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NON-PLANAR, NON-LINEAR OSCILLATIONS OF A BEAM—I. FORCED MOTIONS†

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Abstract—Large amplitude whirling motions of a simply supported beam constrained to have a fixed length are investigated. Equations of motion taking into account bending in two planes and longitudinal deformations are developed. Using the method of harmonic balance, response curves for certain planar and non-planar steady state, forced motions are obtained. Another approximate scheme is used to study the stability of these motions. Stable regions corresponding to non-planar motions are found, thus confirming the existence of whirling motions. Numerical results are presented and discussed for several specific cases.

1. INTRODUCTION

Many investigations into the non-linear motions of strings and beams have been made. Only relatively recently however has it been pointed out that planar forcing of such structures could occasionally generate nonplanar response. This observation was first made for strings. Considerable theoretical and experimental work has been done by Ames *et al.*[1], [2], [3] on a traveling string undergoing a planar, transverse excitation. Not only was so-called "ballooning" of the string observed, but also some new jump phenomena were noticed. For example, jumps from the ballooning state to a planar one occurred. Further work in this general area was done by Shih[4]. For non-traveling strings, theoretical and experimental studies have been done by, for example, Narasimha[5], Anand[6] and Eller[7].

Allied work on elastic beams has also begun to emerge. Haight and King[8] examined the plane harmonic excitation of the base of a cantilever beam and found, both theoretically and experimentally, that in certain circumstances the response was nonplanar. Somewhat related studies of Rodgers and Warner[9] should also be mentioned. They looked at the dynamic out-of-plane stability of thin curved rods excited by harmonic end forces applied normal to the cross-sections. The present paper is also concerned with non-linear motions of elastic beams, with the major focus being on ballooning or whirling motions. Specifically, the structure treated is a simply supported beam, with two perpendicular axes of symmetry, which is constrained to have a fixed length. Forced motions of this structure are considered, the forcing function being harmonic and planar. In a later paper, it is planned to look at free motions.

Approximate equations of motion are derived using Hamilton's extended principle and are reduced to ordinary differential equations by Galerkin's procedure. Several approximate schemes are then used to obtain analytical and numerical results on response and stability, for planar and nonplanar motions.

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2. EQUATIONS OF MOTION

Although the inputs to the system are transverse, an issue is whether torsional motions are likely to arise. Minorsky[10] in discussing finite-degree-of-freedom systems, pointed out that parametric coupling between modes having very different frequencies was quite small. This is also taken to be true for the infinite-degree-of-freedom system at hand. Based on this, and the fact that in general the frequencies of torsional modes are considerably larger than bending-mode frequencies, it is concluded that torsional motions do not arise. Note that this may not be true for thin-walled, open sections. Then with the usual beam theory assumptions,

$$\bar{u} = u(z, t) \tag{1}$$

$$\bar{v} = v(z, t) \tag{2}$$

$$\bar{w} = w(z, t) + \varepsilon_0 z - xu_{,z} - yv_{,z} \tag{3}$$

where the z-axis is the neutral axis of the beam and the x, y-axes are axes of symmetry for the cross-section, u, v, and w are the displacement components of a point on the neutral axis in the x, y, and z directions, respectively, $\bar{u}$, $\bar{v}$, $\bar{w}$, are the displacement components of an arbitrary point in the cross-section, a comma denotes differentiation w. r. t. the subscript, and ε_0 is any initial strain (constant) that may be present. Using equations (1), (2), and (3), and Green's strain measure (in the z-direction) to account for the non-linearity induced by the fixed length requirement, it can be shown that the only non-zero strain component is:

$$\begin{aligned} \varepsilon_{zz} &= \bar{w}_{,z} + \tfrac{1}{2}[(\bar{u}_{,z})^2 + (\bar{v}_{,z})^2 + (\bar{w}_{,z})^2] \\ &= w_{,z} + \varepsilon_0 - xu_{,zz} - yv_{,zz} + \tfrac{1}{2}[(\bar{u}_{,z})^2 + (\bar{v}_{,z})^2 + (\bar{w}_{,z})^2]. \end{aligned} \tag{4}$$

It is assumed that the longitudinal motions are small, and in the sequel $(w_{,z})^2$ in equation (4) will be deleted.

Taking the beam material to be isotropic, the strain energy and kinetic energy can be calculated from equations (1), (2), (3), and (4). Assuming any damping mechanism to be viscous, the work done can readily be found. Then Hamilton's extended principle yields the equations of motion. Assuming no external forces in the z-direction, one of the equations is, on neglecting longitudinal inertia,

$$[w_{,z} + \varepsilon_0 + \tfrac{1}{2}(u_{,z})^2 + \tfrac{1}{2}(v_{,z})^2]_{,z} = 0$$

Integrating this, using the boundary conditions $w(0) = 0$, $w(L) = 0$, where L is the beam length, and substituting the results into the other two equations of motion can be shown to give, on neglecting Poisson effects,

$$\hat{u}_{,\tau\tau} + \frac{I_{yy}}{AL^2}(\hat{u}_{,ssss} - \beta\pi^2 \hat{u}_{,ss}) + k\hat{u}_{,\tau} - \tfrac{1}{2}\hat{u}_{,ss} \int_0^1 [(\hat{u}_{,s})^2 + (\hat{v}_{,s})^2]\,ds = \frac{L}{EA} F_x(s, \tau) \tag{5}$$

$$\hat{v}_{,\tau\tau} + \frac{I_{yy}}{AL^2}(\gamma\hat{v}_{,ssss} - \beta\pi^2 \hat{v}_{,ss}) + k\hat{v}_{,\tau} - \tfrac{1}{2}\hat{v}_{,ss} \int_0^1 [(\hat{u}_{,s})^2 + (\hat{v}_{,s})^2]\,ds = \frac{L}{EA} F_y(s, \tau) \tag{6}$$

where

$$s=\frac{z}{L},\quad \tau=\sqrt{\frac{E}{\rho}}\frac{t}{L},\quad \hat{u}=\frac{u}{L},\quad \hat{v}=\frac{v}{L}$$

