y_2, a_3^0) \succsim (a_1, y_2, a_3^0)$ for every $y_2 \in A_2^+$.

A_1 and $A_2 \times A_3^0$ are mutually sign dependent if and only if A_1 is sign dependent on $A_2 \times A_3^0$ and $A_2 \times A_3^0$ is sign dependent on A_1.

DEFINITION 6. For every pair of elements $a_2, b_2 \in A_2$, $a_2 \approx b_2$ if and only if $(a_1, a_2, a_3) \sim (a_1, b_2, a_3)$ for every $a_1 \in A_1$ and $a_3 \in A_3$. A_2 is essential if and only if there exist $a_2, b_2 \in A_2$ such that $a_2 \not\approx b_2$. Similar definition holds for A_3.

DEFINITION 7. $\succsim$ is a weak ordering of $\{a_1\} \times A_2 \times A_3$. For any set N of consecutive integers, $N = \{i, i+1, i+2, \ldots, j, j+1, j+2, \ldots\}$, a set $\{a_2^j \mid a_2^j \in A_2, j \in N\}$ is a standard sequence on A_2 iff there exist $x_3 \not\approx y_3$ such that $(a_1, a_2^j, x_3) \sim (a_1, a^{j+1}, y_3)$ for all $j, j+1 \in N$. A standard sequence on A_2 is strictly bounded if and only if for any $a_3 \in A_3$, there exist $\bar{a}_2$ and $\underline{a}_2$ such that $\bar{a}_2 \not\approx \underline{a}_2$ and $(a_1, \bar{a}_2, a_3) \succsim (a_1, a_2^j, a_3) \succsim (a_1, \underline{a}_2, a_3)$ for all $j \in N$. A bounded standard sequence is finite if and only if the integer set N defined above is finite.

Thomsen condition, independence, sign dependence, and standard sequence were defined by Krantz et al. (Krantz, Luce, Suppes, and Tvresky, 1971).

DEFINITION 8. A relation $\sim$ on $A_1 \times A_2 \times A_3$ satisfies unrestricted solvability provided that for all $a_2^0 \in A_2^0$, $a_3^0 \in A_3^0$, a_1, $b_1 \in A_1 - A_1^0 - A_1^{00}$, a_2, $b_2 \in A_2$, a_3, $b_3 \in A_3$, there exist $b_2' \in A_2$, $b_3' \in A_3$ and a_1', $a_1'' \in A_1$ such that

$$(a_1, a_2, a_3) \sim (b_1, b_2, b_3'),$$
$$(a_1, a_2, a_3) \sim (b_1, b_2', b_3),$$
$$(a_1, a_2, a_3^0) \sim (a_1', b_2, a_3^0),$$
$$(a_1, a_2^0, a_3) \sim (a_1'', a_2^0, b_3).$$

DEFINITION 9. A relation $\sim$ on $A_1 \times A_2 \times A_3$ satisfies the nonadditive weighting condition provided that for all a_1, $b_1 \in A_1$, $c_1 \in A_1 - A_1^0$, $d_1 \in A_1 - A_1^{00}$, a_2, b_2, d_2, e_2, f_2, $g_2 \in A_2$, $c_2 \in A_2 - A_2^0$, a_3, b_3, d_3, e_3, f_3, $g_3 \in A_3$, and $c_3 \in A_3 - A_3^0$, $c_2 \not\sim g_2$, and $c_3 \not\sim g_3$, if

$$(a_1, e_2, c_3) \sim (b_1, c_2, e_3), \tag{1}$$
$$(d_1, f_2, a_3) \sim (c_1, g_2, d_3), \tag{2}$$
$$(c_1, d_2, d_3) \sim (c_1, b_2, f_3), \tag{3}$$
$$(a_1, a_2, g_3) \sim (b_1, d_2, b_3), \tag{4}$$
$$(c_1, c_2, f_3) \sim (d_1, f_2, c_3), \tag{5}$$
$$(a_1, e_2, g_3) \sim (b_1, g_2, e_3), \tag{6}$$
$$(c_1, g_2, f_3) \sim (d_1, f_2, g_3), \tag{7}$$

then

$$(a_1, a_2, a_3) \sim (b_1, b_2, b_3).$$

This complicated property is necessary for all the simple models as well as for the nonsimple model. To show the necessity for the nonsimple model, $\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3)$, we translate the above equivalences into equations. We drop the notations for the functions and use a_1 to represent $\omega_1(a_1)$, a_1' to represent $\omega_2(a_1)$, and a_2 and a_3 to represent $\phi_2(a_2)$ and $\phi_3(a_3)$, respectively.

The fourth condition yields

$$a_1a_2 + a_1'g_3 = b_1d_2 + b_1'b_3.$$

The conclusion is that

$$a_1a_2 + a_1'a_3 = b_1b_2 + b_1'b_3.$$

Therefore, we need to show that the other six relations will provide

$$a_1'a_3 + b_1d_2 = b_1b_2 + a_1'g_3,$$

or

$$a_1'(a_3 - g_3) = b_1(b_2 - d_2).$$

The first and sixth condition can be translated into

$$a_1 e_2 + a_1' c_3 = b_1 c_2 + b_1' e_3$$

and

$$a_1 e_2 + a_1' g_3 = b_1 g_2 + b_1' e_3 .$$

Subtracting the latter from the former, we have

$$a_1'(c_3 - g_3) = b_1(c_2 - g_2). \tag{4.1}$$

Similarly, the second and seventh conditions will provide that

$$c_1'(d_3 - f_3) = d_1'(a_3 - g_3).$$

The third relation can be translated and rewritten as

$$c_1(b_2 - d_2) = c_1'(d_3 - f_3).$$

It follows that

$$d_1'(a_3 - g_3) = c_1(b_2 - d_2). \tag{4.2}$$

The fifth and seven relations yield

$$c_1(c_2 - g_2) = d_1'(c_3 - g_3). \tag{4.3}$$

Multiplying (4.1), (4.2), and (4.3) together, we have

$$a_1' d_1' c_1 (c_3 - g_3)(a_3 - g_3)(c_2 - g_2) = b_1 c_1 d_1'(c_2 - g_2)(b_2 - d_2)(c_3 - g_3).$$

Since $c_1 \neq 0$, $d_1' \neq 0$ (c_1 is not in A_1^0, and d_1 is not in A_1^{00}), $c_2 \neq g_2$, and $c_3 \neq g_3$, we cancel $c_1 d_1'(c_2 - g_2)(c_3 - g_3)$ and obtain

$$a_1'(a_3 - g_3) = b_1(b_2 - d_2),$$

as required.

From a practical point of view this property is too complicated to test directly. The conditions can be simplified by letting $e_2 = f_2 = g_2 = a_2^0$ and $e_3 = f_3 = g_3 = a_3^0$. The theorems which we want to obtain will not be affected by this simplification. The simplified condition is somewhat easier to test empirically.

We now want to list all the conditions which together are sufficient for the nonsimple model $\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3)$.

It is clear that $\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3)$ is an additive model for $\{a_1\} \times A_2 \times A_3$. Therefore, we need conditions which are sufficient for the additive model. When $\phi_3(a_3) = 0$, $\omega_1(a_1)\,\phi_2(a_2)$ is a multiplicative model for $A_1 \times A_2$. Therefore, we need an axiom which states that A_1 and $A_2 \times A_3^0$ are mutually sign dependent. Similarly, A_1 and $A_2^0 \times A_3$ are mutually sign dependent, but this can be asserted by other axioms. Finally, the weighting condition functions in the nonsimple distributive model the same as the distributive cancellation axiom functions in the simple distributive model.

DEFINITION 10. A relation $\geqslant$ on $A_1 \times A_2 \times A_3$ is a *nonsimple distributive model* iff it satisfies the following axioms.

Axiom 1. $\geqslant$ is a weak order.

Axiom 2. A_2 and A_3 are essential. $A_2{}^0$ and $A_3{}^0$ are not empty.

Axiom 3. A_1 and $A_2 \times A_3{}^0$ are mutually sign dependent.

Axiom 4. For any $a_1 \in A_1$, $\{a_1\} \times A_2$ and $\{a_1\} \times A_3$ are mutually independent.

Axiom 5. Unrestricted solvability of Definition 8 holds.

Axiom 6. For any induced order on $A_2 \times A_3$, every strictly bounded standard sequence in one component is finite (Definition 7).

Axiom 7. The nonadditive weighting condition of Definition 9 holds.

THEOREMS

THEOREM 1. *If a relation $\geqslant$ on $A_1 \times A_2 \times A_3$ is a nonsimple distributive model then the Thomsen condition holds.*

Proof. The proof has to be done separately for the different conditions on the elements. The proof for a representative case is shown below.