$$F_x(s,\tau)=f_x\left(\frac{z}{L},\sqrt{\frac{E}{\rho}}\frac{t}{L}\right),\quad F_y(s,\tau)=f_y\left(\frac{z}{L},\sqrt{\frac{E}{\rho}}\frac{t}{L}\right)$$

$$\gamma=\frac{I_{xx}}{I_{yy}},\quad k=\frac{cL}{A\sqrt{(E\rho)}},\quad \beta=\frac{\varepsilon_0}{\varepsilon_{cr}}$$

$$I_{xx}=\int_A\int y^2\,\mathrm{d}A,\quad I_{yy}=\int_A\int x^2\,\mathrm{d}A$$

f_x, f_y are components of the external force, $cu_{,t}$ and $cv_{,t}$ are the viscous forces per unit length, t is time, E is Young's modulus, A is cross-sectional area, ρ denotes density, and ε_{cr} is the strain in the first buckling mode, i.e., $\varepsilon_{cr}=I_{yy}\pi^2/AL^2$.

Equations (5) and (6) are non-linear and cannot be treated exactly. It is assumed that, for simply supported ends,

$$\hat{u}=\sum_{m=1}^{\infty}\sin m\pi s\xi_m(\tau) \tag{7}$$

$$\hat{v}=\sum_{n=1}^{\infty}\sin n\pi s\eta_n(\tau) \tag{8}$$

Applying Galerkin's method to equations (5) and (6) then yields:

$$\frac{\mathrm{d}^2\xi_m}{\mathrm{d}\tau^2}+k\frac{\mathrm{d}\xi_m}{\mathrm{d}\tau}+P_{xm}\xi_m+\frac{m^2\pi^4}{4}\left[\sum_{j=1}^{\infty}(\xi_j^2+\eta_j^2)j^2\right]\xi_m=Q_{xm} \tag{9}$$

$$\frac{\mathrm{d}^2\eta_n}{\mathrm{d}\tau^2}+k\frac{\mathrm{d}\eta_n}{\mathrm{d}\tau}+P_{yn}\eta_n+\frac{n^2\pi^4}{4}\left[\sum_{j=1}^{\infty}(\xi_j^2+\eta_j^2)j^2\right]\eta_n=Q_{yn} \tag{10}$$

where

$$P_{xm}=h\alpha^2m^2\pi^4(m^2+\beta),\quad P_{yn}=h\alpha^2n^2\pi^4(\gamma n^2+\beta)$$

$$Q_{xm}=\frac{2L}{AE}\int_0^1 F_x(s,\tau)\sin m\pi s\,\mathrm{d}s$$

$$Q_{yn}=\frac{2L}{AE}\int_0^1 F_y(s,\tau)\sin n\pi s\,\mathrm{d}s$$

$$h=\frac{I_{yy}}{A\mathrm{d}^2},\quad \alpha=\frac{\mathrm{d}}{L}$$

d being the depth of the beam (maximum dimension in the xs-plane).

3. RESPONSE

Information on the steady-state response to harmonic inputs is the primary goal. The complex, multi-mode structure of equations (9) and (10) precludes obtaining such information exactly, and approximate techniques must be employed. Here harmonic balance is used. Moreover, only one mode interactions are treated, since it is felt that they are the ones most likely to arise.

Plane harmonic forcing, that is,

$$Q_{xm} = F_{xm} \cos \omega\tau, \quad Q_{yn} = 0 \tag{11}$$

where F_{xm} is a constant, will be first investigated. It is initially assumed that the beam response is planar and involves only one mode, the m-th, say, that is:

$$\xi_m \gg \xi_i, \quad i \neq m, \quad i = 1, 2, \ldots, \infty$$
$$\xi_m \gg \eta_j, \quad j = 1, 2, \ldots, \infty$$

Using equation (11), equations (9) and (10) then reduce to the Duffing equation:

$$\frac{d^2\xi_m}{d\tau^2} + k\frac{d\xi_m}{d\tau} + P_{xm}\,\xi_m + \frac{m^4\pi^4}{4}\xi_m^3 = F_{xm}\cos\omega\tau \tag{12}$$

Seeking harmonic response, one takes

$$\xi_m = A_{xm}\cos(\omega\tau + \theta_{xm}) \tag{13}$$

where A_{xm} and θ_{xm} are constants. Inserting equation (13) into equation (14), equating to zero the coefficients of $\cos(\omega\tau+\theta_{xm})$ and $\sin(\omega\tau+\theta_{xm})$, and eliminating θ_{xm} gives the frequency-amplitude relation

$$A_{xm}^2\left(P_{xm} - \omega^2 + \frac{3m^4\pi^4}{16}A_{xm}^2\right)^2 + (k\omega A_{xm})^2 = F_{xm}^2 \tag{14}$$

Some numerical results for this equation will be presented later.

Nonplanar response to the same input will now be explored by assuming that in addition to the single mode in the xs-plane (ξ_m), a single mode (n-th say) in the ys-plane also occurs. Then equations (9) and (10) reduce to

$$\frac{d^2\xi_m}{d\tau^2} + P_{xm}\xi_m + \frac{m^2\pi^4}{4}(m^2\xi_m^2 + n^2\eta_n^2)\xi_m = F_{xm}\cos(\omega\tau + \theta_{xm}) \tag{15}$$

$$\frac{d^2\eta_n}{d\tau^2} + P_{yn}\eta_n + \frac{n^2\pi^4}{4}(m^2\xi_m^2 + n^2\eta_n^2)\eta_n = 0 \tag{16}$$

To facilitate discussion of the many possible responses, some comments on notation should be made. By i-th order parametric excitation is meant motions with a frequence i times the driving frequency, (some writer's, for example, Efstathiades and Williams[11], call this i-th order internal resonance). For nonplane motions, the phrase $r-l$ parametric excitations is used. It means that the response frequency in the xs-plane (the plane of forcing) and the ys-plane are r and l times the driving frequency, respectively. The term "$m-n$ mode" is used to designate the spatial modes operating. It means that the m-th mode is active in the xs-plane, whereas, the n-th mode is active in the ys-plane.