Given $(a_1, a_2, k_3) \sim (a_1, k_2, b_3)$ and $(a_1, k_2, a_3) \sim (a_1, b_2, k_3)$, we need to show that $(a_1, a_2, a_3) \sim (a_1, b_2, b_3)$.

We consider the case for which

$$k_2 \notin A_2{}^0,\ k_3 \notin A_3{}^0, \qquad \text{and} \qquad a_1 \notin A_1{}^0 \cup A_1^{00}.$$

Since we have

$$\begin{aligned}
&(a_1, a_2, k_3) \sim (a_1, k_2, b_3) && \text{from the given condition,}\\
&(a_1, k_2, a_3) \sim (a_1, k_2, a_3) && \text{from Axiom 1,}\\
&(a_1, k_2, a_3) \sim (a_1, b_2, k_3) && \text{from the given condition,}\\
&(a_1, a_2, k_3) \sim (a_1, k_2, b_3) && \text{from the given condition,}\\
&(a_1, k_2, k_3) \sim (a_1, k_2, k_3) && \text{from Axiom 1,}\\
&(a_1, a_2, k_3) \sim (a_1, k_2, b_3) && \text{from the given condition}
\end{aligned}$$

and

$$(a_1, k_2, k_3) \sim (a_1, k_2, k_3) \qquad \text{from Axiom 1,}$$

the nonadditive weighting condition implies that

$$(a_1, a_2, a_3) \sim (a_1, b_1, b_2).$$

THEOREM 2. *If a relation* $\succcurlyeq$ *on a set* $A_1 \times A_2 \times A_3$ *is a nonsimple distributive model, then there exist real-valued functions* ω_1 *and* ω_2 *on* A_1, ϕ_2 *on* A_2, *and* ϕ_3 *on* A_3 *such that*

$$(a_1, a_2, a_3) \succcurlyeq (b_1, b_2, b_3)$$

iff

$$\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3) \geqslant \omega_1(b_1)\,\phi_2(b_2) + \omega_2(b_1)\,\phi_3(a_3),$$

for all (a_1, a_2, a_3) *and* $(b_1, b_2, b_3) \in A_1 \times A_2 \times A_3$.

Moreover, the real value functions satisfying this property are unique up to the following transformations.

$$\omega_1(a_1) \to \alpha_1\omega_1'(a_1),$$

$$\omega_2(a_1) \to \alpha_2\omega_2'(a_1),$$

$$\phi_2(a_2) \to \beta_2\phi_2'(a_2),$$

$$\phi_3(a_3) \to \beta_3\phi_3'(a_3),$$

where $\alpha_1\beta_2 = \alpha_2\beta_3 > 0$.

Proof. For the case in which $A_1 \neq A_1^0 \cup A_1^{00}$, we define two operations $\oplus$ and $*$. We will then prove that there is an isomorphism from $\langle A_1 \times A_2 \times A, \succcurlyeq, \oplus, *\rangle$ to a subring of $\langle R, \geqslant, +, \cdot\rangle$, provided that Axioms 1–7 hold. The representation theorem for the case in which $A_1 = A_1^0 \cup A_1^{00}$ will be discussed at the end.

Let $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ denote the equivalence class of (a_1, a_2, a_3) for all $(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$. Clearly, if $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$, then the equivalence class of (a_1, a_2, a_3) is exactly the same as the equivalence class of (b_1, b_2, b_3), i.e., $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$. Let $\succcurlyeq$ denote the order relation on $A_1 \times A_2 \times A_3/\sim$, and $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \succcurlyeq (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ if and only if $(a_1, a_2, a_3) \succcurlyeq (b_1, b_2, b_3)$. Define $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, and $a = (a_1, a_2, a_3)$. We let $\mathbf{a} \succ \mathbf{b}$ represent $\mathbf{a} \succcurlyeq \mathbf{b}$ and not $\mathbf{b} \succcurlyeq \mathbf{a}$. Clearly, the relation $\succcurlyeq$ on the equivalence classes is a simple order.

Select $a_1^i, a_1^{ii} \in A_1 - A_1^0 - A_1^{00}$, $a_2^i \in A_2 - A_2^0$, $a_3^i \in A_3 - A_3^0$, such that $(a_1^i, a_2^i, a_3^0) \sim (a_1^{ii}, a_2^0, a_3^i)$. For any $a \in A_1 \times A_2 \times A_3$, there exist $a_1', a_1'' \in A_1$ such that $a \sim (a_1', a_2^i, a_3^0) \sim (a_1'', a_2^0, a_3^i)$; and there exist $a_2' \in A_2$ and $a_3' \in A_3$ such that $a \sim (a_1^i, a_2', a_3^0) \sim (a_1^{ii}, a_2^0, a_3')$ (using unrestricted solvability axiom). There also exist a_2'' of A_2 and a_3'' of A_3 such that $a \sim (a_1, a_2'', a_3^0) \sim (a_1, a_2^0, a_3'')$.

We now define the operations $\oplus$, $*_1$, and $*_2$ as follows.

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3^0) \oplus (\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{a}_3) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3),$$

$$(\mathbf{a}_1, \mathbf{a}_2^i, \mathbf{a}_3^0) *_1 (\mathbf{a}_1^i, \mathbf{a}_2, \mathbf{a}_3^0) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3^0),$$

$$(\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{a}_3^i) *_2 (\mathbf{a}_1^{ii}, \mathbf{a}_2^0, \mathbf{a}_3) = (\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{a}_3).$$

It is necessary to show that $*_1 = *_2 = *$ first.

Step 1. We are going to show that $\mathbf{a} *_1 \mathbf{b} = \mathbf{a} *_2 \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in A_1 \times A_2 \times A_3/\sim$. The unrestricted solvability axiom quarantees that there exist $a_1 \in A_1$ and $a_3 \in A_3$ such that (a_1, a_2^0, a_3^i) is in the equivalence class $\mathbf{a}$ and (a_1^{ii}, a_2^0, b_3) is in the equivalence $\mathbf{b}$, or $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{a}_3^i)$ and $\mathbf{b} = (\mathbf{a}_1^{ii}, \mathbf{a}_2^0, \mathbf{b}^3)$. Similarly, we can find $a_1' \in A_1$, $b_2 \in A_2$, and $b_3' \in A_3$ such that

$$(a_1, a_2^0, a_3^i) \sim (a_1', a_2^i, a_3^0),$$

$$(a_1^{ii}, a_2^0, b_3) \sim (a_1^i, a_2^0, b_3'),$$

$$(a_1^i, a_2^0, b_3') \sim (a_1^i, b_2, a_3^0).$$

But

$$(a_1, a_2^0, a_3^0) \sim (a_i', a_2^0, a_3^0),$$

$$(a_1^i, a_2^i, a_3^0) \sim (a_1^{ii}, a_2^0, a_3^i),$$

$$(a_1, a_2^0, a_3^0) \sim (a_1', a_2^0, a_3^0),$$

$$(a_1^i, a_2^0, a_3^0) \sim (a_1^{ii}, a_2^0, a_3^0).$$

Applying Axiom 7 to the above equations, we have

$$(a_1, a_2^0, b_3) \sim (a_1', b_2, a_3^0).$$

Because $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{a}_3^i) = (\mathbf{a}_1', \mathbf{a}_2^i, \mathbf{a}_3^0)$ and $\mathbf{b} = (\mathbf{a}_1^{ii}, \mathbf{a}_2^0, \mathbf{b}_3) = (\mathbf{a}_1^i, \mathbf{b}_2, \mathbf{a}_3^0)$, $\mathbf{a} *_2 \mathbf{b} = (\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{b}_3)$ and $\mathbf{a} *_1 \mathbf{b} = (\mathbf{a}_1', \mathbf{b}_2, \mathbf{a}_3^0)$. We just prove that $(\mathbf{a}_1', \mathbf{b}_2, \mathbf{a}_3^0) = (\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{b}_3)$, that is $\mathbf{a} *_1 \mathbf{b} = \mathbf{a} *_2 \mathbf{b}$.

We are now going to show that $\langle A_1 \times A_2 \times A_3/\sim, \succcurlyeq, \oplus, *\rangle$ is an Archimedean ordered ring. In other words, we need to show four things:

(a) $\langle A_1 \times A_2 \times A_3/\sim, \succcurlyeq, \oplus\rangle$ is an Archimedean ordered group,
(b) $*$ is associative,
(c) $*$ is distributive to $\oplus$,
(d) there is a zero element θ, if $\mathbf{a} \succ \theta$ and $\mathbf{b} \succ \mathbf{c}$, then $\mathbf{a} * \mathbf{b} \succ \mathbf{a} * \mathbf{c}$ and $\mathbf{b} * \mathbf{a} \succ \mathbf{c} * \mathbf{a}$.