Consider 1–1, m–n, mode parametric excitation. For these one may take:

$$\xi_m = A_{xm}\cos\omega\tau \tag{17}$$

$$\eta_n = A_{yn}\cos(\omega\tau + \theta_{yn}) \tag{18}$$

where A_{xm} and A_{ym} do not depend on τ and θ_{yn} is a constant. Substituting equations (17) and (18) into equations (15) and (16) gives, on equating the coefficient of $\cos\omega\tau$ and $\cos(\omega\tau+\theta_{yn})$ to zero,

$$A_{xm}\left[P_{xm}-\omega^2+\frac{3m^4\pi^4}{16}A_{xm}^2+\frac{m^2n^2\pi^4}{16}A_{yn}^2(2+\cos 2\theta_{yn})\right]=F_{xm}\cos\theta_{xm} \tag{19}$$

$$\frac{m^2n^2\pi^4}{16}A_{xm}A_{yn}^2\sin 2\theta_{yn}=F_{xm}\sin\theta_{xm} \tag{20}$$

$$A_{yn}\left[P_{yn}-\omega^2+\frac{3n^4\pi^4}{16}A_{yn}^2+\frac{m^2n^2\pi^4}{16}A_{xm}^2(2+\cos 2\theta_{yn})\right]=0 \tag{21}$$

$$\frac{m^2n^2\pi^4}{16}A_{xm}^2A_{yn}\sin 2\theta_{yn}=0 \tag{22}$$

Inspection of equations (19), (20), (21) and (22) show that two cases arise, namely: (i) $A_{yn}=0$, $\theta_{xm}=i\pi$, $i=0,\ \pm1,\ \pm2\ldots$. The motion is planar with a frequency-amplitude relation given by equation (14). (ii) $\theta_{xm}=i\pi$, $\sin 2\theta_{yn}=0$, which leads to the subcases: (a) $\theta_{yn}=q\pi$, $q=0,\ \pm1,\ldots$. It emerges later that all such motions are unstable, and so they will not be pursued any further. (b) $2\theta_{yn}=\pm(2i+1)\pi$, $i=1,2,\ldots$. Equations (19), (20), (21), and (22) yield the frequency-amplitude relations:

$$P_{yn}-\omega^2+\frac{n^2\pi^4}{16}(3n^2A_{yn}^2+m^2A_{xm}^2)=0 \tag{23}$$

$$P_{xm}-\omega^2+\frac{m^2\pi^4}{16}(3m^2A_{xm}^2+n^2A_{yn}^2)=\pm\frac{F_{xm}}{A_{xm}} \tag{24}$$

The plus and minus signs in equation (24) correspond to motions in the xs-plane that are in phase and out of phase with the driver, respectively. For A_{yn} to exist, it follows from equation (23) that

$$\omega^2\geqslant P_{yn}+\frac{m^2n^2\pi^4}{16}A_{xm}^2 \tag{25}$$

To examine 1–2, m–n parametric excitations, one would take

$$\xi_m=A_{xm}\cos\omega\tau,\quad \eta_n=A_{yo}+A_{yn}\cos(2\omega\tau+\theta_{yn})$$

and proceed as before. For brevity, such an analysis is not pursued here.

4. STABILITY

Letting

$$\xi_m\to\xi_m+\xi_m',\quad \eta_n\to\eta_n+\eta_n'$$

where the primed quantities denote small perturbations, equations (9) and (10) to the first order yield the so-called variational equations

$$\frac{\mathrm{d}^2\xi_m}{\mathrm{d}\tau^2}+k\frac{\mathrm{d}\xi_m'}{\mathrm{d}\tau}+P_{xm}\xi_m'+\frac{m^2\pi^4}{4}\xi_m'\sum_{j=1}^{\infty}j^2(\xi_j^2+\eta_j^2)$$
$$+\frac{m^2\pi^4}{2}\xi_m\sum_{j=1}^{\infty}j^2(\xi_j\xi_j'+\eta_j\eta_j')=0,\quad m=1,2,\ldots \tag{26}$$

$$\frac{\mathrm{d}^2\eta_n'}{\mathrm{d}\tau^2}+k\frac{\mathrm{d}\eta_n'}{\mathrm{d}\tau}+P_{yn}\eta_n'+\frac{n^2\pi^4}{4}\eta_n'\sum_{j=1}^{\infty}j^2(\xi_j^2+\eta_j^2)$$
$$+\frac{n^2\pi^4}{2}\eta_n\sum_{j=1}^{\infty}j^2(\xi_j\xi_j'+\eta_j\eta_j')=0,\quad n=1,2,\ldots \tag{27}$$

For the planar motions given by equation (13), the relations (26) and (27) reduce to the uncoupled, damped Mathieu equations:

$$\frac{\mathrm{d}^2\xi_m'}{\mathrm{d}\tau^2}+k\frac{\mathrm{d}\xi_m'}{\mathrm{d}\tau}+[P_{xm}+\varepsilon_{xm}+\varepsilon_{xm}\cos(2\omega\tau+2\theta_{xm})]\xi_m'=0 \tag{28}$$

$$\frac{\mathrm{d}^2\xi_j'^2}{\mathrm{d}\tau^2}+k\frac{\mathrm{d}\xi_j'}{\mathrm{d}\tau}+[P_{xj}+\varepsilon_{xj}+\varepsilon_{xj}\cos(2\omega\tau+2\theta_{xm})]\xi_j'=0\qquad j\neq m \tag{29}$$

$$\frac{\mathrm{d}^2\eta_j'}{\mathrm{d}\tau^2}+k\frac{\mathrm{d}\eta_j'}{\mathrm{d}\tau}+[P_{yj}+\varepsilon_{yj}+\varepsilon_{yj}\cos(2\omega\tau+2\theta_{xm})]\eta_j'=0 \tag{30}$$

where

$$\varepsilon_{xm}=\frac{3m^4\pi^4}{8}A_{xm}^2,\quad \varepsilon_{xj}=\frac{m^2j^2\pi^4}{8}A_{xm}^2,\quad \varepsilon_{yj}=\frac{m^2j^2\pi^4}{8}A_{xm}^2 \tag{31}$$