Step 2. The proof of part *a* is contained in Krantz et al. (1971, pp. 257–266).

Step 3. For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A_1 \times A_2 \times A_3/\sim$, we will prove that $\mathbf{a} * (\mathbf{b} * \mathbf{c}) = (\mathbf{a} * \mathbf{b}) * \mathbf{c}$. Let

$$\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2^{i}, \mathbf{a}_3^{0}),$$

$$\mathbf{b} = (\mathbf{b}_1, \mathbf{a}_2^{0}, \mathbf{a}_3^{i}),$$

$$\mathbf{c} = (\mathbf{a}_1^{ii}, \mathbf{a}_2^{0}, \mathbf{c}_3).$$

The unrestricted solvability axiom provides a $b_2 \in A_2$ such that $(b_1, a_2^0, a_3^i) \sim (a_1^i, b_2, a_3^0)$. By unrestricted solvability again, we have the following relations.

$$(a_1', a_2^0, a_3^i) \sim (a_1, b_2, a_3^0), \quad \text{for an} \quad a_1' \in A_1,$$

$$(b_1, a_2^0, c_3) \sim (a_1^i, a_2^0, c_3'), \quad \text{for a} \quad c_3' \in A_3,$$

$$(a_1^i, a_2^0, c_3') \sim (a_1^i, c_2, a_3^0), \quad \text{for a} \quad c_2 \in A_2,$$

but

$$(a_1', a_2^0, a_3^0) \sim (a_1, a_2^0, a_3^0),$$

$$(a_1^i, b_2, a_3^0) \sim (b_1, a_2^0, a_3^i).$$

From definition 3, we have

$$(a_1', a_2^0, a_3^0) \sim (a_1, a_2^0, a_3^0),$$

$$(a_1^i, a_2^0, a_3^0) \sim (b_1, a_2^0, a_3^0).$$

Applying Axiom 7 to the above seven formulas, it results

$$(a_1', a_2^0, c_3) \sim (a_1, c_2, a_3^0).$$

From the definitions of $*$, we have $\mathbf{b} * \mathbf{c} = (\mathbf{b}_1, \mathbf{a}_2^0, \mathbf{c}_3)$. From the transitive property of the equivalence relation $\sim$, $\mathbf{b} * \mathbf{c} = (\mathbf{a}_1^{i}, \mathbf{c}_2, \mathbf{a}_3^0)$. Therefore, $\mathbf{a} * (\mathbf{b} * \mathbf{c}) = (\mathbf{a}_1, \mathbf{c}_2, \mathbf{a}_3^0)$. Similarly, $\mathbf{a} * \mathbf{b} = (\mathbf{a}_1, \mathbf{b}_2, \mathbf{a}_3^0) = (\mathbf{a}_1', \mathbf{a}_2^0, \mathbf{a}_3^{i})$, and hence $(\mathbf{a} * \mathbf{b}) * \mathbf{c} = (\mathbf{a}_1', \mathbf{a}_2^0, \mathbf{c}_3)$. We proved previously that $(a_1', a_2^0, c_3) \sim (a_1, c_2, a_3^0)$, consequently, we have $(\mathbf{a} * \mathbf{b}) * \mathbf{c} = \mathbf{a} * (\mathbf{b} * \mathbf{c})$.

Step 4. We will now prove that $*$ is distributive to $\oplus$. In other words, it has to be shown that $\mathbf{a} * (\mathbf{b} \oplus \mathbf{c}) = (\mathbf{a} * \mathbf{b}) \oplus (\mathbf{a} * \mathbf{c})$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A_1 \times A_2 \times A_3/\sim$. Because of Axiom 5, we can always let $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2^0, \mathbf{a}_3^{i})$, $\mathbf{b} = (\mathbf{a}_1^{i}, \mathbf{b}_2, \mathbf{a}_3^0)$ and $\mathbf{c} = (\mathbf{a}_1^{ii}, \mathbf{a}_2^0, \mathbf{c}_3)$. By the same axiom, we have the following relations:

$$(a_1, a_2^0, a_3^i) \sim (a_1', a_2^i, a_3^0), \quad \text{for an} \quad a_1' \in A_1,$$

$$(a_1^{ii}, a_2^0, c_3) \sim (a_1^i, a_2^0, x_3), \quad \text{for a} \quad x_3 \in A_3,$$

$$(a_1^i, b_2, x_3) \sim (a_1', b_2', a_3^0), \quad \text{for a} \quad b_2' \in A_2,$$

$$(a_1', b_2, a_3^0) \sim (a_1, b_2'', a_3^0), \quad \text{for a} \quad b_2'' \in A_2.$$