Equations (28), (29) and (30) can of course be treated exactly. However, an approximate approach is adopted here, since the results so obtained tie-in better with the approximate response results. Moreover, the scheme is known to give excellent results elsewhere (see References [11], [12] and [13]). From Floquet theory (see 5.6, Bolotin[14]), the solutions to equations (28), (29) and (30) have the form:

$$(\xi_m',\xi_j',\eta_j')=\mathrm{e}^{\lambda\omega\tau}[Q_1(\tau),Q_2(\tau),Q_3(\tau)] \tag{32}$$

where λ is the so-called characteristic exponent, and the Q's are periodic functions. Seeking results for first and second order parametric excitations, they can be approximated by

$$Q_1=a_1\cos\omega\tau+a_3\sin\omega\tau+a_5\cos 2\omega\tau+a_7\sin 2\omega\tau+a_9 \tag{33}$$

$$Q_2=a_2\cos\omega\tau+a_4\sin\omega\tau+a_6\cos 2\omega\tau+a_8\sin 2\omega\tau+a_{10} \tag{34}$$

$$Q_3=a_{11}\cos\omega\tau+a_{12}\sin\omega\tau+a_{13}\cos 2\omega\tau+a_{14}\sin 2\omega\tau+a_{15} \tag{35}$$

where the a's are constants. Since the system is weakly non-linear and since damping is small, physically one would expect the perturbed motions to have slowly varying amplitudes. Hence λ in equation (32) is taken to be small. Substituting equations (32), (33), (34) and (35) into equation (28), retaining only the first order terms in λ, and using the method of harmonic balance, it is found that the equations (homogenous) for the determination of a_1 and a_3 are not coupled with those determining a_5, a_7 and a_9. The vanishing of the determinant of the coefficients in the first set of equations gives:

$$(4\omega^2+k^2\omega^2)\lambda^2+2k\omega(\omega^2+P_{xm}+\varepsilon_{xm})\lambda+(P_{xm}-\omega^2+\tfrac{1}{2}\varepsilon_{xm})(P_{xm}-\omega^2+\tfrac{3}{2}\varepsilon_{xm})+k^2\omega^2=0 \quad (36)$$

Using equations (31), it can be shown that equation (36) is satisfied provided A_{xm}^2 lies outside the region defined by:

$$2(\omega^2-P_{xm})-[(\omega^2-P_{xm})^2-3k^2\omega^2]^{\frac{1}{2}} \leqslant \tfrac{9}{16}m^2\pi^4A_{xm}^2 \leqslant 2(\omega^2-P_{xm}) +[(\omega^2-P_{xm})^2-3k^2\omega^2]^{\frac{1}{2}} \quad (37)$$

Following the same procedure, equations (29) and (30) yield further first order parametric information, namely, the motion is stable provided A_{xm}^2 lies outside the zones defined by:

$$\tfrac{8}{3}(\omega^2-P_{xj})-\tfrac{4}{3}[(\omega^2-P_{xj})^2-3k^2\omega^2]^{\frac{1}{2}} \leqslant \frac{m^2j^2\pi^4}{4}A_{xm}^2 \leqslant \tfrac{8}{3}(\omega^2-P_{xj})+\tfrac{4}{3}[(\omega^2-P_{xj})^2-3k^2\omega^2]^{\frac{1}{2}}, \quad j \neq m, \quad j=1,2,\ldots \quad (38)$$

$$\tfrac{8}{3}(\omega^2-P_{yj})-\tfrac{4}{3}[(\omega^2-P_{yj})^2-3k^2\omega^2]^{\frac{1}{2}} \leqslant \frac{m^2j^2\pi^4}{4}A_{xm}^2 \leqslant \tfrac{8}{3}(\omega^2-P_{yj})+\tfrac{4}{3}[(\omega^2-P_{yj})^2-3k^2\omega^2]^{\frac{1}{2}}, \quad j=1,2,\ldots \quad (39)$$

The equations involving a_5, a_7 and a_9, stemming from equation (28) yield

$$b_0\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0 \quad (40)$$

where

$$\begin{aligned} b_0 &= k\omega(k^2\omega^2+16\omega^4)\\ b_1 &= 8k^2\omega^4+(P_{xm}+\varepsilon_{xm})(16\omega^4+3k^2\omega^2)\\ b_2 &= 3k\omega(P_{xm}+\varepsilon_{xm})^2+4\omega^3k^3-\tfrac{1}{2}k\omega\,\varepsilon_{xm}^2+16k\omega^5\\ b_3 &= \frac{\varepsilon_{xm}^2}{2}(4\omega^2-P_{xm}-\varepsilon_{xm})+4k^2\omega^2(P_{xm}+\varepsilon_{xm})+(4\omega^2-P_{xm}-\varepsilon_{xm})(P_{xm}+\varepsilon_{xm}) \end{aligned}$$

Applying the Routh–Hurwitz criterion, the roots of equation (40) have negative real parts, and so correspond to stable motions, provided all the b's are positive, and $b_1b_2 > b_0b_3$. It can readily be shown that the only condition that restricts A_{xm} is the satisfaction of $b_3 > 0$. For stable motions, it requires that A_{xm}^2 lie outside the region defined by the positive roots of

$$\left(\frac{3m^4\pi^4}{8}A_{xm}^2\right)^3+(5P_{xm}-12\omega^2)\left(\frac{3m^4\pi^4}{8}A_{xm}^2\right)^2 +[2(4\omega^2-P_{xm})(4\omega^2-3P_{xm})+8k^2\omega^2](\tfrac{3}{8}m^4\pi^4A_{xm}^2) +2P_{xm}[(4\omega^2-P_{xm})^2+4k^2\omega^2]=0 \quad (41)$$

So far only equation (28) has been treated. Other second order parametric stability zones are obtained in a similar way from equations (29) and (30). The results are, the motions are stable provided A_{xm}^2 lies outside the regions defined by

$$\left(\frac{m^2j^2\pi^4}{8}A_{xm}^2\right)^3+(5P_{xj}-12\omega^2)\left(\frac{\pi^4m^2j^2}{8}A_{xm}^2\right)^2+[2(4\omega^2-P_{xj})(4\omega^2-3P_{xj})+8k^2\omega^2]\left(\frac{\pi^4m^2j^2}{8}A_{xm}^2\right)+2P_{xj}[(4\omega^2-P_{xj})^2+4k^2\omega^2]=0 \quad (42)$$

and a similar equation, with P_{xj} replaced by P_{yj}.