It was chosen that

$$(a_1^{\,i}, a_2^{\,i}, a_3^{\,0}) \sim (a_1^{ii}, a_2^{\,0}, a_3^{\,i}),$$
$$(a_1, a_2^{\,0}, a_3^{\,0}) \sim (a_1', a_2^{\,0}, a_3^{\,0}),$$
$$(a_1^{\,i}, a_2^{\,0}, a_3^{\,0}) \sim (a_1^{ii}, a_2^{\,0}, a_3^{\,0}).$$

Again, Axiom 7 implies

$$(a_1, b_2'', c_3) \sim (a_1', b_2', a_3^{\,0}).$$

Since $\mathbf{c} = (\mathbf{a}_1^{ii}, \mathbf{a}_2^{0}, \mathbf{c}_3) = (\mathbf{a}_1^{i}, \mathbf{a}_2^{0}, \mathbf{x}_3)$, it follows from the definition $\oplus$ that $\mathbf{b} \oplus \mathbf{c} = (\mathbf{a}_1^{i}, \mathbf{b}_2, \mathbf{x}_3)$. But $(\mathbf{a}_1^{i}, \mathbf{b}_2, \mathbf{x}_3) = (\mathbf{a}_1^{i}, \mathbf{b}_2', \mathbf{a}_3^{0})$, and $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2^{0}, \mathbf{a}_3^{i}) = (\mathbf{a}_1', \mathbf{a}_2^{i}, \mathbf{a}_3^{0})$, hence $\mathbf{a} * (\mathbf{b} \oplus \mathbf{c}) = (\mathbf{a}_1', \mathbf{b}_2', \mathbf{a}_3^{0})$. Clearly, $\mathbf{a} * \mathbf{c} = (\mathbf{a}_1, \mathbf{a}_2^{0}, \mathbf{c}_3)$, and $\mathbf{a} * \mathbf{b} = (\mathbf{a}_1', \mathbf{b}_2, \mathbf{a}_3^{0}) = (\mathbf{a}_1, \mathbf{b}_2'', \mathbf{a}_3^{0})$, the definition of $\oplus$ provides that $(\mathbf{a} * \mathbf{b}) \oplus (\mathbf{a} * \mathbf{c}) = (\mathbf{a}_1, \mathbf{b}_2'', \mathbf{c}_3)$. Since $(a_1, b'', c_3) \sim (a_1', b_2', a_3^{0})$, it follows that $(\mathbf{a} * \mathbf{b}) \oplus (\mathbf{a} * \mathbf{c}) = \mathbf{a} * (\mathbf{b} \oplus \mathbf{c})$.

We thus proved that $\langle A_1 \times A_2 \times A_3/\sim, \oplus, *\rangle$ is a ring with zero element $\theta = (\mathbf{a}_1, \mathbf{a}_2^{0}, \mathbf{a}_3^{0})$. It is easy to show that Axiom 3 leads to the conclusion that $\mathbf{a} \succ \theta$ and $\mathbf{b} \succ \mathbf{c}$ imply $\mathbf{a} * \mathbf{b} \succ \mathbf{a} * \mathbf{c}$ and $\mathbf{b} * \mathbf{a} \succ \mathbf{c} * \mathbf{a}$. We thereby conclude that $\langle A_1 \times A_2 \times A_3/\sim, \succcurlyeq, \oplus, *\rangle$ is an Archimedean ordered ring.