The stability of the nonplanar motions will now be investigated. Substituting equations (17) and (18) into equations (26) and (27) gives the Hill equations

$$\frac{d^2\xi_m'}{d\tau^2}+\left(P_{xm}+\frac{3m^4\pi^4}{4}\xi_m^2+\frac{m^2n^2\pi^4}{4}\eta_n^2\right)\xi_m'+\frac{m^2n^2\pi^4}{2}\xi_m\eta_n\eta_n'=0 \quad (43)$$

$$\frac{d^2\eta_n'}{d\tau^2}+\left(P_{yn}+\frac{m^2n^2\pi^4}{4}\xi_m^2+\eta_n^2\frac{3n^4\pi^4}{4}\right)\eta_n'+\frac{m^2n^2\pi^4}{2}\xi_m\eta_n\xi_m'=0 \quad (44)$$

$$\frac{d^2\xi_j'}{d\tau^2}+\left[P_{xj}+\frac{j^2\pi^4}{4}(m^2\xi_m^2+n^2\eta_n^2)\right]\xi_j'=0,\quad j\neq m \quad (45)$$

$$\frac{d^2\eta_j'}{d\tau^2}+\left[P_{yj}+\frac{j^2\pi^4}{4}(m^2\xi_m^2+n^2\eta_n^2)\right]\eta_j'=0,\quad j\neq n \quad (46)$$

where ξ_m and η_n as are given by equations (17) and (18), respectively.

Seeking only first order parametric stability zones, and working first with equations (43) and (44), it is assumed that

$$\xi_m'=(a_{16}\cos\omega\tau+a_{17}\sin\omega\tau)\,e^{\lambda\omega\tau} \quad (47)$$

$$\eta_n'=(a_{18}\cos\omega\tau+a_{19}\sin\omega\tau)\,e^{\lambda\omega\tau} \quad (48)$$

where the a's are constants. Proceeding as before, noting that an approximate scheme is now a necessity, homogenous equations in a_{16}, a_{17}, a_{18} and a_{19} are obtained. Their satisfaction requires, for $2\theta_{yn}=\pm(2l+1)\pi$, $l=0,1,2\ldots$

$$(2\omega^2\lambda)^4+P(2\omega^2\lambda)^2+q=0 \quad (49)$$

where

$$P=d_1d_3+d_2d_4-2g_1^2$$

$$q=(d_1d_4-g_1^2)(d_2d_3-g_1^2)$$

$$g_1=\mp\frac{m^2n^2\pi^4}{8}A_{xm}A_{yn}$$

$$d_1=P_{xm}-\omega^2+\frac{m^2}{n^2}(\omega^2-P_{yn})+\frac{m^4\pi^4}{8}A_{xm}^2$$

$$d_2=2(\omega^2-P_{yn})-\frac{m^2n^2\pi^4}{8}A_{xm}^2$$

$$d_3=P_{xm}-\omega^2+\frac{m^2}{3n^2}(\omega^2-P_{yn})+\tfrac{13}{24}m^4\pi^4A_{xm}^2$$

$$d_4=\frac{m^2n^2\pi^4}{8}A_{xm}^2$$

Stability requires that all the roots of equation (49) be pure imaginary, which requires $p > 0$, $q > 0$ and $p^2 > 4q$. The requirement $q > 0$ can be written in a much simpler form, but since it does not lead to any new stability zones, the analysis will not be presented here. It appears that no simple algebraic relation emerges from the conditions $p > 0$, $p^2 > 4q$ and each case requires individual numerical treatment. However, for $\gamma = 1$, $m = n$, explicit information can be found, namely, the motions are stable provided A_{xm}^2 lies in the regions defined by:

$$0{\cdot}323(\omega^2 - P_{xm}) \leqslant \frac{m^4\pi^4}{4} A_{xm}^2 \leqslant \tfrac{1}{3}(\omega^2 - P_{xm}) \tag{50}$$

$$4(\omega^2 - P_{xm}) \geqslant \frac{m^4\pi^4}{4} A_{xm}^2 \geqslant \omega^2 - P_{xm} \tag{51}$$

Equations (45) and (46) must still be examined. They are uncoupled Mathieu equations, and can be treated in the same way as equations (28), (29) and (30). However, they will not be pursued any further, since it turns out that no new stability regions arise. The counterparts in the $A_{ym} - \omega$ plane to the above zones can be obtained from either equation (23) or (24).

5. NUMERICAL RESULTS

Some sample results on the response and stability of planar motions are shown in Figs. 1 and 2. Various amplitudes are plotted against ω/ω_0, where ω_0 is the fundamental bending frequency in the xs-plane. Two values of the forcing function are used here (and throughout), namely $F_{x1} = 10^{-6}$ and 5×10^{-6}. Occasionally curves corresponding to the latter value may be off scale. Also, α is taken as 0·0005 throughout. The values used for the other parameters are given on the figure legends.

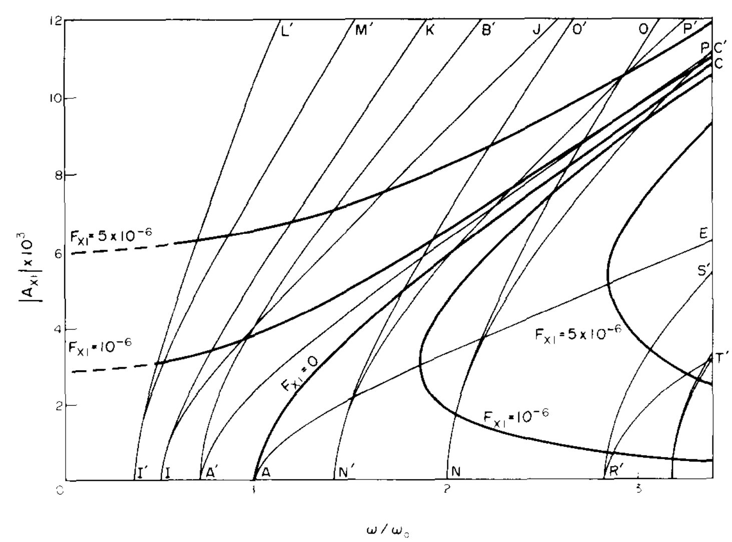

Fig. 1. Response curves and stability zones for planar motions. $m = 1{\cdot}0$, $\gamma = 0{\cdot}5$, $k = 0{\cdot}0$, $\beta = 0{\cdot}0$.

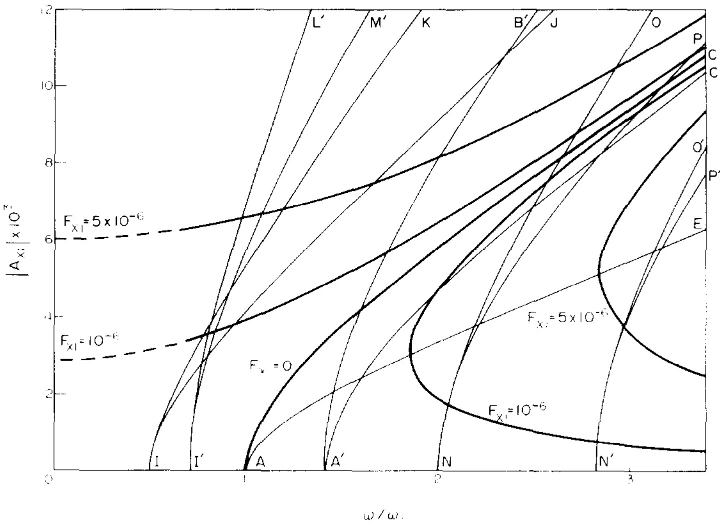

Fig. 2. Response curves and stability zones for planar motions. $m = 1{\cdot}0$, $\gamma = 2{\cdot}0$, $k = 0{\cdot}0$, $\beta = 0{\cdot}0$.

Unprimed and primed letters designate planar and nonplanar instability zones, respectively, with points inside the zone corresponding to the instability. Region AEC comes from expression (37) with $m = 1$. Its lower boundary AE is the locus of vertical tangents on the out-of-phase (w.r.t. the driver) portions of the response curves. On such portions, as the frequency decreases, the amplitude slowly increases until the point of vertical tangency is reached. Then, in what may be called a planar amplitude jump, the motion changes abruptly to an in-phase one with a larger amplitude. Region $A'B'C'$ stems from expression (39) with $m = j = 1$. It represents possible 1–1, 1–1 parametric excitation. To see whether these motions are steady state or not, the two-mode stability analysis described by equation (49) must be investigated. As will be shown later, the motions do indeed turn out to be stable, and so whirling motions can occur. Another interesting item is that the planar jumps corresponding to AEC end up in $A'B'C'$ (at least for the cases considered in Figs. 1 and 2), so that nonplanar motions result. For some values of γ $(= Ixx/Iyy)$, this may not occur. The region IJK comes from expression (41) with $m = 1$. It represents a planar second order parametric excitation of the first mode. Expression (42), with $P_{xj} \rightarrow P_{yj}$ and $m = j = 1$, gives the region $I'L'M'$. It represents possible 1–2, 1–1 mode parametric excitation. The zone NOP, which describes possible second order parametric excitation of the second mode is obtained from expression (38) with $m = 1$, $j = 2$. The regions $N'O'P'$, $R'S'T'$ and $U'V'W'$ represent possible 1–2, 1–2; 1–1, 1–2; 1–2, 1–3 parametric excitations, respectively.

It can be seen from the figures that in all cases the larger the forcing amplitude the more likely whirling motions occur. As the frequency is increased, the types of whirling motion that are encountered (for the cases treated) are 1–2, 1–1; 1–1, 1–1; 1–2, 1–2; 1–1, 1–2; 1–2, 1–3. Some effects of changing γ, should be noted. The dimensionless response curves and the planar instability regions remain unchanged. However, the nonplanar instability zones do depend on γ. As it increases, all the zones shift to the right and the amount of response curve lying in the zones decreases, i.e., as γ increases higher frequencies are required to excite the nonplanar motions. It should also be noted that some of the stability zones overlap. Presumably the motion corresponding to these regions is a complex one, with the possibility, for example, of beating between plane-plane, plane-nonplane, and nonplane-nonplane excitations.

The effects of β (preload) and k (damping) on the results have not been recorded here other than to note the following: For negative values of β (compression), the instability zones shift towards lower frequencies. For nonzero values of k, the zones become narrower and their tips are raised off the frequency axis (making parametric instability impossible below certain amplitude levels). A more extensive discussion, and result for the second mode response can be found in a thesis of the first author[15].

Some typical results for 1–1, 1–1 nonplanar motions are shown in Figs. 3, 4, 5, and 6. In addition to the previous notation, double primed letters designate regions in which the nonplanar motions given by equations (23) and (24) are stable. These stable regions are denoted by hatching (some zones are so narrow as to be not apparent in the scale of the sketch). Planar response curves are included for comparison purposes. They are plotted in solid lines, with the dashed lines standing for nonplanar responses (the out-of-phase portions are sometimes off scale for $F_{x1} = 5 \times 10^{-6}$).

In connection with equation (49), setting $q = 0$, gives three curves $A''B''$, $F''G''$ and $H''I''$. The curve $A''B''$ coincides with $A'B'$. The curves $F''G''$ and $H''I''$ give the nonplanar free vibration response and the locus of vertical tangents to the nonplanar forced response curve, respectively. The requirement $p^2 > 4_q$ gives the curve $H''J''$ in Fig. 3 and $Q''J''$ in Fig. 5. Using equation (23), the various stability boundaries in the $A_{x1} - \omega$ plane can be transferred to the $A_{y1} - \omega$ plane. One result is that $A''B''$ maps onto the frequency axis in the $A_{y1} - \omega$ plane. A general requirement for non-planar motions to exist is

$$\frac{n^4\pi^4}{11} A_{yn}^2 \geqslant \tfrac{4}{3}(\omega^2 - P_{yn})$$

The equality sign in this last expression gives the curve $R''S''$, which maps onto the frequency axis in the $A_{x1} - \omega$ plane.