An Archimedean ordered ring is uniquely isomorphic to a subring of $\langle R, \geqslant, +, \cdot\rangle$ (Krantz et al., 1971, p. 58). Hence there exists a function Ψ such that

$$\mathbf{a} \succcurlyeq \mathbf{b} \quad \text{iff} \quad \Psi(\mathbf{a}) \geqslant \Psi(\mathbf{b}),$$
$$\Psi(\mathbf{a} \oplus \mathbf{b}) = \Psi(\mathbf{a}) + \Psi(\mathbf{b}),$$
$$\Psi(\mathbf{a} * \mathbf{b}) = \Psi(\mathbf{a}) \cdot \Psi(\mathbf{b}).$$

By the definitions of $\oplus$ and $*$, we have

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (\mathbf{a}_1, \mathbf{a}_2^{i}, \mathbf{a}_3^{0}) * (\mathbf{a}_1^{i}, \mathbf{a}_2, \mathbf{a}_3^{0}) \oplus (\mathbf{a}_1, \mathbf{a}_2^{0}, \mathbf{a}_3^{i}) * (\mathbf{a}_1^{ii}, \mathbf{a}_2^{0}, \mathbf{a}_3).$$

Therefore, we have the following relation:

$$\Psi(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \Psi(\mathbf{a}_1, \mathbf{a}_2^{i}, \mathbf{a}_3^{0})\, \Psi(\mathbf{a}_1^{i}, \mathbf{a}_2, \mathbf{a}_3^{0}) + \Psi(\mathbf{a}_1, \mathbf{a}_2^{0}, \mathbf{a}_3^{i})\, \Psi(\mathbf{a}_1^{ii}, \mathbf{a}_2^{0}, \mathbf{a}_3).$$

Let us define ω_1 and ω_2, ϕ_2 and ϕ_3 such that

$$\omega_1(a_1) = \Psi(\mathbf{a}_1, \mathbf{a}_2^{i}, \mathbf{a}_3^{0}), \qquad \text{for all} \quad a_1 \in A_1,$$
$$\omega_2(a_1) = \Psi(\mathbf{a}_1, \mathbf{a}_2^{0}, \mathbf{a}_3^{i}), \qquad \text{for all} \quad a_1 \in A_1,$$
$$\phi_2(a_2) = \Psi(\mathbf{a}_1^{i}, \mathbf{a}_2, \mathbf{a}_3^{0}), \qquad \text{for all} \quad a_2 \in A_2,$$
$$\phi_3(a_3) = \Psi(\mathbf{a}_1^{ii}, \mathbf{a}_2^{0}, \mathbf{a}_3), \qquad \text{for all} \quad a_3 \in A_3.$$

Finally, we have

$$a \succcurlyeq b \quad \text{iff} \quad \omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3) \geqslant \omega_1(b_1)\,\phi_2(b_2) + \omega_2(b_1)\,\phi_3(b_3).$$

Step 5. If ω_1', ω_2', ϕ_2', and ϕ_3' are any other functions satisfying the above equation, we have the following relation

$$\omega_1(a_1)\,\phi_2(a_2) + \omega_2(a_1)\,\phi_3(a_3) = \alpha[\omega_1'(a_1)\,\phi_2'(a_2) + \omega_2'(a_1)\,\phi_3'(a_3)]$$

for all $(a_1, a_2, a_3) \in \{a_1\} \times A_2 \times A_3$, where α is a positive constant (see Krantz, Luce, Suppes, and Tversky, 1971, p. 257).

Let $a_3 = a_3^0$, we have $\phi_2(a_2) = \alpha\omega_1'(a_1)\,\phi_2'(a_2)/\omega_1(a_1)$ and

$$\phi_2(a_2^i) = \alpha\omega_1'(a_1)\,\phi_2'(a_2^i)/\omega_1(a_1).$$

It follows that $\phi_2(a_2) = [\phi_2(a_2^i)/\phi_2'(a_2^i)]\,\phi_2'(a_2)$. Similarly, it can be shown that $\phi_3(a_3) = [\phi_3(a_3^i)/\phi_3'(a_3^i)]\,\phi_3'(a_3)$.

We now want to show that $\omega_1(a_1) = [\omega_1(a_1^i)/\omega_1'(a_1^i)]\,\omega_1'(a_1)$. Unrestrictive solvability guarantees that there exist $b_2 \in A_2$ such that