Those portions of the plane in which only nonplanar or planar, out-of-phase, 1–1, 1–1 motions exist will first be discussed. Overlap zones in which both planar and nonplanar motions can occur are possible. They will be discussed later. In Figs. 3 and 4, which correspond to $\gamma = 1/2$, possible stable motions are represented by the narrow strip bounded by $H''I''$ and $H''J''$. In that zone, as the frequency decreases, A_{x1} increases and A_{y1} decreases, until the point of vertical tangency is reached. Then A_{x1} and A_{y1} jump to higher and lower values, respectively. Thus the particle motion changes from a broad eclipse to a narrow one. For all values of γ less than one, A_{y1} is considerably larger than A_{x1}.

Figs. 5 and 6, in which $\gamma = 2$, show that the situation is different for $\gamma > 1$. A stable region $A''Q''U''$ exists for small values of the forcing function. It is interesting to note that the locus of vertical tangents $H''I''$ falls outside the stability region, so that no nonplanar jumps occur. Instead, as ω decreases, the motion changes to a planar one and then planar jumps occur. For moderate to large forcing functions, only planar out-of-phase motions are stable. When they reach a point of vertical tangency, a jump to a nonplanar state occurs.

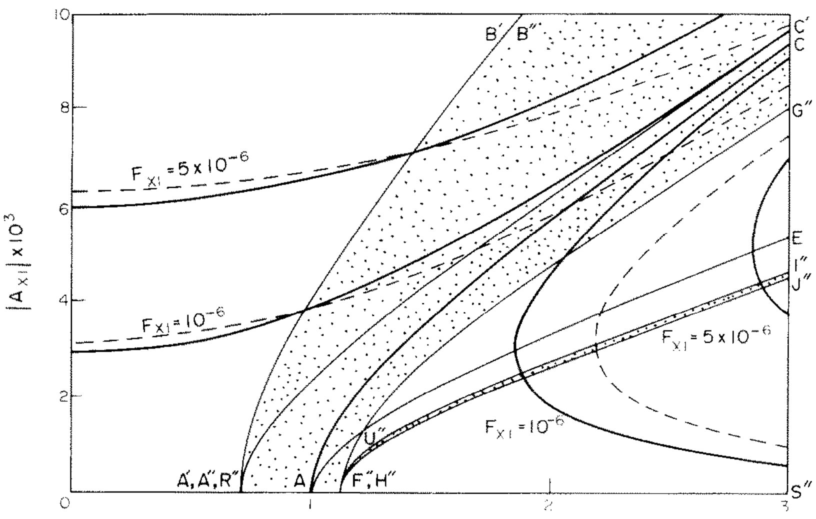

Fig. 3. Response curves and stability zones in the $A_{x1}-\omega$ plane for planar first mode and 1–1, 1 1 motions. $\gamma = 0{\cdot}5$, $k = 0{\cdot}0$, $\beta = 0{\cdot}0$.

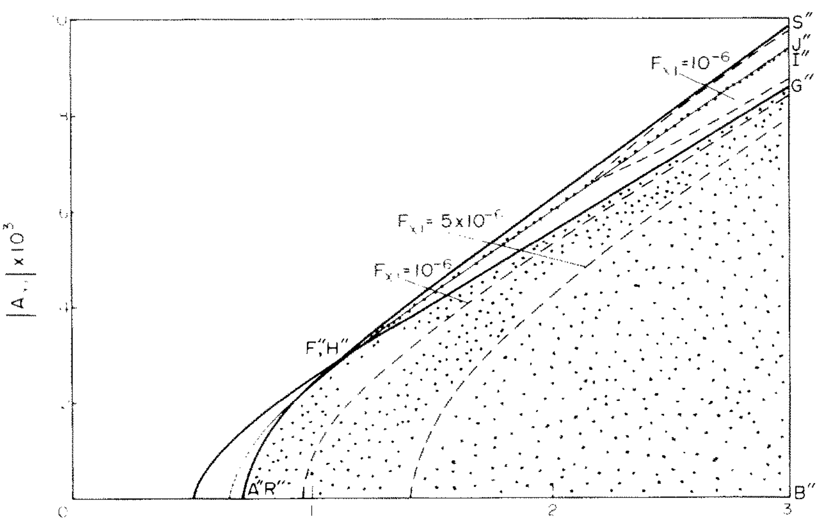

Fig. 4. Response curves and stability zones in the $A_{y1}-\omega$ plane for 1 1, 1–1 motions. $\gamma = 0{\cdot}5$, $k = 0{\cdot}0$, $\beta = 0{\cdot}0$.

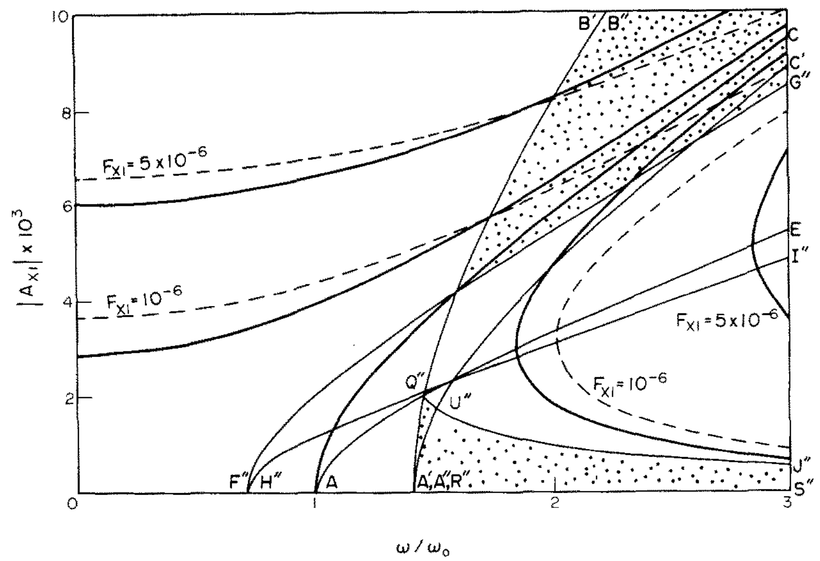

Fig. 5. Response curves and stability zones in the $A_{x1}-\omega$ plane for planar first mode and 1–1, 1–1 motions. $\gamma = 2{\cdot}0$, $k = 0{\cdot}0$, $\beta = 0{\cdot}0$.