$$(a_1, a_2, a_3^0) \sim (a_1^i, b_2, a_3^0).$$

It implies that $\omega_1(a_1)\,\phi_2(a_2) = \omega_1(a_1^i)\,\phi_2(b_2)$ and $\omega_1'(a_1)\,\phi_2'(a_2) = \omega_1'(a_1^i)\,\phi_2'(b_2)$. But $\omega_1(a_1)\,\phi_2(a_2) = \alpha\omega_1'(a_1)\,\phi_2'(a_2)$, hence $\omega_1(a_1^i)\,\phi_2(b_2) = \alpha\omega_1'(a_1^i)\,\phi_2'(b_2)$. Since $\phi_2(x_2) = [\phi_2(a_2^i)/\phi_2'(a_2^i)]\,\phi_2'(x_2)$ for either $x_2 = a_2$ or $x_2 = b_2$, it follows that $\omega_1(a_1) = \alpha[\phi_2'(a_2^i)/\phi_2(a_2^i)]\,\omega_1'(a_1)$ and $\omega_1(a_1^i) = \alpha[\phi_2'(a_2^i)/\phi_2(a_2^i)]\,\omega_1'(a_1^i)$. Consequently, we have $\omega_1(a_1) = [\omega_1(a_1^i)/\omega_1'(a_1^i)]\,\omega_1'(a_1)$.

By the similar procedure, it can be shown that $\omega_2(a_1) = [\omega_2(a^{ii})/\omega_2'(a^{ii})]\,\omega_2'(a_1)$. Let $\alpha_1 = \omega_1(a_1^i)/\omega_1'(a_1^i)$, $\alpha_2 = \omega_2(a^{ii})/\omega_2'(a^{ii})$, $\beta_2 = \phi_2(a_2^i)/\phi_2'(a_2^i)$, and $\alpha_3 = \phi_3(a_3^i)/\phi_3'(a_3^i)$. Clearly, we have $\alpha_1\beta_2 = \alpha_2\beta_3 = \alpha > 0$.

Therefore,

$$\begin{aligned}
\omega_1'(a_1) &= \alpha_1\omega_1'(a_1),\\
\omega_2(a_1) &= \alpha_2\omega_2'(a_1),\\
\phi_2(a_2) &= \beta_2\phi_2'(a_2),\\
\phi_3(a_3) &= \beta_3\phi_3'(a_3),
\end{aligned}$$

where $\alpha_1\beta_2 = \alpha_2\beta_3 > 0$.

We now consider a special case in which $A_1 = A_1^0 \cup A_1^{00}$. Since A_2 is essential, there are $a_2, b_2 \in A_2$ such that $(a_1, a_2, a_3) \not\sim (a_1, b_2, a_3)$ for some $a_1 \in A_1$ and $a_3 \in A_3$. If $a_1 \in A_1^0$, from definition of A_1^0 we know that $(a_1, a_2, a_3) \sim (a_1, b_2, a_3)$. Therefore, a_1 must be in A_1^{00}. Clearly, there is no $b_3 \in A_3$ such that $(a_1, a_2, a_3) \sim (a_1, b_2, b_3)$. Hence the unrestricted solvability is not satisfied, and $\succcurlyeq$ on $A_1 \times A_2 \times A_3$ is not a nonsimple distributed model.

Conclusions

The weighting condition of Definition 9 provides a crucial test of the nonsimple distributive model against other nonsimple models. However, it cannot discriminate nonsimple models from the simple models. The weighting condition is also a necessary condition for the simple models presented by Krantz *et al.* (1971). It is the sign dependence which can diagnose the nonsimple models from the simple models. The nonsimple models satisfy neither joint independence nor proper sign dependence for each pair of factors. The solvability axiom of the nonsimple model is weaker than the solvability axiom of the simple models. For any $a_1 \in A_1$, $a_2, b_2 \in A_2$, $a_3, b_3 \in A_3$, the unrestricted solvability axiom of the simple models guarantees that there exist $b_1 \in A_1$ such that $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$. However the solvability axiom of the nonsimple model does not imply the existence of b_1.

The major difference between the nonsimple distributive model and the simple distributive model is on the different multiplicative effects of A_1 with A_2 and A_3. In the simple distributive model, the multiplicative effect of A_1 on A_2 is the same as the multiplicative effect of A_1 on A_3. Therefore, there is only one identity element of A_1. In the nonsimple distributive model, the multiplicative effect of A_1 on A_2 may be different from the multiplicative effect of A_1 on A_3. Therefore we need two identity elements a_1^i and a_1^{ii} of A_1 which in turn restrict the uniqueness character of the real-valued functions of A_2 and A_3. In the distributive model, the real-valued functions of A_2 and A_3 are unique up to the interval scale, whereas in the nonsimple model the functions are unique up to the ratio scale.

The key concept in this paper is the multiple identity elements for one factor. This concept will allow the construction of axiom systems for the more complicated models.

Reference

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. *Foundations of measurement.* Vol. I. New York: Academic Press, 1971.

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