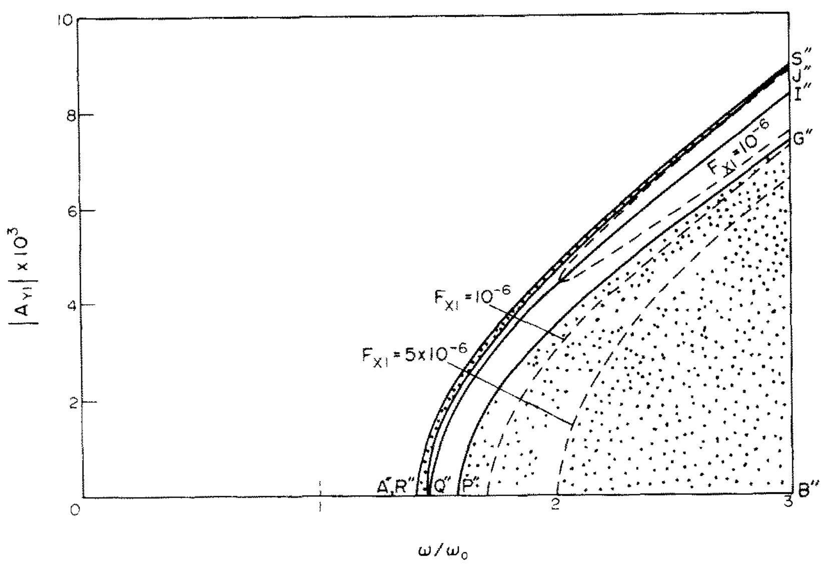

Fig. 6. Response curves and stability zones in the $A_{y1}-\omega$ plane for 1–1, 1–1 motions. $\gamma = 2{\cdot}0$, $k = 0{\cdot}0$, $\beta = 0{\cdot}0$.

Those portions of the plane in which only nonplanar or planar, in-phase, 1–1, 1–1 motions exist will now be considered. They are designated by $A''B''F''G''$ in Figs. 3 and 4, and $P''B''G''$ in Figs. 5 and 6. For small values of ω, the motions are planar and remain so until $A''B''$ is encountered. Then they smoothly become nonplanar. As ω increases further, the particle motion changes from a narrow ellipse to a broad one. Circular motion is never reached for $\gamma \geqslant 1$. However, for $\gamma < 1$, ellipses with A_{x1} smaller than A_{y1} may be obtained. This can be seen, if attention is focused on the free-vibration curve $H''G''$, which can be thought of as giving results for very small values of the applied force. It is interesting to note that for $\gamma > 1$, nonplanar motions can occur only for $\omega > \omega_0$.

Overlapping stability regions will now be discussed. They are denoted by $H''I''J''$ $A'ACC'$, $AU''F''$ in Fig. 3 and $A''U''J''S''$ in Fig. 5. The type of motion that arises depends on the initial conditions.

Consider first in-phase motions. They correspond to the zone $A'ACC'$. If the motion is initially planar, it remains so as ω is decreased. Then either $A'C'$ is or is not met. If it is not, planar motions always occur. If $A'C'$ is met, then a jump to nonplanar motion occurs, with A_{x1} decreasing. The region $AU''F''$ is a zone where nonplanar in-phase and planar out-of-phase stability zones overlap. Such zones occur only for $\gamma < 1$. The main features of the nonplanar motion have been described above. The figure would seem to indicate that initially planar motions can jump to either a planar or a nonplanar state. However the result of the jump must be a planar motion, since in the zone in question, perturbations into the ys-plane cannot grow.

The zones $H''I''J''$ and $A''U''J''S''$ correspond to out-of-phase motions. Nonplanar motions in $H''I''J''$ have been discussed before, so that only the passage of planar motions through the zone will be treated. As ω decreases, A_{x1} increases until $H''J''$ is met. On entering the region, the motions remain planar since perturbations out of the plane cannot grow. Plane motion continues until AE is encountered. Then a jump occurs which always leads to a nonplanar motion. Consider now initially planar motions in $A''U''J''S''$. As ω decreases, the amplitude increases until $A'C'$ is met. Since inside $A'B'C'$, planar motions are unstable and nonplanar motions are stable, a jump to a nonplanar state occurs. The situation for initially nonplanar motions has been discussed before in connection with the region $A''U''Q''$.

Some regions are instability zones for both planar and nonplanar motions. They are $U''G''E''$ in Fig. 3 and $AQ''EG''P''$ in Fig. 5. Presumably this region corresponds to beating motions or to steady state motions with a more complex modal structure.

It should be noted in finishing that a brief discussion of 1–1, 1–2 motions can be found in [15].

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Zusammenfassung—Die Wirbelbewegungen mit grossen Amplituden eines einfach gleagerten Trägers konstanter Länge werden untersucht. Die Bewegungsgleichunge, die Biegung in zwei Ebenen und Längsdeformationen berücksichtigen, werden aufgestellt. Die Verhaltenskurven für bestimmte ebene und nichtebene, stationäre, erzwungene Bewegungen werden unter Verwendung der Methode des harmonischen Ausgleichs hergeleitet. Ein weiteres Näherungsverfahren wird benutzt, um die Stabilität dieser Bewegungen zu untersuchen. Stabilitätsbereiche, die den nichtebenen Bewegungen zugehören, werden gefunden und es wird damit das Vorhandesein von Wirbelbewegungen bestätigt. Zahlenwerte fur einige besondere Fälle werden angegegeben und besprochen.